The Influences of Fringing Fields of an Electrostatic Sector Analyzer in Fifth Order Aberration Coefficients

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Ion trajectories in the extended fringing fields of an electrostatic sector analyzer have been calculated algebraically. The results of the calculations are represented by the transfer matrices for the entrance and exit fringing fields, respectively. The matrix coefficients yield a shift and bend of ion trajectories on a transformation plane at field boundaries of the sector. The effects of the extended fringing fields on image aberration can be estimated from the matrix coefficients. Large effects from the fifth order matrix coefficients have been found for the fringing fields of an electrostatic sector analyzer. The image aberration arising from the fifth order coefficients is one of the problems in obtaining high resolution with a double focusing mass spectrometer.

1. Introduction

The focusing properties and image aberration of a mass spectrometer are conveniently expressed by a transfer matrix.¹⁾ The coefficients of the matrix can be obtained with multiplying transfer matrices for respective ion-optical components in the spectrometer. The influences of an extended fringing field of the components are also expressed as a transfer matrix which yield a shift and bend of ion trajectories at boundaries of the components. The second order matrix coefficients for extended fringing fields of an electrostatic sector analyzer have been calculated by Wollnik.²⁾ The third order coefficients for the fringing fields of a toroidal condenser have been obtained by Matsuda.³⁾

In order to estimate image aberration of a mass spectrometer, second and higher order trajectory calculations are important. In the transfer matrix for the fringing field of electrostatic sector, the magnitude of the terms is not always equal to the order of a power series expansions. The terms having second order quantity have been calculated in the third order coefficients.³⁾ Second order quantity of a fifth order term for the fringing field of a parallel condenser has been evaluated with numerical calculations by Ishihara.⁴⁾ For the fourth and higher order coefficients, the estimation of the terms with such large quantity is therefore important. In this paper, the higher order coefficients for the fringing fields of an electrostatic sector analyzer have been algebraically calculated up to fifth order using the theory of the extended fringing field.

2. Trajectory Calculations

2.1 Field distribution in fringing field

The effects of the fifth order matrix coefficients in a toroidal condenser are exactly

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Fig. 1. The rectangular coordinate system (ξ, η, ζ) describing the ion trajectories in the entrance fringing field of an electrostatic sector analyzer. The projections of the ideal and real trajectory are shown in the ξ - η plane. The ζ axis is perpendicular to the plane.

Fig. 2. A distribution function, $E(\eta)$, expressing the real electrostatic field in the fringing field.

the same as those in a cylindrical one. Hence the cylindrical sector will be adopted the following calculations. The way of the trajectory calculations in the extended fringing field follows ref. 3. To describe the extended fringing field at an entrance side of an electrostatic sector, we use a cartesian coordinate system, $(R_0\xi, R_0\eta, R_0\zeta)$ as shown in Fig. 1, where R_0 is the radius of the central trajectory (optic axis) in the sector analyzer. The variation of a field strength of the real electric field in the fringe can be expressed using a distribution function, $E(\eta)$, which is the field strength along the optic axis. As shown in Fig. 2, the value of the function $E(\eta)$ vanishes at $\eta=\eta_a$ in field free region, and gradually increases with η , and reaches to a constant value E_0 at sufficiently inside the

sector at $\eta = \eta_b$. We choose the origin of the coordinate system using the relation:

$$\int_{\eta_a}^{\eta_b} E(\eta) d\eta = E_0 \eta_b. \quad (1)$$

The electric potential, $\phi(\xi, \eta, \zeta)$, in the fringing region is obtained by solving the Laplace equation, $\nabla^2 \phi = 0$. To describe the effects of the fifth order coefficients, the potential is expressed by a power series expansion around η -axis up to seventh order terms as:

$$\begin{aligned} \frac{1}{R_0} \phi(\xi, \eta, \zeta) = & \left[\int_{\eta_a}^{\eta_b} \int_{\eta_a}^{\eta} E(\eta) d\eta d\eta - \eta \int_{\eta_a}^{\eta} E(\eta) d\eta - I_{1a} E(\eta) \right] - E(\eta) \xi + \left[\frac{1}{2} E(\eta) \right. \\ & + \frac{1}{2} \eta E'(\eta) + \frac{1}{2} I_{1a} E''(\eta) \left. \right] \xi^2 + \frac{1}{6} E''(\eta) \xi^3 + \left[-\frac{1}{8} E''(\eta) \right. \\ & - \frac{1}{24} \eta E'''(\eta) - \frac{1}{24} I_{1a} E'''(\eta) \left. \right] \xi^4 - \frac{1}{120} E'''(\eta) \xi^5 \\ & + \left[\frac{1}{144} E'''(\eta) + \frac{1}{720} \eta E''''(\eta) + \frac{1}{720} I_{1a} E''''(\eta) \right] \xi^6 \\ & + \frac{1}{5040} E''''(\eta) \xi^7, \end{aligned} \quad (2)$$

where I_{1a} is the definite integral of the distribution function from η_a to η_b , given in Eq. (14) of ref. 3 as:

$$I_{1a} = \frac{1}{E_0} \int_{\eta_a}^{\eta_b} \int_{\eta_a}^{\eta} E(\eta) d\eta d\eta - \frac{1}{2} \eta_b^2. \quad (3)$$

The components of the real electric field vector, E_ξ , E_η , E_ζ , can be obtained using the relation $\mathbf{E} = -\text{grad } \phi$, where ϕ is given in Eq. (2):

$$\begin{aligned} E_\xi(\xi, \eta, \zeta) = & E(\eta) + \left[-E(\eta) - \eta E'(\eta) - I_{1a} E''(\eta) \right] \xi - \frac{1}{2} E''(\eta) \xi^2 + \left[\frac{1}{2} E''(\eta) \right. \\ & + \frac{1}{6} \eta E'''(\eta) + \frac{1}{6} I_{1a} E'''(\eta) \left. \right] \xi^3 + \frac{1}{24} E'''(\eta) \xi^4 \\ & + \left[-\frac{1}{24} E'''(\eta) - \frac{1}{120} \eta E''''(\eta) - \frac{1}{120} I_{1a} E''''(\eta) \right] \xi^5 \\ & - \frac{1}{720} E''''(\eta) \xi^6, \end{aligned} \quad (4)$$

$$\begin{aligned} E_\eta(\xi, \eta, \zeta) = & \left[\eta E(\eta) + I_{1a} E'(\eta) \right] + E'(\eta) \xi + \left[-E'(\eta) - \frac{1}{2} \eta E''(\eta) \right. \\ & - \frac{1}{2} I_{1a} E''(\eta) \left. \right] \xi^2 - \frac{1}{6} E'''(\eta) \xi^3 + \left[\frac{1}{6} E'''(\eta) + \frac{1}{24} \eta E''''(\eta) \right. \\ & + \frac{1}{24} I_{1a} E''''(\eta) \left. \right] \xi^4 + \frac{1}{120} E''''(\eta) \xi^5, \end{aligned} \quad (5)$$

$$E_\zeta(\xi, \eta, \zeta) = 0. \quad (6)$$

Note that the component E_ζ is zero, and the other components E_ξ and E_η are independent of the coordinate ζ , because the cylindrical sector is considered.

2.2 Trajectory equations in the fringing field

The vector of the particle velocity, $\mathbf{v}$, inclines to the optic axis with the angles, α, β shown in Fig. 1. The time derivatives, $d\xi/dt$, $d\eta/dt$ satisfy the following relationships:

$$R_0 \frac{d\xi}{dt} = v \cos \beta \sin \alpha, \quad (7)$$

$$R_0 \frac{d\eta}{dt} = v \cos \beta \cos \alpha. \quad (8)$$

The trajectories of a charged particle in the fringing field can be obtained by solving the equations of motion. The equation of motion for ξ -direction is:

$$mR_0 \frac{d^2\xi}{dt^2} = eE_\xi(\xi, \eta, \zeta). \quad (9)$$

We integrate Eqs. (7) and (9) once, then we change the integral element from dt to $d\eta$ using the relation Eq. (8) as:

$$\begin{aligned} R_0 \xi &= R_0 \xi_a + \int_{\eta_a}^{\eta} v \cos \beta \sin \alpha d\eta \\ &= R_0 \xi_a + R_0 \int_{\eta_a}^{\eta} \tan \alpha(\eta) d\eta. \end{aligned} \quad (10)$$

$$\begin{aligned} mR_0 \frac{d\xi}{dt} &= mR_0 \left(\frac{d\xi}{dt} \right)_a + e \int_{\eta_a}^{\eta} E_\xi(\xi, \eta, \zeta) dt \\ &= mR_0 \left(\frac{d\xi}{dt} \right)_a + eR_0 \int_{\eta_a}^{\eta} \frac{E_\xi(\xi, \eta, \zeta)}{v(\eta) \cos \beta(\eta) \cos \alpha(\eta)} d\eta, \end{aligned} \quad (11)$$

where the quantities having the subscript “a” denote the value of the quantities at $\eta = \eta_a$. We eliminate the time derivative ($d\xi/dt$) in Eq. (11) using the relation of Eq. (7). Then we obtain the trajectory equations which should be solved to describe ion trajectory $\xi(\eta)$ and inclination angle $\alpha(\eta)$ as functions of η :

$$\xi(\eta) = \xi_a + \int_{\eta_a}^{\eta} \tan \alpha(\eta) d\eta, \quad (12)$$

$$\begin{aligned} \sin \alpha(\eta) &= \frac{v_a \cos \beta_a \sin \alpha_a}{v(\eta) \cos \beta(\eta)} \\ &+ \frac{eR_0}{mv(\eta) \cos \beta(\eta)} \int_{\eta_a}^{\eta} \frac{E_\xi(\xi, \eta, \zeta)}{v(\eta) \cos \beta(\eta) \cos \alpha(\eta)} d\eta. \end{aligned} \quad (13)$$

The initial velocity of an arbitrary ion having mass m and charge e , is v_a at $\phi = 0$. The kinetic energy of the particle at any position is derived from the energy conservation law as:

$$\frac{1}{2} m \{v(\eta)\}^2 = \frac{1}{2} m v_a^2 - e\phi(\xi, \eta, \zeta). \quad (14)$$

Hence the velocity of the ion, $v(\eta)$, can be expressed as:

$$v(\eta) = v_a \sqrt{1 - \frac{2e\phi(\xi, \eta, \zeta)}{m v_a^2}}, \quad (15)$$

where the electric potential $\phi(\xi, \eta, \zeta)$ is given in Eq. (2).

The energy to charge ratio of the ion at $\phi = 0$ is expressed using the relative energy deviation, δ , compared from that of the reference particle, U_0/e_0 , as:

$$\frac{\frac{1}{2} m v_a^2}{e} = \frac{U_0}{e_0} (1 + \delta). \quad (16)$$

where the energy to charge ratio of reference particle can be expressed by radius of the optic axis, R_0 as:

$$\frac{U_0}{e_0} = -\frac{1}{2} R_0 E_0. \quad (17)$$

Using a para-axial approximation, Eqs. (12) and (13) can be solved by successive approximations. Note here the size of the fringing field region, $\eta_b - \eta_a$, is assumed to be a first order small quantity; for instance, the integral $\int E(\eta) d\eta$ is considered to be a first order small quantity, double integral $\iint E(\eta) d\eta d\eta$ is a second order, and the n -th derivative of $E(\eta)$ is n -th order small. To obtain the ion trajectory and inclination angle with introducing Eqs. (15)–(17) into Eq. (13), the results of the successive approximations from Eqs. (12) and (13) up to second order small quantity are as follows:

$$\begin{aligned} \xi(\eta) = & -\frac{1}{E_0} \int_{\eta_a}^{\eta} \int_{\eta_a}^{\eta} E(\eta) d\eta d\eta + \xi_a + \{\eta - \eta_a\} \alpha_a + \frac{1}{2E_0} E(\eta) \xi_a^2 \\ & - \frac{1}{24E_0} E''(\eta) \xi_a^4, \end{aligned} \quad (18)$$

$$\begin{aligned} \alpha(\eta) = & -\frac{1}{E_0} \int_{\eta_a}^{\eta} E(\eta) d\eta + \left[\frac{1}{E_0} \int_{\eta_a}^{\eta} E(\eta) d\eta + \frac{1}{E_0} \int_{\eta_a}^{\eta} \eta E'(\eta) d\eta + \frac{1}{E_0} I_{1a} E'(\eta) \right. \\ & - \frac{1}{E_0^2} E(\eta) \int_{\eta_a}^{\eta} E(\eta) d\eta - \frac{1}{E_0^2} \int_{\eta_a}^{\eta} \{E(\eta)\}^2 d\eta - \frac{1}{E_0^2} \int_{\eta_a}^{\eta} E''(\eta) \\ & \times \left. \left\{ \int_{\eta_a}^{\eta} \int_{\eta_a}^{\eta} E(\eta) d\eta d\eta \right\} d\eta \right] \xi_a + \alpha_a + \frac{1}{E_0} \int_{\eta_a}^{\eta} E(\eta) d\eta \delta + \frac{1}{2E_0} E'(\eta) \xi_a^2 \\ & + \left[\frac{1}{E_0} E(\eta) + \frac{1}{E_0} \int_{\eta_a}^{\eta} \eta E''(\eta) d\eta - \frac{1}{E_0} \eta_a E'(\eta) \right] \xi_a \alpha_a + \left[-\frac{1}{2E_0} E'(\eta) \right. \\ & - \frac{1}{6E_0} \int_{\eta_a}^{\eta} \eta E'''(\eta) d\eta - \frac{1}{6E_0} I_{1a} E'''(\eta) + \frac{1}{2E_0^2} E'(\eta) E(\eta) \\ & + \frac{1}{6E_0^2} E''(\eta) \int_{\eta_a}^{\eta} E(\eta) d\eta + \frac{7}{6E_0^2} \int_{\eta_a}^{\eta} E''(\eta) E(\eta) d\eta + \frac{1}{6E_0^2} \int_{\eta_a}^{\eta} E''''(\eta) \\ & \times \left. \left\{ \int_{\eta_a}^{\eta} \int_{\eta_a}^{\eta} E(\eta) d\eta d\eta \right\} d\eta \right] \xi_a^3 - \frac{1}{2E_0} E'(\eta) \xi_a^2 \delta - \frac{1}{24E_0} E'''(\eta) \xi_a^4 \\ & + \left[-\frac{1}{6E_0} E''(\eta) - \frac{1}{6E_0} \int_{\eta_a}^{\eta} \eta E''''(\eta) d\eta + \frac{1}{6E_0} \eta_a E'''(\eta) \right] \xi_a^3 \alpha_a \\ & + \frac{1}{24E_0} E'''(\eta) \xi_a^4 \delta + \left[\frac{1}{24E_0} E'''(\eta) + \frac{1}{120E_0} \int_{\eta_a}^{\eta} \eta E''''(\eta) d\eta \right. \\ & + \frac{1}{120E_0} I_{1a} E''''(\eta) - \frac{1}{12E_0^2} E''(\eta) E'(\eta) - \frac{1}{24E_0^2} E'''(\eta) E(\eta) \\ & - \frac{1}{120E_0^2} E''''(\eta) \int_{\eta_a}^{\eta} E(\eta) d\eta - \frac{1}{8E_0^2} \int_{\eta_a}^{\eta} \{E''(\eta)\}^2 d\eta \\ & - \frac{2}{15E_0^2} \int_{\eta_a}^{\eta} E''''(\eta) E(\eta) d\eta \\ & \left. - \frac{1}{120E_0^2} \int_{\eta_a}^{\eta} E''''''(\eta) \left\{ \int_{\eta_a}^{\eta} \int_{\eta_a}^{\eta} E(\eta) d\eta d\eta \right\} d\eta \right] \xi_a^5. \end{aligned} \quad (19)$$

2.3 Transformation for the entrance fringing field

To obtain the position and inclination angle at η_b , we put $\eta = \eta_b$ into Eqs. (18) and (19) using the relations $E(\eta_b) = E_0$, $E'(\eta_b) = 0$, $E''(\eta_b) = 0$, $E'''(\eta_b) = 0 \dots$. The results are:

$$\xi(\eta_b) = -\left[I_{1a} + \frac{1}{2} \eta_b^2 \right] + \xi_a + \{\eta_b - \eta_a\} \alpha_a + \frac{1}{2} \xi_a^2, \quad (20)$$

$$\alpha(\eta_b) = -\eta_b + [-2I_{4a} - \eta_b] \xi_a + \alpha_a + \eta_b \delta - \frac{4}{3} I_5 \xi_a^3 - \frac{4}{15} I_9 \xi_a^5. \quad (21)$$

Where I_{1a} , I_{4a} , I_5 , and I_9 are the following definite integrals^{3), 4)} from η_a to η_b given in Eqs. (22)–(25). The other integrals in Eqs. (18) and (19) can be expressed using the four integrals, and the relationships are given in Appendix.

$$I_{1a} = \frac{1}{E_0} \int_{\eta_a}^{\eta_b} \int_{\eta_a}^{\eta_b} E(\eta) d\eta d\eta - \frac{1}{2} \eta_b^2, \quad (22)$$

$$I_{4a} = \frac{1}{E_0^2} \int_{\eta_a}^{\eta_b} \{E(\eta)\}^2 d\eta - \eta_b, \quad (23)$$

$$I_5 = \frac{1}{E_0^2} \int_{\eta_a}^{\eta_b} \{E'(\eta)\}^2 d\eta, \quad (24)$$

$$I_9 = \frac{1}{E_0^2} \int_{\eta_a}^{\eta_b} \{E''(\eta)\}^2 d\eta. \quad (25)$$

In the ordinary ion optics,³⁾ the effects of a fringing field have been expressed by transformations on the ideal field boundary, *i.e.* between a field-free region and an ideal electrostatic field region. We try to construct the transformation of the position and inclination, $(R_0 u_1, \alpha_1) \rightarrow (R_0 u_2, \alpha_2)$ shown in Fig. 1 using the above results in Eqs. (20) and (21) and following relationships.

$$u_1 + \Delta u = \xi_a - \eta_a \tan \alpha_a, \quad (26)$$

$$\alpha_1 = \alpha_a, \quad (27)$$

$$u_2 = \lim_{\eta_b \rightarrow 0} \xi(\eta_b), \quad (28)$$

$$\alpha_2 = \lim_{\eta_b \rightarrow 0} \alpha(\eta_b). \quad (29)$$

Here we obtained the final expression of the transformations for u_2 and α_2 with $\Delta u = I_{1a}$:

$$u_2 = (u|u)u_1 + (u|uu)u_1^2, \quad (30)$$

$$\alpha_2 = (\alpha|u)u_1 + (\alpha|\alpha)\alpha_1 + (\alpha|uuu)u_1^3 + (\alpha|uuuuu)u_1^5, \quad (31)$$

where

$$(u|u) = 1, (u|uu) = \frac{1}{2}, (\alpha|u) = -2I_{4a}, (\alpha|\alpha) = 1, (\alpha|uuu) = -\frac{4}{3}I_5,$$

$$(\alpha|uuuuu) = -\frac{4}{15}I_9. \quad (32)$$

2.4 Transformation for the exit fringing field

The fringing field of the exit side of an electrostatic sector also affects ion trajectories. The effects can also be expressed by transformations of position and inclination angle on an ideal field boundary of the exit side as $(R_0 u_3, \alpha_3) \rightarrow (R_0 u_4, \alpha_4)$. The transformations can be calculated from Eqs. (30)–(32), with replacing u_1 to u_4 , u_2 to u_3 , α_1 to α_4 and α_2 to α_3 as:

$$u_4 = (u|u)u_3 + (u|uu)u_3^2, \quad (33)$$

$$\alpha_4 = (\alpha|u)u_3 + (\alpha|\alpha)\alpha_3 + (\alpha|uuu)u_3^3 + (\alpha|uuuuu)u_3^5, \quad (34)$$

where

Table 1. The Values of the Four Definite Integrals Defined in Eqs. (22)–(25) for Three Electrode Structures^{4), 5)}

b/k	I_{1a}	I_{4a}	I_5	I_9
0.72	0.000159	−0.00969	17.7	42700
0.30	0.000101	−0.00769	22.5	99700
0.20	0.000094	−0.00740	23.9	142000

b/k : The ratio of the aperture width of the Herzog shunt⁶⁾ to the gap distance of the condenser electrodes. The half distance of the gap, k , is assumed to $0.032 R_0$ in this table.

$$\begin{aligned}
 (u|u) &= 1, (u|uu) = -\frac{1}{2}, (\alpha|u) = -2I_{4a}, (\alpha|\alpha) = 1, (\alpha|uuu) = -\frac{4}{3}I_5, \\
 (\alpha|uuuuu) &= -\frac{4}{15}I_9.
 \end{aligned} \tag{35}$$

3. Results and Discussion

We calculated algebraically the ion trajectories in the extended fringing fields of an electrostatic sector analyzer for obtaining the fifth order matrix coefficients. The effects of the fifth order coefficients on image aberration can be estimated using the results for a double focusing mass spectrometer. The results are represented by the transformations with matrix coefficients for the entrance and exit fringing fields given in Eqs. (30)–(35). The obtained coefficients except the fifth order ones agree with the cylindrical case of Matsuda's results³⁾ of the second order quantity. The fifth order coefficients, $(\alpha|uuuuu)$ are algebraically obtained for the first time for the fringing fields of electrostatic sector analyzer.

Computer simulation of the field distribution and ion trajectories in the fringing field of a parallel plate condenser has been performed by Ishihara.⁴⁾ In that study, the electric field distribution has been numerically calculated for an electrode structure, and the value of the integral I_9 has been evaluated. Several ion paths through the field have been traced numerically, and the value of $(\alpha|uuuuu)$ coefficient has been estimated. By comparing both values, they have obtained the relationships:

$$(\alpha|uuuuu) = (-0.25 \pm 0.01)I_9. \tag{36}$$

The value of -0.25 ± 0.01 is good agreement with our theoretical value, $-4/15$, obtained in Eq. (32).

We obtained the fifth order coefficients for the fringing fields of the electrostatic sector analyzer in Eqs. (30)–(35). The magnitude of the terms, $(\alpha|uuuuu) u_1^5$ and $(\alpha|uuuuu) u_3^5$ in these equations is a second order small quantity, since the integral I_9 is a minus third order small quantity with the size of fringing field. The value of definite integrals in Eqs. (22)–(25) can be evaluated, when we know the shape of the distribution function $E(\eta)$. Three examples of the value are given in Table 1,^{4), 5)} where the structure of the electrodes is given in ref. 5. The value of the integral I_9 become large with a reduction of the size of the fringing field using a shield electrode (Herzog shunt).⁶⁾ In a double focusing mass spectrometer, the deviations, u_1 and u_3 , are mainly derived from the spread of ion injection angle, α_0 , at the source slit. The image aberration of a double

focusing mass spectrometer caused from the effects of the fifth order coefficients is therefore derived from the terms of $(\alpha|uuuuu)\alpha_b^5$.

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Keywords

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Appendix

The relationships of the definite integrals from η_a to η_b in Eqs. (18) and (19).

$$\begin{aligned}
 \int_{\eta_a}^{\eta_b} E(\eta) d\eta &= E_0 \eta_b, \\
 \int_{\eta_a}^{\eta_b} \int_{\eta_a}^{\eta} E(\eta) d\eta d\eta &= E_0 \left(I_{1a} + \frac{1}{2} \eta_b^2 \right), \\
 \int_{\eta_a}^{\eta_b} \{E(\eta)\}^2 d\eta &= E_0^2 (I_{4a} + \eta_b), \\
 \int_{\eta_a}^{\eta_b} \eta E'(\eta) d\eta &= 0, \\
 \int_{\eta_a}^{\eta_b} \{E'(\eta)\}^2 d\eta &= E_0^2 I_5, \\
 \int_{\eta_a}^{\eta_b} \eta E''(\eta) d\eta &= -E_0, \\
 \int_{\eta_a}^{\eta_b} E''(\eta) E(\eta) d\eta &= -E_0^2 I_5, \\
 \int_{\eta_a}^{\eta_b} \{E''(\eta)\}^2 d\eta &= E_0^2 I_9, \\
 \int_{\eta_a}^{\eta_b} E''(\eta) \left\{ \int_{\eta_a}^{\eta} \int_{\eta_a}^{\eta} E(\eta) d\eta d\eta \right\} d\eta &= E_0^2 I_{4a}, \\
 \int_{\eta_a}^{\eta_b} \eta E'''(\eta) d\eta &= 0, \\
 \int_{\eta_a}^{\eta_b} \eta E''''(\eta) d\eta &= 0,
 \end{aligned}$$

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$$\int_{\eta_*}^{\eta_*} \eta E''''''(\eta) d\eta = 0 ,$$

$$\int_{\eta_*}^{\eta_*} E''''(\eta) E(\eta) d\eta = E_0^2 I_9 ,$$

$$\int_{\eta_*}^{\eta_*} E''''(\eta) \left\{ \int_{\eta_*}^{\eta} \int_{\eta_*}^{\eta} E(\eta) d\eta d\eta \right\} d\eta = -E_0^2 I_6 ,$$

$$\int_{\eta_*}^{\eta_*} E''''''(\eta) \left\{ \int_{\eta_*}^{\eta} \int_{\eta_*}^{\eta} E(\eta) d\eta d\eta \right\} d\eta = E_0^2 I_9 .$$