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ORIGINAL PAPERS**Momentum Free Focusing System Using
AG-Electromagnetic Field**

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A focusing system achromatic for the momentum deviations has been investigated. The system consists of a pair of alternate gradient electro-magnetic fields whose n -values, n_e and n_m , were designed ± 5 , respectively. Experiments were carried out using electrons with energies from 5 to 11 keV for various combinations of the electric field with the magnetic field. The momentum free condition was obtained when and only when the ratio of the electric force to the centrifugal force was equal to -1 .

1. Introduction

As a guide field of various types of focusing devices such as a mass spectrometer, an accelerator or a beam transport system, a combining use of an electric field with a magnetic field has many interesting and useful focusing properties that can not be expected by the use of the magnetic field alone.

A mass spectrometer employing a uniform magnetic field with an electric field was first investigated by Henneberg¹⁾ and Herzog.²⁾ Afterwards Fischer^{3), 4)} improved their configurations by the use of a radially decreasing magnetic field of a weak focusing type combining with a cylindrical electric field to obtain the double focusing properties, i.e. the radial and vertical focusings, as well as the velocity focusing.

Recently Miyamoto *et al.*^{5), 6)} proposed to superimpose an electric guide field on a

magnetic field concerning the particle accelerators, by which the focusing properties were found to be much improved. By this method an axially symmetric isochronous cyclotron valid for even the relativistic energy range can be realized⁶⁾ and was verified experimentally by the electron model machine.⁷⁾

An achromatic beam transport system using an appropriate combination of the electric and magnetic fields is also possible to construct. One of the most serious conditions that should be imposed on such beam transport systems would be the conservation of the emittance area⁸⁾ during the beam transportation. However, it is often compulsory to carry out the experiment with the beam in the presence of considerable momentum spreads. It is well-known that finite momentum spreads always change into the spatial spreads and result in the loss of particles after passing through the

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transpot system.

In this paper it will be shown that a momentum free system can be also realized in a strong focusing magnetic field⁹⁾ by combining it with an appropriate radial electric field. The advantage of using the strong focusing magnetic field is that we can make the n -value to specify the degree of the field gradient in the radial direction much greater ($n > 1$) than that of the weak focusing magnetic field as used by Fischer, where the n -value is $1 > n > 0$. Consequently, the focusing property can be much enhanced in the strong focusing field. However, to achieve the radial and vertical focusings simultaneously, it requires employing a pair of alternate gradient magnets as is conventional in the AG synchrotron.¹⁰⁾ Then the experiment was carried out using the simplest unit of focusing magnets, i.e. a radial focusing magnet (vertical defocusing) and a radial defocusing magnet (vertical focusing), combined with an appropriate radial electric field corresponding to the n -values of the magnetic field. Under this configuration, particles will oscillate radially or vertically around the equilibrium orbit defined by the relation

$$\varepsilon + \mu = 1, \quad (1.1)$$

where ε and μ represent the ratios of the electric force and the magnetic force to the centrifugal force, respectively. To realize the achromatic condition, the equilibrium orbit must be defined independent of the variation of the momentum of particles. Such condition would be possible only when we superimpose an appropriate electric field on the magnetic guide field.

Experiments were carried out using the electron beam with an energy range from 5 to 11 keV. In the next section we will present a brief survey of the theoretical description of the motion of the charged particle in the alternate gradient electric and magnetic fields

deriving the transfer matrices of the system.

In the following section the experimental apparatus is described in association with the design concepts and the measuring schemes. Experimental results and some discussions are given in the last section.

2. Theoretical Description

In this section we will describe the motion of the charged particle briefly. More detailed description will be presented in Appendix I.

In axially symmetric electric and magnetic fields, the radial deviation x of a charged particle, oscillating around the equilibrium orbit $r = R$ and having a momentum $p + \Delta p$ slightly different from the equilibrium momentum p , is expressed by the equation of motion as^{6), 11)}

$$\frac{d^2 x}{ds^2} + \frac{f}{R^2} x = \frac{h}{R} \cdot \frac{\Delta p}{p}, \quad (2.1)$$

with

$$f = \varepsilon^2 + \varepsilon + 1 - n, \quad (2.2)$$

$$h = \varepsilon + 1, \quad (2.3)$$

where ε is the ratio of the electric force to the centrifugal force.

In Eq. (2.1), s is the distance measured along the equilibrium orbit, and the combined n -value of the electric and magnetic fields, n , is defined by⁶⁾

$$n = \varepsilon n_e + \mu n_m, \quad (2.4)$$

where μ is the ratio of the magnetic force to the centrifugal force, and n_e and n_m denote the n -values of the electric field and the magnetic field, respectively, and are defined as

$$n_e = -\frac{R}{E} \cdot \frac{dE}{dr}, \quad (2.5)$$

and

$$n_m = -\frac{R}{B} \cdot \frac{dB}{dr}, \quad (2.6)$$

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where E and B are r -component of the electric field and z -component of the magnetic field, respectively.

In a similar way, the vertical deviation z of a charged particle from the median plane is described by the equation

$$\frac{d^2 z}{ds^2} + \frac{g}{R^2} z = 0, \quad (2-7)$$

with

$$g = -\varepsilon + n, \quad (2-8)$$

where g , as well as f defined by Eq. (2-2), is the focusing factor specifying the degree of the focusing strength of oscillations around the equilibrium orbit.

In experiments, both n -values, n_e and n_m , were designed to be equal for the experimental convenience. Therefore, the combined n -value is rewritten as

$$n = (\varepsilon + \mu) n_e = (\varepsilon + \mu) n_m. \quad (2-9)$$

From Eq. (1-1), the sum of $\varepsilon + \mu$ is equal to unity on the equilibrium orbit, and then the combined n -value is reduced simply to n_e or n_m , i. e. $n = n_e = n_m$.

To turn to the problem of the motion of charged particles, it is convenient to consider their motions in the phase space, i. e. the displacement x (or z) and the gradient dx/ds (or dz/ds), employing the transfer matrix^[2] of the system.

From Eq. (2-1) we can get the transfer matrix for the radial motion

$$\mathbf{F}_x = \begin{pmatrix} \cos \phi_f & \frac{s_0}{\phi_f} \sin \phi_f & \frac{s_0^2 h}{R \phi_f^2} (1 - \cos \phi_f) \\ -\frac{\phi_f}{s_0} \sin \phi_f & \cos \phi_f & \frac{s_0 h}{R \phi_f} \sin \phi_f \\ 0 & 0 & 1 \end{pmatrix}. \quad (2-10)$$

with $\phi_f = (\varepsilon^2 + \varepsilon + 1 + n)^{1/2} s_0 / R$, for the radial focusing sector magnet, and

$$\mathbf{D}_x = \begin{pmatrix} \cosh \phi_d & \frac{s_0}{\phi_d} \sinh \phi_d & \frac{s_0^2 h}{R \phi_d^2} (\cosh \phi_d - 1) \\ \frac{\phi_d}{s_0} \sinh \phi_d & \cosh \phi_d & \frac{s_0 h}{R \phi_d} \sinh \phi_d \\ 0 & 0 & 1 \end{pmatrix} \quad (2-11)$$

with $\phi_d = (\varepsilon^2 + \varepsilon + 1 - n)^{1/2} s_0 / R$, for the radial defocusing sector magnet, where s_0 is the length of both sector magnets, and ϕ_f or ϕ_d denotes the phase advance after passing the sector magnet.

The transfer matrix for the straight section is also given by

$$\mathbf{L}_i = \begin{pmatrix} 1 & l_i & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad (2-12)$$

where l_i ($i=0, 1, 2$) denotes the separation between two sector magnets, the distance from the source point to the sector ($\mathbf{F}_x$) or the distance from the sector ($\mathbf{D}_x$) to the image point, respectively.

In the present arrangement of the sector magnets, **FOD**, the combined transfer matrix is calculated by multiplying these matrices to give

$$\mathbf{X} = \mathbf{D}_x \mathbf{L}_0 \mathbf{F}_x \quad (2-13)$$

Let the matrix $\mathbf{X}$ be

$$\mathbf{X} \equiv \begin{pmatrix} X_{11} & X_{12} & X_{13} \\ X_{21} & X_{22} & X_{23} \\ 0 & 0 & 1 \end{pmatrix}. \quad (2-14)$$

The displacement x , the gradient dx/ds , and the momentum deviation $\Delta p/p$ at the image point after passing through the focusing system are related to the initial quantities at the source point, x_0 , $(dx/ds)_0$ and $(\Delta p/p)_0$, by the equation

$$\begin{pmatrix} x \\ \frac{dx}{ds} \\ \frac{\Delta p}{p} \end{pmatrix} = \mathbf{L}_2 \mathbf{X} \mathbf{L}_1 \begin{pmatrix} x \\ \frac{dx}{ds} \\ \frac{\Delta p}{p} \end{pmatrix}_0. \quad (2.15)$$

From Eq. (2.15), the displacement x at the focal point is given by

$$x = (X_{11} + l_2 X_{21}) x_0 + \{l_1 X_{11} + X_{12} + l_2 (l_1 X_{21} + X_{22})\} \times \left(\frac{dx}{ds} \right)_0 + (X_{13} + l_2 X_{23}) \left(\frac{\Delta p}{p} \right)_0. \quad (2.16)$$

In the above equation, if we could make the coefficient of $(\Delta p/p)_0$ to be zero, the displacement x at the focal point becomes momentum free, and we have the focal distance l_2 as

$$l_2 = -X_{13}/X_{23}. \quad (2.17)$$

However, both signs of X_{13} and X_{23} always turn out to be positive (Appendix II), and we fail to obtain the positive value of l_2 from Eq. (2.17), i.e. the focal point is always located inside the sector magnet. Apparently this situation would be inconvenient from the experimental point of view.

Fortunately, however, if and only if the electric field satisfies the relation, $\epsilon = -1$, both X_{13} and X_{23} always vanish simultaneously, and we can make the coefficient

$$X_{13} + l_2 X_{23} = 0 \quad (2.18)$$

independent of the focal distance l_2 .

Then by equating the coefficient of $(dx/ds)_0$ as zero, we can achieve the radial focusing independent of the momentum deviations.

The relation between l_1 and l_2 is given by

$$l_2 = -\frac{l_1 X_{11} + X_{12}}{l_1 X_{21} + X_{22}}. \quad (2.19)$$

This focusing property, free from momentum deviations, can not be expected by means of the magnetic field alone, but it is a specific feature of superimposing the electric field on the magnetic guide field.

Next we consider the vertical focusing property, where the motion does not depend on the momentum deviation as far as its first order as is obvious from Eq. (2.7). Then the transfer matrix for the vertical motion is represented by a 2×2 matrix resulting in

$$\mathbf{F}_z = \begin{pmatrix} \cos \phi_f & \frac{s_0}{\phi_f} \sin \phi_f \\ -\frac{\phi_f}{s_0} \sin \phi_f & \cos \phi_f \end{pmatrix}, \quad (2.20)$$

with $\phi_f = (-\epsilon + n)^{1/2} s_0/R$, for z -focusing sector (r -defocusing sector), and

$$\mathbf{D}_z = \begin{pmatrix} \cosh \phi_d & \frac{s_0}{\phi_d} \sinh \phi_d \\ \frac{\phi_d}{s_0} \sinh \phi_d & \cosh \phi_d \end{pmatrix}, \quad (2.21)$$

with $\phi_d = (\epsilon + n)^{1/2} s_0/R$, for z -defocusing sector (r -focusing sector), where ϕ_f and ϕ_d are again the phase advances after passing through the corresponding sector magnets, respectively.

Multiplying these matrices together in the appropriate way gives the combined transfer matrix for the vertical motion as

$$\mathbf{Z} = \mathbf{F}_z \mathbf{L}_0 \mathbf{D}_z, \quad (2.22)$$

and for simplicity we put $\mathbf{Z}$ as

$$\mathbf{Z} = \begin{pmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{pmatrix}. \quad (2.23)$$

The vertical displacement z and the gradient dz/ds at the vertical focal point are related to their initial quantities by

$$\begin{pmatrix} z \\ \frac{dz}{ds} \end{pmatrix} = \mathbf{L}_2 \mathbf{Z} \dot{\mathbf{L}}_1 \begin{pmatrix} z \\ \frac{dz}{ds} \end{pmatrix}_0 \quad (2.24)$$

Then the vertical displacement z is given by

$$z = (Z_{11} + l_2 Z_{21}) z_0 + \{ (Z_{11} l_1 + Z_{12}) + l_2 (Z_{21} l_1 + Z_{22}) \} \left(\frac{dz}{ds} \right)_0, \quad (2.25)$$

and the vertical focusing condition is obtained by equating the coefficient of $(dz/ds)_0$ as zero, which leads to the relation of the vertical focal length l_2 to the source distance l_1 as

$$l_2 = - \frac{l_1 Z_{11} + Z_{12}}{l_1 Z_{21} + Z_{22}}. \quad (2.26)$$

From the crossing point of two curves given by Eqs. (2.19) and (2.26), we can decide the focal length of the system and the source distance. Under this condition, all particles that are emitted from the source into a large solid angle with a range of energy are refocused on this image point.

3. Experimental Apparatus

In this section we will describe the experimental apparatus in association with the

designing processes and measuring procedures in some details.

The main part of the transport system consists of a pair of sector magnets (a radial focusing sector **F** and a radial defocusing sector **D**) and a set of electrodes to provide a desired radial electric field on the median plane. The schematic view of the apparatus for the arrangement **FOD** is shown in Fig. 1.

The sector magnets of a C-type were employed for radial and vertical focusing of the beam. A cross sectional view of the magnet is shown in Fig. 2. As shown in Fig. 2a, we assume that the profile of the pole face of the sector magnet is given by⁽³⁾

$$xz + a(z \pm b) = 0, \quad (3.1)$$

where a and b are distances from the neutral plane and the median plane to the equilibrium orbit, respectively. Then the scalar potential ϕ of the magnetic field between pole pieces can be expressed as

$$\phi = \pm k(x + a)z, \quad (3.2)$$

where k is a constant which specifies the strength of the magnetic field. Then the z -component, B_z , of the magnetic field on the

Fig. 1. Schematic view of the apparatus for FOD configuration.

equilibrium orbit, i.e. $x=z=0$, is given by

$$B = \mp k a. \quad (3.3)$$

In the present sector magnet, the radius of the equilibrium orbit R , the distance a and the gap $2b$ are designed to be 300 mm, 60 mm and 50 mm, respectively. Using these parameters the n -value n_m defined by Eq. (2.6) is approximately reduced to

$$n_m \cong \mp \frac{R}{a} = \mp 5, \quad (3.4)$$

where the negative sign corresponds to the radial focusing sector and the positive one to the radial defocusing sector.

However, the actual n -values of the magnets were slightly different from the designed values. The reason is that, in Eq. (3.4), we have ignored a small deviation of the n -value due to the effect of the curvature of the sector magnet. This effect is estimated to be about less than 2 % as will be shown in the next section. The other errors in n_m come from the effect of the fringing magnetic field due to the finite dimension of the pole pieces. These errors were eliminated by adding appropriate shims to the edges of pole pieces. Then the n -values of both sector magnets can be regarded as 5 within the experimental errors.

The strength of the magnetic field was adjusted according to the strength of the applied electric field so that the particle may satisfy the equilibrium condition defined by Eq. (1.1). For electrons of the energy 10 keV, the magnetic fields necessary on the equilibrium orbit were 11.3 gauss for the case $\varepsilon=0$ and 22.6 gauss for $\varepsilon=-1$, respectively. Other parameters of the sector magnet are given in Fig. 2b.

To determine the n -value of the electric field, we assume the electric field of the type given by

Fig. 2. (a) Design parameters of the magnet for the radial focusing sector (F). $a=6$ cm, $b=2.5$ cm and $R=30$ cm. Surfaces of pole pieces are defined by $(x+6)z = \pm 30$ cm². (b) Cross sectional view of the sector magnet. Pole faces shown by solid lines are for the radial focusing sector (F) and those shown by dashed lines are for the radial defocusing sector (D). The designed n -values are $n_m = -R/a = -5$ for F and $n_m = R/a = 5$ for D, respectively.

$$E = \rho k^{-1}, \quad (3.5)$$

with $\rho = r - R + r_n$, where k is a constant specifying the strength of the electric field and other parameters are presented in Fig. 3a. In this case the n -value on the equilibrium orbit defined by Eq. (2.5) is reduced to

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$$n_e = \mp \frac{R}{r_0} \quad (3-6)$$

We may choose any combination of n_e and n_m , however, it turns out experimentally convenient to choose n_e to be equal to n_m due to the fact that the combined n -value is reduced to n_e or n_m , i.e. 5.0 in this case. Substituting these values into Eq. (3-6), we can determine the distance r_0 to be 60 mm.

The electric potential of Eq. (3-5) is given by

$$V(\rho) = \frac{V_m}{\ln(d/c)} \ln(\rho/c), \quad (3-7)$$

where c and d are the positions of the innermost and outermost electrodes measured from the point, $r = R - r_0$, and V_m denotes the maximum potential to be supplied on the outermost electrode.

For the versatility of choosing various operational conditions, concentrically divided electrodes were employed to realize the best fitted potential presented by Eq. (3-7). Each electrode was a silver-painted strip along the equilibrium orbit on the inner wall of the vacuum vessel which was made of epoxy-resin. The optimum width and number of electrodes and each separation between them were decided by means of the electrolytic trough method of the actual size of electrodes. The inner wall of the vacuum vessel in the region of the straight section was connected to the potential of the equilibrium orbit. Cross sectional views of the vacuum vessels are shown in Fig. 3.

The desired n -value of the electric field n_e can be realized by applying appropriately divided potentials on each electrode. The profile of the electric potential on the median plane measured by the electrolytic trough method is shown in Fig. 4 comparing with the ideal cylindrical potential given by Eq. (3-7). In experiments the potential to be supplied to

Fig. 3. (a) Ideal cylindrical electric potential for the radial focusing sector and design parameters of the vacuum vessel. $R=30$ cm, $r_0=6$ cm, $c=2.5$ cm and $d=9.5$ cm. The n -values of the electric field are designed equal to those of the magnetic field: $n_e = -R/r_0 = -5$ for F and $n_e = R/r_0 = 5$ for D. (b) Cross sectional view of the vacuum vessel and the electrode arrangement for the radial defocusing sector. Electrodes n and n' are electrically connected to the same potential. (c) Cross sectional view of the vacuum vessel for the straight section. All electrodes are floated on the potential of the equilibrium orbit.

Fig. 4. Electric potential used in the experiment for the radial focusing sector (F). The solid line denotes the ideal electric field defined by Eq. (3-7) in the text. The dashed line represents the approximated potential measured by the electrolytic trough method, and open circles denote potentials on the median plane at each radial position of electrodes.

each electrode was so adjusted that the averaged n -value near the equilibrium orbit was as exact as possible to the ideal n -value on the equilibrium orbit. The maximum potential V_m supplied to the outermost electrode was 646 volts for electrons of 10 keV in this case, $\varepsilon = -1$.

The focusing factors of the combined electromagnetic field, f and g , are listed in Table 1, and the variations of focusing factors along the equilibrium orbit are shown in Fig. 5. The transfer matrices, for example, in the radial motion derived in Eqs. (2-10) and (2-11) are numerically given by

$$F_x = \begin{pmatrix} -0.05970 & 12.23 \\ -0.08151 & -0.05969 \end{pmatrix}, \quad (3-8)$$

and

$$D_x = \begin{pmatrix} 2.028 & 26.47 \\ 0.1176 & 2.028 \end{pmatrix}, \quad (3-9)$$

Then the combined transfer matrices for both cases, **FOD** and **DOF**, given by Eq. (2-13) are

$$X_{\text{FOD}} = \begin{pmatrix} -2.278 - 0.1726 I_0 & 23.25 - 0.1212 I_0 \\ -0.1722 - 0.009584 I_0 & 1.317 - 0.007021 I_0 \end{pmatrix}, \quad (3-10)$$

and

$$X_{\text{DOF}} = \begin{pmatrix} 1.317 - 0.00702 I_0 & 23.22 - 0.1211 I_0 \\ -0.1723 - 0.009588 I_0 & -2.278 - 0.01653 I_0 \end{pmatrix}, \quad (3-11)$$

where I_0 is the length of the straight section between both sector magnets.

By the substitution of these matrix elements into Eq. (2-19), we can obtain the relations between the source distance l_1 and the focal length l_2 . In Fig. 6 l_1 are plotted against l_0 for three values of l_2 in the arrangement of **FOD**. In the figure it should be noted that the source distance l_1 does not depend strongly on the variation of l_0 , while the focal length l_2 is rather sensitive to l_0 . For the arrangement of **DOF**, the focal length is obtained by replacing l_2 by l_1 in the figure. Therefore, the value of l_0 was chosen as 60 mm taking the experimental convenience into account.

In a similar way, the vertical focal length can be obtained from Eq. (2-26). Typical relations between l_2 and l_1 for the radial and vertical cases are shown in Fig. 7. The focal point and the source distance used in experiments were decided from the crossing points of these two pair of curves.

The electron source was inserted from the side port of the vacuum vessel so that the source may be located at an appropriate position on the equilibrium orbit in front of the sector magnet as shown in Fig. 1. Since, in the arrangement of **FOD**, the focal length l_2

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Table 1. Focusing factors, n -values and phase advances for the sector magnets.

	Radial focusing sector (F)	Radial defocusing sector (D)
n -Values :		
n_e	-5	+5
n_m	-5	+5
n	-5	+5
Focusing factors :		
f	$\epsilon^2 + \epsilon + 6$	$\epsilon^2 + \epsilon - 4$
g	$-(\epsilon + 5)$	$-(\epsilon - 5)$
Phase advances :		
ϕ	$2(\epsilon^2 + \epsilon + 6)^{1/2}/3$	$2(4 - \epsilon^2 - \epsilon)^{1/2}/3$
ψ	$2(\epsilon + 5)^{1/2}/3$	$2(5 - \epsilon)^{1/2}/3$

Fig. 5. Variations of focusing factors, f and g , along the equilibrium orbit for the case $\epsilon = -1$. Dashed lines represent effects of the fringing electric and magnetic fields. S is the source position and D_x or D_z denotes the focal point in the radial or vertical direction.

Fig. 6. Diagram relating the source distance l_1 to the length of the straight section l_0 as a parameter of the radial focal length l_2 for FOD configuration. Diagram for DOF configuration can be obtained simply by replacing the source distance with the focal length.

Fig. 7. Diagrams relating the source distance l_1 with the focal length l_2 in both radial and vertical directions for FOD configuration. The experimental condition can be decided from the crossing points of these two pair of curves.

is very sensitive to the source position, i.e. dl_2/dl_1 is very large, only a slight adjustment of the source position by means of a bellows was adequate as is obvious from Fig. 7. A slit system was employed to define the emerging angles of the electron beam which was performed simply by sliding the position of the slit back and forward along the path of the electron beam. The energy of electrons was varied from 5 to 11 keV.

To verify the validity of the theoretical prediction that the system should be free from momentum deviations under an appropriate combination of the electric field with the magnetic field, i.e. $\epsilon = -1$ and $\mu = 2$, the displacements of the focal point in the radial direction were measured by varying the incident energy of the electron beam.

The radial displacement Δx is given by

$$\Delta x = (X_{13} + X_{23} l_2) \left(\frac{\Delta p}{p} \right)_0, \quad (3.12)$$

where X_{13} and X_{23} are the matrix elements defined by Eq. (2.14). The amounts of Δx were measured experimentally from the position

of the beam spot on the fluorescent plate located exactly at the radial focal point rather than the vertical focal point. To do so is important for the measurement of Δx , because the displacement Δx does not depend on the vertical focal length but depends explicitly on the radial focal length as shown in Eq. (3.12). According to the values of ϵ to be used, the focal plane moves along the equilibrium orbit, since the matrix elements involved in the focusing condition, Eq. (2.19), are all the function of ϵ . Then in experiments the position of the fluorescent plate was adjusted by carefully sliding its supporting rod to make the size of the beam spot to be minimum.

The system was evacuated by using a 4" oil diffusion pump from the opposite port of the electron source. The pressure inside the vacuum vessel was maintained about 10^{-6} Torr throughout the experiment.

4. Experimental Results and Discussion

Experiments were mainly carried out with the FOD configuration, though either configuration FOD or DOF was possible. In this configuration the radial focal length was fairly sensitive to the variations of both the source distance and the length of the straight section, while the vertical focal length was not so sensitive as obvious from Figs. 6 and 7.

The present purpose is to verify the validity of the theoretical prediction that the system should be free from momentum deviations under the appropriate condition ($\epsilon = -1$). So it may be adequate only to examine how radial focal points will be affected by the momentum deviation when the electric field strength is altered from $\epsilon = 0$ to -1 , because the vertical motion does not depend on the momentum deviation up to the first order approximation as shown in Eq. (2.7).

First we consider how fringing fields along the equilibrium orbit actually have an influence

on the motion of charged particles. The effective length of the straight section l''_0 differs slightly from the length of the ideal free space l_0 by the relation $l_0 - l''_0 = 2 l'_0$, where l'_0 denotes the length of the transient region measured along the equilibrium orbit. Strictly we should take l''_0 as the length shown in Fig. 5, but for simplicity of the calculation we have assumed the fringing field exists only on the side of the straight section. The transfer matrix of this transient section is then approximately given by⁽⁴⁾

$$\mathbf{T}_{t,d} \cong \begin{pmatrix} 1 - \phi'^2_{t,d}/6 & l'_0(1 - \phi'^2_{t,d}/12) \\ -\phi'^2_{t,d}/2 l'_0 & 1 - \phi'^2_{t,d}/3 \end{pmatrix}, \quad (4.1)$$

where $\phi'_{t,d} = \phi_{t,d} l'_0 / s_0$, suffices f and d refer to the radial focusing and defocusing sectors, respectively.

Now the transfer matrix of the straight section given by Eq. (2.12) has to be replaced by the matrix,

$$\mathbf{L}_{0, \text{eff}} = \mathbf{T}_d \mathbf{L}'_0 \mathbf{T}_f. \quad (4.2)$$

This matrix can be expressed as

$$\mathbf{L}_{0, \text{eff}} = \begin{pmatrix} 1 + \delta_{11} & l_0 + \delta_{12} \\ \delta_{21} & 1 + \delta_{22} \end{pmatrix}, \quad (4.3)$$

where δ_{ij} denotes a small deviation of each matrix element from ideal ones. By the substitutions of $l'_0 = 10$ mm and $l''_0 = 40$ mm into Eq. (4.2), we have the numerical expression of the matrix of the effective straight section as

$$\mathbf{L}_{0, \text{eff}} = \begin{pmatrix} 1 - 0.004 & l_0 - 0.004 \\ -0.007 & 1 - 0.013 \end{pmatrix}. \quad (4.4)$$

The maximum deviation involved in the straight section is then estimated to be at most 1.3 %, and we may safely ignore this effect in the actual experiment.

The fringing field in the radial direction can be compensated by adding an appropriate shim at the edges of the sector magnets as mentioned before. The more stringent error

in the focal length arises from the deviations of the n -value due to the effect of the curvature of the sector magnets. The relative error in the n -value of the magnetic field is given by¹³⁾

$$\frac{\Delta n_m}{n_m} = -\frac{a^2}{4\sqrt{\pi}\phi R} \int_0^\pi dx \times \int_{-\infty}^{+\infty} \frac{\sinh\left(\frac{\pi a^2}{2\phi} - x\right) \sin x \sqrt{x^2 + y^2} - y}{\left\{ \cosh\left(\frac{\pi a^2}{2\phi} - x\right) - \cos x \right\}^2 \sqrt{x^2 + y^2}} dy, \quad (4.5)$$

where ϕ denotes ϕ_f or ϕ_d defined by Eqs. (2.10) and (2.11), respectively. Substituting experimental parameters and calculating the integral numerically, the relative error $\Delta n_m/n_m$ can be estimated to be less than 2 %. The relative error in the n -value of the electric field, $\Delta n_e/n_e$, can be also estimated as much the same as that of the magnetic field.

By these deviations of n -values from the designed values, the focal length in the radial direction differed slightly from that of the vertical direction. Then in the experiment, the radial displacements of the focal point due to the momentum deviation were measured by locating the fluorescent plate exactly on the radial focal plane, and hence the beam spots on the fluorescent plate were strictly a line focus. Results are shown in Fig. 8 for various values of ϵ .

The radial displacement Δx is proportional to the momentum deviation $(\Delta p/p)_0$, and the slope of the curve varies as $1+\epsilon$ as given in Eq. (3.12). The condition $\epsilon=0$ corresponds to the case of the magnetic field alone, and $\epsilon=-1$ to the case of momentum free where the slope of the curve is zero. However, the experimental results differed slightly from those linear curves due to the second order effect with respect to the momentum deviation. In this case the radial displacement is

expressed by

$$\Delta x = A \left\{ 1 + \epsilon - (1 + 2\epsilon) \left(\frac{\Delta p}{p} \right)_0 \right\} \left(\frac{\Delta p}{p} \right)_0, \quad (4.6)$$

where the coefficient A relates to experimental parameters of the apparatus and can be readily calculated by the help of Eqs. (II.1) and (II.2) in Appendix II. It should be noted here that the coefficient A is also a function of ϵ through the phase advances ϕ_f and ϕ_d . Especially when $\epsilon=-1$, Δx is simply reduced to

$$\Delta x = A \left(\frac{\Delta p}{p} \right)_0^2. \quad (4.7)$$

Taking this second order deviation into account, we can say that both experimental and theoretical results show good agreement.

The results described above concerning the configuration of FOD are also valid for

Fig. 8. Radial displacement Δx versus initial momentum deviation $(\Delta p/p)_0$ for four different values of ϵ . Solid lines denote the theoretical displacements given by Eq. (4.6) in the text and experimental results are shown by open circles with estimated error bars.

the configuration of DOF simply by interchanging the source distance l_1 with the focal length l_2 . As is obvious from Figs. 6 and 7, in this configuration, the radial focal length is weakly dependent on the length of the straight section and the source distance, however, the vertical focal length depends strongly on an even small deviation of the source distance. Therefore, this configuration would not be convenient to the experiment where the vertical focusing is essentially required.

Some applications of this system were investigated concerning the guide field of the accelerator^{61, 11)} and various types of transport systems for rather high energy particles¹⁵⁾. An application to the mass spectrometer will be discussed in a subsequent paper.

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Appendix I

In axially symmetric electric and magnetic fields, the relativistic Lagrangian of a charged particle in the cylindrical coordinate (r, θ, z) is given by

$$L = -m_0 c^2 [1 - (\dot{r}^2 + \dot{z}^2 + r^2 \dot{\theta}^2) / c^2]^{1/2} + e r \dot{\theta} A - e \dot{\phi}, \quad (I-1)$$

where m_0 and e are the rest mass and the charge of the particle, and dot denotes the differentiation with respect to time.

The scalar potential of the electric field ϕ and the θ -component of the vector potential of the magnetic field A are expanded around the equilibrium orbit to the second order in x and z as¹⁶⁾

$$\begin{aligned} \phi(r, z) = & \phi_0(R) + x \phi'_0(R) + \frac{x^2}{2} \phi''_0(R) \\ & \dots\dots\dots + \frac{z^2}{2} \phi_1(R) + \dots\dots\dots, \end{aligned} \quad (I-2)$$

and

$$\begin{aligned} A(r, z) = & \phi_0(R) + x \phi'_0(R) + \frac{x^2}{2} \phi''_0(R) \\ & \dots\dots\dots + \frac{z^2}{2} \phi_1(R) + \dots\dots\dots, \end{aligned} \quad (I-3)$$

with

$$\phi_1(R) = -\frac{1}{r} \cdot \frac{d}{dr} \left[r \frac{d}{dr} (r \phi_0) \right], \quad (I-4)$$

$$\phi_1(R) = -\frac{1}{r} \cdot \frac{d}{dr} \left[r \frac{d}{dr} (r \phi_0) \right], \quad (I-5)$$

where prime denotes the differentiation with respect to r .

Now let R denote the function $L - \dot{\theta} \frac{\partial L}{\partial \dot{\theta}}$,¹⁷⁾ we have

$$\begin{aligned} R = & -m_0 c^2 \gamma - e \phi_0 + \frac{1}{2} m_0 \gamma (x^2 + z^2) \\ & + (m_0 \gamma \omega^2 R + e \omega \phi_0 + e \omega R \phi'_0 - e \phi'_0) x \\ & - \frac{1}{2} [(m_0 \gamma \omega + e \phi_0 / R + e \phi'_0)^2 / m_0 \gamma^3 \\ & + 2 \omega R (m_0 \gamma \omega / R + e \phi_0 / R - \frac{1}{2} e \phi''_0) + e \phi''_0] x^2 \\ & - \frac{1}{2} (-e \omega R \phi_1 + e \phi_1) z^2, \end{aligned} \quad (I-6)$$

where ω is the angular frequency of the charged particle and $\gamma = (1 - \omega^2 R^2 / c^2)^{-1/2}$.

From the definition of the n -value, we can relate n_e and n_m to the respective electric field and the magnetic field as

$$n_e = \frac{R \phi''_0}{\phi'_0}, \quad (I-7)$$

$$n_m = \frac{\phi_0 - R \phi'_0 - R^2 \phi''_0}{\phi_0 + R \phi'_0}, \quad (I-8)$$

and the n -value of the combined electric and

magnetic fields is defined by

$$n = \varepsilon n_e + \mu n_m, \quad (\text{I-9})$$

where $\varepsilon = -eE/m_0\gamma\omega^2 R$ and $\mu = -e\omega RB/m_0\gamma\omega^2 R$.

Substituting R into the Lagrangian equations of motion, we have

$$\frac{d^2 x}{ds^2} + \left(f + f_1 \frac{dp}{p} \right) \frac{x}{R^2} = \left(h + h_1 \frac{dp}{p} \right) \frac{1}{R} \cdot \frac{dp}{p}, \quad (\text{I-10})$$

for the radial motion with the relations

$$f = \frac{\varepsilon^2}{\gamma^2} + \varepsilon + 1 - n, \quad (\text{I-11})$$

$$f_1 = n - 1 - \frac{\varepsilon}{\gamma^2} (4\varepsilon - 2 - n_e), \quad (\text{I-12})$$

$$h = \frac{\varepsilon}{\gamma^2} + 1, \quad (\text{I-13})$$

and

$$h_1 = - \left\{ 1 + \frac{\varepsilon}{\gamma^2} \left(\frac{5}{2} - \frac{1}{2} \gamma^2 \right) \right\}. \quad (\text{I-14})$$

The vertical equation of motion is also calculated from the new Lagrangian R as

$$\frac{d^2 z}{ds^2} + \left(g + g_1 \frac{dp}{p} \right) \frac{z}{R^2} = 0, \quad (\text{I-15})$$

with

$$g = -\varepsilon + n, \quad (\text{I-16})$$

$$g_1 = \varepsilon - n + \frac{\varepsilon}{\gamma^2} (1 - n_e). \quad (\text{I-17})$$

In Eqs. (I-10) and (I-15) if we consider the nonrelativistic case $\gamma=1$, taking up to the first order with respect to the momentum deviation $\Delta p/p$, we have Eqs. (2-1) and (2-7) in the text.

Appendix II

The matrix elements, X_{13} and X_{23} , are given by

$$X_{13} = \frac{s_0^2 h}{R \phi_f^2} \left[\left\{ (1 - \cos \phi_f) + \frac{l_0 \phi_f}{s_0} \sin \phi_f \right\} \right. \\ \left. \times \cosh \phi_d + \frac{\phi_f}{\phi_d} \sin \phi_f \sinh \phi_d + \frac{\phi_f^2}{\phi_d^2} \right. \\ \left. \times (\cosh \phi_d - 1) \right], \quad (\text{II-1})$$

and

$$X_{23} = \frac{s_0 h \phi_d}{R \phi_f^2} \left[\left\{ (1 - \cos \phi_f) + \frac{l_0 \phi_f}{s_0} \sin \phi_f \right\} \right. \\ \left. \times \sinh \phi_d + \frac{\phi_f}{\phi_d} \sin \phi_d \cosh \phi_d + \frac{\phi_f^2}{\phi_d^2} \sinh \phi_d \right]. \quad (\text{II-2})$$

If, therefore, we take the case $\varepsilon = -1$, the term h involved in X_{13} and X_{23} vanishes, so we can make both X_{13} and X_{23} to be zero simultaneously for any positive values of ϕ_f and ϕ_d .

Now we consider the signs of X_{13} and X_{23} for the case ε being not equal to -1 . Let α , β , x and y be as follows:

$$\alpha = \frac{\phi_f}{\phi_d} \quad (\alpha > 0),$$

$$\beta = \frac{l_0 \phi_f}{s_0} \quad (\beta > 0),$$

$$x = \cos \phi_f \quad (|x| < 1),$$

$$y = \cosh \phi_d \quad (y > 1).$$

Then Eqs. (II-1) and (II-2) are reduced to

$$X_{13}(x, y) = \frac{h l_0^2}{R \beta^2} \left[\{-yx + (\alpha^2 + 1)y - \alpha^2\} \right. \\ \left. + \{\beta y + \alpha(y^2 - 1)^{\frac{1}{2}}\} (1 - x^2)^{\frac{1}{2}} \right], \quad (\text{II-3})$$

and

$$X_{23}(x, y) = \frac{h l_0}{R \alpha \beta} \left[\{-(y^2 - 1)^{\frac{1}{2}} x + (\alpha^2 + 1)(y^2 - 1)^{\frac{1}{2}}\} \right. \\ \left. + \{\beta(y^2 - 1)^{\frac{1}{2}} + \alpha \beta\} (1 - x^2)^{\frac{1}{2}} \right]. \quad (\text{II-4})$$

Substituting $x = \pm 1$ into Eq. (II-3), we have inequalities

$$X_{13}(1, y) = \alpha^2(y - 1) > 0, \quad (\text{II-5})$$

and

$$X_{13}(-1, y) = \alpha^2(y-1) + 2y = X_{13}(1, y) + 2y > 0, \quad (\text{II} \cdot 6)$$

which always hold for $y > 1$.

In a similar way we have inequalities concerning $X_{23}(\pm 1, y)$ as

$$X_{23}(1, y) = \alpha^2(y^2 - 1)^{\frac{1}{2}} > 0, \quad (\text{II} \cdot 7)$$

and

$$\begin{aligned} X_{23}(-1, y) &= (\alpha^2 + 2)(y^2 - 1)^{\frac{1}{2}} \\ &= X_{23}(1, y) + 2(y^2 - 1)^{\frac{1}{2}} > 0. \end{aligned} \quad (\text{II} \cdot 8)$$

Then we have the curves as shown in Fig. 9, so X_{13} and X_{23} are always positive for any positive values of ϕ_f and ϕ_d .

Fig. 9. Variations of matrix elements X_{13} and X_{23} .

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