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An approximate analytical description of the unidirectional growth of arrays of needles is proposed for the solidification under high temperature gradients. Temperature and solute distribution for a needle are described by use of functions with exponential increase/decrease and integral exponential functions. Those for arrayed needles are described by the addition of distributions. Then, a modeling of the growth of arrays of needles is developed in order to predict needle dimensions. Local equilibrium conditions are applied to determine the tip radius and unknown coefficients included in descriptions of temperature and solute distribution. Minimum undercooling of needle tip is also applied to select the primary spacing of growing needles. Predicted dimensions are studied and compared with those given by other theoretical predictions. The model predicts the growth rate of transition from planar interface to cellular, which is very close to the growth rate given by the Mullins-Sekerka theory. The predicted dependency of tip radius of curvature on growth rate is similar to the dependency given by the Kurz-Fisher model for needle growth.

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1. Introduction

In the unidirectional rapid solidification under high temperature gradients, new properties of material, which are different from those given by solidification under low temperature gradients, can be obtained. Engineers are very much interested in microstructure and microsegregation of solidified alloys, because these are primarily responsible for the mechanical properties of materials. Therefore, it is desirable to have models to predict dimensions of dendrite and the growth temperature in solidification.

Several models for the growth of dendrites have been proposed. Analytical models[1–3] have given guidance to engineers to solve various problems in solidification. Numerical models[4–7] have been developed to study the arrayed growth of dendrites or cells. Analytical models[8,9] study the underlying physics of pattern formation and stability of interface.

Experimental studies of the growth of needles in the rapid solidification under high temperature gradients tell us that needles have close neighbors and have little secondary arms. The purpose of this article is to give approximate analytical model for the growth of these needles, where the interaction between primary arms would determine dimensions of such growth.

An approximate description of arrayed needles is proposed and a modeling of arrayed growth will be given. The key ingredient is the description of field by the functions with exponential increase/decrease and integral exponential function (Ivantsov function[10]) for a needle. Their additions are used to describe the fields of arrayed needles. Solute trapping will be taken into account through the redistribution coefficient of solute. Local equilibrium conditions are applied to determine the tip radius and unknown coefficients included in descriptions of temperature and solute distribution. Minimum undercooling of needle tip is also applied to select a unique solution. And the dimensions of growing needles are predicted.

Assumptions such as solvability[11,12] or overgrowth/tip splitting condition is not used to select a steadily growing needle, because the Ivantsov function itself satisfies the diffusion equation for the steady growth. The minimum undercooling criterion is adopted to select a solution, instead of the use of other conditions, such as the marginal stability. The marginal stability criterion says that the tip radius of curvature of needle will decrease with the square root of growth rate. But this would not be necessarily true near the absolutely stable growth rate. This is a reason why the minimum undercooling criterion is used.

The treatment is limited because fields (for temperature and solute concentration) are described by the addition of integral exponential function corresponding to the primary arms, and other arms (secondary and ternary arms) are not modeled in this treatment. It has been experimentally shown that the primary arm spacing decreases with increase of temperature gradient in the unidirectional solidification. Therefore, predictions of the model are compared with experimental results with high temperature gradient.

2. Description of Arrayed Needle

2.1 Interface and fields

Consider arrayed needles (Fig. 1) growing positive z-direction with the same growth rate, V, and the same tip radius of curvature, \( \rho \), where coordinates \( (x, y, z) \) are made dimensionless by the tip radius of curvature of needles. The position of the tip of arrayed needles is assumed to be \( (i\lambda, j\lambda, 1/2) \), where \( \lambda \) is a positive number, and \( i \) and \( j \) are integers. The distance between needles, that is, the primary spacing of needles is given by \( \lambda_1 = \lambda \rho \).

The interface near the tip of each needle is assumed to be a paraboloid of revolution.
\[ \xi_0^2 = \sqrt{(x-i\lambda)^2 + (y-j\lambda)^2 + z^2} + z = 1. \]  
(1)

This is an extension of the paraboloidal interface for single needle positioned at (0, 0, 1/2)

\[ \xi^2 = \sqrt{x^2 + y^2 + z^2} + z = 1 \]

(2)

where \( \xi \) is a coordinate of paraboloid of revolution \((\xi, \eta, \varphi)\), related to \((x, y, z)\) as

\[ x = \xi \eta \cos \varphi, \quad y = \xi \eta \sin \varphi, \quad z = (\xi^2 + \eta^2)/2. \]  
(3)

Temperature distributions, \( T_L \) and \( T_S \) for liquid and solid, and solute distributions, \( C_L \) and \( C_S \) for liquid and solid, respectively, for steadily growing arrayed needles are assumed to be described by the addition of those for needles.

\[ T_L = T_0^L + A_0^L \sum_{i,j} E_1(P_L \xi_0^2) \text{or}\ B_0^L e^{-2P_L z} \]

\[ + B_0^L e^{-2\lambda_1 \xi_0^2} \frac{1}{2} [\cos(\omega x) + \cos(\omega y)], \]  
(4)

\[ T_S = T_0^S + B_0^S e^{-2P_L z} + B_0^S e^{2\lambda_1 z} \frac{1}{2} [\cos(\omega x) + \cos(\omega y)], \]  
(5)

\[ C_L = C_0^L + A_0^L \sum_{i,j} E_1(P_L \xi_0^2) \text{or}\ B_0 e^{-2P_L z} \]

\[ + B_0 e^{-2\lambda_1 \xi_0^2} \frac{1}{2} [\cos(\omega x) + \cos(\omega y)], \]  
(6)

\[ C_S = C_0^S + B_0^S e^{2P_L z} \frac{1}{2} [\cos(\omega x) + \cos(\omega y)], \]  
(7)

where

\[ \lambda_L = \frac{1}{2} (\sqrt{P_L + \omega^2} + P_L), \quad \lambda_S = \frac{1}{2} (\sqrt{P_S + \omega^2} - P_S), \]

(8)

\[ \lambda_F = \frac{1}{2} (\sqrt{P + \omega^2} + P), \quad \lambda_C = \frac{1}{2} (\sqrt{P_C + \omega^2} - P_C) \]

\[ E_1(x) \text{ is the integral exponential function defined by} \int_0^\infty e^{-x t} dt, \quad \text{where} \quad P_L = \frac{\rho V}{2\alpha_L} \text{ and } P_S \text{ are thermal Peclet numbers for liquid and solid, respectively, and} \quad P \text{ and } P_C \text{ are solute Peclet numbers for liquid and solid, respectively.} \]

And \( \omega = 2\pi/\lambda \), \( T_0^L \), \( A_0^L \), \( B_0^L \), \( T_0^S \), \( A_0^S \), \( B_0^S \), \( C_0^L \), \( \lambda_F \), \( C_0^S \), \( B_0^C \), \( \lambda_C \), and \( \lambda_1 \) are constant numbers to be determined by boundary conditions. Distribution for solid, \( T_S \) and \( C_S \), cannot be described by the integral exponential function, \( E_1(x) \), because the function diverges at \( x = 0 \).

\( B_0^L (B_0^S) \text{ term with } A_0^L (A_0^S) \text{ term in the description of } T_L (C_L) \text{ will give precise expression for the temperature (solute) distribution. But the number of restrictions given by boundary conditions is not enough to determine the values of } B_0^L \text{ and } A_0^L \text{ (} B_0^S \text{ and } A_0^S \text{) separately. Then, two models are constructed (we call the model composed of } A_0^L \text{ and } A_0 \text{ as } AB \text{-model, and that of } B_0^L \text{ and } B_0 \text{ as } BB \text{-model.)} \]

Integral exponential functions and functions with exponential decrease/increase are solutions of the diffusion equation for steady growth. Summations with respect to \( i \) and \( j \) are taken for all integers, and functions with exponential decrease/increase are periodic with length \( \lambda \) in \((x, y)\) plane. Then every needle has same thermal and solutal distribution near the tip of the needle.

General solutions of diffusion equations are given by hypergeometric functions and Laguerre polynomials, and the integral exponential function and the function with exponential decrease/increase are the lowest rank of those solutions. Therefore, the above expressions for fields are generally rude approximations for distributions.

Experimental studies on microstructure in rapid solidification tell us that needles have very small secondary arms and neighboring needles are very close to each other. Then the microstructure would be controlled by interactions between primary arms. In these solidification, where the Peclet number is usually large, the summation with respect to \( i \) and \( j \) will be easily converged by nearby needle neighbors, and above expressions will become good approximations for fields of temperature and solute (for convergence of the summation, see Part II of this work).

2.2 Boundary conditions

Parameters to be determined by boundary conditions are coefficients of fields, \( T_0^L \), \( B_0^L \) (or \( A_0^L \)), \( B_0^S \), \( T_0^S \), \( B_0^C \), \( B_0^S \), \( C_0^L \), \( B_0 \) (or \( A_0 \)), \( A_0 \), \( C_0^S \), \( B_0^C \), tip radius, \( \rho \), and the primary spacing, \( \lambda_1 = \lambda \rho \). There are altogether 13 unknown parameters in each model.

Twelve conditions are derived by boundary conditions for fields and one condition is given by the minimum undercooling assumption at the tip of needle.

The boundary conditions for fields to be satisfied are:

(a) temperature gradient at the tip of needle in liquid, \( G_L \), (b) the average solute concentration, \( C_{0k} \), and at the interface, (c) continuity of temperature, (d) conservation of heat flux, (e) redistribution of solute, (f) conservation of solute flux and (g) interface temperature (Gibbs-Thomson condition).

Let us consider a needle positioned \((x, y) = (0, 0)\) in order to study how restrictions are given among parameters in the expressions of temperatures and solute concentrations. Dimesions of needle would be controlled by the behavior of fields near the tip of needle \((x = y = 0 \text{ and } z = 1/2)\), then we expand the expression of fields with respect to small distance, \( r^2 = x^2 + y^2 \) from tip of needle, that is, near the tip of needle, and apply boundary conditions to each order of \( r^0 \) and \( r^2 \). For example, \( T_L \) and \( T_S \) are expanded to be
\[ T_L = T_0^L + A_0^L \varepsilon_1(P_L) \text{ or } B_0^L e^{-P_L} + B_0^L e^{-\lambda_L} + r^2 \left\{ A_0^L \beta_1(P_L) \text{ or } P_L B_0^L e^{-P_L} + \left( \lambda_L - \frac{\omega^2}{4} \right) B_0^L e^{-\lambda_L} \right\} \]  

(9)

\[ T_S = T_0^S + B_0^S e^{-P_S} + B_0^S e^{\lambda_S} + r^2 \left\{ P_S B_0^S e^{-P_S} + \left( -\lambda_S - \frac{\omega^2}{4} \right) B_0^S e^{\lambda_S} \right\} \]

(10)

where

\[ \varepsilon_1(P_L) = \sum_{i,j} E_1(P_L \cdot r_{ij}) \]

and

\[ \beta_1(P_L) = \sum_{i,j} \frac{e^{-P_{II} r_{ij}}}{r_{ij}} \left[ \frac{1}{2} - \frac{1}{4r_S} + \frac{r_{ij}^2}{4r_S^2} + \frac{r_{ij}^2}{4r_S^2} \right] (P_L + \frac{1}{r_{ij}}) \]

and \( r_S, r_L \) and \( r_{ij} \) are defined in Table 1. Then conditions for the temperature continuity to be satisfied are:

For \( r^0 \) order,

\[ T_0^L + A_0^L \varepsilon_1(P_L) \text{ or } B_0^L e^{-P_L} + B_0^L e^{-\lambda_L} = T_0^S + B_0^S e^{-P_S} + B_0^S e^{\lambda_S}. \]  

(11)

For \( r^2 \) order,

\[ A_0^L \beta_1(P_L) \text{ or } P_L B_0^L e^{-P_L} + \left( \lambda_L - \frac{\omega^2}{4} \right) B_0^L e^{-\lambda_L} = P_S B_0^S e^{-P_S} + \left( -\lambda_S - \frac{\omega^2}{4} \right) B_0^S e^{\lambda_S}. \]

(12)

The eq. (11) means that the temperature should be continuous at the tip of needle from liquid to solid, and the second eq. (12) says that the temperature should be continuous at a point close to the tip. Derivation of restrictions given by heat flux conservation is given in Appendix A, paying attention that the normal direction of the interface is generally different from the direction of heat flow.

In this way, twelve restrictions to be satisfied among coefficients are derived and tabulated in Table 1.

### 2.3 How to predict tip radius and primary spacing

Parameters to be determined by boundary conditions are coefficients of fields, \( (T_0^L, B_0^L) \text{ or } (A_0^L), B_0^L, T_0^S, B_0^S, C_0^L, B_0 \) \text{ or } (A_0), B_0^L, C_0^S, B_0^S \text{, tip radius, } \rho, \text{ and the dimensionless primary spacing, } \lambda. \text{ Now, we have 12 restrictions among coefficients of fields (Table 1(A) and 1(B)). Then, for a given primary spacing (dimensionless primary spacing, } \lambda, \text{ we can solve these 12 equations and get the corresponding tip radius of curvature. If one more requirement is given, then we can determine the primary spacing. The very requirement adopted in this article is the minimum tip undercooling condition for a growing needle.}

![Table 1](image-url)
(B) Constraints among parameters given by boundary conditions on the interface in AB-model.

Temperature gradient: \( G_L = -\frac{2}{\rho} (A_{0L} \alpha_L(P_L) + \lambda_L B_0^{e-L}) \)  
\( \text{Average solute concentration: } C_L^0 = C_L^0 \)  
\( \text{Temperature continuity: } T_L = T_S; \text{Coefficient of } r^0; T_L^0 + A_L^0 \beta_L(P_L) + B_0^{e-L} = T_S^0 + B_0^{e-P_S} + B_0^{e-e_S} \)  
\( \text{Coefficient of } r^2; A_L^0 \beta_L(P_L) + \left( \lambda_L - \frac{\omega_0^2}{4} \right) B_0^{e-L} = P_L B_0^{e-P_S} + \left( -\lambda_S - \frac{\omega_0^2}{4} \right) B_0^{e-e_S} \)  
\( \text{Heat flux: } r^0; \frac{\Delta H}{c_p} P_L = A_L^0 \alpha_L(P_L) + \lambda_L B_0^{e-L} - \frac{K_S}{K_L} (P_S B_0^{e-P_S} - \lambda_S B_0^{e-e_S}) \)  
\( r^2; A_L^0(P_L) \beta_L(P_L) - \gamma_0(P_L) - \delta_L(P_L) + B_0^{e-L} \left( P_L - \lambda_L \right) \left( \lambda_L - \frac{\omega_0^2}{4} \right) - \frac{\omega_0^2}{4} \)  
\( = B_0^{e-L} \left[ P_L + \frac{K_S}{K_L} \lambda_S \left( -\lambda_S - \frac{\omega_0^2}{4} \right) \right] \)  
\( \text{Solute redistribution: } r^0; k C_L^0 = C_L^0 + B_0^{e-e_S} \)  
\( r^2; k \left[ A_0 \beta_L(P) + \left( \lambda_P - \frac{\omega_0^2}{4} \right) B_0^{e-L} \right] = \left( -\lambda_C - \frac{\omega_0^2}{4} \right) B_0^{e-e_C} \)  
\( \text{Solute flux: } r^0; (1 - k) P C_L^0 = A_0 \alpha_L(P) + \lambda_P B_0^{e-L} \gamma_0(P) + D_S \frac{D_{e-C}}{D_{e-C}} B_0^{e-C} \)  
\( r^2; A_0 \beta_L(P) - \gamma_0(P) - \delta_L(P) + B_0^{e-L} \left( P - \lambda_P \right) \left( \lambda_P - \frac{\omega_0^2}{4} \right) - \frac{\omega_0^2}{4} \)  
\( = B_0^{e-L} \left[ P + D_{e-C} \frac{D_{e-C}}{D_{e-C}} \right] \left( -\lambda_C - \frac{\omega_0^2}{4} \right) - \frac{D_S \omega_0^2}{4} \)  
\( \text{Gibbs-Thomson condition: } T_0^0 + B_0^{e-P_S} + B_0^{e-e_S} = T_M - \gamma \frac{2}{\Delta S \rho} + \frac{m_0}{\lambda} (C_S + B_0^{e-C}) \)  
\( r^2; P_S B_0^{e-P_S} + \left( -\lambda_S - \frac{\omega_0^2}{4} \right) B_0^{e-e_S} = \frac{\gamma}{\Delta S \rho} + \frac{m_0}{\lambda} \left( -\lambda_C - \frac{\omega_0^2}{4} \right) B_0^{e-e_C} \)  

where, \( \alpha_L(P) \)'s are defined as follows: Denote 
\( \lambda_{ij} = \sqrt{(i^2 + j^2) \lambda^2}; \quad r_s = \sqrt{\frac{\lambda_{ij} + 1}{4}}; \quad r_L = r_s + \frac{1}{2} \)  

then 
\( \alpha_L(P) = \sum_{i,j} \frac{e^{-P r_L}}{2 - r_S} \beta_L(P) = \sum_{i,j} \frac{e^{-P r_L}}{r_L} \left( \frac{1}{2} - \frac{1}{4} - \frac{r_i^2}{r_S^2} + \frac{r_j^2}{r_S^2} \right) \left( P + \frac{1}{r_L} \right) \)  
\( \gamma_0(P) = \sum_{i,j} \frac{e^{-P r_S}}{2 - r_S} \left( P - \frac{1}{2} - \frac{r_i^2}{r_S^2} + \frac{r_j^2}{r_S^2} \right) \left( P^2 + \frac{1}{r_S^2} + \frac{3}{r_S} \right) \)  
\( \delta_L(P) = \sum_{i,j} \frac{e^{-P r_L}}{2 - r_S} \left( 1 - \frac{r_i^2}{r_S^2} + \frac{r_j^2}{r_S^2} \right) \)  
\( \epsilon_L(P) = \sum_{i,j} E_{i,j} (P_L - r_L) \)  
\( \alpha_L(P_L), \beta_L(P_L), \gamma_0(P_L), \delta_L(P_L) \) and \( \epsilon_L(P_L) \) are defined if \( P \) is replaced by \( P_L \).

Coefficients of fields can be solved, for example, as functions of Peclet numbers as follows: By eqs. (1), (4), (5) and (6) in Table 1(A), coefficients of temperature are expressed as 
\( B_0^{e-e_S} = \left( \frac{4}{\omega_0^2} \right) \cdot B_0^{e-e_S} \)  
\( B_0^{e-L} = B_0^{e-e_S} + \left( \frac{4}{\omega_0^2} \right) \cdot (G_L - \frac{\rho}{2} G_L) \)  
\( B_0^{e-P_L} = \left[ -\frac{\rho}{2} G_L - \lambda_L B_0^{e-e_L} \right] / P_L \)  
\( B_0^{e-e_S} = [-G_L + \lambda_S B_0^{e-e_S}] / P_S \)  

where 
\( \tilde{G}_L = \left[ \frac{\Delta H}{c_p} P_L + \frac{\rho}{2} G_L \right] / (K_S) \)  
\( B_0^{e} = \left[ (P_L - \lambda_L) (\lambda_L - \frac{\omega_0^2}{4}) - \frac{\omega_0^2}{4} \cdot G_L - \frac{\rho}{2} G_L \right] \)  
\( B_0^{e} = \left[ (P_L - \lambda_L) (\lambda_L - \frac{\omega_0^2}{4}) - \frac{\omega_0^2}{4} \right] - \left[ (P_L + K_S \lambda_S) (\lambda_S - \frac{\omega_0^2}{4}) - K_S \omega_0^2 \right] / K_L \)  

By eqs. (2), (8), (9) and (10) in Table 1(A), coefficients of
solute are expressed

\[
B_0^e e^{\lambda p} = B_0^e / B_0^C,
\]

\[
B_0 e^{-\lambda p} = a_{00}^p + a_{01}^p B_0^e e^{\lambda c},
\]

\[
B_0 e^{-\lambda p} = \left[ - \left( \lambda p - \frac{\omega^2}{4} \right) + \frac{1}{k} \left( -\lambda c - \frac{\omega^2}{4} \right) B_0^c e^{\lambda c} \right] / P
\]

where

\[
B_0^c = (1 - k) P C_L^c \left( 1 - \frac{\omega^2}{4} \right) - \frac{\omega^2}{4}
\]

\[
B_0^p = \left( 1 - k \right) P \frac{C_L^p}{\omega^2} \left( \lambda p - \frac{\omega^2}{4} \right) - \frac{\omega^2}{4} + \frac{D_c}{c} \lambda c
\]

\[
- \left( 1 - k \right) P - \lambda p + k \left( \lambda p - \frac{\omega^2}{4} \right)
\]

and

\[
a_{02}^p = (1 - k) P - \omega^2 / 4 + k \left( \lambda p - \omega^2 / 4 \right)
\]

\[
a_{01}^p = \left( -\lambda c - \frac{\omega^2}{4} + \frac{D_c}{D} \lambda c \right) / a_{02}^p.
\]

Insert these into eq. (12) in Table I(A), then the equation

\[
P_S^c e^{-\lambda p} + \left( -\lambda c - \frac{\omega^2}{4} \right) B_0 e^{-\lambda p}
\]

\[
= \frac{\gamma}{\Delta S} \frac{2}{\rho} + \frac{m_0}{k} \left( -\lambda c - \omega^2 / 4 \right) B_0^c e^{\lambda c}
\]

where \( \gamma \) is the interfacial energy and \( \Delta S \) the entropy of fusion, becomes a function between growth rate and tip radius, and gives tip radius of curvature for a given growth rate and dimensionless distance \( \lambda \). Tip undercooling of needle is given as follows:

\[
\Delta T = \frac{\gamma}{\Delta S} \frac{2}{\rho} + \frac{m_0}{k} (C_L^c - C_L^c)
\]

where \( C_L^c = C_L^c + B_0 e^{-\lambda p} + B_0 e^{-\lambda p} \).

The dimensionless primary spacing corresponding to the minimum tip undercooling is selected if the tip undercooling is calculated for other values of dimensionless distance \( \lambda \).

In this way, all unknown parameters included in the expressions of fields and needle dimensions are determined. Solutions corresponding to the experimental observations (small secondary arms and close neighbors) will be given for small dimensionless distance \( \lambda \).

3. Property of Solution and Comparison with Other Models

Procedure to predict needle dimensions will be shown first in the solidification of Al-0.55 mass% Cu alloy with temperature gradient \( G_t = 2.7 \times 10^4 \) K/cm and properties of the solution will be studied. Then predictions will be compared with those predicted by other models. Physical properties used in the calculation are given in Table 2. The flow chart of calculation is also shown in the Appendix B.

3.1 Property of solution given by BB-model (model without the integral exponential function)

Procedure to give predictions of dimensions will be studied easily in this model. In Fig. 2, example solutions of tip radius of curvature corresponding to a fixed value of dimensionless distance \( \lambda \) are shown for the solidification of Al-0.55 mass% Cu alloy. Loop-type solutions are generally given for a fixed dimensionless distance.

For a given growth rate we can find minimum value of tip undercooling, changing the value of dimensionless distance (or the value of tip radius, because each dimensionless distance determines its individual tip radius). An example rela-

![Fig. 2 Tip radius vs. growth rate for the fixed \( \lambda \) (dimensionless spacing) in BB-model.](image-url)
tion between the undercooling and tip radius is shown for a given growth rate in Fig. 3. A tip radius of curvature and a dimensionless distance can be selected corresponding to the minimum tip undercooling by this figure. Corresponding primary spacing is given by $\lambda_1 = \rho \lambda$.

Selected tip radius and primary spacing are shown with the growth rate in Fig. 4.

3.2 Property of solution given by AB-model (model with addition of the integral exponential function)

Example solutions of tip radius of curvature are shown in Fig. 5 corresponding to a fixed value of dimensionless distance, $\lambda$. In Fig. 5(a) all of the solutions are shown, and they are grouped into types of solutions in Figs. 5(b) and (c). Loop-type solutions ($C$ and $D$ in Fig. 5(b)) and a line-type solution ($P$ in Fig. 5(c)) are generally given. $C$ solutions are given by loops with small dimensionless distance ($\lambda \leq 2$), while $D$ solutions are by loops with larger value of $\lambda$ ($\geq 2$). When the temperature gradient becomes small we have many larger loops with larger $\lambda$ for $D$-solution. Loops with small value of $\lambda$ will tell us that $C$ solutions predict the tip radius corresponding to one given by BB-model (this will be studied in the next subsection). $D$ and $P$ solutions are solutions newly born by the addition of the integral exponential function.

Reasons why we have different types of solutions are:

a. different dominant component of fields
b. what contribution balances the decrease of temperature along the interface from tip?

Fields are expressed by addition of the integral exponential function, $\sum_j E_j (P_j^{\alpha_j})$, and the exponentially increase/decrease function, $\frac{1}{2} \cos(\omega x) + \cos(\omega y) e^{-2x^2-y^2}$. The dominant component of $D$-solution is the integral exponential function, whereas $C$-solution is dominated by the exponential function. The difference between $P$-solution and others comes from what contribution equates the decrease of temperature along the interface. In the $C$- and $D$-solution the decrease is equated by the contribution of curvature of the interface and by the pile-up of solute along interface. When the pile-up of solute is almost saturated, $C_{tip} \approx \frac{C_s}{2}$, then the decrease of temperature along the interface is balanced by the contribution of curvature alone. This is the case of $P$-solution.

For a given growth rate we can find local minimum value of tip undercooling corresponding to each group of solutions ($C$, $D$ and $P$). An example is shown in the undercooling/tip radius relation for a given growth rate in Fig. 6. A tip radius of curvature, a dimensionless distance and corresponding primary spacing can be selected corresponding to the minimum tip undercooling by this figure.

Selected tip radius, primary spacing and tip undercooling are shown with growth rate in Fig. 7. Tip undercooling given by solutions of $P$-group is predicted larger than that given by
solutions of $C$- or $D$-group except in low growth rate.

3.3 Cellular and cellular-dendritic growth

In the modeling the fields are expanded by the power of $r^2$ and the local equilibrium conditions are applied up to terms in order of $r^4$. Then we have artificially separated the model into two models to determine the unknown coefficients included in fields. If we have enough number of restrictions, (for example, conditions in order of $r^4$), two above models could be unified into a model. There, the solution given by BB-model would correspond to some solution in AB-model.

The common solution in AB-model with that in BB-model would be $C$-solution. This identification is suggested that both solutions are given only for small dimensionless distance $\lambda = \lambda_1/\rho$. This is also supported by the correspondence between predicted dimensions (tip radius of curvature and primary spacing) by two models (Fig. 4 and Fig. 7), though predicted dimensions given by two models differ in the small growth rate. The reason why two models give similar prediction is that the growth is primarily controlled by the interaction between primary arms and that the effect of tip shape is small. In the modeling the tip shape of interface is assumed to be paraboloid of revolution, $\xi^2 = 1$. The effect of tip shape on prediction will be studied if we assume the tip shape to be $\xi^2 = 1 + \alpha \eta^2$, where $\alpha$ expresses the departure from the paraboloid. In this way the effect of tip shape is studied, but the above correspondence has remained true. These studies tell us that the corresponding growth to $C$-solution (or solution of BB-model) will be cellular.

$D$-solution in AB-model will be a new solution induced by the field described by the addition of integral exponential functions. Because the diffusion field from secondary arms of dendrite is not modeled in AB-model, a candidate of the growth expressed by $D$-solution would be cellular-dendrite, which has only small secondary-arms.

The growth corresponding to $P$-solution would not be realized because of the larger undercooling of $P$-solution than that of $C$- and $D$-solution.

3.4 Comparison with other models

The growing region of growth rate given by Mullins-Sekerka theory is shown for solidification of Al-0.55 mass%Cu alloy in Fig. 4, together with the predicted tip radius of curvature, $\rho$, and primary spacing, $\lambda_1$, given by BB-model.

The critical growth rate from planar interface to cellular/dendritic interface is predicted by BB-model. Below the critical growth rate there is no solutions as shown in Fig. 2. The predicted critical growth rate is a little small than that of Mullins-Sekerka theory. Therefore, the model allows the subcritical bifurcation of planar interface, because of small redistribution coefficient of this alloy.

The absolutely stable growth rate is also predicted by the model and almost same with that given by Mullins-Sekerka theory (Fig. 4). The growth rate is given by the minimum undercooling criteria of growing needle in this model. There, the planar interface is reached by neighbors approaching closer and closer with infinite tip radius.

In Fig. 8 the predicted tip radius of curvature is compared with that given by Kurz-Fisher model, where the marginal stability ($\rho = \lambda_{MS}$; $\lambda_{MS}$ is the wave-length given by Mullins-Sekerka theory) is assumed. The figure shows that the cor-
The difference between the numerical model given by S. Z. Lu and J. D. Hunt and the model in this article is the way in which the steadily growing needle is selected. They select a needle by the overgrowth/tip-splitting condition, but the Ivantsov function (the integral exponential function) is used in this text. The Ivantsov function itself is a solution of steady growth. Both models give similar prediction for needle dimensions in high growth rate.

5. Conclusion

A simple analytical description and modeling for the growth of array of needle are given by addition of fields. Predicted needle dimensions are compared with those given by other modeling of arrayed solidification, and correspondences between these models are studied. Positive features of the model are:
(a) prediction of tip radius of curvature and primary spacing of needles of array,
(b) range of growth rate from the critical growth rate of planar interface to the absolutely stable growth rate,
(c) similar prediction of tip radius of curvature with that of Kurz-Fisher model (based on the marginal stability),
(d) dependence of dimensions on the growth rate,
(e) increase of dimensions near the absolutely stable growth rate.

These features show the addition of the Ivantsov function well describes the field of temperature and solute in the rapid solidification with a high temperature gradient.

The model should be improved for better prediction, (1) by determining all coefficients in the described fields simultaneously, (2) by taking the proper tip shape of needle into account, and (3) by taking account of effects of diffusion from secondary arms into the modeling. A parabolic tip shape of needle is assumed in the model, but the cellular interface does not have necessarily the shape of paraboloid of revolution. Stability of interface will play an important role in the selection of solutions, but what kind of perturbation should be applied to the tip shape of arrayed needle remains in question.

REFERENCES

Appendix A: Derivation of Constraints by Heat Flux Conservation Condition

Heat flows normal to the isothermal plane, and this direction is not necessarily the direction of the normal to the interface. In order to derive constraints by heat flux conservation, the temperature gradient normal to the isothermal plane is first calculated and it should be projected to the direction normal to the interface.

When the angle between the growth direction and the normal to the isothermal plane is denoted by $\alpha$, and the angle between the growth direction and the normal of the interface is by $\theta$, then one should calculate (Fig. A1)

$$
\cos(\theta - \alpha) \left\{ \cos \alpha \frac{\partial T_L}{\partial z} + \sin \alpha \frac{\partial T_L}{\partial x} \right\}, \quad (A \cdot 1)
$$

and this is related to the latent heat. Near the tip of needle these angles and derivatives of fields are related to the unknown coefficients of fields

$$
\tan \theta = \frac{A_L\delta_L(P_L) + \frac{\omega^2}{4} B_0^L e^{-\lambda_L}}{A_L\alpha_L(P_L) + \lambda_L B_0^L e^{-\lambda_L}}, \quad (A \cdot 2)
$$

and

$$
\frac{\partial T_L}{\partial x} = -2 \left\{ A_L\delta_L(P_L) + \frac{\omega^2}{4} B_0^L e^{-\lambda_L} \right\} x,
$$

$$
\frac{\partial T_L}{\partial z} = -2 \left( A_L\delta_L(P_L) + \lambda_L B_0^L e^{-\lambda_L} \right) x^2
$$

Introduce them into eq. (A-1) and pick up the terms proportional to power $x^0$ and $x^2$, then one can obtain the constraints in the text. In the Table 1, the constraint is simplified by using relations given by the condition of temperature continuity at the interface.

Appendix B: Flow of Calculation

<table>
<thead>
<tr>
<th>Physical Property</th>
<th>$G_L, C_L^*$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Growth Rate, $V$</td>
<td></td>
</tr>
<tr>
<td>Dimensionless Distance, $\lambda$</td>
<td></td>
</tr>
<tr>
<td>Tip Radius, $\rho$</td>
<td></td>
</tr>
<tr>
<td>Coefficients of $T_L, T_S, C_L, C_S$</td>
<td></td>
</tr>
<tr>
<td>Boundary Condition eq.(12) at the interface</td>
<td></td>
</tr>
<tr>
<td>YES</td>
<td></td>
</tr>
<tr>
<td>Tip Radius, $\rho$</td>
<td></td>
</tr>
<tr>
<td>Minimum Undercooling of Tip, $\Delta T$</td>
<td></td>
</tr>
<tr>
<td>YES</td>
<td></td>
</tr>
<tr>
<td>Tip Radius, $\rho$</td>
<td></td>
</tr>
<tr>
<td>Primary Spacing, $\lambda_1 = \rho \lambda$</td>
<td></td>
</tr>
<tr>
<td>Undercooling of Tip, $\Delta T$</td>
<td></td>
</tr>
</tbody>
</table>