Thermal Oscillation Modes of the Solid-Liquid Interface
Solidification and Melting

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Oscillation modes of the solid-liquid (S/L) interface using a non-linear equation for thermal transfer are analysed. Thus, the solidification is achieved by dark cnozial oscillation modes and the melting by bright cnozial oscillation modes of this interface. We show the S/L interface self-structures as a non-linear Toda lattice and then specify some of its properties.

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1. Introduction

Different models which describes the solid (S)/liquid (L) phase transition were developed. The oldest approach is modelling at the S/L interface as a surface, whose shape is changing in time. It is, thus, assumed that the interface has zero thickness and such models are sometimes referred to as "sharp-interface" or "classical" models. Implicitly in this idea is the fact that the microscopic structure of the interface, involving a description on the atomic scale, is much smaller than the characteristic length scale of the diffusion field. On the surface representing the interface, boundary conditions are established which permit discontinuities in the temperature or solute fields, or their gradients, across the interface. These boundary conditions express, in part, the microscopic dynamics of the interface as well as the conservation laws for heat and mass.1,2

The discovery of the spinodal decomposition led to a theoretical model developed by Cahn and Hilliard3,4 which represents the interface between two regions of different compositions as diffuse, i.e., of finite thickness. The solution of this theory is the construction of a gradient-weighted free-energy functional, and of a model which ensures that the free energy decreases monotonically in time. This approach does not distinguish the interface from the rest of the system by treating it differently, as is the case in the classical formulation, it so leads to a more coherent description.

Another approach discussed by Kurtz et al.5 is the phase field model of a S/L phase transition. This model extends the ideas used by Cahn and Hilliard for spinodal decomposition, by formulating a S/L phase transition in terms of a gradient-weighted free energy functional. They employ an additional variable, called the phase field, to act as an order parameter to identify regions as solid and liquid.

In the present paper thermal oscillation modes of S/L interface are obtained and, on its basis, we specify the structure of the interface.

2. The Correspondence Force Field-Thermal Gradient in Interface

The temperature gradient in the S/L interface, directed from L to S, generates the force field $\vec{F}$ directed from S to L. The expression of this force field will be obtained by annulling the current density $\vec{j}$.

In order to calculate the current density $\vec{j}$ in the interface, we use the Boltzmann equation$^3$ for stationary regime

$$\vec{v} \cdot \vec{\nabla} f + \frac{\vec{F}}{m} \cdot \vec{\nabla} f = -\frac{f_1}{\tau}$$

where $f$ defines the function of distribution, $f_1$ the first order correction for the distribution of equilibrium $f_0 = e^{-W/T}$, $\vec{v}$ the velocity field of the particles, $\vec{F}$ the force field which perturbs the interface, $m$ the particles mass and $\tau$ the relaxation time. Since at the S/L transition the quantum mechanical parameters present a classical behaviour, $f_1$ differs only slightly from the equilibrium function of distribution, i.e. $f = f_0 + f_1 \approx f_0$, then

$$\partial_t f \approx \delta_t f_0 = -\frac{W}{T} \delta_T \frac{\partial f_0}{\partial W}$$

It follows that:

$$- \frac{\vec{v} \cdot \partial f_0}{\partial W} = \left( -\frac{W}{T} \delta_T + \vec{F} \right) \vec{v} = -\frac{f_1}{\tau}$$

where:

$$f_1 = -\tau \frac{\partial f_0}{\partial W} \left( -\frac{W}{T} \delta_T + \vec{F} \right) \vec{v}$$

Under these circumstances, the current density has the expression:

$$\vec{j} = m \int n f_1 \vec{v} d^2 \vec{v} \cdot \vec{v}$$

By giving the integrals in (6) for the one-dimensional case explicitly, we obtain successively:
\[-m \int_0^\infty N r_\alpha \frac{\partial f_0}{\partial W} \left( -\frac{W}{T} \frac{\partial T}{\partial x} \right) v_x^2 dv_x = \frac{n_0 k_B}{\sqrt{\pi}} \int_0^\infty r \frac{\partial T}{\partial x} e^{\alpha^2 / 2} d\alpha = \frac{3n_0 k_B}{4} \left( \frac{\partial T}{\partial x} \right) \]

and
\[-m \int_0^\infty N r_\alpha \frac{\partial f_0}{\partial W} F_x v_x^2 dv_x = \frac{n_0}{\sqrt{\pi}} \int_0^\infty \tau F_x e^{\alpha^2 / 2} d\alpha = \frac{n_0}{2} (\tau F_x) \]

with \( \alpha = W/k_B T \), therefore (6) getting the form:
\[ j_x = \frac{n_0}{2} \left( \frac{3k_B}{2} \frac{\partial T}{\partial x} + \tau F_x \right) \]

Since at equilibrium \( j_x = 0 \), the last expression gives:
\[ \langle F_x \rangle = -\frac{3k_B}{2} \left( \frac{\partial T}{\partial x} \right) \]

This means that for a negative average temperature gradient in the interface, \( (\partial T/\partial x) \), directed from L to S, will correspond an average force field from S to L. In other words, the mean fluctuations of the thermal field from the interface, \( (T) \), are going to be substituted be mean energy fluctuations of the particle, \( \langle \delta E \rangle = \langle F_x \delta x \rangle \).

3. Thermal Field Variation in S/L Interface

The thermal field variation in S/L interface is obtained by numerical integration of thermal field equations:
\[ \partial_t T_i = k_i (\rho C_i)_{-1} \alpha_i T_i, \quad i = L, S \]

with the frontier condition:
\[ k_L \alpha_i T_L = k_S \alpha_i T_S + \lambda \rho _L \alpha_i x \]

where \( T_i \) are the temperatures of the solid and liquid phases respectively, \( k_i \) the thermal conductivity coefficients, \( \rho_i \) the densities, \( C_i \) the specific heats, \( \lambda \) the solidification heat and \( \alpha_i x \) the moving speed of the solidification front. For this let us admit the following conditions: the initial temperature \( T_f = 293 \text{ K} \), the thermal properties and size of the moulding box \( k_f = 397 \text{ W/m/K, } d = 0.01 \text{ m} \), the environment coefficient of thermal transfer \( \alpha = 6 \times 10^4 \text{ W/m}^2 \text{ K, } T_{\text{solid, Al}} = 933 \text{ K, } \rho_S = 2700 \text{ kg/m}^2, \rho_L = 2385 \text{ kg/m}^3, k_L = 94 \text{ W/m/K, } k_S = 238 \text{ W/m/K, } C_L = C_S = 1080 \text{ J/kg K} \) (for details on the equality \( C_L = C_S \) see Ref. 2).

If one performs the integration in the one-dimensional case (the solidification take place in one direction), we get an additional symmetry — connected condition — the thermal gradient in the symmetry plan is zero. The used scheme is with finite differences, known as triangle explicite scheme or Schmidt formula:\[ T_{j+1}^+ = \beta T_{j+1}^- + (1 - 2\beta) T_j^+ + \beta T_{j-1}^- \]

with \( \beta = \frac{\alpha_{SL} \Delta \alpha}{\Delta \alpha^2} \), where \( \alpha_{SL} \) denotes the fraction in front of the \( \alpha_x x, \Delta x, \Delta t \) being the intervals of space and time division.

The resulting space-time configuration of the thermal field is presented in Fig. 1. This representation confirm the existence of a maximum value of temperature where the S/L interface is located at a given moment, obviously determined by the latent heat elimination (for details on the definition of S/L interface see Ref. 2) and, at the same time, the temporal dependence of the solidification front coordinates agrees with the known relation \( x(t) = k t^{1/2} \) where \( k \) is a constant, depending on the properties of the studied material.

From the spatial variation of the thermal field for a fixed \( t \) given in Fig. 2 results that the temperature fluctuations in the interface are non-linear.

4. Oscillation Modes of the S/L Interface

From the energy conservation law \( dW/dt = 0 \) (see Section 2) results \( dT/dt = 0 \), where we omitted the symbol ( ). For giving this equation explicitly in the one-dimensional case, let us observe the following: i) The thermal fluctuations from the interface are non-linear (see Section 3) implying the functional dependence, \( v_x = v_x(T) \); ii) The dissipation from the interface involves the term \( \partial^2 T/\partial x^2 \) (for more details see Refs. 9, 10)); iii) The dispersion from the interface involves the term \( \partial^2 T/\partial x^3 \) (for more details see Refs. 9, 10)). Then eq. \( dT/dt = 0 \) in the non-dimensional coordinates
\[ \omega_0 \delta t = \tau, \quad \frac{k_0}{T_0} = u, \quad v_x \left( \frac{T}{T_0} \right) \frac{k_0}{\omega_0} = c(u), \]

\[ \frac{dx^2}{dt} \frac{k_0}{\omega_0} = -\mu, \quad \frac{dx^3}{dt} \frac{k_0}{\omega_0} = -\sigma \]

becomes
\[
\frac{\partial u}{\partial \tau} + c(u) \frac{\partial u}{\partial \xi} - \mu \frac{\partial^2 u}{\partial \xi^2} - \sigma \frac{\partial^2 u}{\partial \xi^2} = 0 \tag{15}
\]

with \(c_0, k_0, T_0\) some parameters characterising S/L interface, \(\mu\) the dissipative coefficient, and \(\sigma\) the dispersive coefficient. Equation (15) expresses the fact that non-linear effects, \(c(u) \partial\xi/\partial\xi\), dissipative, \(\mu \partial^2 u/\partial\xi^2\), and dispersive, \(\sigma \partial^2 u/\partial\xi^2\), ones are responsible for the time dependence of the thermal field, \(\partial u/\partial \tau\).

The primary phases (of generating the interface) are dominated by non-linear and dissipative effects (for more details see Refs. 1, 2)). Then eq. (15) takes the form

\[
\frac{\partial u}{\partial \tau} + c(u) \frac{\partial u}{\partial \xi} - \mu \frac{\partial^2 u}{\partial \xi^2} = 0 \tag{16}
\]

If \(c\) is constant, i.e. the linear case, the solution of eq. (16),

\[
u = u_0 e^{-\mu x^2} e^{i(\xi - c\tau)}
\]

describes the “thermal waves” slightly damped, satisfying the dispersion relation \(\omega = ck - i\mu k^2\). If \(c(u) \equiv u\), i.e. the linear approximation (for more details see Refs. 9, 10), eq. (16) becomes

\[
\partial u + u \partial u - \mu \partial^2 u = 0 \tag{17}
\]

Equation (17) corresponds to the modified Burgers (mB) eq. (for more details see Refs. 9, 10). The solution of this equation, with the initial condition

\[
u(\xi, 0) = u_0(\xi) \tag{18}
\]

has the expression:

\[
u = -2\mu \partial_\xi \ln \varphi(\xi, \tau) \tag{19}
\]

where

\[
\varphi(\xi, \tau) = \frac{1}{\sqrt{4\pi \mu \tau}} \cdot \exp \left[ -\frac{(\xi - \eta)^2}{4\mu \tau} - \frac{1}{2\mu} \int_0^\eta u_0(\eta') d\eta' \right] d\eta \tag{20}
\]

The integral (20) is convergent if the relation (18) satisfies the condition:

\[
\int_0^\infty u_0(x')dx' \leq \text{const.} \quad \text{as} \quad x \to \infty \tag{21}
\]

Considering that such a condition is satisfied, one can study the development of an initial disturbance, which disappears for \(\xi \to \pm \infty\). We assume that

\[
\int_{-\infty}^{+\infty} u_0(\xi) d\xi = M < \infty \tag{22}
\]

and then, for any \(\tau\)

\[
\int_{-\infty}^{+\infty} u_0(\xi, \tau) d\xi = \int_{-\infty}^{+\infty} u_0(\xi) d\xi = M \tag{23}
\]

The validity of this relation is easy to prove if (17) is written as

\[
\partial u + u \left( \frac{1}{2} \frac{\partial u^2}{\partial \xi} - \mu \partial_\xi u \right) = 0 \tag{24}
\]

and after integrating both terms on the variable \(\xi\) between the limits \(-\infty\) and \(+\infty\). The surface bounded by \(u(\xi, \tau)\) does not depend on \(\tau\), consequently it is a motion’s integral.

The solution of eq. (17) has, for \(\tau \to \infty\) with the condition (23), the asymptotic form:

\[
u(\xi, \tau) \approx -2\mu \partial_\xi \ln F\left( \frac{\xi}{\sqrt{2\mu \tau}} \right) \tag{25}
\]

with

\[
F(x) = \frac{1}{\sqrt{\pi}} \left[ e^{-\frac{M}{4\mu}} \int_x^\infty e^{-\eta^2} d\eta + e^{\frac{M}{4\mu}} \int_{-\infty}^x e^{-\eta^2} d\eta \right] \tag{26}
\]

The result is interesting since it proves the universality of the asymptotic shape of the profile \(u(\xi, \tau)\) for great values of \(\tau\).

If the dissipative parameter \(\mu\) is zero, from (25) and (26) for \(M > 0\) we can get:

\[
\lim u(\xi, \tau) = \begin{cases} \frac{\xi}{\tau}, & 0 < \xi < (2M\tau)^{1/2} \\ 0, & \xi < 0, \quad \xi > (2M\tau)^{1/2} \end{cases} \tag{27}
\]

For \(M < 0\), from (27) we can obtain the same value, using the transformation \(u \to -u, \xi \to -\xi, \tau \to \tau\).

The asymptotic shape of \(u\) for great values of \(\tau\) is plotted in Fig. 3. If \(\mu \to 0\) then \(u\) has the shape of a triangle with the shock wave placed in front of, or behind the profile, depending on the sign of \(M\). The discontinuity magnitude of the shock wave is \(\sqrt{2M}/\tau\), i.e. the decrease is proportional with \(\tau^{1/2}\); the bottom of its profile increases with \(\tau^{1/2}\), hence the surface remains unchanged.

We can obtain a special (travelling wave) solution of (17) by setting:

\[
u(\xi, \tau) = u(\theta = \xi - w\tau) \tag{28}
\]

It results the shock wave:

\[
u(\theta) = w - A \tanh \left( \frac{A\theta}{2\mu} \right) \tag{29}
\]

with the amplitude \(A = (a + w^2)^{1/2}, a\), a constant of integration, and the passing region width \(\delta = 2\mu/A\). The thickness of this shock wave decreases with the increasing of the amplitude \(A\) — see Fig. 4.

The final phases (of the personalizing interface as a stable structure) are dominated by non-linear and dispersive effects.
(for more details see Refs. 9, 10)). Then eq. (15) becomes
\[
\frac{\partial u}{\partial \tau} + c(u) \frac{\partial u}{\partial \xi} - \sigma \frac{\partial^3 u}{\partial \xi^3} = 0
\] (30)

If \(c\) is constant and \(\partial u/\partial \tau = 0\), i.e. the stationary and linear case, eq. (30) with substitutions
\[
\frac{c}{\sigma} = \frac{1}{\lambda^2}, \quad \frac{\partial u}{\partial \xi} = G
\] (31)
becomes
\[
\frac{d^2 G}{d\xi^2} - \frac{1}{\lambda^2} G = 0
\] (32)

As a result the thermal gradient is expelled from the interface.
We call this effect “thermal Meissner effect” by analogy with the one from superconductivity.11) Correspondingly \(\lambda\) defines the “thermal penetration depth”. Now some consequences are obvious: i) The thermal Meissner effect is associated to the over-cooling phenomenon (for details see Ref. 1, 2)). Thus, conditions for solidification starting are created; ii) The S/L interface behave as a generalized superconductor (for details see Ref. 12)). Thus, considering that at the S/L contact a “solid plasma” forms (for details see Refs. 12, 13)), for a certain value of the thermal gradient it structures itself like an electric double layer (DL) (for details see Refs. 12, 13)), forming the S/L interface. In such a context the Cooper pairs are replaced by the electron-ion pairs from the DL; iii) By analogy with the classical superconductivity and generalized superconductivity (for details see Refs. 12, 13)), the S/L interface in an external thermal gradient is subjected to a force. This force assures the interface dynamics; iv) The solution of eq. (32) (see method from Ref. 11)), i.e.
\[
G = G_0 \cdot e^{-\frac{x}{\lambda}}
\] (33)

where \(G_0\) is the thermal gradient at the surface of the interface. indicates that on a particle moving with speed \(v\), acts the specific force:
\[
\frac{d^2 \xi}{d\tau^2} = -A_0 e^{-\frac{x}{\lambda}}, \quad A_0 = -\frac{3}{2} \frac{k_b v_0 v_0 G_0}{\text{moc}^2}
\] (34)

The particularities of such motion are presented in Ref. 13).

If \(c(u) = u\), i.e. the non-linear case, eq. (30) becomes
\[
\frac{\partial u}{\partial \tau} + \frac{\partial u}{\partial \xi} - \sigma \frac{\partial^3 u}{\partial \xi^3} = 0
\] (35)

Equation (35) corresponds to the modified Korteweg-de Vries (mKdV) equation (for details see Refs. 9, 10)) and the solution has the form (see the method from the Ref. 12))
\[
u(z) = u = A e \text{cn}^2(z, s)
\] (36)

where
\[
A = e_2 - e_1, \quad s^2 = \frac{e_2 - e_1}{e_1 - e_1}, \quad z = \frac{A}{12 \sigma} \left( \frac{\phi}{s} \right)^{1/2} = \xi - \omega \tau
\] (37)

In eqs. (36) and (37) \(e_1 > e_2 \geq e_3\) are the real roots of the cubic
\[
P(u) = \frac{2}{\sigma} \left( \frac{u^3}{6} - \frac{u^2}{2} + Au + B \right)
\] (38)

with \(A, B\) integration constants, cubic obtained by integrating twice eq. (35) for the variable \(\theta\).

It means that in the S/L interface dark cnoidal oscillations modes (see Fig. 5) occur having the following characteristics (see the method from the Ref. 10)):

i) the wave length \(\lambda\),
\[
\lambda = 2 \left( \frac{12 \sigma}{A} \right)^{1/2} s \frac{K(s)}{K(s)}
\] (39)

where \(K(s) = \int_0^{\pi/2} (1 - s^2 \sin^2 \varphi)^{-1/2} d\varphi\) is the first kind elliptic integral;14)

ii) the average temperature
\[
\overline{u} = \lambda^{-1} \int_0^{\pi/2} u(\theta) d\theta = e_3 - \frac{2A E(s)}{s^2} \frac{K(s)}{K(s)}
\] (40)

where \(E(s) = \int_0^{\pi/2} (1 - s^2 \sin^2 \varphi)^{1/2} d\varphi\) is the second kind elliptic integral;14)

iii) the speed \(w\), i.e. the speed of the solidification front
\[
w = e_3 - \frac{A}{3s^4} (2 - s^2)
\] (41)

We distinguish the degenerations:

i) if the dispersive phenomena occur before the nonlinear ones, i.e. \(s \to 0\), then the solution (36) is reduced to the dark waves packet:
\[
u(\theta) = \overline{u} - A \left[ \cos k \theta + \frac{1}{8} s^2 \cos 2k \theta + O(s^4) \right]
\] (42)

with parameters, from (39) and (41),
\[
\frac{1}{k^2} = \frac{3\sigma}{A} s^2 \left( 1 + \frac{s^2}{2} \right) + O(s^6), \quad k = \frac{2\pi}{\lambda}
\] (43)

respectively
\[
w = \overline{u} + \sigma k^2 \left[ 1 - \frac{A^2}{96\sigma^2 k^4} + O(A^2) \right]
\] (44)

The terms which are not dependent on amplitude, give through (44) the phase speed expression for the dark waves packet in linear approximation, or by \(k\) multiplication, the dispersion equation (see Fig. 6),
\[
\omega(k) = \overline{u} k + \sigma k^3
\] (45)

ii) if nonlinear effects occur before the dispersive ones, i.e. \(s \to 1\), then the solution (36) is reduced to dark solitons packet with amplitudes \(A = \text{const.} \) (compared to \(u = \overline{u}\)), with parameters, from (39) and (41),

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Fig. 5 Dark cnoidal oscillations modes.
\[ \lambda = \left( \frac{12\sigma}{A} \right)^{1/2} \left[ \ln(1 - s^2) \right] \]  

(46)

respectively

\[ w \rightarrow \bar{u} - 2A/3 \]  

(47)

iii) if dispersive and non-linear effects are comparable, i.e. \( s = 1 \), then the solution (36) is reduced to the dark soliton (compared to \( u = \bar{u} \)),

\[ u(\theta) = \bar{u} - \text{Asech}^2 \left[ \left( \frac{A}{12\sigma} \right)^{1/2} \theta \right] \]  

(48)

— see Fig. 7, with the amplitude \( A \), the width \( D = (12\sigma/A)^{1/2} \) and speed

\[ w = \bar{u} - 2A/3 \]  

(49)

As a result: i) The solidification is achieved by dark cnoidal oscillations modes of the interface S/L; ii) The degenerations dark waves packet, dark solitons packet and dark soliton are sequences of the solidification process.

The dominance of the non-linear and dispersive effects with \( \sigma < 0 \) brings eq. (15) to the form

\[ \frac{\partial u}{\partial \tau} + c(u) \frac{\partial u}{\partial \xi} + \sigma \frac{\partial^3 u}{\partial \xi^3} = 0 \]  

(50)

If \( c \) is constant and \( \partial u/\partial \tau = 0 \), i.e. the linear and stationary case, eq. (50) with substitution \( -c/\sigma = 1/\lambda^2 \) and \( \partial u/\partial \xi = G \) becomes

\[ \frac{d^2 G}{d\xi^2} + \frac{1}{\lambda^2} G = 0 \]  

(51)

As a result in the S/L interface the thermal Meissner effect is absent, \( \lambda \) defining the fundamental thermal length. The absence of this effect is associated with the over-heating phenomenon. Thus, the melting conditions are created.

If \( c(u) \equiv u \), i.e. the non-linear case, eq. (50) takes the form

\[ \frac{\partial u}{\partial \tau} + u \frac{\partial u}{\partial \xi} + \sigma \frac{\partial^3 u}{\partial \xi^3} = 0 \]  

(52)

The solution of eq. (52) (see the method from the Ref. 10) is

\[ u(\xi) = e_1 + Acn^2(\xi; s) \]  

(53)

i.e. bright cnoidal oscillations modes — see Fig. 8, with the following characteristics (see the method from the Ref. 10): i) the wave length of the same form as eq. (39); ii) the average temperature:

\[ u(\theta) = \bar{u} + \text{Asech}^2 \left[ \left( \frac{A}{12\sigma} \right)^{1/2} \theta \right] \]  

(54)

\[ \bar{u} = e_3 + \frac{A}{3s^2} \frac{E(s)}{K(s)} \]  

(55)

We distinguish the degenerations:

i) if \( s \to 0 \) then the solution (53) is reduced to the bright waves packet:

\[ u(\theta) = \bar{u} + A \left[ \cos k\theta + \frac{1}{8} \cos 2k\theta + O(s^4) \right] \]  

(56)

with parameters of the same form as eq. (43), and respectively

\[ w = \bar{u} - \sigma k^2 \left[ 1 - \frac{A^2}{96\sigma^2 k^4} + O(A^3) \right] \]  

(57)

The terms which are not depending on amplitude, give through (57) the phase speed expression of the bright waves packet in linear approximation. From here, by multiplication with \( k \), results the dispersion relation — see Fig. 9.

\[ \omega(k) = \bar{u}k - \sigma k^3 \]  

(58)

ii) if \( s \to 1 \) then the solution (53) is reduced to bright solitons packet with amplitudes \( A = e_1 - e_2 = \text{const.} \) (compared to \( u = \bar{u} \)), with parameters of the same form as eq. (46), and respectively

\[ w \to \bar{u} + 2A/3 \]  

(59)

iii) if \( s = 1 \) then the solution (53) is reduced to the bright soliton (compared to \( u = \bar{u} \))

\[ u(\theta) = \bar{u} + \text{Asech}^2 \left[ \left( \frac{A}{12\sigma} \right)^{1/2} \theta \right] \]  

(60)
— see Fig. 10, with the amplitude $A$, the width $D = (12\sigma/A)^{1/2}$ and speed

$$w = \bar{u} + 2A/3$$

(61)

Then: i) The melting is achieved by bright cnoidal oscillation modes of the interface S/L; ii) The degenerations bright waves packet, bright solitons packet and bright soliton are sequences of the melting process.

5. S/L Interface Excitation Through a Localized Pulse. Primary and Secondary Dendritic Branches Growing

When the initial condition $u(\xi, 0)$ is a localized pulse given by

$$u(\xi, 0) = a_i \phi(\xi/d)$$

(62)

where $a_i$ and $d$ are the amplitude and width of the pulse respectively, (52) can be transformed to an equation which contains a single dimensionless parameter $U$

$$U = a_i d^2 / 12\sigma$$

(63)

This parameter measures the ratio non-linearity — dispersion. For $U \ll 1$ the perturbation is softly non-linear and for $U \gg 1$ strongly non-linear.

Let us give these situations explicitly. Thus: i) for $U = 0, 3$ and initial condition $\phi(\xi) = e^{-\xi^2}$ the numerical solution for $\tau = 0, 5$ is presented in Fig. 11. The perturbation is decomposed in a soliton and a wave packet. i) for $U = 22, 6$ and initial condition $\phi(\xi) = e^{-\xi^2}$ the numerical solution for $\tau = 0, 5$ is presented in Fig. 12 (we choose this particular value to emphasize the strongly non-linear case). The perturbation will be decomposed into six solitons. Since the soliton speed $w$ is proportional to its amplitude, $w \sim A/3$ (see eq. (61)), it will propagate in agreement with the increasing amplitude rate, i.e. the soliton with maximum amplitude and speed will be placed in front of the solitons, the peaks of the solitons being localized on a line. The waves packet remains behind because the group speed which results from the linear dispersion equation $\omega \approx -\beta k^3$, i.e. $\psi = d\omega/dk = -3\beta k^2$ is negative. In Fig. 12 the waves packet is not visible because the amplitude is very small.

The $a_n$ amplitude of the solitons generated by the initial condition (62) can be correlated with the eigenvalues of Schrodinger equation:

$$\partial_{\xi^2} \Psi + 2U(\phi(\xi) + E_n)\Psi = 0$$

through the relation

$$a_n = -2a_nE_n$$

(65)

If $\phi(\xi) < 0$, for a given $U$, there are no solitons and if $\int^{+\infty} \psi(\xi) d\xi = 0$ solitons exist only for $U \gg 1$. If $\int^{+\infty} \psi(\xi) d\xi > 0$ the Schrodinger equation admits a discrete spectrum.

For $U \ll 1$ there is only one eigenvalue $E = -U[\int^{+\infty} \psi(\xi) d\xi]^2/2$, producing at least one soliton with the amplitude, $a \approx a_iU[\int^{+\infty} \psi(\xi) d\xi]^2$.

For $U \gg 1$, the number of solitons will be described by the
asymptotic relation \( N = \frac{2D}{\pi} \int_{\eta(\xi)<0} \{\phi(\xi)\}^{1/2} d\xi \).

We will explicate the previous results for \( \phi(\xi) = \text{sech}^2(\alpha \xi) \). First it results:

\[
\delta_{\xi} \Psi + [-k^2 + u_0 \text{sech}^2(\alpha \xi)] \Psi = 0 \quad (66)
\]

where

\[
2UE_n = -k^2, \quad 2U = u_0 \quad (67)
\]

If we set \( \eta = \tanh(\alpha \xi) \), so \( \delta_\eta = a \text{sech}^2(\alpha \xi) \delta_\eta = a(1-\eta^2) \delta_\eta \), then (64) with \( k = \sigma \alpha > 0, u_0 = N(N+1)a^2 \) becomes

\[
\delta_\eta [(1-\eta^2) \partial_\eta \Psi] + \left\{ N(N+1) - \frac{s^2}{1-\eta^2} \right\} \Psi = 0 \quad (68)
\]

It results the eigenvalues \( k_x = \sigma \alpha \) and the eigen-functions \( \Psi_N(\eta) = \beta_N P^\prime_N(\eta) \), where \( P^\prime_N(\eta) \) are the associated Legendre polynomial and \( \beta_N \), a normalisation constant. In such a context the number of solitons is \( N = \frac{1}{2}(-1 + [1 + (\frac{2\sigma a}{3})^{1/2}]) \).

The interface evolution is conditioned by the force field present inside it (see the eq. (10)). Its increase with the increase of the thermal gradient from the interface, changes permanently the initial pulse hence the bright cnoidal oscillation modes are gradually excited: bright waves for \( U < 1 \), the bright soliton for \( U = 1 \) and bright solitons packet for \( U \to 1 \). Since for \( U = 1 \) and \( U \to 1 \) the modulus of the elliptic function cn is \( s \equiv 1 \) or \( s \to 1 \), the bright cnoidal oscillation modes will be locked on the bright solitonic or bright solitonic packet components.

For \( U \gg 1 \) bright cnoidal modes will exist if the coordinates axis are rotated by \( \pi/2 \) (for details see Ref. 13). In the new conditions the gradient excitation of the bright oscillation modes (bright soliton and bright solitons packet) according to the initial pulse is resumed but according to a perpendicularly direction. Obviously the delimitation of the “components” oscillation modes is not strictly; practically, they are coupled.

The results obtained correspond both to melting and solidification. In case of solidification the excitation of the any component of the dark oscillation waves (dark waves packet, dark soliton and dark solitons packet) will induce a “deformation” of the S/L interface. The top of this “deformation” (corresponding to the dark soliton) is placed in an under-cooled liquid more intensely than the rest of the plane surface; it will advance in the adjacent melt faster than the rest of the front (corresponding to the dark waves packet), constituting the primary branch of the future dendritic structure (for details see Ref. 8).

The escape of the latent solidification heat along the primary dendrite direction leads to the decrease of the undercooling degree and to the decrease of its corresponding increasing process, i.e. the conditions \( U \to 1 \) is satisfied. Further one, on finds an accelerated increase at the level of a new crystallographic direction, thus developing secondary branches of the dendritic structure, orthogonal to the primary branch, i.e. the condition \( U > 1 \) is satisfied. But the development of the secondary dendrite structure will take place until the diminish of the undercooling degree placed in front of their tops. The dendritic growth will continue (tertiar, quatern branchs) until the solidification of the entire liquid. Thus the fractal character of this process is emphasized (see Ref. 16)).

6. Interface S/L. Ordering as a Lattice

Cnoidal oscillation modes indicate that the interface S/L is ordered as a Toda lattice (for details see Ref. 17). With this end in view, let us admit in the solution (53) the transformation \( \theta = \xi - wT \to \theta = \xi + i\omega T \), i.e. to confer space like characteristics to the temporal coordinate. Then solution (53) becomes

\[
u(x_r + iy_r) = e_2 + A a n^2 (x_r + iy_r; k) \quad (69)
\]

where \( x_r = (\frac{k_x}{12\sigma})^{1/2} \frac{k_x}{k} \) and \( y_r = (\frac{k_y}{12\sigma})^{1/2} \frac{k_y}{k} \) are relative coordinates.

We present in Figs. 13(a)-(b) the thermal field for \( e_2 = 0 \) and \( A = 1 \), i.e.

\[
|u| = \left\{ 1 - sn^2[x_r; k]dn^2[y_r; k'] \right\}^{1/2}
\]

\[
1 - dn^2[x_r; k]sn^2[y_r; k']
\]

as function of relative coordinates \( x_r, y_r \) for \( k = k' = 1/\sqrt{2} \). For various values of \( k, i.e. \) for different values of the thermal gradient, this result may describe the evolution of the S/L interface. Note that in Ref. 1) similar result is obtained. Also, note a periodicity of the thermal field (minima and maxima). Consequently, the solid plasma (see paragraph 4) self-structures as a 2D lattice of vortices (the particles — ions and electrons — are associated with objects of vortex type structurated in a vortex lattice — for details see Refs. 18, 19)).

Since minima and maxima of the thermal field overlap with zeroes \( (2m + 1)K + 2mK' \) and poles \( 2mK + i(2n + 1)K' \), \( m, n = 0, \pm 1, \pm 2, \ldots \) of the elliptic function cn of complex argument \( u \) (for details see Ref. 14)).

Fig. 13 a-b Normalized thermal field on the relative coordinates \( x_r \) and \( y_r \) for \( k^3 = k'^3 = 1/2 \).
where
\[ K = \int_0^{\pi/2} \frac{d\varphi}{(1 - k^2\sin^2 \varphi)^{3/2}} \]
\[ K' = \int_0^{\pi/2} \frac{d\varphi}{(1 - k'^2\sin^2 \varphi)^{3/2}}, \quad k^2 + k'^2 = 1 \] (72)
and \(a, b\) are lattice constants along \(Ox\) and \(Oy\) directions, the real part of the complex action
\[ S = \frac{2mD}{2} \ln|u| = mD \ln|cn(u; k)| \] (73)
will define the potential of the vortex lattice. In eq. (73) \(D\) defines diffusion coefficient of Nottale (for details see Ref. 20).

Having in view the interface characteristics (solid plasma) one can choose for the diffusion coefficient the expression \(D = \epsilon \lambda_D/2\), with \(\lambda_D\) the screening length of Debye.

The dynamics of such a lattice is given in Fig. 14 for \(k^2 = 0.1; 0.5; 0.9\) making \(u = \text{const.}\) in eq. (73). Note that in the limit cases \(k \to 0\) and \(k \to 1\) the ‘vortex streets’ are generated along \(Ox\) direction, and \(Oy\) direction, respectively. Only in these limits the lattice gets superconducting properties (for details see Refs. 12, 13). Let us explicate this idea. The generalized impulse of the particle is
\[ P = \frac{dS}{dz} \] (74)
or explicitly, taking into account (71) and (73)
\[ P = mc\lambda_D \frac{K(k)}{2a} \frac{sn(u; k)dn(u; k)}{cn(u; k)} \] (75)
Since only real quantities have direct physical meaning, in what follows we consider only the real part of the expression (75). Using now the relations of transformation for the elliptic function of complex argument into elliptic functions of real argument (for details see Ref. 14) and introducing the notations
\[ s = \text{sn}(\alpha, k); \quad s_1 = \text{sn}(\beta, k'); \]
\[ c = \text{cn}(\alpha, k); \quad c_1 = \text{cn}(\beta, k'); \]
\[ d = \text{dn}(\alpha, k); \quad d_1 = \text{dn}(\beta, k'); \]
\[ \alpha = \frac{K}{a}x; \quad \beta = \frac{K}{a}y \] (76)
eq (75) becomes
\[ P = mc\lambda_D \frac{K(k)}{2a} \text{Re} \left[ \frac{sn(u; k)dn(u; k)}{cn(u; k)} \right] \]
\[ = mc\lambda_D \frac{K(k)}{2a} \frac{c(c^2 - k^2d^2) + s(s^2 - k'^2d'^2)}{(1 - d^2s^2)(c^2s^2 + s^2d'^2d'^2d'^2)} \] (77)
The components of \(P\) are obtained thinking of the vortex distribution as changing into vortex streets. Then, in agreement with the previous observations, the following degenerations are imposed (for details see Refs. 12, 13):

i) \(k = 1, k' = 0, K = \infty, K' = \pi/2\) for \(P_x\), i.e.
\[ P_x = \frac{\pi mc\lambda_D}{2b} \frac{\sin \alpha \cosh \alpha}{\cosh^2 \alpha - \sin^2 \beta} \] (78)

with
\[ \alpha = \frac{\pi x}{2b}, \quad \beta = \frac{\pi y}{2a} \] (79)

ii) \(k = 0, k' = 1, K = \pi/2, K' = \infty\) for \(P_y\), i.e.
\[ P_y = \frac{\pi mc\lambda_D}{2a} \frac{\sin \gamma \cos \gamma}{\sin^2 \gamma + \sin^2 \gamma \sinh^2 \delta} \] (80)
with
\[ \gamma = \frac{\pi x}{2a}, \quad \delta = \frac{\pi y}{2b} \] (81)
The background vortex field intensity \(\mathbf{\Omega}\) will have, through \(\mathbf{\Omega} = m^{-1} \nabla \times \mathbf{\hat{P}}\), the non-zero component
\[ \Omega_z = \frac{\pi^2 c \lambda_D}{4a} \left[ \frac{1}{a} \times \frac{2 \cos 2\gamma \cos^2 \frac{\alpha}{a} \cosh^2 \frac{\alpha}{b} \sinh^2 \frac{\beta}{b} + \sin^2 \gamma \sinh^2 \frac{\alpha}{b}}{(\cos^2 \gamma \cos^2 \frac{\alpha}{a} \cosh^2 \frac{\alpha}{b} \sinh^2 \frac{\beta}{b})} \right] \]

We display in Fig. 15 the background vortex field dependence on the reduced coordinates \( x_r = x/a \) and \( y_r = y/b \). This field is concentrated along the vortex streets.

Averaging, i.e.

\[ \langle \Omega_z \rangle = \int_0^b \int_0^a \Omega_z dz / \int_0^b \int_0^a dS \] (83)

relation (83) becomes

\[ \langle \Omega_z \rangle = \frac{\pi c \lambda_D}{2ab} \left\{ \frac{b}{a} - \frac{1}{\pi} \ln \left[ \frac{\cosh \left( \frac{\pi a}{b} \right) + \cos \left( \frac{\pi b}{a} \right)}{2 \cosh \frac{\pi b}{2a} \cosh \frac{\pi a}{2b}} \right] \right\} \] (84)

In Fig. 16 one can see the dependence of the mean background vortex field on the lattice lengths \( a, b \). The field diverges for \( a = b \), specifying an intrinsic anisotropy of S/L interface. The existence of this anisotropy determines the mechanism of the dendritic growth. For \( a \gg b \) (84) takes the approximate form

\[ \langle \Omega_z \rangle \approx \frac{\pi c \lambda_D}{2a^2} \] (85)

Evaluating (85) for \( a \approx \lambda_D \), we find \( \langle \Omega_z \rangle \approx 10^{14} \text{s}^{-1} \). This means the S/L interface behaves like a nonlinear lattice of oscillators, where the acoustic component of the phononic spectrum is missing.

The background vortex flux is obtained by multiplying (84) with \( S = ab \). From here results

\[ \frac{\Phi}{\Phi_0} = \left\{ \frac{b}{a} - \frac{1}{\pi} \ln \left[ \frac{\cosh \left( \frac{\pi a}{b} \right) + \cos \left( \frac{\pi b}{a} \right)}{2 \cosh \frac{\pi b}{2a} \cosh \frac{\pi a}{2b}} \right] \right\} \] (86)

In Fig. 17 one shows the dependence of the ratio \( \Phi/\Phi_0 \) on the lengths ratio \( a/b \). For \( a \gg b \) (86) takes the approximate form

\[ \Phi \approx \delta \Phi_0, \quad \delta < 1 \] (87)

or through (74)

\[ \oint p_A d\psi \approx \delta (2mc \lambda_D) \] (88)

This means that the quantization of the fluxoid is equivalent with a fractional angular momentum quantization of quasi-particles. It results that in S/L interface anyons appear, i.e., quasi-particles which respect fractional statistics.

For \( a/b < 1 \) the ratio \( \Phi/\Phi_0 \) increases — see Fig. 17, except for the sequences \( a/b = 1/(2n + 1) \), \( n = 0, 1, 2, \ldots \) when

\[ \frac{\Phi}{\Phi_0} = \frac{1}{2n + 1} \] (89)

— see Fig. 18 for \( a/b = 1/7 \). In this way anyons are generated, i.e., quasi-particles with the following characteristics:

i) through (89) with \( \Phi = 2mc \lambda_D \), \( \Phi_0 = 2mc \lambda_D \) having the fractional charge

\[ \lambda_D^* = \frac{\lambda_D}{2n + 1} \] (90)

ii) they satisfy the ‘anyon statistics’, that is a complete loop of a quasi-particle around another produces a phase factor \( e^{i2\pi \theta} \) where \( \theta = 1/(2n + 1) \). This definition of statistics has to do with the dynamical phase acquired by the function as quasi-particles wind around each other. We call it ‘anyon statistics’ to distinguish it from the usual kinematical exchange statistics, though it is a common practice to refer to the net phase simply as ‘statistics’ without qualification. In such a context the presence of anyons indicates a Cantorian-fractal structure of space in the S/L interface.
7. Conclusions

The main results of the present work are as follows:
i) Has been constructed a mathematical model for explaining the solidification and the melting, using a non-linear equation for thermal transfer in S/L interface;

ii) The solidification is associated with the dark cnoidal oscillation modes of the S/L interface, and the melting with the bright cnoidal oscillation modes of the S/L interface. In such a context, the existence of thermal Meissner effect is associated to an over-cooling of the interface, and its absence to an over-heating. Thus, conditions of the development of either solidification or melting are created; iii) The degeneration of the dark cnoidal oscillation modes (dark waves packet, dark soliton, dark solitons packet) are sequences of the solidification process, and the degeneration of bright oscillation modes (bright waves packet, bright solitons, bright solitons packet) are sequences of the melting process;

iv) From the analysis of the S/L interface excitation by means of a localised pulse results the fractal character of the dendrite forming process;

v) One shows the S/L interface self-structurates as a nonlinear Toda lattice and one specifies some of its properties (intrinsic anisotropy, the absence of the acoustic component from the phononic spectrum and the Cantorian-fractal character.

REFERENCES