Linear Intercept Length Distribution in
a Grain Structure Model with Diameter Distribution
of Log-normal Form

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The linear intercept length distribution has been calculated with a grain structure model based on
the assumption that the grain diameter has a log-normal distribution. The grain shapes of the solids
adopted are spherical, cubic and tetrakaidecahedral. Results obtained are as follows.

1. The linear intercept length distribution is not log normal in every model.
2. The difference from a log-normal form in the linear intercept length distribution depends on
the standard deviation of the logarithm of grain diameter, \( \ln \sigma_g \). The difference tends to decrease with the
increment of \( \ln \sigma_g \).
3. The linear intercept length distribution is influenced by the grain shape.
4. The calculated results of the linear intercept length distribution in the present model appreciably differ from the experimental results in the actual grain structure. The difference is, however, attributed to the limit of measurability for the linear intercept length in the experiments.

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distribution, log-normal distribution, grain diameter, grain shape

I. Introduction

Grain size is one of the important factors influencing various properties of polycrystalline metals and alloys. In general, however, the grain size is described only by the mean value and is roughly evaluated. It is, therefore, necessary to consider the distribution of the grain size in order to estimate the grain size more exactly.

Accordingly to Schücker(1), the grain size for a spherical grain can be expressed in several ways, depending on two numbers of dimensions in expression and measurement, as listed in Table 1. The grain size distribution has different forms for the number of dimensions in measurement. The grain size distribution in three dimensions has been found to be approximately log normal in several studies(1)–(5). Furthermore, it has been considered to be closest to the theoretical log-normal distribution among one-, two- and three-dimensional.

While the grain size distribution in one dimension, or linear intercept length distribution, was concluded to be of log-normal type(3)-(6), the distribution was pointed out not to be log normal(6). It was also found by the present authors experimentally that the distribution deviated from log-normal form(7).

In this study, for the purpose of clarifying the relation between the linear intercept length

Table 1 Notations of variables in the most common
distributions (Example: spherical grain shape)(3).

| Number of dimensions in which the measurement has been carried out | Number of dimensions in which the grain size is expressed |
|---|---|---|
| (1) Line | \( l \), length of intercept within the sphere |
| (2) Plane | \( d_A \), diameter of circular section |
| (3) Space | \( D_s \), diameter of sphere |

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distribution and the log-normal distribution, the former distribution is calculated and discussed using a grain structure model with the assumption that the grain diameter has a log-normal distribution.

II. Grain Structure Model

The following assumptions are made for the present grain structure model.

(i) The grain structure consists of the aggregate of grains of which diameters are log-normally distributed.

(ii) The grains have random distribution and random orientation in space, and besides, fill space.

(iii) All of the grains have the same shape, that is, the grain structure is made up of geometrically similar grains.

The assumptions of (i) and (ii) are appropriate due to consistency with the actual grain structure, though “space filling” in the assumption (ii) admits of some discussion as mentioned below. The assumption (iii) is assumed to simplify the model.

An important factor in present model is grain shape. The grain shape was discussed based on its space filling; the objects were such as the cube, the hexagonal prism, the rhombic dodecahedron and the tetrakaidecahedron. It is, however, difficult to discuss space filling in the present model because grains do not have the same size. Further, space filling of grain shape itself is not very important for the purpose of this study. Therefore, “space filling” is considered here to mean that all intersecting lines pass through only insides of grains.

In the consideration above, the following three kinds of solids are adopted as grain shape: (a) sphere, which is completely isotropic, (b) cube, which is appreciably anisotropic, and (c) tetrakaidecahedron (which is abbreviated to TKD at times below), which has been often accepted as the approximative shape of grain.

In the grain structure model, the grain diameter \( D \) has a log-normal distribution from the assumption (i). Thus, the diameter of a sphere circumscribing the grain (which is called “circumscription diameter” below) \( D_{cs} \) is also log-normally distributed. The geometric means of \( D \) and \( D_{cs} \) are denoted by \( D_g \) and \( D_{cs,g} \), respectively, and the geometric standard deviation is denoted by \( \sigma_{D,g} \). The natural logarithm of circumscription diameter \( D_{cs} \) is expressed as \( X \) in eq. (1).

\[
X = \ln D_{cs} \quad \text{or} \quad D_{cs} = e^X \tag{1}
\]

The probability density function of \( X, f_X(X) \) is given by the next equation.

\[
f_X(X) = \frac{1}{\ln \sigma_g \sqrt{2\pi}} \exp \left\{ -\frac{(X - \ln D_{cs,g})^2}{2(\ln \sigma_g)^2} \right\} \tag{2}
\]

The volume of a grain of the circumscription diameter \( D_{cs} \), \( V_D \) is represented as

\[
V_D = K_1 D_{cs}^3 = K_1 e^{3X}, \tag{3}
\]

where \( K_1 \) is a constant. Denoting the total volume and the total number of grains in the model as \( V \) and \( N \), respectively, the mean grain volume \( \bar{V} \) is derived from eqs. (1), (2) and (3) as

\[
\bar{V} = V/N = \frac{1}{N} \int_{-\infty}^{\infty} V_D f_X(X) \, dX = \int_{-\infty}^{\infty} V_D f_X(X) \, dX = K_1 D_{cs,g}^3 e^{36\ln \sigma_g^2/2}. \tag{4}
\]

Then, the volume fraction for grains of circumscription diameter \( D_{cs} \) in the model, \( dF \) is expressed as follows:

\[
dF = \frac{V_D f_X(X) \, dX}{V} \tag{5}
\]

\( \dagger \) In this study, “grain diameter” always refers to “equivalent volume diameter” defined as the diameter of a sphere having the same volume as the grain. It is one of grain sizes in three dimensions.

\( \dagger\dagger \) The diameter (equivalent volume diameter) \( D \) is related to the grain volume \( V \) as \( D = (6V/\pi)^{1/3} \). Thus, if the grain diameter distribution is log normal, the grain volume distribution is also the same.

\( \dagger\dagger \dagger \) The values of \( \ln D_g \) and \( \ln D_{cs,g} \) are the (arithmetic) means of \( \ln D \) and \( \ln D_{cs} \), respectively. The geometric standard deviation of \( D_{cs} \) is equal to that of \( D \) because the value of \( D_{cs} \) is the product of a constant and the value of \( D \). The value of \( \ln \sigma_g \) is the (arithmetic) standard deviation of \( \ln D \) and \( \ln D_{cs} \).
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The standard deviation of the natural logarithm of grain diameter, $\ln \sigma_g$ is set in the range of 0.2 to 0.9, on the basis of the previous results(3)(5) for the grain volume distribution.

III. The Derivation of the Linear Intercept Length Distribution

The calculation of the linear intercept length distribution in the grain structure model was proceeded from the derivation for a single grain in consideration of every direction, and then, on the basis of this distribution, it was developed into the derivation for the grain structure model with the log-normal distribution of grain diameter. Now, for the sake of simplifying the expression in equations, the probability density function and the distribution function in the standard normal distribution are given by eqs. (6) and (7), respectively.

$$\varphi_0(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$  \hspace{1cm} (6)

$$\Phi_0(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-t^2/2} \, dt$$  \hspace{1cm} (7)

To begin with, the natural logarithm of linear intercept length $l$ is denoted by $\lambda$.

$$\lambda = \ln l \text{ or } l = e^{\lambda}$$  \hspace{1cm} (8)

The probability density function and the distribution function of $\lambda$ for a single grain are represented as $f_\lambda(\lambda, X)$ and $F_\lambda(\lambda, X)$, respectively, because the linear intercept length distribution for a single grain depends on the circumscription diameter $D_{cs}$.

Consider the distribution function in the grain structure model $F_\lambda(x)$, which refers to the probability that the value of $\lambda$ is not more than $x$. Straight lines of total length $L$ pass through the grains to the number of $n_L$. The linear intercepts of $\lambda \leq x$ consist of all intercepts for grains of $\ln D_{cs} = X \leq x$ and intercepts of $\lambda \leq x$ for grains of $X > x$. Therefore, the distribution function $F_\lambda(x)$ is

$$F_\lambda(x) = \frac{1}{n_L} \left( \int_{-\infty}^{x} dn + \int_{X=x}^{\infty} F_\lambda(x, X) \, dn \right)$$

where $dn$ is the number of grains of circumscription diameter $D_{cs}$ through which the intersecting lines pass. The sum of the length of intercepts occupied by grains of circumscription diameter $D_{cs}$ is expressed as $LdF$ using the volume fraction $dF$. On the other hand, the length of the lines per a grain of $D_{cs}$, or the mean linear intercept length $\bar{l}$ for grains of $D_{cs}$ is given by the next equation:

$$S_V = 2N_L = 2\bar{l}$$

holds here. Substituting eq. (5) into the arranged eq. (11),

$$dn = \frac{LdF}{K_2 D_{cs}}$$

is obtained. Furthermore, substituting eq. (2) into the eq. (12) and integrating it, the total number of grains through which the intersecting lines pass, $n_L$ is gotten as

$$n_L = \int \frac{L}{K_2 D_{cs}} e^{(\ln \sigma_g)^2/2} \times \int_{-\infty}^{\infty} e^{2\lambda f_\lambda(X) \, dX} \, dX$$

The distribution function of $\lambda$, $F_\lambda(x)$ is, therefore, derived from the substitution of eqs. (12) and (13) in eq. (9) as follows:
1. The sphere grain structure model

First, the linear intercept length distribution for a single grain is derived as follows. The circumscription diameter $D_{c_{s}}$ is equal to the grain diameter $D$ for the sphere. In consideration of only a certain direction, the distribution function of linear intercept $l$, $F_{s}(l, D)$ and the probability density function of $l$, $f_{s}(l, D)$ are expressed as the next equations.

$$F_{s}(l, D) = \frac{l^2}{D^2}$$  \hspace{1cm} (15) \\

$$f_{s}(l, D) = 2l/D^2$$  \hspace{1cm} (16)

Then, for the logarithm of $l$, $\lambda$,

$$F_{s}(\lambda, X) = e^{2\lambda}/e^{2\lambda X}$$  \hspace{1cm} (17) \\

$$f_{s}(\lambda, X) = 2e^{2\lambda}/e^{2\lambda X}.$$  \hspace{1cm} (18)

The distribution function of $\lambda$ for the sphere model, $F_{s}(\lambda)$ is derived from the substitution of eq. (17) into eq. (14) as

$$F_{s}(\lambda) = \Phi_{0}((\lambda - \ln D_{c_{s}})/\ln \sigma_{\phi} - 2 \ln \sigma_{\phi})$$

$$+ \frac{e^{2\lambda}}{D_{c_{s}}^2 e^{2\ln \sigma_{\phi}}} \int_{X}^{\infty} f_{s}(X) dX$$

$$= \Phi_{0}((\lambda - \ln D_{c_{s}})/\ln \sigma_{\phi} - 2 \ln \sigma_{\phi})$$

$$+ \frac{e^{2\lambda}}{D_{c_{s}}^2 e^{2\ln \sigma_{\phi}}} \left[ 1 - \Phi_{0}((\lambda - \ln D_{c_{s}})/\ln \sigma_{\phi}) \right]$$

$$= \Phi_{0}((\lambda - \ln D_{c_{s}})/\ln \sigma_{\phi} - 2 \ln \sigma_{\phi}) + \frac{e^{2\lambda}}{D_{c_{s}}^2 e^{2\ln \sigma_{\phi}}} \left[ 2 - \frac{1}{\ln \sigma_{\phi}} \Phi_{0}((\lambda - \ln D_{c_{s}})/\ln \sigma_{\phi}) \right]$$

$$/\ln \sigma_{\phi} - 2 \Phi_{0}((\lambda - \ln D_{c_{s}})/\ln \sigma_{\phi}).$$

2. The cube grain structure model

The linear intercept length distribution for a single cubic grain is derived as follows. Consider a cube of edge length $a$ and the orthogonal coordinates, $x$, $y$ and $z$ with origin at the center of the cube. Figure 1 shows the projection drawing of the cube in a certain direction, which is described by $\theta$ and $\phi$ of the polar coordinates. The length distribution for the linear intercepts given by the lines perpendicular to the figure is calculated. The linear intercept length in the projection drawing is obtained relatively easily; for example, in the trapezoid ABIE, the linear intercept lengths are 0 and $a \sec \phi$ on the lines AB and IE, respectively, and the linear intercept length at a point is proportional to the distance between the point and the line AB. The distribution function of the linear intercept length, $F_{c}(l, a)$ is, therefore, expressed as

$$F_{c}(l, a) = \Phi_{0}((l - \ln D_{c_{c}})/\ln \sigma_{\phi} - 2 \ln \sigma_{\phi})$$

$$+ \frac{e^{2l}}{D_{c_{c}}^2 e^{2\ln \sigma_{\phi}}} \int_{X}^{\infty} f_{c}(X) dX$$

$$= \Phi_{0}((l - \ln D_{c_{c}})/\ln \sigma_{\phi} - 2 \ln \sigma_{\phi})$$

$$+ \frac{e^{2l}}{D_{c_{c}}^2 e^{2\ln \sigma_{\phi}}} \left[ 1 - \Phi_{0}((l - \ln D_{c_{c}})/\ln \sigma_{\phi}) \right]$$

$$= \Phi_{0}((l - \ln D_{c_{c}})/\ln \sigma_{\phi} - 2 \ln \sigma_{\phi}) + \frac{e^{2l}}{D_{c_{c}}^2 e^{2\ln \sigma_{\phi}}} \left[ 2 - \frac{1}{\ln \sigma_{\phi}} \Phi_{0}((l - \ln D_{c_{c}})/\ln \sigma_{\phi}) \right]$$

$$/\ln \sigma_{\phi} - 2 \Phi_{0}((l - \ln D_{c_{c}})/\ln \sigma_{\phi}).$$

Fig. 1 Projection drawing of the cube with the orthogonal coordinates in a certain direction.
where \( S_l(\theta, \phi) \) is the area of the region in which the linear intercept length is not more than a value of \( l \), as \( S_{\text{tot}}(\theta, \phi) \) is the total area of the projection drawing. From the troublesome calculations using eq. (21), the distribution function \( F_{i}(l, a) \) and the probability density function \( f_{i}(l, a) \) are derived as the following equations:

When \( 0 \leq l < a \),

\[
F_{i}(l, a) = - \frac{l^2}{2\pi a^2} + \frac{8l}{3\pi a},
\]
\[
f_{i}(l, a) = - \frac{l}{\pi a^2} + \frac{8}{3\pi a}.
\]

When \( a \leq l < \sqrt{2}a \),

\[
F_{i}(l, a) = \frac{4}{\pi} \arctan \left( \frac{\sqrt{l^2-a^2}}{a} \right) - \frac{4a}{\pi l^2} \frac{\sqrt{l^2-a^2}}{\pi l^2} - 16(l^2-a^2)^{3/2} \left( 1 - \frac{1}{6\pi} \right) \frac{a^2}{l^2} + \frac{l^2}{\pi a^2} + 1 + \frac{1}{\pi},
\]
\[
f_{i}(l, a) = - \frac{2}{\pi a^3} + \frac{2l}{\pi a^3}.
\]

When \( \sqrt{2}a \leq l \leq \sqrt{3}a \),

\[
F_{i}(l, a) = - \frac{4a}{\pi l^2} \frac{\sqrt{l^2-a^2}}{\pi l^2} - \frac{4a^4}{\pi l^2(l^2-a^2)} - \frac{4}{\pi} \left( \frac{a^2}{l^2} \right) \arctan \left( \frac{a}{\sqrt{l^2-a^2}} \right)
\]
\[
- \left( 1 + \frac{19}{6\pi} \right) \frac{a^2}{l^2} + \frac{40a^2}{3\pi l^2} \frac{28l^2}{3\pi l^2} - \frac{2}{\pi a^2} + 2 + \frac{25}{3\pi},
\]
\[
f_{i}(l, a) = - \frac{16a}{3\pi l^3} \frac{\sqrt{l^2-a^2}}{\pi l^3} + \frac{8a}{3\pi l^3} \frac{\sqrt{l^2-a^2}}{\pi l^3} - \frac{8a^2}{3\pi l^3} \frac{28l^2}{3\pi l^3} - \frac{2}{\pi a^2} + 2 + \frac{25}{3\pi}.
\]

The distribution function and the probability density function for the logarithm of the linear intercept length are obtained based on the relation \( D_{c} = \sqrt{3}a \) as follows. For \( X < \ln \sqrt{3} \),

\[
F_{i}(\lambda, X) = - \frac{3e^{2\lambda}}{2\pi e^{2X}} + \frac{8}{3\pi e^{2X}},
\]
\[
f_{i}(\lambda, X) = - \frac{3e^{2\lambda}}{3\pi e^{2X}} + \frac{8}{3\pi e^{2X}}.
\]

For \( \ln \sqrt{3} \leq \lambda < \ln \sqrt{3}/2 \),

\[
F_{i}(\lambda, X) = \frac{4}{\pi} \arctan \left( \frac{\sqrt{3e^{2\lambda} - e^{2X}}}{e^{X}} \right) - \frac{4e^X \sqrt{3e^{2\lambda} - e^{2X}}}{3\pi e^{2X}} - \frac{16(3e^{2\lambda} - e^{2X})^{3/2}}{9\pi e^{X} e^{2\lambda}}
\]
\[
- \left( 1 + \frac{1}{6\pi} \right) \frac{e^{2X}}{3e^{2\lambda}} + \frac{3e^{2\lambda}}{\pi e^{2X}} + 1 + \frac{1}{\pi},
\]
\[
f_{i}(\lambda, X) = \frac{8e^X \sqrt{3e^{2\lambda} - e^{2X}}}{3\pi e^{2\lambda}} - \frac{16\sqrt{3e^{2\lambda} - e^{2X}}}{\pi e^{X}} + \frac{32(3e^{2\lambda} - e^{2X})^{3/2}}{9\pi e^{X} e^{2\lambda}} + \left( 2 + \frac{1}{9\pi} \right) \frac{e^{2X}}{\pi e^{2X}} + \frac{6e^{2\lambda}}{\pi e^{2X}}.
\]

For \( \ln \sqrt{3}/2 \leq \lambda \leq X \),

\[
F_{i}(\lambda, X) = \frac{4e^X \sqrt{3e^{2\lambda} - e^{2X}}}{9\pi e^{2\lambda}} + \frac{8}{3\pi e^{2X}} - \frac{4e^X}{3\pi e^{2X}}.
\]
Substituting the eqs. (24) into eq. (14), the following equation is derived for the distribution function of $\lambda$ in the model, $F_{1}(x)$:

$$F_{1}(x) = \Phi_{0} ((x - \ln D_{c3,g})/\ln \sigma_{g} - 2 \ln \sigma_{g})$$

$$+ \int_{(x - \ln D_{c3,g})/\ln \sigma_{g} - 2 \ln \sigma_{g}}^{(x + \ln \sqrt{3} - \ln D_{c3,g})/\ln \sigma_{g} + 2 \ln \sigma_{g}} F_{1}(x, \ln \sigma_{g}(Z + 2 \ln \sigma_{g}) + \ln D_{c3,g}) \phi_{0}(Z) \, dZ$$

$$- \frac{3e^{2x}}{2\pi D_{c3,g}^2 e^{2\ln \sigma_{g}}} \left(1 - \Phi_{0} ((x + \ln \sqrt{3} - \ln D_{c3,g})/\ln \sigma_{g})) \right)$$

$$+ \frac{8 \sqrt{3} e^{x}}{3\pi D_{c3,g}^2 e^{3\ln \sigma_{g}/2}} \left(1 - \Phi_{0} ((x + \ln \sqrt{3} - \ln D_{c3,g})/\ln \sigma_{g} - \ln \sigma_{g}) \right) .$$

Further, the probability density function $f_{1}(x)$ is represented as

$$f_{1}(x) = \frac{1}{\ln \sigma_{g}} \phi_{0} ((x - \ln D_{c3,g})/\ln \sigma_{g} - 2 \ln \sigma_{g})$$

$$+ \frac{d}{dx} \int_{(x - \ln D_{c3,g})/\ln \sigma_{g} - 2 \ln \sigma_{g}}^{(x + \ln \sqrt{3} - \ln D_{c3,g})/\ln \sigma_{g} + 2 \ln \sigma_{g}} F_{1}(x, \ln \sigma_{g}(Z + 2 \ln \sigma_{g}) + \ln D_{c3,g}) \phi_{0}(Z) \, dZ$$

$$- \frac{3e^{2x}}{2\pi D_{c3,g}^2 e^{2\ln \sigma_{g}}}$$

$$\times \left\{2 - \frac{1}{\ln \sigma_{g}} \phi_{0} ((x + \ln \sqrt{3} - \ln D_{c3,g})/\ln \sigma_{g} - 2 \Phi_{0} ((x + \ln \sqrt{3} - \ln D_{c3,g})/\ln \sigma_{g})) \right\}$$

$$+ \frac{8 \sqrt{3} e^{x}}{3\pi D_{c3,g}^2 e^{3\ln \sigma_{g}/2}} \left\{1 - \frac{1}{\ln \sigma_{g}} \phi_{0} ((x + \ln \sqrt{3} - \ln D_{c3,g})/\ln \sigma_{g} - \ln \sigma_{g}) \right\} .$$

The second terms in eqs. (26) and (27) were calculated by the numerical calculations. Simpson's rule was used on the numerical integration and the number of the divided interval was set as 200. The numerical differentiation was derived by the following equation.

$$f'(x) \approx \frac{f(x+h) - f(x-h)}{2h}$$

The value of $h$ was set as 0.002 here.

3. The tetrakaidecahedron grain structure model

In the tetrakaidecahedron grain structure model, the analytical calculation of the linear intercept length distribution is difficult because of the very complicated procedure. The distribution is, therefore, calculated by the numerical analysis as follows.

First, the linear intercept length distribution for a single tetrakaidecahedral grain is derived. Assuming that the intersecting lines in every direction pass through the tetrakaidecahedron, the distribution of linear intercept length is obtained as that of the distance between the two points of the intersection.

Consider the tetrakaidecahedron represented in the orthogonal coordinates with origin.
at the center of it as shown in Fig. 2. When
the distance between square faces is taken
as \( d \), the equations of the fourteen faces com-
posing the tetrakaidecahedron is expressed
as follows:

\[
\begin{align*}
  x &= \pm \frac{d}{2}, \quad y = \pm \frac{d}{2}, \quad z = \pm \frac{d}{2} \\
  x+y+z &= \pm \frac{3}{4} d, \quad x+y-z = \pm \frac{3}{4} d \\
  x-y+z &= \pm \frac{3}{4} d, \quad x-y-z = \pm \frac{3}{4} d \\
\end{align*}
\]

(29a–n)

A unit vector of a certain direction, \( \mathbf{n} \) is given by

\[
\mathbf{n} = (\cos \theta \sin \phi, \sin \theta \sin \phi, \cos \phi),
\]

where \( \theta \) and \( \phi \) are presented as in Fig. 2. The
lines distributed uniformly of the direction vec-
tor \( \mathbf{n} \) are expressed by the parameter \( t \) as the
next equations.

\[
\begin{align*}
  x &= p \sin \theta + q \cos \theta \cos \phi + t \cos \theta \sin \phi, \\
  y &= -p \cos \theta + q \sin \theta \cos \phi + t \sin \theta \sin \phi,
\end{align*}
\]

(31a, b)

\[
  z = -q \sin \theta + t \cos \phi,
\]

(31c)

where \( p \) and \( q \) are constants determining the position of the line; the distance between the
origin and the line is \( \sqrt{p^2 + q^2} \).

In a direction of given values of \( \theta \) and \( \phi \),
the approximate distribution function of the
logarithm of linear intercept length, \( \lambda \), \( F_{\lambda, X}(\theta, \phi) \) is derived from the distances between the inter-
secting points obtained by varying \( p \) and \( q \).
The smaller interval of variation of \( p \) and \( q \)
gives a more accurate solution of \( F_{\lambda, X}(\theta, \phi) \).
The interval of \( p \) and \( q \) was set as \( D_{\text{tot}}/200 \).

The objective distribution function \( F_{\lambda}(\lambda, X) \)
is the average value of \( F_{\lambda, X}(\theta, \phi) \) on \( \theta \) and \( \phi \).
Thus, the distribution function \( F_{\lambda}(\lambda, X) \) is ex-
pressed as the next equation:

\[
F_{\lambda}(\lambda, X) = \frac{\int \int F_{\lambda, X}(\theta, \phi) S_{\text{tot}}(\theta, \phi) \sin \theta \, d\theta \, d\phi}{\int \int S_{\text{tot}}(\theta, \phi) \sin \theta \, d\theta \, d\phi},
\]

(32)

where \( S_{\text{tot}}(\theta, \phi) \) is the total area of the projec-
tion drawing of the tetrakaidecahedron.

The integral of the numerator in eq. (32) was
calculated by the numerical integration with
Simpson's rule. The divided intervals of \( \theta \) and
\( \phi \) on the numerical integration were equal to
\( \pi/160 \) rad. The numerical calculation above
provided about four effective places of
decimals for \( F_{\lambda}(\lambda, X) \). The probability density
function \( f_{\lambda}(\lambda, X) \) was derived from numerical
differentiation of \( F_{\lambda}(\lambda, X) \). The distribution
function and the probability density function
for the linear intercept length were also
calculated in the same way.

Next, based on the result for a single tet-
trakaidecahedral grain, the linear intercept
length distribution is derived in the grain struc-
ture model. The distribution function of \( \lambda \) is
obtained by arranging eq. (14) as the next equa-
tion:

\[
F_{\lambda}(x) = \phi_0((x - \ln D_{\text{cs}}, \phi)/\ln \sigma_\phi - 2 \ln \sigma_\phi) + \int_{x}^{\infty} F_s(x, \ln \sigma_\phi(Z + 2 \ln \sigma_\phi) + \ln D_{\text{cs}}, \phi_0(Z) \, dZ
\]
Simpson’s rule was also adopted here for the integral. The domain of the integration was within \([-5, 5]\) in consideration for the character of the integrated function; the error due to this is less than \(5.6 \times 10^{-7}\). The divided interval of \(Z\) in the numerical integration was set as \(0.02/\ln \sigma_g\). The numerical calculation above was supposed to give three or four effective places of decimals for the distribution function \(F_\lambda(x)\).

IV. The Calculated Results and the Discussion

1. The linear intercept length distribution for a single grain

First, the probability density functions of the linear intercept length \(l\), for a single grain, \(f_{ls}(l, D_{cs})\) are shown in Fig. 3. The probability density function for a spherical grain is represented simply as a straight line, as in the figure. The function for a cubic grain is remarkably different from that for a sphere, as it is discontinuous at \(l/D_{cs}=1/\sqrt{3}\) and one of the limiting values is infinity. The function for a tetrakaidecahedral grain is appreciably different from that for a spherical grain, although it has a similar distribution to that of a spherical grain in the range of \(0.3 \sim 0.7\). The function also seems to have some discontinuous points, which is not obviously conformed in the figure owing to the intermittent plots by the numerical analysis. Consequently, the linear intercept length distribution considerably depends on the grain shape.

Then, Fig. 4 shows the distribution function of the logarithm of the linear intercept length, \(\lambda\), \(F_\lambda(\lambda, X)\), plotted on a normal probability paper. This function is more important than the above function \(f_{ls}(l, D_{cs})\) in the grain structure model. The figure also reveals that the grain shape affects the linear intercept length distribution. For a cubic grain, a break in the curve of \(F_\lambda(\lambda, X)\) appears in stead of the continuity in that of \(f_{ls}(l, D_{cs})\). Further, the spread of the distribution is observed to extend in the order as spherical, tetrakaidecahedral and cubic grains.
2. The linear intercept length distribution in the grain structure model

Figure 5 shows the probability density function of $\lambda$, $f_\lambda(x)$ in the sphere grain structure model. The distribution of $\lambda$ is asymmetric, having the peak toward right, at the standard deviation of $\ln D$, $\ln \sigma_g=0.2$. As $\ln \sigma_g$ increases, the spread of the distribution increases and the asymmetry is relieved, and moreover, the value of $\lambda$ at the peak increases appreciably. It seems that the distribution of $\lambda$ is close to normal form at $\ln \sigma_g=0.9$.

Figure 6 shows the probability density function $f_\lambda(x)$ in the cube model. At $\ln \sigma_g=0.2$, the distribution of $\lambda$ is skewed in the opposite sense and the marked difference of the slopes in both sides of the curve is found. The distribution spreads with the relief of the asymmetry, as $\ln \sigma_g$ increases, in the same way as in the sphere model.

Figure 7 shows the probability density function $f_\lambda(x)$ in the TKD model. The distribution of $\lambda$ at $\ln \sigma_g=0.2$ is asymmetric as in the two models mentioned above. The configuration of the curve in this model is intermediate between those in the sphere- and cube-models. The distribution in the TKD model is considered to be somewhat close to that in the sphere model from the configuration and the height of the peak in the curve. The peak of the distribution moves toward right and the distribution approaches to symmetric form with increment of $\ln \sigma_g$. The significant difference between the distributions in the TKD model and in the sphere model is the fact that the spread of the distribution to the left side of the peak is considerably larger than to the right side even at $\ln \sigma_g=0.9$ in the TKD model. In this regard, the distribution in the TKD model is rather similar to that in the cube model.

To exhibit the influence of the grain shape on the distribution more exactly, the probability density functions of $\lambda$ at $\ln \sigma_g=0.6$ in three kinds of the grain structure models are shown in Fig. 8. They are plotted against $x-\ln D_g$,
where $D_g$ is the geometric mean of grain diameter. As in the figure the spread of the distribution to the left side increases in this order: the sphere-, the TKD- and the cube-models. This results from the anisotropy and angularity of the grain shape. On the other hand, the peak location, which varies in the models as shown in Figs. 5, 6 and 7, is independent of the grain shape by plotting against the expression normalized by grain diameter compared with that by circumscription diameter. It is, therefore, considered that "grain diameter" is more useful in the expression of grain size than the "circumscription diameter".

The distribution functions of the logarithm of the linear intercept length, $F_l(x)$ in the three models are shown on normal probability papers in Figs. 9, 10 and 11, to investigate the relationship between the linear intercept length distribution and log-normal distribution. The distribution function of $\lambda$ is not represented as a straight line but a concave curve at every value of $\ln \sigma_g$ in every model, as in the figures. Consequently, it is concluded that the linear intercept length distribution is not log normal. Furthermore, the influence of the standard deviation of $\ln D$, $\ln \sigma_g$ on the distribution function is considerably large; the curve tends to approach to a straight line of the smaller slope with increment of $\ln \sigma_g$. The distribution function also varies with the grain shape. The curves of the function in the sphere model are relatively gentle, while those in both cube- and

![Fig. 8 Probability density functions of the logarithm of linear intercept length, $f_l(x)$ at the standard deviation of the logarithm of grain diameter, $\ln \sigma_g=0.6$ in the three grain structure models.](image)

![Fig. 9 Distribution functions of the logarithm of linear intercept length, $F_l(x)$ at the various standard deviations of the logarithm of grain diameter, $\ln \sigma_g$ in the sphere grain structure model.](image)

![Fig. 10 Distribution functions of the logarithm of linear intercept length, $F_l(x)$ at the various standard deviations of the logarithm of grain diameter, $\ln \sigma_g$ in the cube grain structure model.](image)
TKD-models have clear bends near \( F_{\theta}(x) = 50\% \). The dependence of the configurations in the distribution function curves on the grain shape is corresponding to that in a single grain as shown in Fig. 4.

As described above, the linear intercept length distribution in the model with grain diameter distribution of log-normal form cannot be regarded as log-normal distribution. It is considered that the approximation of the grain shape by a sphere is not very available for discussion of the linear intercept length distribution because of the difference from the actual grain structure.

### 3. Comparison of calculated and experimental results

The assumption of the grain diameter distribution of the log-normal form, used for the calculation of the linear intercept length distribution in this study, is consistent with the actual grain structure. Thus, it is interesting to compare the calculated result with the experimental one for the linear intercept length distribution. Hereafter the calculated result in the TKD model is compared with the experimental one in the previous study.

The experimental results of the grain size distributions in both one- and three-dimensions for the same sample is required to compare with the calculated result in the present grain structure model. Okazaki and Conrad\(^3\) have obtained useful results on grain size distributions measured in one-, two- and three-dimensions for \(\alpha\)-titanium. According to their results, the grain volume distributions are lognormal for the five samples. Then, converting them into the grain diameter distributions, the standard deviations of the logarithm of the grain diameter, \(\ln \sigma_g\), are evaluated as about 0.28. The linear intercept length distribution for these samples are shown in Fig. 12. It is apparent from the figure that the data of this distribution are to lie on concave curves, although they were considered to lie on straight lines or to approximately obey the log-normal distribution by the authors. Compared with the curve at \(\ln \sigma_g = 0.3\) in Fig. 11, the degree of bending of these data is small which...
cannot be attributed to the slight difference in \( \ln \sigma_g \).

It should be noted that the range of the linear intercept length plotted in Fig. 12 is much narrower than the calculated result. In an ordinary experiment, the measurable minimum value or the limit of measurability is determined by the resolution of the measuring system. Thus the linear intercept of which length is less than the minimum value cannot be measured practically. This fact is considered to be the greatest cause for the difference between the calculated and experimental results. Figure 13 indicates the distribution functions calculated using a certain value as the tentative limit of measurability. As shown in the figure, the configuration of the curve varies remarkably with the limit of measurability and in particular the curve, which must be curved, is almost shown by a straight line in the narrower measurable range. The experimental results for \( \alpha \)-titanium in Fig. 12 correspond approximately to the curve c or d in Fig. 13. Accordingly, the linear intercept length distribution must be determined in consideration of the measurable range.

### V. Summary

The linear intercept length distribution has been calculated in a grain structure model with diameter distribution of log-normal form. Then the results obtained are summarized as follows:

1. The linear intercept length distribution is not log normal and the distribution for the logarithm of linear intercept length is indicated as a concave curve on a normal probability paper in every model.

2. The difference from a log-normal form in the linear intercept length distribution depends on the standard deviation of the logarithm of grain diameter, \( \ln \sigma_g \). The difference tends to decrease with increment of \( \ln \sigma_g \).

3. The linear intercept length distribution is influenced by the grain shape. The asymmetry for the probability density function of the logarithm of the linear intercept length, \( f_\alpha(x) \) increases in the order of the sphere-, TKD- and cube-models.

4. The calculated results of the linear intercept length distribution in present model appreciably differ from the experimental results in the actual grain structure. The difference is, however, attributed to the limit of measurability for the linear intercept length in the experiments.

A part of the numerical calculations in this study was carried out using the HITACHI M-280H/S-810 computer system at the Computer Center of the University of Tokyo.

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