An Examination of the Factors Affecting Cooling Curves for Quenching of Steel*

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The cooling curves at the center and surface of silver and steel specimens with various diameters, quenched into various quenchants, were measured and examined. The effects of size, shape, thermal properties and measuring position were discussed. A curve showing only the cooling characteristics of quenchants independent of size, shape or material of specimen was introduced, which will hereafter be referred to as the "master cooling curve". The curve was valid regardless of whether the specimens obeyed Newton's law of cooling. The concept of the master cooling curve is widely applicable to the quantitative determination of steel hardening.

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I. Introduction

There are three main factors to decide whether the steel parts are hardened by quenching. They are the hardenability of steel, size of steel part and cooling ability of the quenchant. There have been several investigations to clarify the quantitative relationship among these factors for quenching. Typical investigations have been made by M. A. Grossmann and his coworkers(1)(2) and by F. Wever and A. Rose(3)(4).

Grossmann et al derived the relationship between hardenability, size of steel and cooling ability of quenchant from the differential equation of heat transfer based on the Newton's law of cooling and other assumptions that make the half-temperature time the standard for hardening. This relationship was not precisely applicable to quenching practice because of the inadequacy of the assumptions for quenching in spite of their clear mathematical calculations.

F. Wever and A. Rose attempted to clarify the hardening behavior of steel during quenching by comparing a cooling curve or cooling time from 800° to 500° with the continuous cooling transformation diagram of steel. This might be a most reasonable method*. However, they did not point out any general method for deriving the cooling curve in the case of a quenching operation. They showed only some examples of cooling curves and cooling times from 800° to 500°C.

The present work is an attempt to find a general method to derive the cooling curve in the case of a quenching operation.

A cooling curve at any point in a quenched specimen depends not only on cooling characteristics of the quenchant but also on thermal properties, size and shape of the specimen. The cooling curve also depends on measuring position in the specimen. If it is assumed that the cooling curve obeys Newton's law of cooling, the factors of thermal properties, size and measuring position can be evaluated mathematically. It seems, however, that the measured cooling curves deviate from Newton's cooling, because the heat transfer factor actually varies with temperature. Hence, the effects of those factors on the cooling curve must

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(2) M. A. Grossmann: Elements of Hardenability, ASM, 1962.
* There is, however, a problem that the continuous cooling transformation diagram of a steel depends also on cooling characteristics. The diagram is determined by a series of coolings where only the parameter of rate of cooling is varied. Hence, the type of diagram determined from the Newton's cooling series may differ slightly from that determined from linear coolings. But, such a difference may be so small technologically that it can be neglected in this work.

Fig. 1 Silver specimen used.
be quantitatively clarified by experiments.

II. Experimental Procedure

Two experiments were conducted in this work. At first, cylindrical silver specimens of four different diameters, as shown in Fig. 1, were used. Their diameters, D, were 10, 15, 20 and 25 mm. Alumel wires of 0.5mm diameter were brazed at the center and surface of specimen. The other thermo-element was the silver specimen itself. The cooling curves were measured during quenching by an oscillograph. Several quenchants shown in Table 1 were used. These quenchants showing typical cooling characteristics were chosen on the basis of a previous research(5). The quenching temperature was fixed at 800°C.

Cylindrical steel specimens of various diameters were prepared from two types of steel, SUJ 2 (0.98% C, 1.4% Cr) and SK 6 (0.8% C), which have suitable hardenability for the experiment. Two chromel-alumel thermocouples were welded at the center and surface of the specimen and the cooling curves were recorded by the same procedure as for the silver specimens. The steel specimens were austenitized in charcoal and

Table 1 Properties of quenchants used.

<table>
<thead>
<tr>
<th>Quenchant</th>
<th>Sp. gr. at 15°C</th>
<th>Flash point* °C</th>
<th>Viscosity (Redwood)</th>
<th>Acid value</th>
<th>Iodine value</th>
<th>Saponification value</th>
<th>Remarks</th>
</tr>
</thead>
<tbody>
<tr>
<td>Distilled water</td>
<td></td>
<td></td>
<td>1.54</td>
<td>98.0</td>
<td>180.0</td>
<td>30 % at 30°C</td>
<td></td>
</tr>
<tr>
<td>City water</td>
<td></td>
<td></td>
<td>136</td>
<td>80.4</td>
<td>123.3</td>
<td>194.4</td>
<td></td>
</tr>
<tr>
<td>10% NaCl soln.</td>
<td></td>
<td></td>
<td>142</td>
<td>80.4</td>
<td>123.3</td>
<td>194.4</td>
<td></td>
</tr>
<tr>
<td>Rape seed oil</td>
<td>0.916</td>
<td>220</td>
<td>281</td>
<td>1.54</td>
<td>98.0</td>
<td>180.0</td>
<td></td>
</tr>
<tr>
<td>Soya bean oil</td>
<td>0.924</td>
<td>235</td>
<td>220</td>
<td>0.45</td>
<td>123.3</td>
<td>194.4</td>
<td></td>
</tr>
<tr>
<td>Sperm skin oil</td>
<td>0.871</td>
<td>208</td>
<td>136</td>
<td>1.66</td>
<td>80.4</td>
<td>126.8</td>
<td></td>
</tr>
<tr>
<td>Coconut oil</td>
<td>0.911</td>
<td>199</td>
<td>131</td>
<td>13.07</td>
<td>15.4</td>
<td>-</td>
<td></td>
</tr>
<tr>
<td>180 turbine oil</td>
<td>0.898</td>
<td>196</td>
<td>551</td>
<td>0.78</td>
<td>14.5</td>
<td>-</td>
<td></td>
</tr>
<tr>
<td>110-dynamo oil</td>
<td>0.900</td>
<td>185</td>
<td>524</td>
<td>0.062</td>
<td>15.4</td>
<td>-</td>
<td></td>
</tr>
<tr>
<td>60-spindle oil</td>
<td>0.898</td>
<td>145</td>
<td>67</td>
<td>0.062</td>
<td>-</td>
<td>-</td>
<td></td>
</tr>
</tbody>
</table>

*Pensky-Martens close cup method.

![Fig. 2 Center cooling curves of silver specimens in various diameters quenched from 800°C into several quenchants.](image)

the austenitizing temperature was 845°C for SUJ2 and 820°C for SK 6. The effect of heat evolution due to transformation in the specimen was observed sometimes on the cooling curves as shown on the curve of the SUJ 2 specimen of 70 mm diameter quenched into water at 20°C. In such a case, the curve was corrected as indicated in Fig. 5. When the effect of heat evolution was not so obvious, the correction was not made. Concerning the heat evolution due to the transformation of steel, no attention was paid in the present work. However, the problem of heat evolution must be clarified in the near future.

III. Results and Considerations

1. Cooling curves of silver specimen
Fig. 2 shows the cooling curves at the center of specimens and Fig. 3 shows those at the surface. The cooling curves differ from each other in specimen diameter and measuring positions even if quenched into an identical quenchant. The cooling curves at the same position in the specimen quenched into the same quenchant hardly shift along the temperature axis but shift considerably along the time axis with increasing diameter.

For any one of the quenchants the cooling time, $t$, to reach a terminal temperature after quenching can be stated as a function of specimen diameter:

$$t = k_1 D^n$$  \hspace{1cm} (1)

where $D$ is the diameter of specimen (cm), $n$ is a size factor constant, and $k_1$ is a constant at the terminal temperature and varies with terminal temperatures. The values of $n$ and $k_1$ are independent of the diameter. They were calculated for each terminal temperature by the least square method putting into Equation (1) the cooling time, $t$, to
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attain the terminal temperature after quenching specimens of various diameters, \( D \). The \( n \)-values at every terminal temperature showed almost an equal value, but they varied slightly by the quenchants. When all the \( n \)-values were averaged, the \( n \)-value was 1.34 at meters in every quenchant agree so well that they are condensed into the single curve for each quenchant as shown in Fig. 4. These curves are independent of the size of silver specimen.

2. Cooling curves of steel specimens

Fig. 5 Center cooling curves of SUJ2-steel specimens in various diameters quenched from 845°C into several quenchants.

Fig. 6 Surface cooling curves of SUJ2-steel specimens in various diameters quenched from 845°C into several quenchants.

The \( k_1 \)-values at every terminal temperature for each quenchant were recalculated using the average \( n \)-value. The \( k_1 \) vs. temperature curves for four diameters in every quenchant agree so well that they are condensed into the single curve for each quenchant as shown in Fig. 4. These curves are independent of the size of silver specimen.

The cooling curves for SUJ 2 were shown in Fig. 5 for the center and in Fig. 6 for the surface. The values of \( n \) and \( k_1 \) at every terminal temperature for each quenchant were calculated for the center by using
Equation (1) and the same treatment as for the silver specimen. The averaged $n$ value was found to be 1.37. This value coincides with 1.34 for the silver specimens at the center. The $k_1$-temperature curves using the average $n$ value, 1.37, can be aligned in a single curve

3. Size factor, $n$

The size factor, $n$, at the center of the specimen showed almost an equal value for silver and steel. A simple calculation was made assuming Newton's law of cooling. Cooling times to attain various terminal temperatures were calculated from Equations (14), (15) and (16) in Appendix II on the basis of Newton's law, for cooling the silver specimens of 10, 15, 20 and 25mm in diameter from 800°C to 20° or 30°C for various heat transfer ratios, $h$, using Russel's Table(6). The thermal diffusivity of silver is 1.54 cm/sec*.

These calculated cooling times were put into Equation (1) to obtain the $n$ value. The $n$ values are shown in Table 2. The $n$ value was smaller than that measured in this investigation, and the difference increased as heat transfer increased.

French(7) gave the relationship between the cooling velocity, $V_e$ at 720°C at the center of a steel specimen and its diameter, $D$, as

$$VD^m = C_1$$  \( (2) \)

for each quenchant, as shown in Fig. 7*. The curves for SUJ 2 and SK 6 are in good agreement.

In Fig. 6 the surface cooling curves varied only slightly with diameter of the steel specimen. The fluctuation of the curves was mainly caused by heat evolution due to the phase transformation of the specimen. The resulting small or indistinct effect of diameter on the surface cooling curve differs from that for the silver specimen, and is caused by a low thermal conductivity of steel. Since, due to the low thermal conductivity of steel, the surface temperature of specimen is rapidly decreased, the diameter dependence appears smaller. Weaver and Rose(3) reported similar results as shall be described later. Thus, at the steel surface the $n$ value approaches zero.

### Table 2. Calculated values of for silver specimen based on the Newton's law.

<table>
<thead>
<tr>
<th>Heat transfer ratio $h$ cm$^{-1}$</th>
<th>Final temp. (°C)</th>
<th>Position</th>
<th>Value of $n$</th>
<th>Mean value</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.80 (larger than $h$-value for 10% NaCl soln.)</td>
<td>20</td>
<td>Surface</td>
<td>0.925</td>
<td>1.04</td>
</tr>
<tr>
<td>0.20 (corresponds to $h$-value for water at 20°C (68°F))</td>
<td>20</td>
<td>Surface</td>
<td>0.964</td>
<td></td>
</tr>
<tr>
<td>0.04 (corresponds to $h$-value for fatty oil)</td>
<td>30</td>
<td>Surface</td>
<td>1.018</td>
<td></td>
</tr>
<tr>
<td>0.02 (corresponds to $h$-value for mineral oil)</td>
<td>30</td>
<td>Surface</td>
<td>1.081</td>
<td></td>
</tr>
<tr>
<td>0.90 (larger than $h$-value for 10% NaCl soln.)</td>
<td>20</td>
<td>Centre</td>
<td>1.245</td>
<td>1.16</td>
</tr>
<tr>
<td>0.30 (corresponds to $h$-value for 35°C (95°F))</td>
<td>20</td>
<td>Centre</td>
<td>1.123</td>
<td></td>
</tr>
<tr>
<td>0.04 (corresponds to $h$-value for fatty oil)</td>
<td>30</td>
<td>Centre</td>
<td>1.049</td>
<td></td>
</tr>
<tr>
<td>0.02 (corresponds to $h$-value for mineral oil)</td>
<td>30</td>
<td>Centre</td>
<td>1.243</td>
<td></td>
</tr>
</tbody>
</table>

* In this Figure, the curves were compared to each other by shifting the time axis to the origin of 800°C in order to avoid differences of initial quenching temperature. The possibility for such a treatment was already reported(5).
where \( m \) and \( C_1 \) were constants, and \( m \) was called the size factor. The \( m \)-value was 1.75 for water and 1.40 for oils. Those values are almost equal to the \( n \) values of this investigation, namely 1.73 for 20°C tap water and 1.16 in average for oils at the centre of SUJ 2 specimen. Apparently, \( m \) and \( n \) are of the same nature.

4. Effect of thermal properties of specimen material

From the results described above, it was clarified that the size factor, \( n \), averaged 1.35 at the center of the specimen independently of the specimen material. However, the temperature-\( k_1 \) curves for silver in Fig. 4 differ considerably from those for steels in Fig. 7, even if the specimens are quenched into the same quenchant. It may be considered that such a difference is caused by the thermal properties of the specimen material. Considering a constant, \( K \), which includes both the factors of cooling characteristics of quenchant and thermal properties of specimen material, one obtains

\[
T = F_1(K, iD^n) = F_2(K, k_1),
\]

where \( T \) is temperature of specimen at the center. For simplification,

\[
T = F_3(K/tD^n) = F_4(K \cdot k_1)
\]

is considered instead of Equation (3). If Newton's law is assumed, the variable of \( K/tD^n \) corresponds to time factor, \( \tau \), stated in Appendix II, and hence it must be an exponent of a power function and also a non-dimensional quantity. The \( n \) value is 1.35, that is, at the interface of the specimen and quenchant; \( \lambda \) is the thermal conductivity (cal/(cm·sec·°C)) of the specimen; \( a^2 \) is the thermal diffusivity (cm²/sec); \( h = C/\lambda \) is the heat transfer ratio (cm⁻²) and corresponds to the twofold value of quenching severity, \( H \), proposed by Grossmann et al. \( s \) is the specific heat (cal/(g·°C)); \( \rho \) is the density (g/cm³) of the specimen material. The \( C \) value mainly shows the cooling characteristic of the quenchant and commonly varies with temperature. The \( C \) value is assumed a constant in the case of Newton's cooling. In this work, it does not matter whether \( C \) is a constant or a function of temperature. The relation between the \( C \) value and temperature is being studied. The product of \( s \) and \( \rho \) is the heat capacity of a unit volume of the specimen and is representative of the thermal properties of the specimen. Equation (4) can be restated as

\[
T = F_5(C \cdot t/(s \cdot \rho \cdot D^n)) = F_6(C/(s \cdot \rho) \cdot k_1).
\]

On the basis of this consideration, the \( T-k_1 \) curves for silver in Fig. 4 and for steels in Fig. 7 were changed into the \( T-(k_1/(s \cdot \rho)) \) type curves by means of dividing the \( k_1 \) value at each terminal temperature by an \( (s \cdot \rho) \) value, where the \( (s \cdot \rho) \) value was 0.626 for silver and 1.104 cal/(cm³·°C) for steels*. The \( T-(k_1/(s \cdot \rho)) \) curves for each quenchant are shown separately in Fig. 8. The curves for silver and two kinds of steel show a good agreement to be aligned into a single curve for each quenchant. These curves show only the cooling characteristic of the quenchant and are not affected by other factors. Each curve is

![Diagram](image)

* See Appendix I.

nearly unity. Hence, \( K \) has a dimension of \( \sqrt{L^2/T} \) and cm/sec in CGS unit. According to the definition of \( K \), mentioned above, it may be stated as

\[
K = \alpha C/\lambda - h \cdot a^2 = (C/\lambda)/(\lambda/s \cdot \rho) = C/\rho
\]

where \( C \) is the heat transfer factor (cal/(cm²·sec·°C)) determined for a quenchant under a given condition. It is reasonable to say from Equation (6) that the curve includes the heat transfer factor, \( C \).
5. Effect of position in specimen on the cooling curve

Wever and Rose(3) reported the cooling times from 800°C to 500°C at various positions of Cr-steel bars (34Cr4), 28, 48 and 95mm in diameter, quenched into water. When the radius of the specimen and the distance from its center to the measuring points are indicated by R and r, respectively, their results can be shown as in Fig. 9. At the surface (r/R = 1), the cooling times are nearly equal; it means that the n value approaches zero. Fig. 10 shows the relation of r/R vs. t/D^n to the data, using n = 1.37. At the center (r/R = 0), three plots show a good agreement; it means that the n value determined above has extensive adaptability.

If Newton's cooling is assumed, the effect of position on the cooling time is indicated by a Bessel or cosine function in the first approximation, as shown in Appendix II. Now, an attempt will be made to indicate the curves in Fig. 10 as:

\[ t/D^n \approx A(r/R) \]

The constants of A and M were determined so as to satisfy the above equation at the center and the surface of the 95mm-diameter. Using the values of A and M, the distribution of the t/D^n-value vs. position, r/R, is also shown in Fig. 10. \( J_0 \) is the Bessel function of the first kind in zero order. The calculated curve in Fig. 10 slightly deviates from the measured curve in the range of r/R = 0.4 to 0.9. In such cases, it will be possible to find the effect of position by a reasonable experimental equation.

6. Effect of shape of specimen

French(7) extended the relation in Equation (2) to the shape of specimen, and gave the following equation:

\[ V = (S/W)^a \cdot C_2 \] (8)

where S and W are the surface area and the volume of specimen, respectively, and C_2 is a constant. The dimension of S/W is the same as that of 1/D. Hence, in a similar manner, Equations (1), (4) and (6) may also be rewritten as follows:

\[ t = k_2 \cdot (S/W)^a, \text{ at a fixed terminal temperature} \] (9)

\[ T = F_k \cdot (S/W)^a \cdot t = F_k \cdot (k_2) \] (10)

\[ T = F_k \cdot (C/sp) \cdot (S/W)^a \cdot t = F_k \cdot (C/sp) \cdot k_2 \] (11)

where k_2 varies with terminal temperatures. The case using k_1 is a particular case of the infinitely long cylindrical bar using k_2. Therefore, there is a relation between k_1 and k_2 as:

\[ k_2 = t \left( \frac{|D|}{(D|/4)} \right)^a = (4)^a \cdot k_1. \] (12)

Fig. 11 shows the comparison of the cooling curves of a silver sphere of 20mm in diameter derived from the T-k_1 curves and the data of Peter(8) and Schultze (9) for the quenchants of water and oils. It appears that the properties and measuring conditions of the quenchants are sufficiently different to warrant the slight deviations in Fig. 11. If the difference in properties and conditions of quenchants are taken into consideration, a good agreement would be found. Thus viewed, the curve irrelevant to the size, shape and material of a specimen may be obtained by transforming the T-k_1(sphere) curve in Fig. 8 in to the T-k_2(sphere) curve.

7. Concept of the master cooling curve

It was shown that the curve irrelevant to the size, shape and material of a specimen at the center could be derived from the above investigations for a quenchant. Further consideration was given to the possibility of an extensively generalized curve which does not depend on the measuring points in the specimen. Such a generalized curve may be called the “master cooling curve” for quenching.

If the master cooling curve for a quenchant is given, it is expected that the cooling curve of a steel part in use can be derived from the master cooling curve. When the cooling curve is compared with the continuous cooling transformation diagram of the steel, the hardening behavior of the steel part can be found clearly. The master cooling curve can be used as a criterion for the cooling ability of a quenchant.

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(8) W. Peter: Arch. Eisenhütten, 20 (1949), 263.
(9) G. R. Schultze: Einiges über Härteöle, June 1953. (Informal pamphlet by Inst. für Erdölorschung in Hannover)
because it shows the heat transfer factor as a function of temperature and is not affected by other factors such as size, shape and material. The concept of the master cooling curve can also be used as the basis for comparing the results of others on the quenchants for hardening steel.

IV. Conclusion

The cooling curves at the center and the surface of silver and steel specimens with various diameters, quenched into various quenchants, were measured and examined. The size factor, \( n \), was almost equal at the centre of the silver and steel specimens, and averaged 1.35. A curve which shows only the cooling characteristic of a quenchant and is independent of the size, shape and material of the specimen was obtained by rewriting the cooling curve into the relation of temperature vs. \( k_2/(s\rho) \), where \( (s\rho) \) was the heat capacity in a unit volume of the specimen. The \( T-k_2/(s\rho) \) curve showed the cooling ability of a quenchant, including the heat transfer factor as a function of temperature, and was valid regardless of whether the cooling curve obeyed Newton's law of cooling. It was also shown that there was a possibility to extend the \( T-k_2/(s\rho) \) curve to a more generalized curve which was independent not only of the size, shape and material of specimen but also of the position in the specimen.

Such a generalized curve may be called the "master cooling curve". The hardening behavior of steel parts, therefore, may be quantitatively found by comparing the cooling curve, which can be derived from the master cooling curve and the size factor, \( n \), with the continuous cooling transformation diagram of the steel.

The authors wish to thank Mr. T. Kondo, Mr. K. Akamatsu and Mr. S. Sakai for their assistance in the experiments.

Appendix I.

The purity of silver used in this work was more than 99.99%. The density \( (\rho) \) of silver is 10.21 g/cm\(^3\) on an average\(^{(10)}\). The mean specific heat \( (s) \) of silver from room temperature to 800°C is 0.0613 cal/(g·°C)\(^{(10)}\). The heat capacity \( (s\rho) \) is 0.626 cal/(cm\(^3\)·°C).

During hardening of steel, there might be both martensite and austenite in the specimen. Grossmann\(^{(2)}\) considered the temperature and thermal diffusivity distributions in the Jominy bar during a hardenability test. On the basis of the same consideration as Grossmann, the thermal conductivity of austenite at 300°C was found to be 0.043 cal/(cm·sec·°C)\(^{(11)}\). Grossmann\(^{(2)}\) also showed the thermal diffusivity of martensite at room temperature to be 0.114 cm\(^2\)/sec. Griffiths et al\(^{(11)}\) gave the mean specific heat of martensite for En8-steel 0.108 cal/(g·°C). The density of hardened tool steel was 7.758 g/cm\(^3\)\(^{(11)}\). Thus, the thermal conductivity of martensite could be calculated as 0.096 cal/(cm·sec·°C). The mean value of both conductivities was \( \lambda_{Fe} = 0.069 \) cal/(cm·sec·°C) during hardening. Grossman\(^{(2)}\) gave the thermal diffusivity of steel\((\alpha_Fet)\) during hardening the value of 0.0625 cm\(^2\)/sec (0.0099 in\(^2\)/sec). Hence, the heat capacity \( (s_{Fe}\cdot\rho_{Fe}) \) was

\[
 s_{Fe}\cdot\rho_{Fe} = \lambda_{Fe}/\alpha_{Fe} = 0.069/0.0625 = 1.041 \text{ cal/(cm·°C)}.
\]

This value seems to be larger than the product of the specific heat and the density of steel, but it is real-

\(^{(10)}\) K. Shiba: Physical Tables, Iwanami, 1943.

\(^{(11)}\) Metals Handbook, ASM, 1948, 313.

\(^{(12)}\) E. Griffiths, P. R. Pallister: J. Iron Steel Inst., 175 (1953), 30.
sonable if the latent heat of martensitic transformation in the specimen is taken into consideration.

**Appendix II.**

The differential equation on heat flow
\[
\frac{\partial^2 T}{\partial r^2} + \frac{1}{r} \cdot \frac{\partial T}{\partial r} = \frac{1}{a^2} \cdot \frac{\partial T}{\partial t} \quad (13)
\]
is solved under the following conditions:
1. the heat flows only in a radial direction in an infinite cylindrical bar (specimen),
2. at the initial state, the temperature of specimen is constant and uniform throughout the specimen,
3. at the boundary of the specimen and the quenchant, the heat transfer obeys Newton's law of cooling, and
4. the thermal properties of specimen are independent of the temperature.

The solution of this equation will be
\[
u = \sum_{n=1}^{\infty} A_n e^{-\lambda_n^2 t} \cdot J_0 (M_n \cdot B) \quad (14)
\]
where \(M\) has all values which satisfy the equation
\[
HJ_0(M) = MJ_1(M) \quad (15),
\]
and
\[
A_n = \frac{2H}{(H^2 + Mn^2) \cdot J_0^2(Mn)} \quad (16)
\]

\[
\tau = \frac{a^2}{R^2} \cdot t \quad (17)
\]
\[
B = \frac{r}{R} \quad (18)
\]
\[
H = hR \quad (19)
\]
and
\[
u = \frac{T - T_1}{T_1 - T_2} \quad (20).
\]

**Notation**

- \(T_1\) : initial temperature of the specimen (a constant)
- \(T_2\) : the constant temperature of the quenchant
- \(T\) : variable temperature at any point of the specimen
- \(t\) : time, measured from the commencement of quenching
- \(a^2\) : thermal diffusivity of the specimen, \(a^2 = \lambda / (s \rho)\)
- \(\gamma\) : thermal conductivity of the specimen
- \(s\) : specific heat of the specimen
- \(\rho\) : density of the specimen
- \(R\) : radius of the specimen
- \(r\) : distance from the centre of specimen
- \(h\) : heat transfer ratio, \(h = C/\lambda\)
- \(C\) : heat transfer factor
- \(\lambda_0\) : the Bessel function of the first kind-order zero
- \(J_1\) : the Bessel function of the first kind-first order