Finite Element Analysis of Debonding and Frictional Sliding in Fiber-Reinforced Brittle Matrix Composites: A Parametric Study

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A comprehensive parametric study on the pullout behavior of a cylindrical model has been carried out using finite element analysis. Interfacial slip-stick phenomena were included in the analysis of a short debond length range. The effects of such important parameters as thermal expansion mismatches between fiber and matrix, fiber volume fraction, material properties of fiber and matrix, and the coefficient of friction at the debond interface, on the debond tip energy release rate, G , and fiber-matrix relative displacement, u , were determined. Effects of these parameters on the applied fiber stress-displacement relation were discussed quantitatively. The results of Hutchinson and Jensen and the present results were compared to understand the accuracy of the shear-lag based analytical results.

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I. Introduction

The fiber pullout behavior as a bridging phenomenon in fiber-reinforced ceramics and an experimental method to infer interfacial properties has been the subject of numerous studies⁽¹⁾⁻⁽¹⁴⁾. While more references on this subject are provided in Refs. (7) and (13), those directly related to the present study will be briefly reviewed below. The analytical model proposed by Hutchinson and Jensen⁽⁶⁾ (hereafter denoted as H & J), close in some respects to those of Gao *et al.*⁽¹⁾, Hsueh⁽⁴⁾, Kerans and Parthasarathy⁽⁷⁾, is expressed in appropriate closed form, and is convenient and general in its treatment of bonding/debonding condition at the interface and anisotropy of the fiber. Finite element analysis (FEA) has also been applied to the pullout problem: Povirk and Needleman⁽¹¹⁾ used this to solve the pullout process; Sigl and Evans⁽²⁾ performed FEA and the obtained result was compared with shear lag analysis. Focusing on interface debonding initiation.

Koss *et al.*⁽⁸⁾ analyzed fiber pullout and pushout models for negligible and large thermally induced stresses. The effect of thermal stresses induced during fabrication on the debonding behavior has also been shown in Honda and Kagawa's work⁽¹⁰⁾. These studies produced fruitful results and better understanding of the pullout behavior.

The analyses based on the shear-lag model are known to be a good approximation for long debond lengths⁽⁶⁾⁽⁷⁾⁽⁹⁾⁽¹⁴⁾ in evaluating the pullout behavior. Effect of important parameters in the pullout model was documented in the literature. However, the fiber debond length is believed to be small when the fiber is just left behind a main matrix crack tip and begins to serve as

bridging. In particular, in a composite consisting of large SCS-6 fibers, the fiber debond length and subsequent slip length could be fairly small⁽¹⁵⁾⁽¹⁶⁾. The shear-lag models may be inappropriate in such situations and the extent of approximation in these models is not well identified. Therefore, the short debond length regime are investigated in this study with stick-slip behavior considered. The stick-slip phenomena sometimes occur in fiber pullout process when the pullout test is repeatedly interrupted for observation or measurement or when the coefficient of static friction is sufficiently greater than the coefficient of dynamic friction. How it influences the stress transfer at the fiber-matrix interface in a pullout test seems to be less explored, with exception of Ref. (11) where fiber-matrix bonding/debonding phenomena have been ignored. Moreover, the effect of thermal stress originating from thermal expansion mismatches of the fiber and matrix, which shows strong singularity at the debond crack tip, was not included in proper form in shear-lag models⁽¹⁾⁻⁽⁹⁾. It is, therefore, essential that more attention should be given to the thermal stress and the parametric dependencies occurring during this process.

The purpose of this paper focuses on a comprehensive parametric study of the cylindrical pullout model to identify the parametric dependencies which take place in the pullout behavior when slip and stick conditions at the debond interface are considered. We particularly look at the following points: (i) effects of the thermal mismatch strain between fiber and matrix, fiber volume fraction, and Young's modulus ratios of the fiber and matrix on the debond tip energy release rate, (ii) effects of these parameters including the coefficient of friction on the matrix crack opening displacements, (iii) simulation of stress-displacement curve and dependence of the curve

on various parameters, (iv) comparison between debond lengths obtained by the present study and shear-lag model to understand the accuracy of the shear-lag analysis in the short debond range.

II. Finite Element Analysis of Pullout Model

The FEA model and its boundary conditions are shown in Fig. 1 and are completely identical to those of our previous report⁽¹⁴⁾. The outer boundary conditions of the cylindrical model are set to stress-free, as the condition is appropriate for the pullout test specimen containing a single fiber with the same geometry. This analysis completely satisfied stress and strain continuities at the bond interface. Interfacial singular shear stress distributions along the bond interface due to both thermal loading and applied stresses were obtained. By considering large-scale contact (stick and slip) conditions in the FEA, changes of sliding areas over the debond interface with the applied stress were simulated and the effects of the coefficient of friction on the debond tip energy release rate were identified. A method to determine the debond length based upon an energy-type criterion was proposed and debonding process of a pullout test was discussed. The authors' previous study⁽¹⁴⁾ showed that the interface debond energy release rate, G_i , as a function of normalized debond length, l/R_f , has physical meanings in fiber bridging conditions when $l/R_f \geq 1 \sim 1.5$, and therefore only this range of the debond length was analyzed in this paper.

A general-purpose finite element code (MENTAT-MARC system) was used to obtain numerical results. Only half of the model shown in Fig. 1 is discretized by

Fig. 1 Axisymmetric cylindrical model, definition of coordinates and forces: (a) longitudinal section, (b) transverse section.

4-node quadrilateral axisymmetric elements because of symmetry. A typical mesh and details at the debond crack tip are shown in Fig. 2. The total length of the pullout model, L , was set to 0.75 mm and the debond crack length, l , ranged from 0.025 to 0.30 mm and this condition satisfied $L \gg l$. Analysis was carried out using 12 different meshes with debond length ratios, l/R_f , ranging $0 < l/R_f \leq 6.0$. Each of the models consisted of 2244 nodes and 1904 elements with the smallest radial length of the elements being $0.00101R_f$ and that for z direction $0.00182R_f$ which corresponded to the actual length of 0.0505 and 0.091 μm , respectively. The element sizes were selected by continuously refining the mesh until further refinement resulted in only slight changes in the numerical solution.

It is assumed that the pullout test is repeatedly interrupted at every increment of the debond length after the initial debonding begins to propagate beyond $l/R_f \geq 1$ and that the stick condition at the fiber-matrix interface predominates at first. Slip is allowed in the process of applying the pullout stress. The computation procedure was as follows. In the first step, thermally induced stress during processing of the specimen was obtained using temperature change ΔT (assumed to be the temperature difference between processing temperature and room temperature). Then, tensile stress, σ , was applied to the end of the fiber in 15 incremental steps with an equal

Fig. 2 Analytical model of FEA: (a) FEA mesh and (b) details of A.

increment fiber stress of $\Delta\sigma=30$ MPa.

Energy release rate G_i of the debond tip was calculated in the following manner. The potential energy of an elastic body, P , from the finite element solution is expressed as⁽¹⁷⁾:

$$P = \frac{1}{2} \{u\}^T [K] \{u\} - \{u\}^T \{f\}, \quad (1)$$

where $[K]$, $\{u\}$ and $\{f\}$ are the master stiffness matrix, the vectors of the nodal displacement and prescribed nodal loads, respectively. The superscript “ T ” denotes transpose. By differentiating eq. (1) with respect to crack length or using an extended J-integral formulation along a closed contour enclosing the crack tip region, the following equation is obtained⁽¹⁹⁾⁽²⁰⁾:

$$G_i = -\frac{\partial P}{\partial A} \Big|_{\text{load}} = -\frac{1}{2} \{u\}^T \frac{\partial [K]}{\partial A} \{u\} + \{u\}^T \frac{\partial \{f\}}{\partial A}, \quad (2)$$

where $A=2\pi R_f l$. The first term of eq. (2) can be obtained by the stiffness derivative method⁽¹⁸⁾⁽¹⁹⁾, while the second term represents work change due to the crack increment Δl from tractions on the crack surface which is equivalent to that of normal and frictional forces in the pullout model. The crack tip node was advanced by 1% of the axial size of the smallest mesh to obtain the first term of eq. (2). The 1% crack tip movement was found to give convergent results⁽¹⁴⁾⁽¹⁸⁾. The second term was equivalent to $\{u_z\}^T (\partial \{f_z\} / \partial A)$ where the subscript z indicates quantities along the fiber axis as the crack surface was closed. u_z and f_z were obtained in the finite element analysis. Once the results of the debonding energy release rate corresponding to various levels of applied stresses and debond lengths were obtained, an interface fracture criterion was imposed on the results to determine the appropriate debond length under a given applied stress. In this way the crack growth behavior during a pullout test was calculated.

III. Effects of Variables on Debond Energy Release Rate and Matrix Crack Opening Displacement

The material properties listed in Table 1⁽²¹⁾ are referred

to as a standard set, which essentially correspond to a combination of SiC matrix reinforced with continuous polymer-derived SiC fiber. Some of the material parameters are varied independently in the following parametric study and the altered parameters are indicated in the corresponding figure.

The debond energy release rate, G_i , is defined and calculated as explained in Section 2. The matrix crack opening displacement, u , is defined as the relative displacements in the fiber axial direction of the fiber and matrix and it is comprised of two portions, as schematically shown in Fig. 3. The matrix crack opening, u , obtained in the above FEA process is expressed as:

$$u = u_T + u_a \quad (3)$$

where u_T is the contribution of the thermal stress and u_a is that of the applied fiber stress (Fig. 3). It should be noted that the above definition of the matrix crack opening displacement is half of the total crack opening displacement for a fiber bridging problem.

1. Effect of the thermal expansion mismatch between fiber and matrix

Taking the coefficients of thermal expansion for the fiber and matrix as α_f and α_m , respectively, an indicator for the thermal expansion mismatch between the fiber and matrix is defined as $\Delta T \Delta \alpha = \Delta T (\alpha_f - \alpha_m)$, where ΔT is the temperature change from bonding. Figure 4 illustrates the effect of the thermal expansion mismatch between the fiber and matrix, on the debond energy release rate, G_i , which is normalized as

Table 1 Properties of fiber and matrix used in the analysis⁽²¹⁾.

Materials	Fiber	Matrix
Young's modulus, E (GPa)	200	400
Poisson's ratio, ν	0.15	0.19
Coefficient of thermal expansion, α ($\times 10^{-6} \text{ K}^{-1}$)	3.0	6.0
Fiber radius, R_f (μm)	50	—
Fiber volume fraction, f		0.25
Friction coefficient μ		0.10
Temperature change, ΔT ($^{\circ}\text{C}$)		975

Matrix crack opening displacement (COD): $u = u_T + u_a$

Fig. 3 Schematic of the matrix crack opening displacement in the pullout model.

Fig. 4 Effects of $\Delta T\Delta\alpha$ on the debond energy release rate ($\sigma=150$ MPa). $\tilde{G}_i = G_i E_m / (R_f \sigma^2)$.

$\tilde{G}_i \equiv G_i E_m / (R_f \sigma^2)$. Values of $\Delta T\Delta\alpha$ signify that the interface is subject to residual compression due to thermal expansion mismatch between fiber and matrix ($\alpha_f < \alpha_m$). The difference between these curves are entirely due to the contribution of thermal residual stresses. Obviously, this effect is substantial and larger $\Delta T\Delta\alpha$ results in larger G_i . As in the case of the influence of the coefficient of friction⁽¹⁴⁾, G_i reaches a maximum at $l/R_f \approx 1.0$ and then decreases with the increase of the debond length. However, G_i tends to become independent of the debond length when the debond length is large. Such a tendency is due to the fact that the stick effect occurs at the debond interface and stick regions are large because of the large debond length. The stick condition keeps the stress fields at the debond tip from making significant change, irrespective of the increase of the debond length. Consequently, G_i does not change significantly when the debond length is large enough because it reflects the local stress fields at the debond tip in nature. Further inspection of Fig. 4 shows that G_i almost independent of the debond length appears at a smaller debond length due to larger stick friction generated by larger residual compression on the debond interface, which is originated from larger $\Delta T\Delta\alpha$. The role of larger $\Delta T\Delta\alpha$ is similar to a larger coefficient of friction at the debond interface⁽¹⁴⁾. Once the interface fracture criterion $G_i \geq G_{ic}$ (G_{ic} : critical interface debond energy release rate) is introduced, the debond length can be determined⁽¹⁴⁾. Figure 4 means that larger $\Delta T\Delta\alpha$ gives a larger debond length. **Figure 5** shows variations of G_i as functions of the applied stress with $\Delta T\Delta\alpha$, for a fixed debond length. In spite of the increase in the applied stress, G_i depends only slightly on the applied stress until $\sigma=220$ MPa in the case of $\Delta T\Delta\alpha=0.994 \times 10^{-3}$, while the applied stress, σ , is 390 MPa in the case of $\Delta T\Delta\alpha=2.925 \times 10^{-3}$. The inflection

Fig. 5 Relations between the normalized debond energy release rate and the applied stress for two values of $\Delta T\Delta\alpha$ ($l/R_f=5.5$). ST and SL stand for the stick region and the slipping region, respectively.

Fig. 6 Relations between the normalized matrix crack opening displacement and the applied stress for three values of $\Delta T\Delta\alpha$ ($l/R_f=2.5$).

in these curves coincides with transition from stick to slip at the debonded interface; the transition occurs at a higher applied stresses since there is a larger normal force when $\Delta T\Delta\alpha$ is large, as noted in Fig. 4. However, G_i strongly depends on the applied stress above these two values of the applied stress. This tendency is remarkable, especially for larger $\Delta T\Delta\alpha$ because the transition of stick state to slip state at the debond interface occurs at these stress levels⁽¹⁴⁾.

Figure 6 shows the variation of u with the applied stress for three different $\Delta T\Delta\alpha$ values. As shown, larger u is obtained for larger $\Delta T\Delta\alpha$. The matrix crack opening displacement due to the thermal stress alone ($u_a=0$,

Fig. 7 Effects of fiber volume fraction, f , on the normalized debond energy release rate ($\sigma = 150$ MPa).

Fig. 9 Effects of the ratios of fiber and matrix Young's moduli on the normalized debond energy release rate ($\sigma = 150$ MPa).

Fig. 8 Effects of fiber volume fraction, f , on the normalized matrix crack opening displacement ($l/R_f = 2.5$).

Fig. 10 Effects of the ratios of fiber and matrix Young's moduli on the normalized matrix crack opening displacement ($l/R_f = 2.5$).

$u = u_T$) becomes large for a larger $\Delta T \Delta \alpha$ and the total opening displacement, u , increases slightly with the increase in the applied fiber stress. For example, the total opening displacement until $\sigma = 450$ MPa is only about half of that due to the thermal stress alone, which means $u_T \gg u_a$. Larger $\Delta T \Delta \alpha$ always produces larger u_T and the effects of $\Delta T \Delta \alpha$ on u_T are substantial. For the same applied stress level, larger $\Delta T \Delta \alpha$ produces larger thermal deformations and consequently the total deformations become larger.

2. Effect of Fiber Volume Fraction, f

The effects of fiber volume fraction, f , on the debond energy release rate, G_i , and the matrix crack opening displacement, u , are shown in Figs. 7 and 8, respectively. Here, fiber volume fraction, f , is defined as $f = (R_f/R_m)^2$, where R_f and R_m are radii of the fiber and matrix, respectively (Fig. 1). The fiber volume fraction, f , is varied by changing the matrix radius R_m while the

fiber radius, R_f , is held constant. The effect of fiber volume fraction, f , on G_i is larger than its effect on u . However, since G_i dictates the debond length and a larger debond length means larger u , a comprehensive grasp of these parameters is necessary. When the interface fracture criterion $G_i \geq G_{ic}$ is applied to the result of Fig. 7, larger fiber volume fraction always yields a smaller debond length for the same level of the applied stress.

3. Effect of Young's modulus ratio, E_f/E_m

The effects of Young's modulus ratio, E_f/E_m , on the debond energy release rate, G_i , and the matrix crack opening displacement, u , are shown in Figs. 9 and 10,

Fig. 11 Relations between the normalized matrix crack opening displacement and the applied stress for three values of the coefficient of friction, μ ($l/R_f=2.5$).

respectively. In the two figures, keeping E_m fixed, only E_f is changed in the computation; it is clear that E_f/E_m strongly influences G_i and u . According to the debond length determination method described above, a larger ratio of E_f/E_m means a larger debond length at the same level of the applied stress.

4. Effect of the coefficient of Friction, μ , on the matrix crack opening displacement, u

The effect on G_i of the coefficient of friction at the debond interface, μ , has been given in Ref. (16) and only the effect on $u(=u_T+u_a)$ is addressed here (Fig. 11). Non-linearity appears in the figure when μ becomes large and it is intuitively understood that at the same level of the applied stress, larger μ results in smaller u .

IV. Simulation OF Stress-Displacement Curves

Theoretical analysis of the present pullout model is useful for the determination of interface properties. In such cases, it is important that the analytical results be related to measurable macro-scale parameters. The fiber stress-displacement curve in a pullout test is frequently used for that purpose. Thus, knowledge of the stress-displacement behavior in a pullout situation is necessary. In this section, a method to simulate theoretical stress-displacement curves will be discussed.

It is assumed that an initial debond length $l/R_f=1.0$ has already been generated. The interface fracture criterion $G_i \geq G_{ic}$ is applied to the energy release rate results to obtain an initial applied fiber stress that satisfies the condition $G_i = G_{ic}$. Then the applied stress is increased intermittently to result in repeatedly interrupted pullout test at every increment of the debond length so that the stick condition at the fiber-matrix interface

predominates at first. The energy release rate in the range of the interface debond length of the bridging condition generally decreases with the increase in debond length but increases with the increase of the applied stress when the other factors are constant. Accordingly, further debonding needs more applied stress, so the third step is to increase the applied stress until $G_i = G_{ic}$ is satisfied, and a new interface debond length is thus obtained. The matrix crack opening displacement, u , can also be calculated corresponding to the new debond length.

Repeating this process, the theoretical stress-displacement curves are obtained. It is noteworthy that as FEA results of G_i and u are discretized values over incremental steps of applied stresses and debond lengths, interpolation over these two factors is inevitable. Linear interpolation over two nearest neighbor applied stresses and debond lengths is applied. The origin of the loading-point displacement is defined as the position of the thermal deformation, *i.e.*, $u=u_a$ at $l/R_f=1.0$. Total crack opening displacement is obtained by eq. (3).

Material properties shown in Table 1 are used as the basis for the simulation. The interface mechanical properties are known to depend on many variables. Several tens of N/m of the critical debond energy release rate and frictional resistance from 2 MPa to over 100 MPa have been reported⁽²²⁾⁻⁽²⁴⁾ in fiber-reinforced ceramics. In the present study, values of these properties are assumed and indicated in the corresponding figures.

Figure 12 shows typical simulated results for $G_{ic}=10$ and 20 N/m. Before debonding propagation from an initial debond length of $l/R_f=1.0$, the stress-displacement curves for $G_{ic}=10$ and 20 N/m experience the same path up to completion of interface slip, and then they separate due to the propagation of interface debonding. The overall stress-displacement curves show non-linearity, primarily due to interface debonding and friction at the debond interface. Furthermore, when the same level of matrix crack opening displacement, u , is compared, larger applied stresses are required for larger G_{ic} . This

Fig. 12 Simulated results of the stress-displacement curve for different critical debond energy release rates.

Fig. 13 Simulated results of the stress-displacement curve for different $\Delta T\Delta\alpha$.

means that the energy absorption in the interface debonding process increases with the increase of G_{ic} .

Figure 13 shows the stress-displacement curves for $\Delta T\Delta\alpha = 2.925 \times 10^{-3}$, 1.930×10^{-3} and 0.994×10^{-3} with $G_{ic} = 20 \text{ N/m}$. It is clear that at the same level of the applied stress, the displacement is smaller with smaller $\Delta T\Delta\alpha$, because the thermally-induced compressive thermal stress is smaller and the debond length becomes smaller. Conversely, when $\Delta T\Delta\alpha = 2.925 \times 10^{-3}$, the compressive thermal stress is large enough to debond the fiber-matrix interface unstably within the whole calculated debond range without any additional applied stress.

This means that too large $\Delta T\Delta\alpha$ will cause a detrimental effect. These results clearly demonstrate the importance of matching properly the thermal expansion coefficients of the fiber and matrix, as well as other parameters discussed above.

One important application of the stress-displacement curves is to infer interface properties of fiber-reinforced ceramics, using the simulated results shown above. Specifically, once two corresponding points of loads and displacements are measured experimentally, it is possible to measure G_{ic} and μ inversely. The results shown in Figs. 12 and 13 are in qualitative agreement with Kerans and Parthasarathy's parametric study⁽⁷⁾. The stick-slip conditions alter the slopes of the stress-displacement curves at various pullout stages.

V. Comparison with Hutchinson & Jensen's Solution

Based on the shear-lag model and Lamé solution, the analysis by Hutchinson and Jensen⁽⁶⁾ is currently one of the most representative and is widely accepted⁽¹²⁾⁽¹³⁾⁽²³⁾⁻⁽²⁵⁾. This model was therefore, selected for comparison, and the difference between their results and those of the present study, as well as the underlying reasons will be provided. Here, we only deal with the debond length ranging $l/R_f \leq 6$.

According to Hutchinson and Jensen⁽⁶⁾, the normalized debond length, l/R_f , and the normalized matrix crack opening displacement, u/R_f , are respectively expressed as,

$$\frac{l}{R_f} = -\frac{1}{2\mu b_1} \ln \left[b_1 \frac{\frac{(1-fa_1)\bar{\sigma}}{f} + a_2 E_m \varepsilon_z^T - \frac{1}{c_1 c_3} \frac{1-f}{f} \sqrt{\frac{E_m G_{ic}}{R_f}}}{\sigma_r^-} + 1 \right], \quad (4)$$

$$\frac{u}{R_f} = (b_2 + b_3) \left[\frac{1-f}{f} \frac{1}{R_f c_1 c_3} \sqrt{\frac{G_{ic}}{E_m R_f}} - \frac{\sigma_r^-}{2\mu b_1^2} (e^{-\xi_l} + \xi_l - 1) \right], \quad (5)$$

where $\xi_l = 2\mu b_1 l/R_f$ and, a 's, b 's and c 's are material- and geometry-relevant parameters. Further information of the model and symbols in eqs. (4) and (5) was provided in Ref. (6).

Present results for comparison are obtained, based on Fig. 4 and G_i results shown elsewhere⁽¹⁴⁾. Figure 14 is the results of l/R_f for two cases of μ under the same given conditions. The results by the two approaches show similar tendencies for different μ , however, they show large differences in absolute values when l/R_f is small. For $\mu = 0.2$, the discrepancy between the two studies tends to lessen, although the underlying reason for the lessening is different; in the present study, the debond energy release rate at the debond tip is small due to the stick condition at the debond interface (see explanations for Fig. 4 in the text), leading to small debond lengths,

while sliding friction at the debond interface resistance probably accounts for the small debond lengths in Hutchinson and Jensen's analysis. In either friction coefficients at the debond interface, the two studies agree well when the applied stress becomes large. The crack opening displacements obtained by the two studies shown in Fig. 15 show a similar tendency with the increase of debond length. The results in Figs. 14 and 15 identify the typical differences in the short debond length range obtained by the two approaches.

The present analysis confirms the strong dependence of the debond stress on the initial debond length when the initial debond length is short, as pointed out in Refs. (6) and (9). The influence of the thermal stress on the initial debond stress should also be significant (Fig. 4).

Fig. 14 Comparison of the normalized debond length for two values of the coefficient of friction, μ , obtained in the present study and H & J's analysis.

Fig. 15 Comparison of the normalized matrix crack opening displacement for two values of the coefficient of friction, μ , obtained in the present study and H & J's analysis.

VI. Conclusion

A comprehensive parametric study of a fiber pullout model was carried out with FEA. The normalized energy release rate for the debond crack tip as a function of the normalized debond length, l/R_f , decreases with the in-

crease of the debond length and then becomes almost independent of l/R_f when l/R_f is large enough. Such a result is attributed to the influence of stick conditions at the debond interface. Effects of major parameters involved in the model on the energy release rate for the debond crack and matrix crack opening displacement during the pullout process are quantified and discussed. The fiber stress-matrix crack opening displacement curves are simulated and the influence of major parameters on the curves are studied. The extent of approximation in the shear-lag models in the range of short debond length is identified. Although the numerical results are for particular cases, they reveal the general tendencies of how the factors influence the behavior of the pullout model.

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