1. Introduction.

Let $X^\alpha (2 > \alpha > 0)$ be an isotropic stable process with index $\alpha$ on $\mathbb{R}^d$ $(d \geq 2)$. We denote the Lévy measure, the exponent and the Green function of $X^\alpha$ by $N(dy)$, $\mathcal{F}(x)$ and $\phi(x-y)$ respectively. Then it is known that $N(dy) = N(|y|) dy = M_1 |y|^{-d-\alpha} dy$, $\mathcal{F}(s) = M_2 s^\alpha$, and $\phi(r) = M_3 r^{\alpha-d}$, where $M_i$, $i = 1, 2, 3$, are positive constants. Further $C(Q_r) = M_4 r^{d-\alpha}$, where $Q_r$ is a ball with radius $r$ and $C(Q_r)$ denotes the capacity of $Q_r$ relative to $X^\alpha$ (Takeuchi [9]). Clearly we have

$$ (1.1) \quad N(1/r) \approx r^{-d} C(Q_r) \approx (r^d \phi(r))^{-1} \approx r^d N(r), \quad r \to 0. $$

Our purposes in this paper are as follows: One is showing (1.1) for a wider class of isotropic Markov processes with stationary independent increments on $\mathbb{R}^d$ $(d \geq 2)$. The other is giving comparison theorems on capacity between two processes.

In the following we introduce the class of Markov processes and certain terminologies with which we will be concerned in this paper. Let $X = (X_t = (x^1_t, x^2_t, \ldots, x^d_t), t \geq 0)$ be a Markov process with stationary independent increments taking values in $\mathbb{R}^d$ (shortly, a process with s.i.i. on $\mathbb{R}^d$). Assume that $X$ is isotropic: if $\tau: \mathbb{R}^d \to \mathbb{R}^d$ is any rotation about 0, then $\tau X$ and $X$ have the same distribution under $P_0$. Then $E \exp (2\pi \langle x, X_t \rangle) = \exp - t \mathcal{F}(x)$ where

$$ \mathcal{F}(|x|) = \mathcal{F}(x) = i + \int_{\mathbb{R}^d_{-\{0\}}} (1 - \cos 2\pi \langle x, y \rangle) N(\text{dy}), $$

$$ (1.2) \quad \int_{\mathbb{R}^d} \frac{|y|^2}{1+|y|^2} N(\text{dy}) < \infty. $$

$\mathcal{F}$ is called the exponent of $X$, and $N(dy)$ is called the Lévy measure of $X$.

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1) This research was supported in part by Matsunaga Science Foundation.
2) See (1.5) for the definition of “$\approx$”.
Note that $N(dy)$ is isotropic: $N(zA) = N(A)$ for every Borel set $A$ in $\mathbb{R}^d - \{0\}$. Throughout this paper we assume that

\[ A1) \quad \int_{\mathbb{R}^d} N(dy) = \infty \]

and

\[ A2) \quad \lambda = \mathcal{F}(0) > 0. \]

In case $d \geq 2$, under the conditions A1) and A2), $X$ has a transition density relative to Lebesgue measure on $\mathbb{R}^d$ (Zabczyk [11]) and so that there exists a potential kernel $\phi(x, y) = \phi(|x - y|)$ such that for bounded universally measurable functions $f$

\[ E_x \left( \int_0^\infty f(X_t) dt \right) = \int_{\mathbb{R}^d} \phi(|x-y|) f(y) dy. \]

We call $\phi(|x-y|)$ a Green function of $X$. Note that $\phi(|x|)$ is excessive and lower semicontinuous. The following fact plays important roles in potential theory of Markov processes. For every compact set $K \subset \mathbb{R}^d$ there exists a unique measure $\mu_K$ on $K$ (we call $\mu_K$ the capacitary measure of $K$ relative to $X$) such that

\[ (1.3) \quad P_x(\sigma_K < \infty) = \int_K \phi(|x-y|) \mu_K(dy), \]

where $\sigma_K = \inf\{ t > 0, X_t \in K \}$. See Blumenthal-Getoor [1] chapter VI for details and further properties. Set

\[ (1.4) \quad C(K) = \mu_K(K) \]

and we call $C(K)$ the capacity of $K$ relative to $X$. Let $f_1(r)$ and $f_2(r)$ be functions on $(0, \infty)$. We write

\[ (1.5) \quad f_1(r) \preceq f_2(r), \quad r \to a, \]

for $a \in [0, \infty]$, if

\[ 0 \leq \liminf_{r \to a} \frac{f_1(r)}{f_2(r)} \leq \limsup_{r \to a} \frac{f_1(r)}{f_2(r)} < \infty \]

Throughout this paper we use the following symbols:

\[ Q_r = \{ x \in \mathbb{R}^d : |x| \leq r \}, \quad S_r = \{ x \in \mathbb{R}^d : |x| = r \} \]

and $|Q_r|$ denotes the volume of $Q_r$. Note that $C(Q_r)$ is monotone increasing with $\lim_{r \to a} C(Q_r) = 0$ in case $d \geq 2$. This fact will be used effectively in § 6.
Now we are going to explain our result. The conclusion of Theorem 2 is as follows: Assume that the Lévy measure has a density $N(|y|)$ relative to Lebesgue measure. If there exists $\tilde{N}(r)$ such that $\tilde{N}(r) \leq N(r)$, $r \to 0$, $r^{d+\beta} \tilde{N}(r)$ is strictly decreasing and $r^{d+\beta} \tilde{N}(r)$ is strictly increasing for some $2 > \beta > \beta_1 > 0$, then

$$ (1.1') \quad \mathcal{P}(1/r) \preceq r^{-d} C(Q_r) \preceq (r^{d[d_1]}(r))^{-1} \preceq r^{d} N(r), \quad r \to 0, $$

where

$$ [\phi]_d^2(r) = \int_0^r \phi(s)s^{d-1}ds. $$

Although it is open whether we can replace $[\phi]_d^2(r)$ by $\phi(r)$ in (1.1'), that is, (1.1) holds under our assumptions on $N(r)$, it will be shown in Theorem 2 that (1.1) is valid if we assume $N'(r) < 0$ in addition. We will also show that (1.1) (excluding the last term $r^{d} N(r)$) holds under a certain assumption on the exponent even if the Lévy measure has not a density (Theorem 1) and in this case $[\phi]_d^2(r)$ can not be replaced by $\phi(r)$ in general as is shown by Zabczyk's example ($\S$ 7). In $\S$ 5 and $\S$ 6 we are concerned with a problem on comparison between capacity for given isotropic processes $X_0$ and $X$ on $\mathbb{R}^d$ with s.i.i. In particular we will see in Theorem 4 that the following statements are equivalent to each other$^5$ if the exponent $\mathcal{P}_s$ of $X_s$ satisfies that there is $\mathcal{P}_s$ such that $\mathcal{P}_s(s) \preceq \mathcal{P}(s)$, $s \to \infty$, $r^{-\beta} \mathcal{P}_s(s)$ is strictly decreasing and $r^{-\beta} \mathcal{P}_s(s)$ is strictly increasing for some $d > \beta > \beta_1 > 0$:

i) $M_2 C_0(K) \preceq C(K) \preceq M_1 C_0(K)$ for every compact set $K$.

ii) $C(Q_r) \preceq C(Q_r)$, $r \to 0$.

iii) $[\phi]_d^2(r) \preceq [\phi_0]_d^2(r)$, $r \to 0$.

iv) $\mathcal{P}(s) \preceq \mathcal{P}_0(s)$, $s \to \infty$.

v) $C(K) = 0$ if and only if $C(K) = 0$ for every compact set $K$. In the above the suffix "0" denotes the notion about $X_0$. In general we can not replace iii) by

vi) $\phi(r) \preceq \phi_0(r)$, $r \to 0$,

as Zabczyk's example shows, but, if both $\phi$ and $\phi_0$ are decreasing, then iii) is equivalent to vi) under our assumption on $\mathcal{P}_s$. This conclusion would be interesting if we recall our previous result on regular points $[7]$: In case $\phi_0(r) = r^{\alpha-d}$, $2 > \alpha > 1$, that is, $X_0$ is an isotropic stable process with index $\alpha$, vi) is equivalent to the statement: $K_0 = K_{s_0}$ for every compact set $K$, where $K_0$ is the set of all regular points of $K$ relative to $X$ (resp. $X_0$), even if we do not assume apriori that $\phi$ is decreasing.

§ 2 contains preliminary material that will be used in the subsequent sections. § 3 and § 4 will be devoted to the proof of (1.1) and (1.1'). Com-

3) Our proof is based on Taylor's result [10].
comparison theorems will be given in §5 and §6. We gather counter-examples in §7 which show that we can not weaken the assumptions in the above theorems.

2. Preliminaries.

Throughout this section we fix an isotropic process $X=(X_t)_{t \geq 0}=(x_1^t, x_2^t, \cdots, x_d^t)_{t \geq 0}$ with s.i.i. on $\mathbb{R}^d$ ($d \geq 2$) with the exponent $\varphi$ satisfying A1) and A2). We denote the Lévy measure of $X$ by $N(dy)$ and the Green function of $X$ by $\phi(|x-y|)$. Note that

\[(2.1) \quad \int_0^\infty \phi(s)s^{d-1}ds < \infty \]

by A2). In general $\phi(r)$ is not finite on $(0, \infty)$ and not monotone. For convenience we introduce the symbol;

\[(2.2) \quad \phi_k(r) = r^{-d+\varphi} \int_0^r \phi(s)s^{d-\varphi}ds. \]

We write simply $\phi*\mu(x)$ instead of $\int \phi(|x-y|)\mu(dy)$.

**Remark 2.1.** As in §1 we denote the capacity of a compact set $K$ by $C(K)$. Since $\phi(|x-y|)$ satisfies the maximum principle; $\sup(\phi*\mu(x); x \in \mathbb{R}^d) = \sup(\phi*\mu(x); x \text{ is in the support of } \mu)$ for every finite measure $\mu$, $C(K)$ is zero if and only if $\phi*\mu$ is unbounded for every non-zero measure $\mu$ with support in $K$.

The following Lemma plays a key role in this research.

**Lemma 1.**

a) $C(Q_r)^{-1} \asymp [\phi]_r(t), r \to 0$.

b) $C(S_r)^{-1} \asymp [\phi]_r(t), r \to 0$, if $d \geq 3$,

and

\[C(S_r)^{-1} \asymp \int_0^r \phi(t)(4r^2-t^2)^{-\frac{1}{2}}dt, r \to 0, \quad \text{if } d=2\]

and if $C(S_r) > 0$. Further $C(S_r) = 0$ if and only if $[\phi]_r(t) = \infty$ (resp. $\int_0^r \phi(t)(4r^2-t^2)^{-\frac{1}{2}}dt = \infty$) in case $d \geq 3$ (resp. $d=2$).

**Proof.** First we note that

\[1=|Q_r|^{-1} \int_{Q_r} P_y(\sigma_{Q_r} < \infty)dy = |Q_r|^{-1} \int_{Q_r} \int_{Q_r} \phi(|x-y|)d\mu Q_y(dx) \]
by (1.2). Since
\[ \sup \left( \phi \ast d\tilde{y}(x) ; x \in Q_r \right) \leq \int_{Q_r} \phi(|x|) \, dx \]
and
\[ \inf \left( \phi \ast d\tilde{y}(x) ; x \in Q_r \right) \geq M \int_{Q_r} \phi(|x|) \, dx, \]
where \( d\tilde{y} = d\tilde{y}|_{Q_r} \) and \( M \) is a positive constant depending only on the dimension \( d \), there exist positive constants \( M_i, i=1,2 \), such that
\[ (2.4) \quad M_i[\phi]^d(r) \leq C(Q_r)^{-1} \leq M_i[\phi]^d(2r). \]
Combining (2.4) with the fact that \( C(Q_{2r}) \leq C(Q_r) \), we have
\[ M_i[\phi]^d(2r) \geq C(Q_r)^{-1} \geq C(Q_{2r})^{-1} \geq M_i[\phi]^d(2r). \]
The proof of a) is finished.

The statement b) was proved essentially in our previous paper. But we repeat it here, because \( \tilde{\phi}(r) \) is not necessarily monotone at present.

First suppose \( C(S_r) > 0 \). Then \( P_x(\sigma_{S_r} < \infty) = 1 \) on \( S_r \) by the isotropy of \( X \), and so that
\[ \text{where } \phi \ast d\mu_S(x) \text{ is constant on } S_r, \]
we have
\[ (2.5) \quad C(S_r)^{-1} = M(d) r^{d-1} \int_0^{2r} \phi(s) s^{d-2} (1 - (s/2r)^2)^{(d-3)/2} ds \]
with a positive constant \( M(d) \) depending only on \( d \). Denote the right hand side of (2.5) by \( \tilde{\phi}(r) \). In case \( d \geq 3 \), (2.5) implies
\[ (2.6) \quad M(d)(3/4)^{(d-3)/2}[\tilde{\phi}]^d(r) \leq \tilde{\phi}(r) \leq M(d) 2^d [\phi]^d(2r). \]
Here we note that if \( [\phi]^d(r) < \infty \) for some \( r \), then \( [\phi]^d(r) < \infty \) for any \( r \) in view of (2.1). In order to compare \( [\phi]^d(2r) \) with \( [\phi]^d(r) \), we study the capacity of balls \( Q'_r \) in \( R^{d-1} \) with radii \( r \). Then
\[ \int_{Q'_r} \phi(|x-y'|) d\nu = \int_{Q'_r} \phi(|x'|) dx' = M_3[\phi]^{d-1}(2r) |Q'_r| = M_3[\phi]^{d-1}(2r), \]
where \( d\nu \) denotes Lebesgue measure on \( R^{d-1} \). Hence, if \( [\phi]^d(r) < \infty, C(Q'_r) > 0 \) be Remark 2.1, and so that \( P_x(\sigma_{Q'_r} < \infty) = 1 \) for every interior point of \( Q'_r \) by the stationarity of \( X \). Therefore, we can see
\[ C(Q'_r)^{-1} \leq [\phi]^{d-1}(r) \geq [\phi]^{d-1}(2r), \quad r \rightarrow 0, \]
in the same way as in the proof of a) by noting that \( C(Q_{2r}) \geq C(Q'_r) \). Com-
bining (2.6) with the identity \[ [\phi]^2_0(r) = [\phi]^{d-1}_1(r) \], we see from (2.5) and (2.6) that \( C(S_r) > 0 \) implies \( [\phi]^2_0(r) < \infty \) and \( C(S_r) \leq [\phi]^2_0(r)^{-1} \), \( r \to 0 \), in case \( d \geq 3 \).

The proof of the case \( d = 2 \) is trivial by (2.5).

Next suppose that \( [\phi]^2_0(r) < \infty \). Then the measure \( \mu_r(dy) \) defined by

\[
\mu_r(dy) = 1/\bar{\phi}(r)\bar{\phi}(dy)
\]

satisfies \( \phi^*\mu_r(x) = 1 \) for every \( x \in S_r \), and so that \( C(S_r) > 0 \) by Remark 2.1. The proof of the case \( d = 2 \) is same. Now we finish the proof of \( \mathbf{b}) \).

**Remark 2.2.** From the proof of \( \mathbf{b}) \) we see that the capacitary measure \( \mu_{S_r}(dy) \) of \( S_r \) relative to \( X \) is given by the formula;

\[
\mu_{S_r}(dy) = C(S_r)\bar{\phi}(dy) = \bar{\phi}(r)^{-1}\bar{\phi}(dy).
\]

For a measure \( \mu \) with compact support define

\[
[\mu]_\phi = \int_{R^d} \int_{R^d} \mu(dx)\phi(|x-y|)\mu(dy),
\]

and

\[
[\mu]_{\bar{\phi}} = \int_{R^d} \bar{\phi}(\xi)^{-1}\bar{\phi}(\xi)^2 d\xi
\]

where \( \bar{\phi}(\xi) = \int_{R^d} \exp(2\pi i \langle \xi, y \rangle)\mu(dy) \). Then it is known that \( [\mu]_\phi < \infty \) if and only if \( [\mu]_{\bar{\phi}} < \infty \) and

(2.7) \( [\mu]_\phi = [\mu]_{\bar{\phi}} \).

Further, for a compact set \( K \), it holds that

(2.8) \( C(K)^{-1} = \inf [\mu]_\phi \)

where infimum is taken over all measures with unit mass and with support in \( K \). Hence we have

**Lemma 2.** Let \( \Phi_i, i=1,2 \), be the exponents of isotropic processes \( X_i \) with s.i.i. on \( R^d \) (\( d \geq 2 \)) satisfying A1) and A2). If

(2.9) \( \Phi_i(s) \leq M_i\Phi_i(s) \)

for every large \( s \), then

(2.10) \( C_i(K) \leq M_iC(K) \)

for all compact set \( K \subseteq R^d \), where \( C_i(K), i=1,2 \), denote the capacities of \( K \) relative to \( X_i, i=1,2 \), respectively and \( M_i, i=1,2 \), are positive constants.
In [8], Orey showed that “$F_1(x) \leq M F_2(x)$, $x \in \mathbb{R}^d$, $|x|$; sufficiently large” implies (2.10) for a larger class of Markov processes with s.i.i. on $\mathbb{R}^d$ (including unsymmetric case), although he asserted only that $C^1(K) = 0 \Rightarrow C^2(K) = 0$.

Now we consider the projection $X^{(k)}$ on $\mathbb{R}^k$ of $X$, $0 < k < d$, that is, $X^{(k)} = (x_1, \ldots, x_k)$. Then $X^{(k)}$ is also an isotropic process with s.i.i. on $\mathbb{R}^k$. Let $\varphi^{(k)}$ be the exponent of $X^{(k)}$ and $p^{(k)}(t, A)$ be its transition probability. Then

$$P^{(k)}(t, A) = P(t, A \times \mathbb{R}^{d-k}),$$

where $P(t, A)$ denotes the transition probability of $X$. Set

$$\phi^{(k)}(r) = M(d, k) \int_r^\infty \phi(s) s (s^2 - r^2)^{(d-k-2)/2} ds,$$

where $M(d, k)$ is the area of the surface of the unit ball in $\mathbb{R}^{d-k}$. Then

$$\phi^{(k)}(|x-y|) \text{ is the Green function of } X^{(k)}.$$

Indeed, since $\int_0^\infty P(t, B) dt = \int_B \phi(|y|) dy < \infty$ for a Borel set $B$ by A2), we have

$$\int_0^\infty P^{(k)}(t, A) dt = \int_0^\infty P(t, A \times \mathbb{R}^{d-k}) dt = \int_{A \times \mathbb{R}^{d-k}} \phi(|x|) dx < \infty,$$

and so that the function

$$\phi^{(k)}(\sqrt{x_1^2 + \cdots + x_k^2}) = \int_{\mathbb{R}^{d-k}} \phi(\sqrt{x_1^2 + \cdots + x_k^2 + \cdots + x_d^2}) dx_{k+1} \cdots dx_d$$

is integrable on $\mathbb{R}^k$ by Fubini’s theorem and is a Green function of $X^{(k)}$. Note that $\phi^{(k)}(r)$ is continuous and monotone decreasing on $(0, \infty)$ if $d - k - 2 \geq 0$. If the Lévy measure $N(dy)$ of $X$ has the density $N(|y|)$ relative to Lebesgue measure, then, setting

$$N^{(k)}(r) = M(d, k) \int_r^\infty N(s) s (s^2 - r^2)^{(d-k-2)/2} ds,$$

$N^{(k)}(|y|)$ is the density of Lévy measure of $X^{(k)}$ on $\mathbb{R}^k$. We also give a formula for $N^{(k)}(dt) = N(dt \times \mathbb{R}^{d-k-1})$ when $N(dy)$ does not necessarily have a density. Since $N(dy)$ is an isotropic measure on $\mathbb{R}^d - \{0\}$, there exists a measure $N(dr)$ on $(0, \infty)$ such that

$$N(A) = \int_0^\infty \varepsilon_r(A) N(dr),$$
where \( \varepsilon_r(dy) \) is the uniform measure on \( S_r \) with unit mass. Since \( N^{(2)}(A) = N(A \times R^{d-1}), A \subset R^d \), we get

\[
N^{(1)}(dt) = N^{(1)}(t)dt,
\]

\[
N^{(1)}(t) = M(d, 1) \int_{|t|}^{\infty} r^{-1}(1-(t/r)^{(d-3)/2})N(dr).
\]

The above results are used in the proof of the following lemma and Theorem 4.

**Lemma 3.** Assume \( d \geq 3 \). It holds that

\[
\mathcal{F}(cs) \leq c^{-1} \mathcal{F}(s)
\]

for every \( s \) and \( c \) such that \( 0 < c < 1 \).

**Proof.** Since

\[
\mathcal{F}^{(1)}(s) = A + 2(1 - \cos(2\pi st))N^{(1)}(dt),
\]

we have

\[
\mathcal{F}^{(1)}(cs) = A + \int_0^{\infty} (1 - \cos(2\pi st))N^{(1)}(t/c)dt(2/c)
\]

\[
\leq A + \int_0^{\infty} (1 - \cos(2\pi st))N^{(1)}(t)dt(2/c)
\]

\[
\leq \mathcal{F}^{(1)}(s)(2/c).
\]

Noting that \( \mathcal{F}^{(1)}(s) = \mathcal{F}(s) \), the proof is complete.

**3. Relations among capacity of balls, the exponent and the Green functions.**

In this section we study a process \( X \) on \( R^d (d \geq 2) \) as in \( \S 2 \). First we list up the conditions on the exponent \( \mathcal{F} \) used in our theorem. The following A3) always holds in case \( d \geq 3 \) by Lemma 3, but we impose it upon \( \mathcal{F} \) in case \( d = 2 \).

A3) There exists a measurable function \( \mathcal{F}_1 \) on \([0, \infty) \) with the property that \( \mathcal{F}_1(s) \asymp \mathcal{F}(s), s \to \infty \), and \( \mathcal{F}_1(cs) \leq \mathcal{F}_1(s)/c \) for every \( s \in (0, \infty) \) and every \( 0 < c < 1 \).

One or both of the following conditions are imposed on \( \mathcal{F} \) in our theorem.

A4), A5) There exists a measurable function \( \mathcal{F}_2 \) (resp. \( \mathcal{F}_3 \)) on \([0, \infty) \) with the property that \( \mathcal{F}_2(s) \asymp \mathcal{F}(s), s \to \infty \) (resp. \( \mathcal{F}_3(s) \asymp \mathcal{F}(s), s \to \infty \)) and
The capacity for isotropic Markov processes $s^{-\Psi_2}(s)$ (resp. $s^{-\Psi_3}(s)$) is monotone decreasing (resp. monotone increasing).

In the above and in the subsequent sections we agree that "monotone" means "strictly monotone".

Now we state our theorem.

**Theorem 1.** Let $X$ be an isotropic process with s.i.i. on $\mathbb{R}^d$ ($d \geq 2$) satisfying A1) and A2). In case $d=2$, A3) is assumed in addition.

(I) If $A4)_{\beta_1}$ holds for some $\beta_1$ such that $d>\beta_1>0$, then

$$\Psi(1/r) \asymp r^{-d} C(Q_r) \asymp (r^d [\phi^d](r))^{-1}, \quad r \to 0.$$  

In general $A4)_{\beta_2}$, $d>\beta_2>0$, can not be replaced by $A4)_{\beta_1}$.

(II) If $A4)_{\beta_1}$ and $A5)_{\beta_2}$ hold for some $\beta_1$ and $\beta_2$ such that $d>\beta_1>0$ and $d>\beta_2>1$, then

$$\Psi(1/r) \asymp r^{-d} C(S_r) \asymp (r^d [\phi^d](r))^{-1}, \quad r \to 0,$$

in case $d \geq 3$, and

$$\Psi(1/r) \asymp r^{-2} C(S_r) \asymp \left( r \int_0^{2r} \phi(s)(4r^2-s^2)^{-1/2} ds \right)^{-1}, \quad r \to 0,$$

in case $d=2$. In general $A5)_{\beta_2}$, $d>\beta_2>1$, can not be replaced by $A5)_{\beta_1}$.

Before the proof we note that

\begin{equation}
\int_{S_r} \exp(2\pi i \langle x, y \rangle) dy = M_i(d) |x|^{-d/2} J_{d/2}(2\pi |x| r)^{d/2} \tag{3.1}
\end{equation}

and

\begin{equation}
\int_{S_r} \exp(2\pi i \langle x, y \rangle) \varepsilon_r(dy) = M_i(d)(|x| r)^{-(d-3)/2} J_{(d-2)/2}(2\pi |x| r), \tag{3.2}
\end{equation}

where $\varepsilon_r(dy)$ denotes the uniform measure on $S_r$ with unit mass and $M_i(d)$, $i=1,3$, are positive constants depending only on the dimension $d$. Here we denote the Bessel function of degree $\nu$ by $J_{\nu}(x)$; that is

\begin{equation}
J_{\nu}(x) = \frac{(x/2)^\nu}{\Gamma(\nu+1/2)\Gamma(1/2)} \int_0^\pi \cos(x \cos \theta) \sin^{\nu} \theta \, d\theta. \tag{3.3}
\end{equation}

The following properties are used in the proof of our theorems.

\begin{equation}
|J_{\nu}(x)| \leq M_1 |x|^{\nu}, \quad M_1; \text{ a positive constant.} \tag{3.4}
\end{equation}

\begin{equation}
\int_0^\infty |J_\alpha(t)|^\beta t^{-\alpha} \, dt < \infty \quad \text{if } 2\nu + 1 > \alpha > 0. \tag{3.5}
\end{equation}
In the proof we denote absolute positive constants by $M_1, M_2, \ldots$. Note that A3) always holds in case $d \geq 3$ by virtue of Lemma 3.

**Proof of (I) of Theorem 1.** From the proof of Lemma 1 we have

\begin{align}
M_1[\phi](r)[Q_r]^2 \leq & \int_{Q_r} \int_{Q_r} dx \phi(|x-y|) dy \leq M_3[\phi]^2(\rho) |Q_r|^2, \tag{3.6}
\end{align}

and so that

\begin{align}
M_1 \frac{|Q_r|^2}{C(Q_r)} = & \int_{Q_r} \int_{Q_r} dx \phi(|x-y|) dy \leq M_5 \frac{|Q_r|^2}{C(Q_r)}. \tag{3.7}
\end{align}

Using (3.1) and (2.7), we have

\begin{align}
\int_{Q_r} \int_{Q_r} dx \phi(|x-y|) dy = & M_9 \int_0^\infty \frac{1}{t^{d+1}} (J_{\alpha/\gamma}(t)^2 dt. \tag{3.8}
\end{align}

Hence, putting

\[ I_r(A, B) = \int_{B}^{A} \frac{1}{t^{d+1}} (J_{\alpha/\gamma}(t)^2 dt, \]

it holds that

\begin{align}
M_{10} \frac{r^d}{C(Q_r)} \leq & I_r(0, \infty) \leq M_9 \frac{r^d}{C(Q_r)}. \tag{3.9}
\end{align}

By (3.9) we have

\begin{align}
\sup_{0 \leq a \leq \pi} \mathcal{W}(t/2\pi r)^{-1} \int_{\pi}^{\pi} \frac{1}{t} (J_{\alpha/\gamma}(t)^2 dt \leq M_9 \frac{r^d}{C(Q_r)},
\end{align}

and therefore, using A3), we get

\begin{align}
\mathcal{W}(1/r) \geq M_9 r^{-d} C(Q_r), \quad 0 < r < \delta_1. \tag{3.10}
\end{align}

Next we fix $\varepsilon$ such that $M_9/2 > \varepsilon > 0$ and show that there exist $B > 1$ and $\delta_2 > 0$ such that

\begin{align}
I_r(B, \infty) \leq & \varepsilon \frac{r^d}{C(Q_r)}, \quad 0 < r < \delta_2. \tag{3.11}
\end{align}

Indeed, it follows from (3.10) that

\begin{align}
I_r(B, \infty) \leq & M_9 \frac{r^d}{C(Q_{2\pi B})} \int_{B}^{\infty} \frac{1}{t^{d+1}} (J_{\alpha/\gamma}(t)^2 dt
\end{align}

and then by Lemma 1
Using Lemma 1 again and (3.5), we can get (3.11) choosing a sufficiently large $B$.

Fix $\alpha$ such that $1 > \alpha > \beta / d$. Then, by A4)_{\beta}, it holds for sufficiently small every $r$ that

$$I_r(r^\alpha, A) \leq M_{12} \frac{A^{\beta_1}}{\Upsilon(A/2\pi r)} \int_{r^\alpha}^A t^{-(1+\beta_1)}(J_{d/2}(t))^2 dt.$$ 

Therefore, noting (3.4), we can choose $A$, $0 < A < 1$ such that

$$(3.12) \quad I_r(r^\alpha, A) \leq \varepsilon - \frac{A^{\beta_1}}{\Upsilon(A/2\pi r)}, \quad 0 < r < \delta_0.$$ 

In the following we fix $A$ and $B$ so that (3.11) and (3.12) hold. Note that it holds that

$$I_r(0, r^\alpha) \leq M_{13} \int_0^{r^\alpha} t^{d-1} dt = M_{13} r^{d-\beta_1},$$

by (3.4) because $0 < t = \Upsilon(0) \leq \Upsilon(t)$. Then by A4)_{\beta}, and A3) we have

$$(3.13) \quad I_r(0, r^\alpha) \leq M_{14} r^{d-\beta_1} \frac{1}{\Upsilon(1/r)} \leq M_{15} r^{d-\beta_1} \frac{1}{A \Upsilon(A/2\pi r)}.$$

On the other hand

$$(3.14) \quad I_r(A, B) \geq M_{16} \frac{A^{\beta_1}}{\Upsilon(A/2\pi r)} \int_1^A t^{-(\beta_1+1)}(J_{d/2}(t))^2 dt$$

by A4)_{\beta}. Noting that $M_{10}$ and $M_{16}$ are positive constants independent of the choice of $A$ and $B$, it follows from (3.9), (3.12), (3.13) and (3.14) that there exists $\delta_0 > 0$ such that

$$(3.15) \quad \frac{M_5}{2} \frac{r^d}{C(Q_r)} \leq M_9 \frac{r^d}{C(Q_r)} \leq \varepsilon \frac{r^d}{C(Q_r)} \leq 2I_r(A, B)$$

for $0 < r < \delta_0$. Hence

$$\inf_{4 \leq \varepsilon \leq B} \Upsilon(t/2\pi r) \leq M_{17} r^{-d} C(Q_r),$$

and so that, using A3), we have

$$AB^{-1} \Upsilon(A/2\pi r) \leq M_{18} r^{-d} C(Q_{2\pi r/A}) = M_{19} (2\pi r/A)^{-d} C(Q_{2\pi r/A}).$$
Hence
\[(3.16) \quad \Psi(1/r) \leq M_{10} r^{-d} C(Q_r), \quad 0 < r < \delta_5.\]
Combining (3.10) with (3.16), we have
\[\Psi(1/r) \leq r^{-d} C(Q_r), \quad r \to 0.\]
The proof of (I) of Theorem 1 is finished.

REMARK 3.1. (3.10) and (3.11) are valid without the condition A4)\(_{\beta_1}.

Proof of (II) of Theorem 1. Since C(Q_r) \(\nsubseteq\) C(S_r), it follows from (3.10) that
\[(3.17) \quad \Psi(1/r) \geq M_{20} r^{-d} C(S_r).\]
Since \(1 = \int_{S_r} \varepsilon_r(dx) P_x(\sigma_{s_r} < \infty) = C(S_r) \int_{S_r} \phi \ast \varepsilon_r(x) \varepsilon_s(dx)\) by Lemma 1, we have by (3.2)
\[(3.18) \quad C(S_r)^{-1} = M_{21} r^{-d} \int_0^\infty \frac{t}{\Psi(t/2\pi r)} (J_{(d-2)/2}(t))^2 \, dt.\]
Set
\[I_r(A, B) = \int_A^B \frac{t}{\Psi(t/2\pi r)} (J_{(d-2)/2}(t))^2 \, dt.\]
Then it holds that
\[(3.19) \quad M_{22} r^{-d} C(S_r)^{-1} \leq I_r(0, \infty) \leq M_{23} r^{-d} C(S_r)^{-1}.\]
If we fix \(a \) such that \(1 > a > \beta_1/d\), we can prove using (3.17) that that
\[(3.20) \quad I_r(0, A) \leq M_{24} A^{\beta} \Psi(A/2\pi r)^{-1}, \quad 0 < r < \delta_5,\]
just as in the proof of (3.12) and (3.14). For \(B > 1\) we have
\[I_r(B, \infty) \leq M_{25} \frac{B^{\beta_1}}{\Psi(B/2\pi r)} \int_B^\infty t^{-\beta_1} (J_{(d-2)/2}(t))^2 \, dt\]
by A5)\(_{\beta_2}.\) Since \(d > \beta_2 > 1\), it follows from A3) that
\[(3.21) \quad I_r(B, \infty) \leq M_{26} B^{\beta_2+1} A^{-\beta_1} \Psi(A/2\pi r)^{-1}.\]
On the other hand we have by A5)\(_{\beta_3}\)
\[(3.22) \quad I_r(A, B) \leq M_{27} \frac{A^{\beta_3}}{\Psi(A/2\pi r)} \int_A^B t^{-\beta_3} (J_{(d-2)/2}(t))^2 \, dt.\]
Take such A and B. Then we have
\[ M_{28} r^d C(S_r) \leq F(A/2\pi r)^{-1}, \quad 0 < r < \delta, \]
and so that, using A5)$\beta$, we can prove that
\[ M_{29} r^{-d} C(S_r) \geq F(1/r), \quad 0 < r < \delta. \]
Combining (3.23) with (3.17), we get
\[ F(1/r) \geq r^{-d} C(S_r), \quad r \to 0. \]
Now the proof of Theorem 1 is complete.

4. Relations among the Lévy measure, the exponent and the Green function.

In this section we point out how the order of divergence at the origin of the density of the Lévy measure governs the order of divergence at infinity of the exponent and the property of the Green function. As a result we can show that (1.1) or (1.1') holds if the Lévy measure has a good density.

Our conditions on the density $N(|y|)$ of the Lévy measure are as follows:

A6)$\beta$ (A7)$\beta$) There exists a positive measurable function $N_1(r)$ (resp. $N_2(r)$) with the property that $N_1(r) \geq N(r), r \to 0$, and $r^{d+\beta} N_1(r)$ is monotone decreasing (resp. $N_2(r) \geq N(r), r \to 0$, and $r^{d+\beta} N_2(r)$ is monotone increasing).

THEOREM 2. Let $X$ be an isotropic process with s.i.i. on $\mathbb{R}^d$ $(d \geq 3)$ satisfying A1) and A2). Suppose that the Lévy measure has a density $N(|y|)$ relative to Lebesgue measure on $\mathbb{R}^d$.

(I) If A6)$\beta_1$ and A7)$\beta_2$ are satisfied for some $\beta_1$ and $\beta_2$, $2 > \beta_2 > \beta_1 > 0$, then
\[ \psi(1/r) \geq (r^d [\varphi]^2(r))^{-1} \geq r^{-d} C(Q_r) \geq r^d N(r), \quad r \to 0. \]

(II) If we assume that $N(r) \in C'(0, \infty)$ and $N'(r) < 0$ in addition to A6)$\beta_1$ and A7)$\beta_2$, $2 > \beta_2 > \beta_1 > 0$, then we can replace $[\varphi]^2_1(r)$ by $\varphi(r)$ in (4.1), that is, (1.1) holds.

Neither (4.1) nor (1.1) holds in general if we assume "A6)$\beta$ and A7)$\beta", 2 > \beta_2 > 0", or "A6)$\beta$, 2 > \beta_1 > 0, and A7)$\beta"."

In the following we prove
\[ \psi(1/r) \geq M r^d N(r), \quad M > 0, \]
for sufficiently small every $r$ under A6)$\beta_1$, $2 > \beta_1 > 0$, and
\[
\psi(r/r) \asymp r^\alpha N(r), \quad r \to 0,
\]
under A6)_{\beta_1} and A7)_{\beta_2}, \ 2 > \beta_1 > \beta_2 > 0. For the proof we prepare some lemmas.

**Lemma 4.** (1) If \( N(r) \) satisfies A6)_{\beta} for some \( \beta, 2 > \beta > 0 \) then \( \psi \) satisfies A4)_{\beta} and A5)_{\beta}. (2) If \( N(r) \) satisfies A7)_{\beta}, \( 2 > \beta > 0 \), then \( \psi \) satisfies A4)_{\beta}.

**Proof.** We show this dividing into some steps.

1. It does not lose generality that we assume \( M_1 N_1(r) \leq N(r) \leq M_2 N_2(r) \) for every \( r \) (\( M_3 N_2(r) \leq N(r) \leq M_4 N_2(r) \)) under the condition A6)_{\beta} (resp. A7)_{\beta}). Indeed, let \( \psi \) be the exponent corresponding to the Lévy measure obtained by changing \( N(r) \) on the neighborhood of the infinity. Then \( \psi(s) \asymp \psi(s), s \to \infty \).

2. If \( N(r) \) satisfies A6)_{\beta} (resp. A7)_{\beta}), then \( N^{(1)}(r) \) defined by (2.14) has the following properties: a) There exists \( N^{(1)}_{(1)}(r) \) (resp. \( N^{(1)}_{(2)}(r) \)) such that \( M_1 N^{(1)}_{(1)}(r) \leq N^{(1)}(r) \leq M_2 N^{(1)}_{(1)}(r) \) (resp. \( M_3 N^{(1)}_{(2)}(r) \leq N^{(1)}(r) \leq M_4 N^{(1)}_{(2)}(r) \)) for every \( r \) and \( r^{1 + \beta} N^{(1)}(r) \) is monotone decreasing (resp. \( r^{1 + \beta} N^{(1)}(r) \) is monotone increasing) on \((0, \infty)\). Indeed, we have only to note

\[
r^{1 + \beta} N^{(1)}(r) = M(d, 1) \int_1^\infty (rt)^{d + \beta} N(rt)t^{-(d + \beta - 1)}(t^2 - 1)^{(d-3)/2} dt.
\]

3. Since

\[
s^{-\beta} \psi(s) = \lambda s^{-\beta} + \pi^{-1} \int_0^\infty (1 - \cos(t))s^{-(1 + \beta)} N^{(1)}(t/2\pi s)dt,
\]
it follows from step 2. that A6)_{\beta} implies A5)_{\beta} and A7)_{\beta} implies A4)_{\beta}.

4. Define

\[
\psi_{2, \epsilon}(s) = \int_0^{2\epsilon} (1 - \cos(2\pi sr))N^{(1)}(r) dr
\]
for \( \epsilon > 0 \). Then we have, under the condition A6)_{\beta}, \( 2 > \beta > 0 \),

\[
\psi(s) \asymp \psi_{2, \epsilon}(s), \quad s \to \infty
\]
for each fixed \( \epsilon \) and \( s^{-\beta} \psi_{2, \epsilon}(s) \) is monotone decreasing. This is proved as follows: By (2.11)

\[
\psi(s) = \lambda + 2 \int_0^\infty (1 - \cos(2\pi sr))N^{(1)}(r)dr \equiv \lambda + \psi_1(s).
\]

Set
Using the result of step 2., we have
\[
\int_{c/2s}^{\infty} (1-\cos (2\pi sr))N^{(1)}(r) \, dr \leq M_{\delta}N^{(1)}(c/2s) \frac{I(\beta, 0, \infty)}{s}
\]
and
\[
\int_{0}^{c/2s} (1-\cos (2\pi sr))N^{(1)}(r) \, dr \geq M_{\delta}N^{(1)}(c/2s) \frac{I(\beta, 0, c)}{s}
\]
under the condition \(A6)_\delta\). Since \(2 > \beta > 0, \langle \psi_{1,c}(s) \rangle \psi_{1}(s), s \to \infty\), and so that \(\psi(s) \sim \psi_{1,c}(s), s \to \infty\). Noting
\[
\psi_{1,1}(s) = 2\pi \int_{0}^{c/2s} \sin (2\pi sr) rN^{(1)}(r) \, dr - N^{(1)}(1/2s)s^{-1},
\]
we see that
\[
2\psi_{1,1}(s) - s\psi_{2,1}(s) \geq (2\pi s)^{-1} \int_{0}^{\infty} (2 - 2\cos (t) - t \sin (t))N^{(1)}(t/(2\pi s)) \, dt > 0,
\]
which implies that \(s^{-2}\psi_{2,1}(s)\) is monotone decreasing. Hence \(\psi\) satisfies \(A4)_\delta\). The proof is complete.

**Lemma 5.** (4.2) holds under the condition \(A6)_{\beta_1}, 2 > \beta_1 > 0\) and (4.3) also holds if we assume \(A7)_{\beta_2}, 2 > \beta_2 > 0\), in addition.

**Proof.** It holds that
\[
\psi(1/r) \geq \psi_{1,1}(1/r) \geq M_{\delta}rN^{(1)}(r/2)
\]
by (4.7), and, putting \(\beta = \beta_1\) in (4.4), we have
\[
(r/2)N^{(1)}(r/2) \geq M_{\delta}rN^{(1)}(r/2) \int_{1}^{r} t^{-(d+\beta_1-1)(t^2-1)^{(d-3)/2}} \, dt.
\]
Hence (4.2) is proved. Using \(A7)_{\beta_2}\), we can show that
\[
\psi_{1,1}(1/r) \leq M_{\delta}N^{(1)}(r)r
\]
just as in the proof of (4.7). On the other hand, it follows from (4.4) that
\[
rN^{(1)}(r) \leq M_{\delta}r^{d} N(r)M(d, 1) \int_{1}^{r} t^{-(d+\beta_2-1)(t^2-1)^{(d-3)/2}} \, dt.
\]
Combining the above inequalities with (4.6), the proof of (4.3) is finished.
REMARK 4.1. Lemma 4 and Lemma 5 are valid without the condition $N'(r) < 0$.

The following lemmas are concerned with the Green function. We say that a function $\varphi(|x-y|)$ has monotone singularity, if there exists a measurable and monotone decreasing function $\varphi$ on $(0, \infty)$ with $\varphi(r) \leq \varphi(r')$, $r' \rightarrow 0$.

**Lemma 6.** If the density $N(|y|)$ of Lévy measure satisfies $N'(r) < 0$, then the Green function $\varphi(|x-y|)$ has monotone singularity. Indeed, $\varphi(r)$ itself is monotone decreasing and continuous on $(0, \infty)$.

**Proof.** Since

$$
\int_{\mathbb{R}^d} \frac{|y|^2}{1+|y|^2} N(|y|) \, dy < \infty,
$$

it is easily checked by the integral by parts that

(4.8) \quad \lim_{r \rightarrow 0} r^d N(r) = 0 \quad \text{and} \quad \lim_{r \rightarrow \infty} r^{d+2} N(r) = 0,

in case $N'(r) < 0$. Set

$$
n(r) = -\frac{N'(r)}{2\pi r}.
$$

Then, using (4.8), we can show

$$
\int_{\mathbb{R}^{d+2}} \frac{|x|^2}{1+|x|^2} n(|x|) \, dx.
$$

Let $\tilde{X}$ be an isotropic process with s.i.i. on $\mathbb{R}^{d+2}$ whose exponent $\tilde{\psi}$ is given by

$$
\tilde{\psi}(|x|) = \lambda + \int_{\mathbb{R}^{d+2}} (1 - \cos(2\pi \langle x, y \rangle)) n(|y|) \, dy,
$$

where $\lambda = \psi(0) > 0$. Then $X$ with the Lévy measure $N(|y|)dy$ is the projection on $\mathbb{R}^d$ of $\tilde{X}$ by (2.14), and so that

$$
\varphi(r) = 2\pi \int_r^\infty \varphi(s) s \, ds
$$

by (2.12), where $\varphi$ is the Green function of $\tilde{X}$.

**Lemma 7.** Let $\varphi$ be a positive measurable function on $(0, \infty)$ with the property that

a) $\varphi$ has monotone singularity,

and
b) there exists a positive measurable and monotone decreasing function \( \varphi_1 \) with \( \lim_{r \to \infty} \varphi_1(r) = \infty \) such that \( \varphi_1(r) \leq \varphi_1(r) = r^{-\beta} \), \( r \to 0 \), and \( r^{d-\beta} \varphi_1(r) \) is monotone increasing for some \( \beta, d > \beta > 0 \). Then

\[
(4.9) \quad \varphi(r) \leq [\varphi]^d_1(r), \quad r \to 0.
\]

PROOF. For a fixed \( c, 1 > c > 0 \), we have

\[
(4.10) \quad [\varphi]^d_1(cr) c^{d-\beta} \leq M_1[\varphi]^d_1(r)
\]

by b), where \( M_1 \) is a positive constant independent of small \( r \) and the choice of \( c \). Noting that

\[
[\varphi]^d_1(r) \geq r^{-d} \int_{cr}^r \varphi(s)s^{d-1} ds \geq [\varphi]^d_1(r) - c^d[\varphi]^d_1(cr)
\]

it follows from (4.10) that

\[
[\varphi]^d_1(r) \geq r^{-d} \int_{cr}^r \varphi(s)s^{d-1} ds \geq [\varphi]^d_1(r) - M_1 c^d[\varphi]^d_1(r).
\]

Hence we have

\[
(4.11) \quad r^{-d} \int_{cr}^r \varphi(s)s^{d-1} ds \leq [\varphi]^d_1(r), \quad r \to 0,
\]

for a sufficiently small fixed \( c \). The condition a), (4.11) together with (4.6) imply that

\[
\varphi(cr) \geq M_2[\varphi]^d_1(r) \geq \frac{M_2}{M_1} c^{d-\beta} [\varphi]^{d-\beta}_1(cr), \quad r \to 0.
\]

On the other hand \( [\varphi]^d_1(cr) \geq M_3 \varphi(cr) \) by the condition a). We have

\[
\varphi(cr) \leq [\varphi]^d_1(cr), \quad r \to 0,
\]

for a fixed \( c \), so that (4.9) holds. The proof is finished.

Proof of Theorem 2. By Lemma 4 we can apply Theorem 1 to \( X \), and so that we can prove (4.1) using (4.3). Combining (4.1) with A6),, we see that \( \varphi \) satisfies b) of Lemma 7. Since \( \varphi \) also satisfies a) of Lemma 7 by Lemma 6, (4.9) is valid. (4.1) together with (4.9) implies (1.1). The proof is finished.

5. Comparison theorems (I).

For given isotropic processes with s.i.i. \( X_0 \) and \( X \) on \( R^d \) \((d \geq 2)\) with exponents \( \mathcal{F}_i \) and \( \mathcal{F} \) respectively, we consider the following relations: Let
\( \phi_0 (\phi) \) be the Green function of \( X_0 \) (resp. \( X \)) and \( C(K) \) (resp. \( C(K) \)) be the capacity of \( K \) relative to \( X_0 \) (resp. \( X \)).

i) There exist positive constants \( M_i, i=1, 2 \), such that \( M_2 C_0(K) \leq C(K) \leq M_1 C_0(K) \) for every compact set \( K \).

ii) \( C(Q_r) \asymp C_0(Q_r), r \to 0 \).

iii) \( [\phi]^2(r) \asymp [\phi_0]^2(r), r \to 0 \).

iv) \( \mathcal{V}(s) \asymp \mathcal{V}_0(s), s \to \infty \).

v) \( X_0 \) and \( X \) have the same polar sets, that is, \( C(K) = 0 \) if and only if \( C_0(K) = 0 \) for each compact set \( K \).

vi) \( \phi(r) \asymp \phi_0(r), r \to 0 \).

Combining Lemma 1 with Lemma 2, it is clear that iv) \( \Rightarrow \) i) \( \Rightarrow \) ii) \( \Rightarrow \) iii), and, if both \( X_0 \) and \( X \) satisfy A4) \( \beta \), for some \( \beta, d > \beta > 0 \), it follows from Theorem 1 that i) \( \sim \) iv) are equivalent to each other.

In this section we show that i) \( \sim \) iv) are equivalent to each other even if \( X \) does not satisfy A4) \( \beta \). The relations v) and vi) are studied in the next section.

**THEOREM 3.** Fix an isotropic process \( X_0 \) with s.i.i. on \( \mathbb{R}^d \) (\( d \geq 2 \)) with the exponent \( \mathcal{V} \) satisfying A1), A2) and A4) \( \beta \), for some \( \beta, d > \beta > 0 \). Let \( X \) be an isotropic process with s.i.i. on \( \mathbb{R}^d \) with the exponent \( \mathcal{V} \) satisfying A1) and A2). In case \( d=2 \), A3) is assumed for both \( X_0 \) and \( X \). Then i), ii), iii) and iv) are equivalent to each other.

**PROOF.** We have only to show that ii) \( \Rightarrow \) iv). First we note that, if ii) holds, then the following A8) \( \beta \), \( d > \beta > 0 \), holds for \( X \):

**A8) \( \beta \)** There exists a function \( q \) on \( (0, \infty) \) with the property that \( q(r) \asymp C(Q_r), r \to 0 \), and \( r^{\beta - d} q(r) \) is monotone increasing.

Indeed, since \( X_0 \) satisfies A4) \( \beta \), it follows from Theorem 1 that A8) \( \beta \) holds for \( X_0 \), and so that A8) \( \beta \) also holds for \( X \) by ii).

Hence we can finish the proof of Theorem 3, if the following is proved.

**THEOREM 1'.** Let \( X \) be as in Theorem 3. Under the condition A8) \( \beta \) for \( d > \beta > 0 \), we have

\begin{equation}
\mathcal{V}(1/r) \asymp r^{-d} C(Q_r), \quad r \to 0.
\end{equation}

**PROOF.** Since (3.10) is valid without A4) \( \beta \) (Remark 3.1), we have

\begin{equation}
\mathcal{V}(1/r) \geq M_r r^{-d} C(Q_r)
\end{equation}

for \( 0 < r < \delta \). Noting that (3.11) also holds without A4) \( \beta \) (Remark 3.1) and
is valid without $A_4)_\beta$, we can prove that there exist $B>2\pi$ and $\delta_i>0$ such that

$$M_3 r^{-d} C(Q_r) \leq I_r(r^\alpha, A) + I_r(A, B) + M_2 r^d$$

holds for $A, \alpha$ and $r$ satisfying $2\pi > A > 0$, $1 > \alpha > \beta/d$ and $0 < r < \delta_i$. On the other hand, since

$$I_r(r^\alpha, A) \leq M_4 \int_{r^\alpha}^r r^d C(Q_{3r^\alpha/2})^{-1} t^{-(d+1)(J_{d/2}(t))^2} dt$$

holds by (5.2), it follows from $A_8)_\beta$ that

$$I_r(r^\alpha, A) \leq M_5 \frac{r^d}{C(Q_r)} \int_{r^\alpha}^r t^{-(1+\beta)(J_{d/2}(t))^2} dt.$$

Since $0 < \beta < d$, we have

$$I_r(r^\alpha, A) \leq M_3 \frac{r^d}{3C(Q_r)}$$

for $0 < r < \delta_i$ and sufficiently small $A$. Noting that we can choose $\delta_i > 0$ so that

$$M_3 r^d \leq M_4 r^d - \beta r^d \leq M_2 \frac{r^d}{3C(Q_r)}$$

for $0 < r < \delta_i$, we get

$$\frac{M_3}{3} \frac{r^d}{C(Q_r)} \leq I_r(A, B)$$

for $0 < r < \delta_i$. Now we can prove

$$\mathcal{F}(1/r) \leq M_5 r^{-d} C(Q_r)$$

in the same way as in the proof of (3.23). Combining (5.4) with (5.2), we get (5.1). The proof is complete.

Remark 5.1. Using Theorem 1 and Theorem 1', we see that $A_4)_\beta$ is equivalent to $A_8)_\beta$ for $d > \beta > 0$.

6. Comparison theorems (II).

If we impose $A_5)_\beta$, $d > \beta_i > 0$, upon $X_0$ in Theorem 3, we can get a strengthened result as follows.
THEOREM 4. Let $X_0$ and $X$ be as in Theorem 3. If we further assume that $X_0$ satisfies $A5$, for some $\beta_1, d > \beta_1 > 0$, then i), ii), iii), iv), and v) are equivalent to each other.

If we assume in addition that the Green functions $\varphi_0$ and $\varphi$ have monotone singularity (§ 4), then i) ~ v) are equivalent to each other. In general i) ~ v) are not sufficient for vi) if both $\varphi_0$ and $\varphi$ are not monotone.

For the proof we need some results on the Hausdorff measure and the generalized capacity (Frostman capacity [4]), and Taylor's result [10] plays an essential role.

Throughout this section we use the following symbols:

$$D = \{ \varphi; \varphi(r) \text{ is monotone decreasing on } (0, \infty) \text{ with } \lim_{r \to 0} \varphi(r) = \infty \}.$$  

$$\Phi = \{ \varphi; \varphi(r) \text{ is monotone decreasing on } (0, \infty) \}.$$  

$$\Phi^c (\text{resp. } \Phi^c_r \text{)} = \{ \varphi \in \Phi (\text{resp. } \Phi_r), \varphi(r) \text{ is continuous on } (0, \infty) \}.$$  

For each compact set $K \subset \mathbb{Q}^d$, we set

$$\mathcal{M}_K = \{ \mu; \mu \text{ is a measure defined on } K \text{ such that } \mu(K) = 1 \}.$$  

Then the generalized capacity of $K$ relative to $\varphi \in \Phi$, denoted by $C^\varphi(K)$, is defined by

\begin{align*}
\text{a) } & \text{ if } V^\varphi(K) = \infty, \text{ then } C^\varphi(K) = 0, \\
\text{b) } & \text{ if } V^\varphi(K) < \infty, \text{ then } C^\varphi(K) = \varphi^{-1}(V^\varphi(K)),
\end{align*}

where $V^\varphi(K) = \inf_{\varphi \in \mathcal{M}_K} \{ \sup_{x \in \mathbb{Q}^d} \varphi \ast \mu(x) \}$.

The following is known for $\varphi \in \Phi^c$:

(6.1) If $C^\varphi(K) > 0$, then there exists $\mu \in \mathcal{M}_K$ such that $\varphi \ast \mu < M$ everywhere for some constant $M$.

Therefore, in case $\varphi \in \Phi^c$ is the Green function of an isotropic process $X$ with s.i.i.,

(6.2) $C^\varphi(K) = 0 \iff C(K) = 0$,

where $C(K)$ denotes the capacity of $K$ relative to $X$ (§ 1). In this section we are concerned with the problem whether $C(K) = 0$ or not. So we use $C^\varphi(K)$ instead of $C(K)$ frequently in case that $\varphi$ is a Green function.

For $h$ such that $1/h \in \Phi$, the Hausdorff measure $\Lambda_h$ is defined by

$$\Lambda_h(K) = \lim_{r \to 0} \left\{ \inf \sum_{i=1}^{\infty} h(d(Q_i)) \right\},$$  

where $d(Q_i)$ denotes the diameter of $Q_i$ and the infimum is taken over all
coverings of $K$ by sequences $\{Q_i\}$ of spheres with diameter less than $\delta$. Frostmann [4] shows that for $\varphi \in \Phi^c$

\[(6.3) \quad A_{\varphi}(K) = 0 \Rightarrow C^c(K) = 0.\]

Furthermore we have

**Lemma 8.** Let $X$ be an isotropic process on $\mathbb{R}^d$ ($d \geq 2$) with s.i.i. satisfying $A1$ and $A2$). Set $h(r) = C(Q_r)$. Then

\[(6.4) \quad A_h(K) = 0 \Rightarrow C(K) = 0\]

for each compact set $K$.

**Proof.** This is proved by Carleson [2] under a slightly different situation. Let $\varphi$ be the Green function of $X$. If $\mu$ is a measure with support in $Q_r$ such that $\varphi \ast \mu < M_1$ on $Q_r$, then we have

\[(6.5) \quad \mu(Q_r) \leq MM_1h(r),\]

where $M$ is a positive constant independent of $r$ and $\mu$. Indeed, it holds that

\[\int_{Q_r} \varphi \varphi(\mu(x)) \, dx \leq M_1 |Q_r|\]

and so that (6.5) follows from Lemma 1. For a given $\varepsilon > 0$, choose a countable family of spheres $\{Q_k\}$ with radii $r_k$ such that $U_k Q_k \supset K$ and $\sum r_k h(r_k) < \varepsilon$. If $C(K) > 0$, there exists $\mu \in \mathcal{M}_K$ such that $\varphi \ast \mu < M_1$ on $\mathbb{R}^d$. Let $\mu_k$ be the restriction of $\mu$ on $Q_k$. Then, since $\mu_k(Q_k) \leq M_1 M h(r_k)$ by (6.5), it holds that $1 = \mu(K) \leq \sum \mu_k(Q_k) \leq M_1 M \varepsilon$. As $\varepsilon$ is arbitrary, $C^c(K)$ must be zero.

**Remark 6.1.** In case $\varphi \in \Phi^c$, (6.4) is also valid by taking $h(r) = [\varphi]^c_1(\varphi^{-1})$, even if $\varphi$ is not a Green function. Indeed, the Hausdorff measure relative to $[\varphi]^c_1(\varphi^{-1})$ is defined because $[\varphi]^c_1$ is monotone decreasing, and so that the proof is the same as in Lemma 8.

The next lemma is essential in our proof of Theorem 4.

**Lemma 9.** (S. J. Taylor [10] Theorem 4 and Remark) Let $\varphi_i$, $i = 1, 2$, be functions of class $\Phi^c$ such that $\varphi_i \in \Phi_d$ with $\lim_{r \to 0} r^d \varphi_i(r) = 0$ and $[\varphi_i]^c_1(r) \leq M \varphi_i(r)$ for every $r$, $0 < r < \delta$. Suppose

\[(6.6) \quad \lim \inf_{r \to 0} \varphi_i(r) / \varphi_i(r) = 0.\]

Then there exists a compact set $K \subset \mathbb{R}^d$ such that
Remark 6.2. If \( \varphi_2 \) does not satisfy the condition in Lemma 9, (6.7) is not valid in general even if (6.6) holds. For example, choose a function \( \varphi \in \Phi^c \) such that

\[
\lim \inf_{r \to 0} \varphi(2r)/\varphi(r) = 0
\]

and set \( \varphi_2(r) = \varphi(2r) \) and \( \varphi_1(r) = \varphi(r) \). Since \( 2^{-i}[\varphi]_i^2(r) \leq [\varphi]_i^2(2r) \leq [\varphi]_i^2(r) \), we have \( [\varphi_1]_i^2(r) \leq [\varphi_2]_i^2(r) \), and so that \( A_{h_2}(K) > 0 \) if and only if \( A_{h_1}(K) > 0 \), where \( h_i(r) = [\varphi_i]_i^2(r)^{-1} \), \( i = 1, 2 \). Suppose that (6.7) was valid. Then \( A_{\frac{1}{1+1}}(K) = 0 \) and \( A_{h_1}(K) > 0 \) by Remark 6.1 for some compact set \( K \), and so \( A_{h_2}(K) > 0 \). On the other hand \( A_{\frac{1}{1+1}}(K) > 0 \) if \( A_{h_1}(K) > 0 \), because \( \varphi_1(r) \leq d[\varphi_i]_i^2(r) \).

The next lemma is used in the proof of Theorem 4.

Lemma 10. Let \( \varphi \in \Phi \). Suppose that there are constants \( \alpha_i, i = 1, 2, d > \alpha_1 > \alpha_2 > 0 \) such that \( [\varphi]_i^\alpha_i(r) < \infty, i = 1, 2 \), and

\[
[\varphi]^\alpha_i(r) \leq [\varphi]_i^\alpha_i(r), \quad r \to 0.
\]

Then

\[
\varphi(r) \leq [\varphi]^\alpha_i(\alpha), \quad r \to 0.
\]

Proof. Since

\[
[\varphi]^\alpha_i(r) = \int_0^r \varphi(rt)t^{d-\alpha_i} \, dt, \quad i = 1, 2,
\]

it follows from (6.8) that

\[
\int_0^r \varphi(rt)(Mt^{d-\alpha_i} - t^{d-\alpha_i}) \, dt > 0, \quad 0 < r < \delta,
\]

for some \( M > 1 \). Setting \( c = (2M)^{-1/(\alpha_1-\alpha_2)} \), we have

\[
0 < \int_0^r \varphi(rt)t^{d-\alpha_i}(Mt^{\alpha_i-\alpha_2} - 1) \, dt \leq \frac{1}{2} \int_0^r \varphi(rt)t^{d-\alpha_i} \, dt + M \int_0^r \varphi(rt)t^{d-\alpha_i} \, dt,
\]

and so that

\[
\frac{M}{d-\alpha_1+1} \varphi(cr) \geq \frac{1}{2} c^{d-\alpha_1+1} [\varphi]_i^\alpha_i(cr), \quad 0 < r < \delta.
\]

Since \( [\varphi]_i^\alpha_i(cr) \geq (d-\alpha_1+1)^{-1} \varphi(cr) \), we have \( [\varphi]_i^\alpha_i(r) \leq \varphi(r), \, r \to 0. \)
REMARK 6.3. Let $X$ be an isotropic process with s.i.i. on $R^d$ ($d \geq 2$) with the exponent $\psi$ satisfying A1) and A2). Let $\varphi(|x-y|)$ be the Green function. Define

$$\varphi^t(r) = \varphi(tr)$$

for a fixed $t > 0$. Then $\varphi^t(|x-y|)$ is the Green function of $X^t$ with the exponent $\psi^t(s) = \psi(s/t)$. Set $h(r) = C(Q_r)$. Then $1/h \in \Phi$ and $C^{t/h}(K) > 0$ implies that $C^{t'}(K) > 0$ for some $t' > 0$ and each compact set $K$.

**Proof.** The first half is obvious. If $C^{t/h}(K) > 0$, there is a measure $\mu \in \mathcal{M}_K$ such that

$$[\mu]_{1/h} = \int \frac{1}{h(|x-y|)} \mu(dx) \mu(dy) < \infty.$$ 

Noting that $1/h(r) > [\varphi]^T(r)$, $r \to 0$, by Lemma 1 and $[\varphi]^T(r) = \int_0^r \varphi(rt) t^{d-1} dt$, we have $\int_0^r t^{d-1} [\mu]_t dt < \infty$. Therefore $[\mu]_t < \infty$ for some $t > 0$, which implies $C^{t'}(K) > 0$ by (2.8).

Now we establish several properties about $X_0$. In the following we denote a compact set by $K$ and put $h_0(r) = C^{t_0}(Q_r)$.

**Lemma 11.** Let $X_0$ be as in Theorem 4. Then the following are valid:

- a) $C^{t_0}(K) = 0$ if and only if $C^{e}(K) = 0$ for each fixed $t > 0$.
- b) $C^{e}(tK) = 0$ if and only if $C^{e}(K) = 0$ for each fixed $t > 0$.
- c) Set

$$\bar{\varphi}_0(r) = \frac{1}{h_0(r)}$$

for a fixed positive $\gamma$ such that $1 > \gamma > 1 - \beta$. Then

$$\bar{\varphi}_0 \in \Phi_{d - \beta_1 - \gamma + 1}$$

and

$$\bar{\varphi}_0(r) \sim \frac{1}{h_0(r)} \bar{\varphi}_0^T(r), \quad r \to 0.$$ 

**Proof.** Using A4)$_s$ and A5)$_s$, we have

$$\varphi^s(s/t) \sim \varphi^s(s), \quad s \to \infty,$$

and so that a) follows from Remark 6.3 and Lemma 2. Now b) is clear because $C^{t_0}(K) = 0$ if and only if $C^{e}(tK) = 0$. Finally we prove c). For the
proof of (6.9) it is sufficient to show \( \phi_0 \in \Phi \). Since \( 1/h_0 \) is decreasing, we have

\[
(6.11) \quad \frac{1}{d-\beta_1-\gamma+1} \frac{1}{h_0(r)} < \phi_0(r)
\]

by the definition of \( \phi_0 \). Therefore \( \phi'_0(r) = r^{-1}h_0(r)^{-1} - (d-\beta_1-\gamma+1)r^{r-1}\phi_0(r) < 0 \) and so that \( \phi_0 \in \Phi \). We next show

\[
(6.12) \quad \phi_0(r) < \frac{M_1}{1-\gamma} \frac{1}{h_0(r)}.
\]

Applying Theorem 1 to \( X_0 \), we have \( h_0(r) \asymp r^d \phi_0(1/r), r \to 0 \), and so, using A5) \( \phi_1 \), we see that there exists \( h_0 \) with the property; \( h_0(r) \asymp h_0(r), r \to 0 \), and \( r^{d-2}\phi_0(r)^{-1} \) is monotone increasing on \((0, \delta)\). Hence

\[
\phi_0(r) = r^{-(d-\beta_1-\gamma+1)} \int_0^r h_0(t)^{-1}t^{2-\beta_1}t^{-\gamma} dt 
\leq M_1r^{d-2}\phi_0(r)^{-1} \int_0^r t^{-\gamma} dt \leq \frac{M_1}{1-\gamma} h_0(r)^{-1}.
\]

The proof of (6.12) is finished. Combining (6.11) with (6.12) we have \( \phi_0(r) \asymp h_0(r)^{-1}, r \to 0 \). On the other hand

\[
\frac{\phi_0(r)}{d} \geq [\phi_0, \phi_1](r) \geq \frac{\phi_0(r)}{\beta_1+\gamma-1}
\]

and so that we finish the proof of (6.10).

**Proof of Theorem 4.** Since i) \(
\rightarrow iv) \) are equivalent to each other by virtue of Theorem 3 and evidently i) \( \Rightarrow v) \), it is sufficient to show \( v) \Rightarrow iii) \) for the proof of the first half. By making use of Lemma 9, we are to prove the following two inequalities:

\[
(6.13) \quad [\phi_0, \phi_1](r) \geq M_1[\phi, \phi_1](r),
\]

\[
(6.14) \quad \frac{[1/h_0](r)}{h_0(r)} \geq M_2[\phi, \phi_0](r), \quad h_0(r) \equiv C(Q, r),
\]

for each fixed \( \alpha \) such that \( \beta_1+\gamma > \alpha > 1 \), where \( \phi_0(r) \) and \( \gamma \) are those in Lemma 11, c). If (6.13) did not hold, it follows from Lemma 1 and (6.10) that

\[
\lim \inf_{r \to \alpha} h_0(r)\phi_0(r) = 0.
\]

Since we can apply Lemma 9 to \( h^{-1} \) and \( \phi_0 \) by Lemma 11, c), there exists a compact set \( K \) with the property that \( C^{\alpha}(K) > 0 \) and \( A_{\alpha}(K) = 0 \), and therefore \( C^{\alpha}\phi_0^{-1}(K) > 0 \) and \( C^{\alpha}(K) = 0 \) by (6.10) and Lemma 8. Hence \( C^{\alpha}(K) > 0 \) for some
$t > 0$ by Remark 6.3, and so $C^v(K) > 0$ by Lemma 11, a). This contradicts to the assumption v). Hence (6.13) is valid. In the next we prove (6.14). Suppose

$$\lim \inf_{r \to 0} (1/h)^{\alpha}(r) \cdot \varphi(r)^{-1} = 0.$$ 

Noting that

$$[1/h]^\alpha(r) \in \Phi_{d-\alpha+1} \quad \text{and} \quad [(1/h)^\alpha(r)]^\alpha \leq [1/h]^\alpha(r)/\alpha - 1,$$

we can apply Lemma 9, and so there exists a compact set $K$ with the property that $C^{(d+\alpha)}(K) > 0$ and $A_{1\tilde{\varphi}_0}(K) = 0$. Using (6.10) and Lemma 8, it holds that

$$A_{1\tilde{\varphi}_0}(K) = 0 \Rightarrow A_{h_0}(K) = 0 \Rightarrow C^v(K) = 0.$$ 

On the other hand, since $h(r)^{-1} \leq (d-\alpha+1)[h^{-1}]^\alpha(r)$, we have

$$C^{(d+\alpha)}(K) > 0 \Rightarrow C^v(K) > 0.$$ 

Hence $C^v(K) > 0$ for some $t > 0$ by Remark 6.3 and so that $C^v(tK) > 0$. The assumption v) implies $C^v(tK) > 0$. Hence $C^v(K) > 0$ by Lemma 11, b). Consequently we get

$$A_{1\tilde{\varphi}_0}(K) = 0 \quad \text{and} \quad C^v(K) > 0.$$ 

Such $K$ does not exist, because $A_{1\tilde{\varphi}_0}(K) = 0 \Rightarrow A_{h_0}(K) = 0$ by (6.10) and so $C^v = 0$ by Lemma 8. Now the proof of (6.14) is finished. Next we prove

(6.15) $$[1/h]^\alpha(r) \cdot \tilde{\varphi}(r), \quad r \to 0.$$ 

Note that $[\varphi_0]^\alpha(r) \geq M_\delta h(r)^{-1}$ by (6.13) and Lemma 1. Then $[(\varphi_0)^\alpha]^\alpha(r) \geq M_\delta [h^{-1}]^\alpha(r)$. On the other hand, since $[\varphi_0]^\alpha(r) \cdot h(r)^{-1} \cdot \tilde{\varphi}(r), \quad r \to 0$ by Lemma 1, a) and (6.10), we have $[(\varphi_0)^\alpha]^\alpha(r) \leq M_\delta \tilde{\varphi}(r)$ by (6.9). Therefore

$$M_\delta \tilde{\varphi}(r) \geq [1/h]^\alpha(r),$$

which together with (6.14) implies (6.15). Noting that (6.15) holds for arbitrary $\alpha$ such that $\beta_1 + \gamma > \alpha > 1$, it follows from Lemma 10 that

$$1/h(r) \cdot \tilde{\varphi}(r), \quad r \to 0.$$ 

Now the proof of v) \Rightarrow iii) is finished. Finally we show the last half of Theorem 4. Assume that $\varphi_0$ and $\varphi$ have monotone singularity. Then the conditions a) and b) in Lemma 7 hold for both $\varphi_0$ and $\varphi$ if we suppose iii) holds. Indeed b) holds for $\varphi_0$ by A5), $d > \beta_1 > 0$ and it also holds for $\varphi$ by
iii). Hence we can prove that iii) implies vi), because (4.9) is valid for both \( \varphi_0 \) and \( \varphi \). The proof of \( \text{vi)} \implies \text{iii} \) is trivial. We complete the proof.

7. Counter-examples.

In this section we show by examples that the conclusions of Theorem 1, Theorem 2 and Theorem 4 do not always hold if we replace the conditions by weaker ones.

For this purpose we study the subordinated process. Let

\[
F(u) = bu + \int_0^\infty (1 - \exp (-ut)) \mu(dt),
\]

\[
b \geq 0, \quad \int_0^\infty \frac{t}{1 + t} \mu(dt) < \infty.
\]

Define

\[
\psi_1(|x|) = F(2\pi |x|^2), \quad x \in \mathbb{R}^d.
\]

Then it is known that \( \psi_1 \) is the exponent of an isotropic process \( X_1 \) with s.i.i. on \( \mathbb{R}^d \), which we call the subordinated process of Brownian motion by the subordinator \( F \). Further the Lévy measure of \( X_1 \) has the density \( N(|y|) \) relative to Lebesgue measure and

\[
N(|y|) = \int_0^\infty (2\pi t)^{-d/2} \exp \left(-\frac{|y|^2}{2t}\right) \mu(dt).
\]

For details, see for example, Ikeda-Watanabe [5].

**Remark 7.1.** Let \( X \) be an isotropic process with s.i.i. on \( \mathbb{R}^d \) with the exponent \( \psi(|x|) = \lambda + \psi_1(|x|) \), \( \lambda > 0 \). Then \( X \) has a Green function \( \phi(|x-y|) \) such that \( \phi \in \Phi^c \). (For the definition of \( \Phi^c \), see § 6.)

**Proof.** This would be known, but we give a simpler proof here. Since

\[
-\frac{N'(r)}{2\pi r} = \int_0^\infty (2\pi t)^{-(d+2)/2} \exp \left(-\frac{r^2}{2t}\right) \mu(dt) > 0,
\]

the result follows directly from Lemma 6.

**Example 1.** Let us consider the subordinator \( F \) of the form:

\[
F(u) = \int_0^\infty (1 - \exp (-ut))t^{-1}e^{-t} dt = \log (1 + u),
\]

and let \( X \) be a subordinated process on \( \mathbb{R}^d \) (\( d \geq 3 \)) of Brownian motion by \( F \).
Then the density \( N(|y|) \) of the Lévy measure of \( X \) satisfies
\[
N(r) = r^{-d} \int_{0}^{\infty} (2\pi u)^{-d/2} \exp\left(-\frac{1}{2u}\right) u^{-1} \exp(-r^2u) \, du \sim r^{-d}, \quad r \to 0,
\]
and so that \( N(r) \) satisfies A6), for \( \beta = 0 \) but not for \( \beta > 0 \). Further, since \( \Psi \) satisfies A4), for \( 0 < \beta < d \), it follows from Theorem 1 that
\[
[\phi]_{r}^2(r) \sim r^{-d} \log \left(\frac{1}{r}\right)^{-1}, \quad r \to 0.
\]
If the conclusion of (II) of Theorem 2 was valid in this case, we would have
\[
[\phi]_{r}^2(r) \sim \phi(r), \quad r \to 0,
\]
and so that \( \int_{0}^{r} \phi(r) r^{-d-1} \, dr = \infty \), which contradicts the fact that \( \phi(|x|) \in L'(R^d) \). The above example shows that we can not replace A6), \( 2 > \beta > 0 \), by A6), in general.

**Example 2.** Set
\[
F(u) = \int_{0}^{u} u^\beta \, \log s \cdot s \to \infty.
\]
Since \( F(u) \) has a completely monotone derivative, that is, \( F(u) \geq 0, F' (u) \geq 0, F'' (u) \leq 0, \ldots, F^{(2k)} (u) \leq 0, F^{(2k+1)} (u) \geq 0, \ldots, F(u) \) is a subordinator. See G. Choquet [3]. Let \( X_1 \) be the subordinated process on \( R^d (d \geq 3) \) of Brownian motion by \( F \). We calculate the density \( N_1(|y|) \) of the Lévy measure of \( X_1 \). By the known formula
\[
w^\beta = \int_{0}^{\infty} \left(1 - \exp(-ut)\right) \frac{\beta}{(1-\beta)} t^{-(1+\beta)} \, dt,
\]
we have
\[
F(u) = \int_{0}^{u} \left(1 - \exp(-ut)\right) \mu(t) \, dt,
\]
where
\[
\mu(t) = \int_{0}^{t} \frac{\beta}{(1-\beta)} t^{-(1+\beta)} \, d\beta.
\]
Hence
\[
N_1(r) = \int_{0}^{r} \frac{\beta}{(1-\beta)} \int_{0}^{\infty} (2\pi t)^{-d/2} \exp\left(-\frac{y^2}{2t}\right) t^{-(1+\beta)} \, dt \, d\beta
\]
\[
= M \int_{0}^{r} \frac{\beta}{(1-\beta)} r^{-(d+2\beta)} \, d\beta.
\]
Using the formula \( \Gamma(z+1) = \Gamma(z)z \), we have

\[
N_1(r) \asymp r^{-d} \int_0^r (1 - \beta) r^{-2\beta} \, d\beta \asymp r^{-d-1} \left( \log \frac{1}{r} \right)^{-2}, \quad r \to 0,
\]

and so that \( N_1(r) \) satisfies \( A_6)_{\beta_1} \) for \( 2 > \beta_1 > 0 \) and \( A_7) \) but not \( A_7)_{\beta_2} \) for \( 2 > \beta_2 > 0 \). On the one hand, by the result of [6], § 6, \( X \) has a Green function \( \phi \) such that

\[
\phi_1(r) \asymp r^{d-2} \log \frac{1}{r} \left( \varphi_1 \right)(r), \quad r \to 0.
\]

Since \( \phi_1(r) \asymp r^{-2d} N(r)^{-1}, \quad r \to 0 \), this example shows that (I) and (II) of Theorem 2 are not valid in general, if we replace \( A_7)_{\beta_2}, \quad 2 > \beta_2 > 0 \), by \( A_7) \).

**EXAMPLE 3.** Let \( X_1 \) be a process on \( \mathbb{R}^2 \) given in example 2 and \( X_2 \) be a projection on \( \mathbb{R}^2 \) of \( X_1 \). Then \( X_2 \) has a Green function \( \phi_2 \) such that

\[
\phi_2(r) = \pi \int_r^\infty \phi(s) s(s^2 - r^2)^{-1/2} \, ds \asymp \left( \log \frac{1}{r} \right)^2, \quad r \to 0.
\]

Hence

\[
\frac{1}{C(Q_r)} \left[ \phi_2 \right](r) \asymp \left( \log \frac{1}{r} \right)^2, \quad r \to 0,
\]

by Lemma 1. On the other hand, the exponent \( \Psi_2 \) has the property that

\[
\Psi_2 \left( \frac{1}{r} \right) = \Psi_1 \left( \frac{1}{r} \right) \frac{1}{r^2 \log \left( \frac{1}{r} \right)}, \quad r \to 0.
\]

The process \( X_2 \) shows that \( A_4)_{\beta_2}, \quad d > \beta_2 > 0 \), in Theorem 1 can not be replaced in general by \( A_4) \).

**EXAMPLE 4.** Set

\[
N(r) = r^{-d+1} \left( \log \frac{1}{r} \right)^2, \quad r \text{ sufficiently small,}
\]

and assume that \( d \geq 3 \). Let \( X_3 \) be a process on \( \mathbb{R}^d \) with the Lévy measure \( N(|y|) \, dy \) and \( \phi_3 \) be its Green function. Then, by Theorem 2,

\[
\Psi_3(r) \asymp r^{-2d} N(r)^{-1} \asymp r^{1-d} \left( \log \frac{1}{r} \right)^{-2}, \quad r \to 0,
\]

and
Using Lemma 1 and the above estimate of $\phi$, we have

$$C(S_r) \approx (\phi)(r)^{-1} \approx r^{d-1} \log \left( \frac{1}{r} \right), \quad r \to 0.$$  

This example shows that the conclusion of (II) of Theorem 1 is not valid in general if $A_5) f (A, A^T) > 1$, is replaced by $A_5) f (1)$. The next example tells us that, even if the exponent is sandwiched by another "good" exponent, the Green function may behave "badly" near the origin, and the same example also shows that the conclusion of the last half of Theorem 4 is not valid in general without the monotonicity condition on $\phi$ and $\phi_0$.

**Example 5 (Zabczyk [11]).** Choose monotone sequences $\{a_n\}$ and $\{r_n\}$ of positive numbers such that $\sum a_n = 1$ and $r_n \to 0$. We denote the surface of a ball with radius $r_n$ whose center is the origin by $S_n$ and the uniform measure on $S_n$ with unit mass by $\varepsilon_n(dy)$. Set

$$\nu = \sum a_n \varepsilon_n$$

and

$$\Psi(|x|) = |x|^\alpha + \int_{R^d} (1 - \cos (2\pi \langle x, y \rangle)) \nu(dy)$$

$$\equiv |x|^\alpha + \Phi(|x|), \quad 0 < \alpha < 2.$$  

Then clearly $\Psi(s) \approx s^\alpha$, $s \to \infty$, but the isotropic process $X$ with the exponent $\Psi$ has a Green function $\phi$ such that $\phi(r_n) = \infty$ for every $n$, if $0 < \alpha < 1/2$ and $d \geq 3$. This example is given by Zabczyk [11] without proof. We give a proof here in case $0 < \alpha < 1/2$ and $d \geq 3$, for completeness. Let $\tilde{X}$ be an isotropic process with s.i.i. with the exponent $\tilde{\Psi}$. Then $\tilde{X}$ is the compound Poisson process whose semi-group of measures is given by

$$Q_s = e^{-t} \sum_{k=0}^{\infty} \frac{t^k}{k!} \nu^{*k},$$

where $\nu^{*k}$ denotes the $k$-multiple convolution. Therefore

$$\phi(|x|) = \int_0^\infty p(t, |x|) dt \geq \int_0^\infty te^{-t} \int_{R^d} p_s(t, |x-y|) \nu(dy) dt,$$

where $p_s(t, |x|)$ is the transition density of the isotropic stable process with
index $\alpha$, and so that, setting $\phi_\alpha(r) = r^{2\alpha - d}$, it holds for small $x$ and every large $k$ that

$$\phi(|x|) \geq M_1 \int_{S_k} \phi_\alpha(|x-y|) a_k \xi_k(dy).$$

Especially, for $|x| = r_k$, we have

$$\phi(r_k) \geq M_2 \phi_\alpha(r_k) = \infty$$

if $1 > \alpha > 0$.

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