On the index theorem for isometry-invariant geodesics

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§ 1. Introduction

Let $(M, \langle \cdot, \cdot \rangle)$ be a riemannian manifold, and $B \subset M \times M$ be a submanifold. Put $C_B := \{\gamma: [a, b] \rightarrow M; H^1\text{-curves such that } (\gamma(a), \gamma(b)) \in B\}$. We shall consider the energy integral $E(\gamma) := 1/2 \int_a^b \langle \dot{\gamma}(t), \dot{\gamma}(t) \rangle dt$ on C_B . Then $c \in C_B$ is a critical point of $E(\gamma)$ if and only if $c: [a, b] \rightarrow M$ is a geodesic which satisfies the boundary condition $(c(a), c(b)) \in B$, $(\dot{c}(a), -\dot{c}(b)) \perp T_{(c(a), c(b))}B$. At a critical point $c \in C_B$, we may define the second variation $D^2E(c)$ which is a symmetric bilinear form on $T_c C_B = \{X: [a, b] \rightarrow c^*TM; H^1\text{-vector fields along } c \text{ such that } (X(a), X(b)) \in T_{(c(a), c(b))}B\}$. Then the index of c is defined as the dimension of a maximal subspace of $T_c C_B$ over which $D^2E(c)$ is negative definite. Geometrically, this indicates the number of essentially different curves in the neighbourhood of c which are shorter than c and satisfy the boundary condition.

Recently, using the intersection theory of curves in the manifold of Lagrange spaces of a symplectic vector space, J. J. Duistermaat ([3]) obtained the general index theorem for the geodesics in the category of calculus of variations. His theorem asserts that index of c is expressed as the intersection number of the curve of Lagrange spaces $t \rightarrow \Phi(0, t)$ ($a \leq t \leq b$) with a fixed Lagrange space plus a correction term, where $\Phi(0, t)$ is the solution matrix of the linear Hamilton flow with eigenvalue parameter 0.

On the other hand, for the typical cases $B = \text{the diagonal set of } M \times M$ (i.e., closed geodesic case) or $B = N \times P$ ($N, P \subset M$ submanifolds; separate end manifolds case), the index theorem has been obtained directly, by constructing explicitly the Lagrange spaces of $V^{2(\dim M - 1)}(\dot{c}(a))$ which is the orthogonal complement of the geodesic spray $\xi_{\dot{c}(a)}$ in $T_{\dot{c}(a)}UM$ (see [1], [4], [5], [6]). This approach uses more geometrical properties of Jacobi fields and doesn't appeal to the topological method.

In the present note we shall give the index theorem for the case $B = \text{graph } f$, where f is an isometry of M , by an explicit construction of the Lagrange space (i.e., index theorem for isometry-invariant geodesics).

Index of isometry invariant geodesics appeared e.g. in a work of Colin de Verdière ([2]). This generalize a result of Klingenberg ([4], [5]), but the proof heavily depends on the Klingenberg's method. On the other hand, we shall treat the non-degenerate case and degenerate case in a unified manner.

§ 2. Preliminaries

2.1. Let $(M, \langle, \rangle)$ be an $(n+1)$ -dimensional riemannian manifold, and $p: TM \rightarrow M$ be the tangent bundle of M . Let $K: TTM \rightarrow TM$ denotes the connection map with respect to the Levi-Civita connection. Then for $X \in TM$, we have an identification $T_X TM \approx T_x M \oplus T_x M$ ($p(X)=x$) via an isomorphism $\tilde{X}(\in T_X TM) \rightarrow (p_*X, KX)(\in T_x M \oplus T_x M)$. Under the above identification, we may write $\tilde{X} = (X_h, X_v)$. In particular, the geodesic spray ξ takes the form $\xi_x = (X, 0)$ at $X \in TM$. We define the Sasaki metric G on TM by $G(\tilde{X}, \tilde{Y}) := \langle X_h, Y_h \rangle + \langle X_v, Y_v \rangle$.

Next let UM be the differentiable manifold of unit tangent vectors to M . Then for $X \in UM$, we have an identification $T_X UM \approx T_x M \oplus \perp X$ ($x = p(X)$), where $\perp X := \{Y \in T_x M; \langle X, Y \rangle = 0\}$. We shall denote by $V_h^{n+1}(X)$ (resp. $V_v^{n+1}(X)$) the first (resp. second) summand of the above identification of $T_X UM$. Let $V_h^n(X)$ be the orthogonal complement (with respect to $\langle, \rangle$) of X in $V_h^{n+1}(X)$, and we put $V^{2n}(X) := V_h^n(X) \oplus V_v^n(X)$ which may be identified with the orthogonal complement (with respect to G) of the geodesic spray ξ_x in $T_X UM$. Thus we have a vector bundle $\tau^{2n}: V^{2n} \rightarrow UM$.

Now, for every $X \in UM$, $V^{2n}(X)$ has a natural symplectic structure α which is defined by $2\alpha((X_h, X_v), (Y_h, Y_v)) := \langle X_h, Y_v \rangle - \langle X_v, Y_h \rangle$. Then subspaces of $V^{2n}(X)$ on which α vanishes identically will be called isotropic, and n -dimensional isotropic subspaces will be called Lagrange subspaces. For example, $V_h^n(X), V_v^n(X)$ are Lagrange subspaces.

2.2. Let $f: (M, \langle, \rangle) \rightarrow (M, \langle, \rangle)$ be an isometry. Then f induces an isometry $f_*: (UM, G) \rightarrow (UM, G)$, where f_* is the differential of f . In fact, for f_{**} (differential of f_*): $TUM \rightarrow TUM$, we have $Kf_{**} = f_*K$, $p_*f_{**} = f_*p_*$, and consequently

$$(2.1) \quad f_{**}(X_h, X_v) = (f_*X_h, f_*X_v).$$

Especially $f_{**}: V^{2n}(X) \rightarrow V^{2n}(f_*X)$ preserves the symplectic form α .

Next, we put $B = \text{graph } f = \{(x, f(x)) \in M \times M; x \in M\}$. Then we have easily

(2.2) B is a totally geodesic submanifold with respect to the induced metric.

$$(2.3) \quad (T_{(x, f(x))} B)^\perp = \{(u, v) \in T_x M \oplus T_{f(x)} M; f_* u + v = 0\},$$

where $\perp$ means the orthogonal complement.

2.3. In this subsection we shall review the Klingenberg's decomposition theorem of symplectic vector space via the Jordan canonical form of symplectic transformation ([4], [5]). Let (V^{2n}, α) be a symplectic vector space and $P: V \rightarrow V$ be a symplectic transformation. We shall consider the complexification V_c of V , hermitian skew-symmetric extension of α (i.e., $\alpha(X, Y) = -\overline{\alpha(Y, X)}$), and the extension of P on V_c . For the sake of simplicity, we shall denote these by the same notation V, α, P . Let $V(\rho)$ be a generalized eigenspace of the eigenvalue ρ of P . A sequence of vectors $\{x_1, \dots, x_k\}$ will be called a ρ -sequence if $Px_1 = \rho x_1$, $Px_2 = \rho x_2 + x_1$, $\dots$, $Px_k = \rho x_k + x_{k-1}$ hold. Then, by Jordan's theorem, V is a direct sum of $V(\rho)$'s and each $V(\rho)$ has a basis consisting of ρ -sequences. When P is symplectic, if ρ is an eigenvalue of P , then $\bar{\rho}, \rho^{-1}, \bar{\rho}^{-1}$ are also eigenvalues of P . In this subsection 'orthogonal' means α -orthogonal, and a subspace $W \subset V$ will be called non-degenerate if $\alpha|_W$ is non-degenerate. Then we have the following.

(2.4) V is the orthogonal sum of $V(\rho) \oplus V(\bar{\rho}^{-1})$ ($|\rho| \neq 1$) and $V(\rho)$ ($|\rho| = 1$). $V(\rho)$ ($|\rho| = 1, \rho \neq \pm 1$) has a basis consisting of mutually orthogonal non-degenerate ρ -sequences $\{x_1, \dots, x_k\}$ with $\alpha(x_i, x_j) = 0$ for $i + j \leq k$ and $\alpha(x_i, x_j) \neq 0$ for $i + j = k + 1$. $V(\pm 1)$ has a basis consisting of mutually orthogonal $\{e_1, \dots, e_{2k}\}$'s and $\{f_1, \dots, f_{2k-1}; \tilde{f}_1, \dots, \tilde{f}_{2k-1}\}$'s, where $\{e_1, \dots, e_{2k}\}$ is a real non-degenerate ρ -sequence with $\alpha(e_i, e_j) = 0$ for $i + j \leq 2k$ and $\alpha(e_i, e_j) \neq 0$ for $i + j = 2k + 1$, and $\{f_i; \tilde{f}_j\}$ is a non-degenerate subspace generated by real ρ -sequences $\{f_i\}$ and $\{\tilde{f}_j\}$ with odd elements ($\rho = \pm 1$). Moreover $\{f_i; \tilde{f}_j\}_{1 \leq i, j \leq k-1}$ is isotropic and $\{f_i; \tilde{f}_j\}_{1 \leq i, j \leq k}$ is non-degenerate.

In the following, we shall fix these ρ -sequences which are determined up to the symplectic isomorphism.

(2.5) $V^{2n} = V_{co} \oplus V_{nc}$: orthogonal decomposition of real non-degenerate invariant subspaces, where

V_{nc} is generated by $V(\rho) \oplus V(\bar{\rho}^{-1})$ ($|\rho| \neq 1$) and ρ -sequences ($|\rho| = 1$) which consists of more than one elements;

V_{co} is generated by ρ -sequences ($|\rho| = 1$) with only one element.

Next, we shall define real non-degenerate subspaces V_{in}^{2p}, V_{un}^{2q} ($p + q = n$) and real invariant isotropic subspace $V_{in}^p \subset V_{in}^{2p}$. First for ρ ($|\rho| \neq 1$), $V(\rho) \oplus V(\bar{\rho}^{-1})$ belongs to V_{in}^{2p} and isotropic $V(\rho)$ ($|\rho| < 1$) belongs to V_{in}^p . Next, for ρ ($|\rho| = 1, \rho \neq \pm 1$), ρ -sequence $\{x_1, \dots, x_{2k}\}$ with even elements belongs to

V_{in}^{2p} and isotropic $\{x_1, \dots, x_k\}$ belongs to V_{in}^p . On the other hand, for ρ -sequence $\{x_1, \dots, x_{2k-1}\}$ with odd elements ($k > 1$), isotropic $\{x_1, \dots, x_{k-1}\}$ belongs to V_{in}^p . Since $\{x_i\}_{1 \leq i \leq k}$ is non-degenerate, we can take its orthogonal complement $\{x'_{k+1}, \dots, x'_{2k-1}\}$ in $\{x_i\}_{1 \leq i \leq 2k-1}$. Then $\{x_1, \dots, x_{k-1}, x'_{k+1}, \dots, x'_{2k-1}\}$ belongs to V_{in}^{2p} and $\{x_k\}$ belongs to V_{un}^{2q} . Finally for $\rho = \pm 1$, ρ -sequence $\{e_1, \dots, e_{2k}\}$ with even elements belongs to V_{in}^{2p} and isotropic $\{e_1, \dots, e_k\}$ belongs to V_{in}^p . For $\{f_i; \tilde{f}_j\}_{1 \leq i, j \leq 2k-1}$ ($k > 1$), $\{f_i; \tilde{f}_j\}_{1 \leq i, j \leq k-1}$ belongs to V_{in}^p . Since $\{f_i; \tilde{f}_j\}_{1 \leq i, j \leq k}$ is non-degenerate we can take its orthogonal complement $\{f'_{k+1}, \dots, f'_{2k-1}; \tilde{f}'_{k+1}, \dots, \tilde{f}'_{2k-1}\}$ in $\{f_i; \tilde{f}_j\}_{1 \leq i, j \leq 2k-1}$. Then $\{f_1, \dots, f_{k-1}, f'_{k+1}, \dots, f'_{2k-1}; \tilde{f}_1, \dots, \tilde{f}_{k-1}, \tilde{f}'_{k+1}, \dots, \tilde{f}'_{2k-1}\}$ belongs to V_{in}^{2p} and $\{f_k, \tilde{f}_k\}$ belongs to V_{un}^{2q} . On the other hand V_{co} is contained in V_{un}^{2q} . Thus we have,

(2.6) $V^{2n} = V_{in}^{2p} \oplus V_{un}^{2q}$; orthogonal decomposition into real non-degenerate subspaces, where V_{in}^{2p} has an real, invariant subspace V_{in}^p .

§ 3. Index theorem for isometry-invariant geodesics

3.1. Let $B = \text{graph } f$, where f is an isometry of $(M, \langle \cdot, \cdot \rangle)$. Let $C_B := \{\gamma: [a, b] \rightarrow M; H^1\text{-curve with } (\gamma(a), \gamma(b)) \in B\}$ and $E(\gamma)$ be the energy integral on C_B . Then $c \in C_B$ is a critical point of $E(\gamma)$ if and only if c is a geodesic which is invariant under f , i.e., $f(c(a)) = c(b)$, $f_*(\dot{c}(a)) = \dot{c}(b)$ (c.f. § 1 and (2.3)). Note that c is complete, i.e., is defined for all values of parameter. At a critical point c with $|\dot{c}(t)| = 1$ for $t \in [a, b]$, we shall consider the second variation $D^2E(c)$. By (2.1) we have on $T_c C_B$,

$$(3.1) \quad D^2E(c)(X(t), Y(t)) = \int_a^b \{ \langle \nabla X(t), \nabla Y(t) \rangle + \langle R(X(t), \dot{c}(t))Y(t), \dot{c}(t) \rangle \} dt,$$

where ∇ denotes the covariant differentiation with respect to $\dot{c}(t)$, and R denotes the curvature tensor. So, for the index problem it suffices to consider $D^2E(c)$ on the subspace

$$\mathcal{E} = \{X(t); H^1\text{-vector fields along } c \text{ such that } X(t) \perp \dot{c}(t), \text{ and } X(b) = f_* X(a)\}.$$

That is, we have index of $D^2E(c)|_{T_c C_B} = \text{index of } D^2E(c)|_{\mathcal{E}}$.

3.2. Let $c: [a, b] \rightarrow M$ be a normal geodesic with $f(c(a)) = c(b)$ and $f_*(\dot{c}(a)) = \dot{c}(b)$. Then from §§ 2.1, we have a vector bundle $V^{2n}(t) = V_h^n(t) \oplus V_v^n(t) \rightarrow [a, b]$ which is induced from $\tau^{2n}: V^{2n} \rightarrow UM$ via the map $\dot{c}: [a, b] \rightarrow UM$. Let φ_t be the geodesic flow, then $(\varphi_t)_*: V^{2n}(t_0) \rightarrow V^{2n}(t+t_0)$ maps $(A, B) \in V^{2n}(t_0)$ into $(Y(t+t_0), \nabla Y(t+t_0))$, where Y is a Jacobi field with $Y(t_0) = A$, $\nabla Y(t_0) = B$. In the following we shall use the notation $\tilde{Y}(t) := (Y(t), \nabla Y(t))$ for Jacobi field $Y(t)$.

Now, if we define $P_f := f_{**}^{-1}(\varphi_{b-a})_*$, then since geodesic flow respects α and (2,1) hold, P_f is a linear symplectic transformation of $(V^{2n}(a), \alpha)$. Applying §§ 2.3 to this symplectic transformation P_f , we get the (α) -orthogonal decomposition $V^{2n}(a) = V_{in}^{2p} \oplus V_{in}^{2q}$, where V_{in}^{2p} has an isotropic subspace V_{in}^p . Then for the Lagrange space $V_v^n(a)$, the projection $V_v^n(a) \cap (V_{in}^p \oplus V_{un}^{2q}) \rightarrow V_{un}^{2q}$ modulo V_{in}^p determines an isotropic subspace $V_{un}^q \subset V_{un}^{2q}$ (see [3], Lemma 2.3.). We define the Lagrange space $V_e^n = V_{in}^p \oplus V_{un}^q$. Then V_e^n is also decomposed as a direct sum $V_e^n = V_{in}^p \oplus V_v^q$ ($V_v^q \subset V_v^n(a)$).

3.3. Let V_1 be the real even dimensional non-degenerate subspace of V_{co} which is generated by the 1-sequences consisting of one element. Let $V_{\bar{1}}$ be the orthogonal complement of V_1 , and $\pi: V_e^n \rightarrow V_{\bar{1}}$ be the projection modulo $V_{\bar{1}}$. Note that V_{in}^p is a subspace of $\mathcal{V} := \text{Ker } \pi = V_{\bar{1}} \cap V_e^n$. We take $\mathcal{U} \subset V_v^q$ as a complement to $\mathcal{V}$ in V_e^n , and put $u(\tilde{Y}) := (\pi|_{\mathcal{U}})^{-1}\pi(\tilde{Y})$. We shall see that the index theorem turns out to be independent of the choice of $\mathcal{U}$. Then clearly $V_e^n = \mathcal{U} \oplus \mathcal{V}$. $\mathcal{V}$ may be decomposed into the form $\mathcal{V} = V_{in}^p \oplus \mathcal{V}_{un}$ ($\mathcal{V}_{un} \subset V_{un}^q$). Finally we put $v := id - u$. Then $v(V_e^n) = \mathcal{V}$.

3.4. Now we shall define the notion of conjugate points. Put $W(t) := (\varphi_{t-a})_* V_e^n \cap V_v^n(t)$. First we define for $a < t_0 < b$,

$C(t_0) :=$ vector space of once broken Jacobi fields $Z(t)$ defined by $Z(t) = Z_1(t)$, $a \leq t \leq t_0$; $Z(t) = Z_2(t)$, $t_0 \leq t \leq b$, where $Z_1(t), Z_2(t)$ are Jacobi fields such that

- (i) $\tilde{Z}_1(t_0) - \tilde{Z}_2(t_0) (= \tilde{Z}(t_0 - 0) - \tilde{Z}(t_0 + 0)) =: \tilde{Y}(t_0) \in W(t_0)$
- (ii) $(P_f - id)\tilde{Z}_1(t) = (P_f - id)\tilde{Y}(t) + (v\tilde{Y}(a))(t)$ for $a \leq t \leq t_0$.

REMARK 1. From (i), (ii), we have

- (iii) $(p_f - id)\tilde{Z}_2(t) = (v\tilde{Y}(a))(t)$ for $t_0 \leq t \leq b$,

and $C(t_0) \subset \mathcal{E}$. In fact, $(Z(a) - f_*^{-1}Z(b), \nabla Z(a) - f_*^{-1}\nabla Z(b)) = \tilde{Z}_1(a) - f_{**}^{-1}\tilde{Z}_2(b) = \tilde{Z}_1(a) - P_f\tilde{Z}_2(a) = \tilde{Z}_1(a) - \tilde{Z}_2(a) - (P_f - id)\tilde{Z}_2(a) = \tilde{Y}(a) - v\tilde{Y}(a) = u\tilde{Y}(a) \in V_v^q$.

REMARK 2. Let $\mathfrak{N} := \{Y(t); \text{Jacobi field with } f_{**}\tilde{Y}(a) = \tilde{Y}(b)\}$ be the null space of $D^2E(c)$ on $\mathcal{E}$. Then obviously $\mathfrak{N} \subset C(t_0)$.

DEFINITION. Put $\bar{n}(t_0) := \dim(C(t_0)/\mathfrak{N})$. We say that t_0 is a conjugate point if $\bar{n}(t_0)$ is positive and $\bar{n}(t_0)$ will be called the multiplicity of the conjugate point t_0 .

Similary we define

$C(a) :=$ vector space of Jacobi fields $Z(t)$ such that $(P_f - id)\tilde{Z}(a) = v\tilde{Y}(a)$ for some $\tilde{Y}(a) \in W(a)$.

Then it is easy to see $C(a) \subset \mathcal{E}$ and $\mathfrak{N} \subset C(a)$. Note that $\mathcal{U} \subset W(a)$ holds. Now the following lemma is well-known.

LEMMA 1 ([4], [5]). Let $n_0 := n(t_0) = \dim W(t_0) \neq \{0\}$. Choose a basis $\{\tilde{Y}_j(t) = (Y_j(t), \nabla Y_j(t))\}_{1 \leq j \leq n}$ for $(\varphi_{t-a})_* V_e^u$ such that $\{\tilde{Y}_j(t_0)\}_{1 \leq j \leq n_0}$ is a basis for $W(t_0)$. Then we have,

(i) $\nabla Y_i(t_0)$, $1 \leq i \leq n_0$; $Y_j(t_0)$, $n_0 + 1 \leq j \leq n$ form a basis for $\perp \dot{c}(t_0)$.

(ii) For $t \neq t_0$ sufficiently near t_0 , $\{Y_j(t)\}_{1 \leq j \leq n}$ are linearly independent. So $W(t_0) \neq \{0\}$ only for finite values of t .

Next, note that $C(t_0) \cap C(t'_0) = \mathfrak{N}$ ($a \leq t_0 < t'_0 < b$). Then we have

LEMMA 2. (i) $\zeta: \bigoplus_{a < t_0 < b} C(t_0) \rightarrow \bigoplus_{a < t_0 < b} W(t_0)$ which maps $Z(t) \in C(t_0)$ into $\tilde{Z}_1(t_0) - \tilde{Z}_2(t_0) = \tilde{Y}(t_0) \in W(t_0)$ is a surjective homomorphism with kernel $\mathfrak{N}$. (ii) $\zeta_a: C(a) \rightarrow vW(a)$ which maps $Z(t) \in C(a)$ into $v(\tilde{Y}(a))$ is a surjective homomorphism with kernel $\mathfrak{N}$.

PROOF. ζ and ζ_a are surjective because $vW(a) \subset \mathcal{V}$ is contained in $(P_f - id)V^{2n}(a)$.

REMARK 3. Since $\zeta|_{C(t_0)}: C(t_0) \rightarrow W(t_0)$ is a surjective linear map with kernel $\mathfrak{N}$, we have $\bar{n}(t_0) = n(t_0)$. By lemma 1, $\bar{n}(t_0)$ is positive only for finite values of t_0 .

3.5. Now we define $\lambda_f: \bigoplus_{a \leq t_0 < b} C(t_0) \rightarrow \mathcal{V}_{un}$ by $\lambda_f Z(t) := (v(\zeta Z(a)))_{un}$ for $Z(t) \in \bigoplus_{a \leq t_0 < b} C(t_0)$ and $\lambda_f Z(t) := (\zeta_a Z)_{un}$ for $Z(t) \in C(a)$. Then we get,

LEMMA 3. $\lambda_f: \bigoplus_{a \leq t_0 < b} C(t_0) \rightarrow \mathcal{V}_{un}$ is a surjective map and we have

$$(3.2) \quad \dim \text{Ker } \lambda_f = \sum_{a < t_0 < b} \bar{n}(t_0) + \dim (V_v^n(a) \cap V_{in}^p) + \dim \mathfrak{N}.$$

PROOF. Since $\mathcal{V}_{un} \subset V_{in}^p \oplus V_v^q$, for any $\tilde{Y}_{un} \in \mathcal{V}_{un}$, there exist $\tilde{Y}_{in} \in V_{in}^p$ and $\tilde{Y}_q \in V_v^q$ such that $\tilde{Y}_{un} = \tilde{Y}_{in} + \tilde{Y}_q$. Then $\tilde{Y}_q \in W(a)$. Now the surjectivity of λ_f follows from the surjectivity of ζ and ζ_a . Next we shall define a linear map $\Phi: \mathcal{U} \oplus \mathcal{V}_{un} \oplus (V_v^n(a) \cap V_{in}^p)$ (direct sum) $\rightarrow W(a)$ by $(\tilde{Y}_u, \tilde{Y}_{un} = \tilde{Y}_{in} + \tilde{Y}_q, \tilde{Y}_\cap) \rightarrow \tilde{Y}_u + \tilde{Y}_q + \tilde{Y}_\cap \in W(a)$. Then clearly Φ is injective. On the other hand, since $W(a) = \mathcal{U} \oplus W(a) \cap \mathcal{V}$ holds and $\Phi|_{\mathcal{V}_{un} \oplus (V_v^n(a) \cap V_{in}^p)}$ is a surjective map onto $W(a) \cap \mathcal{V}$ by the same reason as above, we see that Φ is surjective. So $\dim W(a) = \dim \mathcal{U} + \dim \mathcal{V}_{un} + \dim (V_v^n(a) \cap V_{in}^p)$. Now,

$$\begin{aligned} \dim \text{Ker } \lambda_f &= \dim \bigoplus_{a \leq t_0 < b} C(t_0) - \dim \mathcal{V}_{un} = \sum_{a < t_0 < b} \dim W(t_0) + \dim vW(a) \\ &\quad + \dim \mathfrak{N} - \dim \mathcal{V}_{un} = \sum_{a < t_0 < b} \dim W(t_0) + \dim W(a) + \dim \mathfrak{N} \\ &\quad - \dim \mathcal{U} - \dim \mathcal{V}_{un} = \sum_{a < t_0 < b} \dim W(t_0) + \dim (V_v^n(a) \cap V_{in}^p) \\ &\quad + \dim \mathfrak{N}. \end{aligned} \quad \text{q.e.d.}$$

Next we define a bilinear form Q_f on $\mathcal{V}_{un}$ by $Q_f(\tilde{Y}(a), \tilde{Y}'(a)) := -2\alpha(\tilde{Y}(a), (P_f - id)^{-1}\tilde{Y}'(a))$ for $\tilde{Y}(a), \tilde{Y}'(a) \in \mathcal{V}_{un}$. Note that $(P_f - id)^{-1}\tilde{Y}'(a)$ is determined modulo eigenvectors in V_{in}^p of eigenvalues 1 and eigenvectors in V_1 which are $(\alpha-)$ orthogonal to $\mathcal{V}_{un}$, so Q_f is well-defined. Q_f is a symmetric bilinear form (c.f. (4.1)).

Now we shall state our theorem.

THEOREM. *Index c ($:=$ index of $D^2E(c)$ on $T_c C_B$) $= \sum_{a < t_0 < b} \bar{n}(t_0) + \dim(V_v^n(a) \cap V_{in}^p) + (\text{index} + \text{nullity})Q_f$.*

§ 4. Proof of the theorem

4.1. Let $W(t_0) \neq \{0\}$. Then by lemma 2, for $\tilde{Y}(t_0) \in W(t_0)$ we have an element $Z(t) \in \zeta^{-1}(\tilde{Y}(t_0)) \subset \bigoplus_{a \leq t < b} C(t)$ which is determined modulo $\mathfrak{N}$. Then, for $Z(t) \in \zeta^{-1}(\tilde{Y}(t_0))$, $Z'(t) \in \zeta^{-1}(\tilde{Y}'(t_0))$ ($t_0 \geq t'_0$), we have

$$(4.1) \quad D^2E(c)(Z, Z') = Q_f(\lambda_f Z, \lambda_f Z').$$

In fact, for e.g. $a < t'_0 \leq t_0 < b$,

$$\begin{aligned} D^2E(c)(Z, Z') &= \int_a^b \{ \langle \nabla Z(t), \nabla Z'(t) \rangle - \langle R(Z(t), \dot{c}(t))\dot{c}(t), Z'(t) \rangle \} dt \\ &= \langle \nabla Z(b), Z'(b) \rangle - \langle \nabla Z(a), Z'(a) \rangle + \langle \nabla Z(t_0 - 0) - \nabla Z(t_0 + 0), Z'(t_0) \rangle \\ &= \langle f_*^{-1} \nabla Z_2(b) - \nabla Z_1(a), Z'(a) \rangle + \langle \nabla Y(t_0), Z'(t_0) \rangle \\ &= -2\{ \alpha((P_f - id)\tilde{Z}_2(a) + \tilde{Z}_2(a) - \tilde{Z}_1(a), \tilde{Z}'_1(a)) + \alpha(\tilde{Y}(t_0), \tilde{Z}'_3(t_0)) \} \\ &= -2\{ \alpha(v\tilde{Y}(a) - \tilde{Y}(a), \tilde{Z}'_1(a)) + \alpha(\tilde{Y}(a), \tilde{Z}'_1(a) - \tilde{Y}'(a)) \} \\ &= -2\alpha(v\tilde{Y}(a), \tilde{Z}'_1(a)) = -2\alpha(v\tilde{Y}(a), (P_f - id)^{-1}v\tilde{Y}'(a)) \\ &= -2\alpha((v\tilde{Y}(a))_{un}, (P_f - id)^{-1}(v\tilde{Y}'(a))_{un}) \\ &= Q_f((v\tilde{Y}(a))_{un}, (v\tilde{Y}'(a))_{un}) \\ &= Q_f(\lambda_f Z, \lambda_f Z'). \end{aligned}$$

In the above equalities, we have used the facts that V_e^n is isotropic, $\alpha(V_{in}^p, \mathcal{V}_{un}) = 0$, and $(P_f - id)$ leaves V_{in}^p invariant.

So there exists a subspace $\mathcal{L} \subset \bigoplus_{a \leq t < b} C(t)$, with $\mathcal{L} \cap \mathfrak{N} = \{0\}$ such that its dimension is equal to $\dim \lambda_f^{-1}(\text{Null space of } Q_f) - \dim \mathfrak{N}$ and $D^2E(c)|_{\mathcal{L}} \equiv 0$. On the other hand, there is a subspace $\mathcal{L}' \subset \bigoplus_{a \leq t < b} C(t)$, which is linearly independent to $\text{Ker } \lambda_f$, is of dimension equal to $\text{index } Q_f$, and such that $D^2E(c)|_{\mathcal{L}'}$ is negative definite. Since $\mathcal{L}'$ and the negative part of $\mathcal{L}$ are independent, we get

$$\begin{aligned}
(4.2) \quad \text{Index of } D^2E(c)|_{\mathcal{E}} &\geq \dim \mathcal{L} + \dim \mathcal{L}' \\
&= \dim \text{Ker } \lambda_f - \dim \mathfrak{N} + (\text{nullity} + \text{index})Q_f \\
&= \sum_{a < t < b} \dim W(t) + \dim (V_v^n(a) \cap V_{in}^p) + (\text{nullity} + \text{index})Q_f,
\end{aligned}$$

by virtue of lemma 3.

4.2. To see that the inverse inequality does hold, from the definition of $\mathcal{L}$ and $\mathcal{L}'$, it suffices to show the following:

$$\begin{aligned}
(4.3) \quad &\text{If a differentiable } \xi \in \mathcal{E} \text{ satisfies } D^2E(c)(\xi, \oplus_{a \leq t < b} C(t)) = 0, \\
&\text{then } D^2E(c)(\xi, \xi) \geq 0.
\end{aligned}$$

To show this let $Z(t) \in \zeta^{-1}(\tilde{Y}(t_0)) \subset C(t_0)$ ($a < t_0 < b$). Then we get

$$D^2E(c)(Z, \xi) = \int_a^b d/dt \langle \nabla Z(t), \xi(t) \rangle dt = \langle \nabla Y(t_0), \xi(t_0) \rangle - \langle \nabla uY(a), \xi(a) \rangle,$$

where $uY(t)$ denotes the Jacobi field defined by $(uY(a), \nabla uY(a)) = u(\tilde{Y}(a))$. Similarly, for $Z(t) \in \zeta_a^{-1}(v\tilde{Y}(a)) \subset C(a)$, we get

$$D^2E(c)(Z, \xi) = \langle \nabla Y(a), \xi(a) \rangle - \langle \nabla uY(a), \xi(a) \rangle.$$

Now $u(\tilde{Y}(a)) \in \mathcal{U} \rightarrow \langle \nabla uY(a), \xi(a) \rangle$ is a linear form on $\mathcal{U}$. Since the subspace generated by $\mathcal{U}$ and V_1 is non-degenerate and $\alpha|_{\mathcal{U}} \equiv 0$, the above linear form may be written in the form $u(\tilde{Y}(a)) \in \mathcal{U} \rightarrow 2\alpha(u(\tilde{Y}(a)), \tilde{Y}_1) = -\langle \nabla uY(a), Y_1(a) \rangle$ for some $\tilde{Y}_1 \in V_1$. Then if we put $\xi_1 = \xi + Y_1$, where Y_1 belongs to $\mathfrak{N}$, we get

$$(4.4) \quad D^2E(c)(Z, \xi_1) = \langle \nabla Y(t_0), \xi_1(t_0) \rangle = -2\alpha(v(\tilde{Y}(a)), \tilde{Y}_1(a)) = 0 \quad (a \leq t_0 < b).$$

Next we choose a basis $\{\tilde{Y}_i(a)\}_{1 \leq i \leq n}$ such that $\{\tilde{Y}_i(a)\}_{1 \leq i \leq p}$ forms a basis of V_{in}^p and $\{\tilde{Y}_a(a)\}_{p+1 \leq a \leq n}$ forms a basis for V_{un}^a . Since $\{P_f \tilde{Y}_j(a) = f_{**}^{-1} \tilde{Y}_j(b)\}_{1 \leq j \leq p}$ generates $P_f V_{in}^p = V_{in}^p$, we have

$$(4.5) \quad f_{**}^{-1} \tilde{Y}_j(b) = \sum_{k=1}^p a_j^k \tilde{Y}_k(a) \quad (j=1, \dots, p),$$

where (a_j^k) is non-singular. Then from (4.4), lemma 1, $\xi_1(b) = f_* \xi_1(a)$, and (4.5), we know that $\xi_1(t)$ may be expressed in the form $\xi_1(t) = \sum_{i=1}^n w_i(t) Y_i(t)$ with differentiable $w^i(t)$.

Now from $\xi_1(b) = f_* \xi_1(a)$, we get $\sum_{\beta=p+1}^n w^\beta(b) Y_\beta(b) = \sum_{k=1}^p \{ \sum_{j=1}^p w^j(a) b_j^k - w^k(b) \} Y_k(b)$, where (b_j^i) denotes the inverse matrix of (a_j^i) , i.e.,

$$\sum_{\beta=p+1}^n w^\beta(b) \tilde{Y}_\beta(b) - \sum_{k=1}^p \left(\sum_{j=1}^p w^j(a) b_j^k - w^k(b) \right) \tilde{Y}_k(b) = \tilde{U}(b) \in W(b).$$

Then, by direct calculation using this and (4.5) we can show that

$$\left\langle \sum_{r=1}^n w^r(b) \nabla Y_r(b), \sum_{s=1}^n w^s(b) Y_s(b) \right\rangle - \left\langle \sum_{r=1}^n w^r(a) \nabla Y_r(a), \sum_{s=1}^n w^s(a) Y_s(a) \right\rangle$$

=0 does hold. Finally we have

$$\begin{aligned} (4.6) \quad D^2 E(c)(\xi, \xi) &= D^2 E(c)(\xi_1, \xi_1) \\ &= \int_a^b |\sum \dot{w}^j(t) Y_j(t)|^2 dt + \int_a^b d/dt \left\langle \sum_{k=1}^n w^k(t) \nabla Y_k(t), \xi(t) \right\rangle dt \\ &\geq \left\langle \sum_{r=1}^n w^r(b) \nabla Y_r(b), \sum_{s=1}^n w^s(b) Y_s(b) \right\rangle \\ &\quad - \left\langle \sum_{r=1}^n w^r(a) \nabla Y_r(a), \sum_{s=1}^n w^s(a) Y_s(a) \right\rangle = 0. \end{aligned}$$

This completes the proof of theorem.

REMARK 4. If c is non-degenerate (i.e., 1 is not an eigenvalue of P_f), then $\mathcal{V} = V_e^n$ and $\mathcal{V}_{un} = V_{un}^q$.

REMARK 5. If $P_f = \text{identity}$, then $V_e^n = V_v^n(a)$, $V_{in}^p = \{0\}$, and $\lambda_f = 0$. So Index $c = \sum_{a < t < b} \dim((\varphi_{t-a})_* V_v^n(a)) \cap V_v^n(t) = \text{sum of the multiplicities of ordinary conjugate points.}$

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