Good torus fibrations with twin singular fibers

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§ 1. Introduction

In [2], the notion of good torus fibration (GTF) was introduced, and the types of singular fibers of good torus fibrations were classified. Matsumoto's problem is as follows.

PROBLEM. Classify all closed, oriented smooth 4-manifolds that admit good torus fibration, up to orientation preserving (but not necessarily fiber preserving) diffeomorphism.

Among the singular fibers which are not multiple, the simplest one seems to be \(I^+_1\) or \(I^-_1\). \(I^+_1\) (resp. \(I^-_1\)) consists of an immersed 2-sphere which intersects itself at one point with intersection number +1 (resp. -1). The following Theorem is known.

THEOREM 1.1 ([2], Theorem 8.1.). Let \(f_i: M_i \rightarrow S^2, i=1,2\), be GTF's over \(S^2\) with at least one singular fiber. Suppose that each singular fiber is of type \(I^+_1, I^-_1\) and that \(\sigma(M_i) \neq 0\). Then \(M_i\) is diffeomorphic to \(M_2\) if and only if \(\sigma(M_1) = \sigma(M_2)\) and \(e(M_1) = e(M_2)\). (The symbols \(\sigma\) and \(e\) represent the signature and the euler characteristic respectively.)

Let \(f: M \rightarrow S^2\) be a GTF satisfying the condition of Theorem 1.1. Let \(a, b\) be the numbers of the singular fibers of type \(I^+_1, I^-_1\) of \(M\) respectively. Then, \(\sigma(M) = -(2/3)(a-b)\) and \(e(M) = a + b\). See [2] Theorem 7.3. Therefore, the diffeomorphism type of the total space is determined by \(a\) and \(b\) if \(a \neq b\).

What happens if \(a=b\)? In this case, it is known that we can deform the projection map slightly so that all the singular fibers are the simple twin singular fibers.

In this paper, we classify all the manifolds that admit GTF structures over \(S^2\) with simple twin singular fibers only. We have

MAIN THEOREM. Let \(m\) be a positive integer and let \(M \rightarrow S^2\) be a GTF. Assume that (A) or (B) is satisfied.

(A) \(M\) has only \(m\) twin singular fibers as singular fibers. At most two of them are multiple.
<table>
<thead>
<tr>
<th>$Z/n$</th>
<th>$\mathbb{Z}$</th>
<th>$S^1 \times S^3 \times S^3$</th>
<th>$S^1 \times S^3 \times S^3 # m(S^1 \times S^3)$</th>
<th>$S^1 \times S^3 \times S^3 # m(S^2 \times S^1)$</th>
<th>$S^1 \times S^3 \times S^3 # m(S^2 \times S^2)$</th>
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<tr>
<td>$n$: even, $\geq 2$</td>
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<td>$L_n # (m-1)(S^1 \times S^3)$</td>
<td>$L_n # (m-1)(S^2 \times S^1)$</td>
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<td>$n$: odd, $\geq 3$</td>
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<td>$\pi_1(M)$</td>
<td>spin type II</td>
<td>not spin type II</td>
<td>not spin type II</td>
<td>spin type II</td>
<td>not spin type II</td>
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<tr>
<td>$e(M)$</td>
<td>$= 2$</td>
<td>$&lt; 2$</td>
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(B) \( M \) has only one multiple torus and only \( m \) twin singular fibers as singular fibers. At most one of them is multiple.

Then, \( M \) is diffeomorphic to one of the manifolds in the table above.

(The symbol \( \chi(M) \) means the euler characteristic of \( M \), which is \( 2m \). The symbol \( kX \) means \( X \# \cdots \# X \) (\( k \)-times). \( S^2 \times S^2 \) means the total space of the non-trivial \( S^2 \)-bundle over \( S^2 \). \( L_n \) and \( L'_n \) are the manifolds defined in [4].)

Conversely, the manifolds in the table above admit such GTF structures.

In §4 we treat the case \( m=1 \). This case was essentially treated in [4], but we repeat it here in a way more convenient for our purpose because we need it in §5 to treat the case \( m=2 \), (A). In §6 we treat the case \( m \geq 3 \), (A); and the case \( m \geq 2 \), (B).

The author would like to express his thanks to Prof. Y. Matsumoto for suggesting the problem and for many helpful suggestions and encouragements.

NOTATIONS. (i) \( N(X) \) means the regular neighborhood of \( X \) in some manifold.

(ii) For sets \( U, V, A_i \), the symbols \( U + V \), \( \sum_{i=1}^n A_i \) mean the disjoint unions.

(iii) We sometimes represent a circle and its homology class in some space by the same symbol.

(iv) Let \( M \) be a manifold and \( \Sigma \) be a sphere embedded in \( \text{Int}(M) \) which has trivial normal bundle. Then, \( \chi(M; \Sigma) \) means the manifold obtained from \( M \) by the Milnor surgery on \( \Sigma \). If \( \dim(M) = 4 \) and \( \dim(\Sigma) = 1 \), there are two framings for \( \Sigma \). If \( \dim(M) = 4 \) and \( \dim(\Sigma) = 2 \), there is only one framing for \( \Sigma \).

§2. Twins

First, we recall some properties of the twin. See [3]. The twin is the manifold which consists of two \( S^2 \times D^2 \)'s plumbed at two points with opposite signs. We assume that the angle is straightened. We denote the twin by the symbol \( Tw \).

Let \( R, S \) be the cores of two \( S^2 \times D^2 \)'s. They generate \( H_2(Tw; Z) \cong Z \oplus Z \). The following lemma is known. See [3].

**Lemma 2.1.** Let \( T^2 \) be an embedded torus in \( S^4 \) which bounds a solid torus in \( S^4 \). Then, \( S^4 - \text{Int}(N(T^2)) \) is a twin. Therefore \( \partial(Tw) \) is diffeomorphic to \( T^2 \).

**Proof.** Regard \( S^4 \) as \( S^1 \times D^3 \cup_h D^2 \times S^2 \). Let \( K_0 \) be an unknotted loop in \( D^3 \). Consider the manifold \( X = S^1 \times (D^3 - \text{Int}(N(K_0))) \cup_h D^2 \times S^2 \). Let \( a \) be the arc properly embedded in \( D^3 - \text{Int}(N(K_0)) \) as in Figure 1.
Fig. 1

Put $R = S^1 \times a \cup D^2 \times \partial a$ and $S = \{0\} \times S^1 \subset D^2 \times S^2$. Then, $R$ and $S$ are 2-spheres which have trivial normal bundles in $X$, and they meet transversely at two points with opposite signs. Therefore, the regular neighborhood $N$ of $R \cup S$ is a twin. We may assume that $N$ is $S^1 \times N(a) \cup D^2 \times S^2$ where $N(a)$ is the regular neighborhood of $a$ in $D^3 - \text{Int}(N(K_0))$. Then, $X - \text{Int}(N)$ is diffeomorphic to $\partial X \times [0, 1]$. Therefore we conclude that $X$ is $Tw$ and $\partial(Tw)$ is $T^n$.

Since $K_0$ bounds a disk in $D^3$, therefore $S^1 \approx K_0$ bounds a solid torus in $S^3$, and $S^1 \approx K_0$ and $T^2$ are isotopic to each other. Now we have the conclusion. □

Let $D(r)$, $D(s)$ be 2-disks properly embedded in $Tw$ such that $R \cdot D(r) = S \cdot D(s) = 1$ and $R \cdot D(s) = S \cdot D(r) = 0$. $\partial D(r)$ and $\partial D(s)$ are circles in $\partial(Tw)$. We call them $r$ and $s$ respectively. Note that their homology classes in $H_1(\partial(Tw); \mathbb{Z})$ are well-defined. Choose a circle $l$ in $\partial(Tw)$ such that $\langle l, r, s \rangle$ is an oriented basis of $H_1(\partial(Tw); \mathbb{Z})$. For the ambiguity of the choice of $l$, see Remark 2.5.

Next, we consider the manifold $D^2 \times T^2 \approx D^2 \times S^1 \times S^1$. Let $l, r, s, \bar{s}$ be the circles $\partial D^2 \times \{s\} \times \{s\}, \{s\} \times S^1 \times \{s\}, \{s\} \times \{s\} \times S^1$ in $\partial(D^2 \times T^2)$ respectively.

For any integer $m \geq 2$, let $F_m$ be the compact, connected planar surface whose boundary has $m$ components $S^1_1, \ldots, S^1_m$. The boundary of the manifold $F_m \times T^2 = F_m \times S^1 \times S^1$ has $m$ copies of $T^2$. Let $l_i, r_i, \bar{s}_i$ be the circles $S^1_i \times \{s\} \times \{s\}, \{s\} \times S^1_i \times \{s\}, \{s\} \times \{s\} \times S^1_i$ in $\partial(F_m \times T^2)$ respectively.

Throughout this paper, $Tw$ means a copy of $Tw$ and $R_i, S_i, l_i, r_i, s_i$ mean the surfaces or curves in $Tw_i$ corresponding to $R, S, l, r, s$ respectively. (But $S^n$, of course, means the $n$-dimensional sphere).

**Lemma 2.2.** (i) $\chi(S^1 \times (S^1 \times D^2); S^1 \times \{s\})$ is diffeomorphic to $Tw$ for any framing of $S^1 \times \{s\}$.

(ii) Let $\lambda$ be a simple loop in $D^2 \times T^2$ such that $[\lambda] \in \pi_1(D^2 \times T^2)$ is a primitive element. Then, $\chi(D^2 \times T^2; \lambda)$ is diffeomorphic to $Tw$ for any framing of $\lambda$.

(iii) $\pi_1(D^2 \times T^2)$ is generated by $\bar{r}, \bar{s}$. If $[\lambda] = \bar{s}$, we can choose a framing of
\[ \lambda \text{ such that the curves } r, -\tilde{l}, s \subset \partial(D^3 \times T^2) \text{ can be taken as } l, r, s \subset \partial(D^3 \times T^2; \lambda) = \partial(Tw) \text{ respectively.} \]

**Proof.** Let \( K_0 \) be an unknot in \( D^3 \).

(i) As is shown in the proof of Lemma 2.1, \( S^1 \times (D^3 - \text{Int}(N(K_0))) \cup_S D^2 \times S^1 \) is \( Tw \) for any diffeomorphism \( h : \partial(S^1 \times D^3) \to \partial(D^3 \times S^1) \). Note that \( D^3 - \text{Int}(N(K_0)) \) is diffeomorphic to \( S^1 \times D^2 - \text{Int}(N[\ast]) \) \((\ast \in \text{Int}(S^1 \times D^3)) \). We have that \( S^1 \times (S' \times D^2 - \text{Int}(N(\ast))) \cup_S D^2 \times S^1 \) is \( Tw \) for any \( h \). Now we have the conclusion.

(ii) Under some identification of \( D^2 \times S^1 \) with \( S^1 \times (S^1 \times D^2) \), \( \lambda \) is homotopic to \( S^1 \times \{ \ast \} \). Since \( D^2 \times S^1 = S^1 \times (S^1 \times D^2) \) is 4-dimensional, \( \lambda \) is isotopic to \( S^1 \times \{ \ast \} \). We have \( (D^2 \times S^1, \lambda) \cong (S^1 \times (S^1 \times D^2), S^1 \times \{ \ast \}) \).

(iii) Identify \( D^2 \times S^1 \) with \( S^1 \times (S^1 \times D^2) \) such that \([S^1 \times \{ \ast \}] = [\tilde{l}] = \tilde{s}\) in \( \pi_1(D^2 \times S^1) \). Let \( i : N(S^1 \times \{ \ast \}) \to S^1 \times N(\{ \ast \}) \) be the natural identification and let \( \phi : N(\{ \ast \}) \to D^3 \) be a diffeomorphism. For the framing of \( N(S^1 \times \{ \ast \}) \), choose the composition of the two maps \( \psi : N(S^1 \times \{ \ast \}) \to S^1 \times N(\{ \ast \}) \) and \( i \circ \phi : S^1 \times N(\{ \ast \}) \to S^1 \times D^3 \). Let \( j : \partial(S^1 \times D^3) \to \partial(D^3 \times S^1) \) be the natural map. Put \( h = j \cdot (i \circ \phi) \).

See Figure 2-a. It shows \( S^1 \times (S^1 \times D^2) \). A meridian (resp. longitude) circle of \( S^1 \times D^2 \) is \( \tilde{l} \) (resp. \( \tilde{r} \)) and \( S^1 \times \{ \ast \} \) \((\ast \in \partial(S^1 \times D^3))\) is \( \tilde{s} \). Figure 2-b shows

![Fig. 2-a](image-url)

![Fig. 2-b](image-url)

\( S^1 \times (D^3 - \text{Int}(N(K_0))) \), which is diffeomorphic to \( S^1 \times (S^1 \times D^3 - \text{Int}(N(\{ \ast \}))) \). We know that \( S^1 \times (D^3 - \text{Int}(N(K_0))) \cup_S D^2 \times S^1 \) is a twin. Let \( D_1 \) be a disk properly embedded in \( \{ \ast \} \times (D^3 - \text{Int}(N(K_0))) \) such that \( \partial D_1 \) is \( \tilde{l} \) and that \( D_1 \) meets \( a \) transversely where \( a \) is the properly embedded arc in Figure 2-b. Let \( D_1 \) be the disk \( S^1 \times b \cup D^2 \times \{ P \} \) where \( b \) is the properly embedded arc and \( P \) is a point of \( \partial b \) in Figure 2-b. The argument in the proof of Lemma 2.1 shows that \( D_1 \) (resp. \( D_2 \)) can be taken as \( D(r) \) (resp. \( D(s) \)) up to orientation. This completes the proof. \( \Box \)
Figure 3 is useful for visualizing twins.

We will introduce another visualization of twins. By the definition, $T_w$ is of the form $(S^2 \times D^2)_1 \cup (S^2 \times D^2)_2$, where $(S^2 \times D^2)_i$ is a copy of $S^2 \times D^2$. $(S^2 \times D^2)_1 \cap (S^2 \times D^2)_2$ is the disjoint union of two $D^2$s, and the closure of its complement in $T_w$ is the union of two $D^3 \times S^1 \times D^2$s which are regarded as round 1-handles. This consideration leads us to:

**Lemma 2.3.** For $i = 1, 2$, let $L_i \cup L'_i$ be a Hopf link in $\partial D_i^4$, where $D_i^4$ is a 4-ball. For $j = 1, 2$, let $h_j$ be a round 1-handle connecting $L_j$ and $L'_j$. We denote by $\mathcal{X}$ the set of all the diffeomorphism types of the manifolds obtained in this way. Suppose that $X$, a member of $\mathcal{X}$, satisfies the following conditions (i) and (ii):

(i) $X$ is orientable.

(ii) The intersection form on $H_4(X; \mathbb{Z})$ is trivial.

Then, $X$ is a twin.

**Proof.** Let $N(L_i)$ be a regular neighborhood of $L_i$. We may assume that $(D_i^4 \cup D_i^4) \cap (h_1 \cup h_2) = \bigcup_{i=1}^{2} N(L_i)$. Orient $X$ and orient $L_i$ and $L'_i$ so that $L_i \cup L'_i$ is the $(\pm)$-Hopf link in $\partial D_i^4$. Then, orient $L_i$ so that $[L_i] = [L'_i]$ in $H_i(h_1^i; \mathbb{Z})$, and orient $L'_i$ so that $L_i \cup L'_i$ is the $(-)$-Hopf link in $\partial D_i^4$.

Choose an oriented meridian-(preferred) longitude pair $(m_i, l_i)$ for $N(L_i)$, $(i = 1, 2; j = 1, 2)$ so that $L_i$ and $l'_i$ are homologous to each other in $N(L_i)$. 
Let $A^i$ be the core of $h^i$. We can regard $m^i_l$ and $l^i_l$ as the curves in $h^i - \text{Int}(N(A^i))$, which is diffeomorphic to $S^1 \times S^1 \times D^1 \times D^1$. Since $\langle m^i_l, l^i_l \rangle$ is a basis of $H_1(h^i - \text{Int}(N(A^i)); Z)$ for $i = 1, 2$, there exist integers $a^i, b^i, c^i, d^i$ such that the formula $[m^i_l, l^i_l] = [m^i_l, l^i_l] \begin{bmatrix} a^i & b^i \\ c^i & d^i \end{bmatrix}$ is satisfied.

The construction above shows that $a^1 = d^1 = 1, a^2 = d^2 = \pm 1$, and $c^1 = c^2 = 0$. Let $E^i$ be a disk in $D^1$ such that $\partial A^i = \partial E^i \cup \partial E^i$ and put $x^i = [A^i + E^i + E^i] \in H_2(X; Z)$. Then, $\langle x^1, x^2 \rangle$ is a basis of $H_2(X; Z)$. We have

$$\begin{align*}
| x^1 \cdot x^1 | &= | b^1 | \\
| x^1 \cdot x^2 | &= 1 - d^2 \\
| x^2 \cdot x^2 | &= | b^2 |
\end{align*}$$

By assumption, we have $b^1 = b^2 = 0$ and $d^1 = 1$. Since the intersection form on $H_2(Tw; Z)$ is trivial, we have the conclusion. □

Let $f^i : T^3 \to T^3$ $(i = 1, 2)$ be a diffeomorphism. By Waldhausen’s theorem [5], $f^1$ and $f^2$ are isotopic to each other if and only if $(f^1)_* = (f^2)_*$, where $(f)_* : H_1(T^3; Z) \to H_1(T^3; Z)$.

The following Proposition is due to [3].

**Proposition 2.4.** (i) For any diffeomorphism $h : \partial(Tw) \to \partial(Tw)$, define $A^h \in GL_3 Z$ by \((h_*(l), h_*(r), h_*(s)) = (l, r, s)A^h\) in $H_1(\partial(Tw); Z)$. Then, $h$ can be extended to a diffeomorphism $h : Tw \to Tw$ if and only if $A^h \in H_i$, where

\[
H_i = \left\{ \begin{bmatrix} \pm 1 & 0 & 0 \\ * & a & b \\ * & c & d \end{bmatrix} \in GL_3 Z \mid a + b + c + d \equiv 0 \pmod{2} \right\}
\]

(ii) $H_i$ is generated by

\[
\begin{bmatrix}
-1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}, \quad
\begin{bmatrix}
1 & 0 & 0 \\
0 & -1 & 0 \\
0 & 0 & 1
\end{bmatrix}, \quad
\begin{bmatrix}
1 & 0 & 0 \\
0 & 0 & 1 \\
0 & 0 & -1
\end{bmatrix}, \quad
\begin{bmatrix}
1 & 0 & 0 \\
0 & 0 & -1 \\
0 & 2 & 1
\end{bmatrix}
\]

(iii) Put

\[
H_\epsilon = \left\{ \begin{bmatrix} \pm 1 & 0 & 0 \\ * & a & b \\ * & c & d \end{bmatrix} \in GL_3 Z \right\}
\]

Then, the set...
Remark 2.5. Recall that $l$ is defined as an element of $H_1(\partial(Tw); \mathbb{Z})$ such that $\langle l, r, s \rangle$ is a basis for $H_1(\partial(Tw); \mathbb{Z})$. Let $l_i$ and $l_j$ be two $l$'s. Then, $l_j = \pm l_i + ar + bs$ for some integers $a, b$. Proposition 2.4 shows that there exists a diffeomorphism $\overline{h} : Tw \to Tw$ such that $\langle \overline{h}_x(l_i), \overline{h}_x(r), \overline{h}_x(s) \rangle = (l_j, r, s)$.

Lemma 2.6. (i) For any diffeomorphism $h : \partial(D^2 \times T^2) \to \partial(D^2 \times T^2)$, define $A^h \in GL_3 \mathbb{Z}$ by $(h(l), h(r), h(s)) = \langle 1, 0, s \rangle A^h$ in $H_1(\partial(D^2 \times T^2); \mathbb{Z})$. Then, $h$ can be extended to a diffeomorphism $\overline{h} : D^2 \times T^2 \to D^2 \times T^2$ if and only if $A^h \in H_2$, where

$$
H_2 = \left\{ \begin{pmatrix} \pm 1 & * & * \\ 0 & * & * \\ 0 & * & * \end{pmatrix} \in GL_3 \mathbb{Z} \right\}.
$$

(ii) $H_2$ is generated by

$$
\left\{ \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & * & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & * \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & * & * \\ 0 & * & * \end{pmatrix} \right\}.
$$

Proof. (ii) Let $A$ be an element of $H_2$. If the $11$-component of $A$ is $-1$, then $A$ is of the form $\begin{pmatrix} \pm 1 & * & * \\ 0 & * & * \\ 0 & * & * \end{pmatrix}$. Since $\begin{pmatrix} 1 & a_{22} & a_{23} \\ 0 & a_{22} & a_{23} \\ 0 & a_{22} & a_{23} \end{pmatrix}$ or $\begin{pmatrix} 1 & a_{22} & a_{23} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, (ii) is proved.

(i) Note that $l$ is null-homologous in $D^2 \times T^2$ and that $\langle r, s \rangle$ is a basis of $H_1(D^2 \times T^2; \mathbb{Z})$. If $h$ extends to $\overline{h}$, then $\overline{h}_x(l)$ is also null-homologous in $D^2 \times T^2$. Therefore $A^h \in H_2$.

To prove the converse, we have only to show that $h$'s for which $A^h$ is of the form in (ii) extends to $\overline{h} : D^2 \times T^2 \to D^2 \times T^2$. If $A^h$ is $\begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ or $\begin{pmatrix} 1 & 0 & 0 \\ 0 & * & * \\ 0 & * & * \end{pmatrix}$, then it is clear that $h$ extends to $\overline{h} : D^2 \times T^2 \to D^2 \times T^2$. If $A^h$ is $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, then it is clear that $h$ extends to $\overline{h} : D^2 \times T^2 \to D^2 \times T^2$. If $A^h$ is $\begin{pmatrix} 0 & * & * \\ 0 & * & * \end{pmatrix}$,
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We may assume that $h$ is of the form $h' \times \text{id}: (\partial D^2 \times S^1) \times S^1 \to (\partial D^2 \times S^1) \times S^1$ where $h'_s$ maps $\mu$ (meridian) and $\lambda$ (longitude) to $\mu$ and $a\mu + \lambda$ respectively. It is well-known that $h'$ extends to $\bar{h'}: D^2 \times S^1 \to D^2 \times S^1$. Put $\bar{h} = \bar{h'} \times \text{id}$. It is an extension of $h$. Similar argument holds for

\[
\begin{bmatrix}
1 & 0 & a \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\]

**Remark 2.7.** Lemma 2.6 (i) shows that $h$ extends to $\bar{h}$ if and only if $h_*(\bar{l}) = \pm \bar{l}$. Therefore, for an arbitrary 4-manifold $M$ and an arbitrary embedding $e: \partial(D^2 \times T^2) \to \partial M$, the diffeomorphism type of $D^2 \times T^2 \cup e \cdot M$ is determined by the curve $e(\bar{l})$. We call $\bar{l}$ the meridian of $D^2 \times T^2$.

**Remark 2.8.** $\chi(Tw; S)$ and $\chi(Tw; R)$ are diffeomorphic to $D^2 \times T^2$. The curve $r$ (resp. $s$) is the meridian of $\chi(Tw; S)$ (resp. $\chi(Tw; R)$).

**Lemma 2.9.** (i) For any diffeomorphism $h: \partial(F_m \times T^2) \to \partial(F_m \times T^2)$ which satisfies $h(S_i \times S^i \times S') = S_i \times S^i \times S^i$ ($i = 1, \ldots, m$), define $A^h = A_{h,1} \oplus \cdots \oplus A_{h,m} \in \bigoplus_m \text{GL}_3\mathbb{Z}$ by $(h_*(l_i), h_*(\bar{r}_i), h_*(\bar{s}_i)) = (l_i, \bar{r}_i, \bar{s}_i)A_{h,i}$ in $H_3(\partial(F_m \times T^2); \mathbb{Z})$. Then, $h$ can be extended to a diffeomorphism $\bar{h}: F_m \times T^2 \to F_m \times T^2$ if and only if $A^h \in H_3$.

(ii) $H_3$ is generated by

$$H_3 = \left\{ \sum_{i=1}^m \begin{bmatrix}
\varepsilon & 0 & 0 \\
p_i & a & b \\
q_i & c & d
\end{bmatrix} \in \bigoplus_{i=1}^m \text{GL}_3\mathbb{Z} : \begin{cases}
\sum_{i=1}^m p_i = 0 \\
\sum_{i=1}^m q_i = 0
\end{cases} \right\}.$$
and \[ \bigoplus_{i=1}^{m} A_i \begin{pmatrix} 1 & 0 & 0 \\ p_i & 1 & 0 \\ q_i & 0 & 1 \end{pmatrix}, \sum_{i=1}^{m} p_i = 0 \sum_{i=1}^{m} q_i = 0 \] generate \( H_3 \). Note that \( \bigoplus_{i=1}^{m} A_i \) above is \( \mathbb{A}^{b_1,p_1,q_1} \mathbb{A}^{b_2,p_2,q_2} \cdots \mathbb{A}^{b_m,p_m,q_m} \). This completes the proof.

(i) Note that \( \bar{r}_1 = \cdots = \bar{r}_m \) and \( \bar{s}_1 = \cdots = \bar{s}_m \) in \( \pi_i(F_m \times T^3) \) and they generate the center of \( \pi_i(F_m \times T^3) \). If \( h \) extends to \( \bar{h} \), then \( h_\#(\bar{r}_1) = \cdots = h_\#(\bar{r}_m) \)

and \( h_\#(\bar{s}_1) = \cdots = h_\#(\bar{s}_m) \). Hence \( A^h \) is of the form \( \bigoplus_{i=1}^{m} \begin{pmatrix} * & x_i & y_i \\ * & a & b \\ * & c & d \end{pmatrix} \). Note that \( h_\#(\bar{r}_1) = \tilde{l}_1(\bar{r}_1) \) is in the center of \( \pi_i(F_m \times T^3) \). This shows that \( x_i = 0 \). Similarly, \( y_i = 0 \). Since \( \tilde{l}_1 + \cdots + \tilde{l}_m = 0 \), therefore \( h_\#(\tilde{l}_1) + \cdots + h_\#(\tilde{l}_m) = 0 \). Now we have that \( A^h \) belongs to \( H_3 \).

To prove the converse, we have only to show that \( h \)'s for which \( A^h \) is of the form in (ii) extend to \( \bar{h} \).

The former is clearly extended to \( \bar{h} \).

For the latter, \( \mathbb{A}^{b_1,p_1,q_1} \mathbb{A}^{b_2,p_2,q_2} \cdots \mathbb{A}^{b_m,p_m,q_m} \), let \( b: [0,1] \times [0,1] \to F_m \) be a proper embedding which satisfies \( b([0] \times [0,1]) \subset S_1 \) and \( b([1] \times [0,1]) \subset S_2 \). Let \( \rho: \mathbb{R} \to \mathbb{R} \) be a smooth function which satisfies \( \rho(x) = 0 \) for \( x \leq 0 \) and \( \rho(x) = 1 \) for \( x \geq 1 \). Let \( (x,y,z) \in F_m \times \mathbb{R}/\mathbb{Z} \times \mathbb{R}/\mathbb{Z} \) be a coordinate of \( F_m \times S^1 \times S^1 \). Define the diffeomorphism \( \Psi: F_m \times S^1 \to F_m \times S^1 \times S^1 \) by

\[
(x,y,z) \begin{cases} (x, (\rho \cdot \text{proj}_2 \cdot b^{-1}(x))p + y, (\rho \cdot \text{proj}_2 \cdot b^{-1}(x))q + z) & \text{if } x \in \text{Image}(b) \\
(x,y,z) & \text{if } x \notin \text{Image}(b) \end{cases}
\]

where \( \text{proj}_2: [0,1] \times [0,1] \to [0,1] \) denotes the projection to the second coordinate. \( \Psi \) or \( \Psi^{-1} \) restricted to \( \partial(F_m \times S^1 \times S^1) \) is isotopic to \( h \). Therefore \( h \) can be extended.\( \square \)

**Lemma 2.10.** Let \( C_1, C_2 \) be two curves in \( \partial(Tw) \) such that there exists a diffeomorphism \( h: \partial(Tw) \to S^1 \times S^1 \times S^1 \) which maps \( C_1, C_2 \) to the curves \( \{\ast\} \times \{\ast\} \times S^1 \) and \( \{\ast\} \times \{\ast\} \times S^1 \) (\( \ast \neq \ast \)). Assume that \( [C_1] = [C_2] = 2a + b + c \in H(\partial(Tw); \mathbb{Z}) \) for some integers \( a, b, c \). Let \( D_i \) be a 2-chain in \( Tw \) such that

(a) \( \partial D_i (\text{mod } 2) = C_i \) (\( i = 1, 2 \))

(b) \( [D_1] = [D_2] \) in \( H_3(Tw, \partial(Tw); \mathbb{Z}) \)

(c) \( D_i \) and \( D_i \) meet transversely.

Then,

(i) \( D_1 \cdot D_2 \) (\( \text{mod } 2 \)) is determined by \( b, c \).

(ii) \( D_1 \cdot D_2 \equiv bc \) (\( \text{mod } 2 \)).

**Proof.** (i) Assume that the pair \( D_i, D_i \) satisfies the conditions above
for \( j = 1, 2 \). Attach a 2-handle \( h^i \) to \( Tw \) along \( C_i \) with framing \( C_i \), i.e. so that \( C_i \) and \( C_j \) span disjoint disks in \( h^i \). Then, \( D_i \cdot D_j = [D_i + D_j] \) where \( D \) is the core of \( h^i \). Since

\[
D_i \cdot D_j - D_i \cdot D_j = [D_i + D_j] - [D_i + D_j] = 0
\]

and \( [D_i - D_j] \in H_2(Tw; Z) \), we have \( [D_i - D_j] = 0 \) (mod 2) and \( D_i \cdot D_j = D_i \cdot D_j \) (mod 2).

(ii) \( D_i \cdot D_j \) is determined by \( a, b, c \). We put \( D_i \cdot D_j = Y(a, b, c) \). Since \( \gcd(2a, b, c) = 1 \), we know that \( b \equiv 0 \) or \( c \equiv 0 \) (mod 2). Put \( b' = b/\gcd(b, c), c' = c/\gcd(b, c) \). Let \( k: S^1 \times S^1 \times S^1 \times [0, 1] \to Tw \) be a collar of \( a(Tw) \), i.e. \( k \) is a smooth embedding which satisfies \( k(S^1 \times S^1 \times S^1 \times [0]) = S^1 \times S^1 \times S^1 \times [0] \to (Tw) \) is a diffeomorphism. Moreover, we assume that \( k([*] \times S^1 \times S^1 \times [0]) = l \), and

\[
k([*] \times [s] \times S^1 \times [0]) = b'r + c's.\]

Choose \( s_1, s_2 \in S^1 \) \((s_1 \neq s_2)\). Let \( t_i: S^1 \times S^1 \to \{s_i\} \times S^1 \times S^1 \times [0] \) be an inclusion map for \( i = 1, 2 \). We may assume that there exists a loop \( C: S^1 \to S^1 \times S^1 \) such that \( C_i = k \cdot t_i \cdot C \). Choose a point \( P \) in \( S^1 \times S^1 \), \( \text{Image}(C) \to (S^1 \times S^1) \) and let \( \rho: (S^1 \times S^1 \times S^1 \times [0, 1]) \to (S^1 \times S^1) \) be a continuous map which satisfies \( \rho(x, 0) = x, \rho(x, 1) \in S^1 \times \{s_i\} \cup \{s_i\} \times S^1 \) and \( \rho(x, t) \in S^1 \times S^1 \times P \) for all \( x \in S^1 \times S^1 \times P, t \in [0, 1] \). Let \( E_1: S^1 \times [0, 1] \to Tw \) be a map defined by \( E_1(x, t) = k(s_i, \rho(C(x), t), t) \). Note that \( \text{Image}(E_1) \cap \text{Image}(E_2) = \emptyset \) and \( \partial E_i = [C_i] + \gcd(b, c)[k([s] \times S^1 \times [1])] \) in \( H_i(k([s] \times S^1 \times S^1 \times [0, 1]); Z/2) \) where \( Z/2 \) is the natural quotient map. This leads us to the fact that \( Y(a, b, c) \equiv Y(0, b, c) \) (mod 2).

Regard \( Tw \) as in Lemma 2.3. If \( a = 0 \), we may assume that \( C_i \) and \( C_j \) are in \( \partial D_i \) \((-h^i \cup h^i)\). The link \( C_i \cup C_j \) in \( \partial D_i \) is as follows. Regard \( \partial D_i \) as \( V_i \cup V_j \) where \( V_i, V_j \) are solid tori. \( C_i \) is the \((b, \pm c)\) (resp. \((c, \pm b)\))-cable of the core of \( V_i \) in \( \text{Int}(V_i) \) and \( C_j \) is the \((c, \pm b)\) (resp. \((b, \pm c)\))-cable of the core of \( V_j \) in \( \text{Int}(V_j) \). Therefore we know that \( Y(0, b, c) \equiv \pm b \equiv 0 \) (mod 2) where \( \pm b \) means the linking number in \( \partial D_i \).

We recall the Gluck-surgeries. This term is due to [3], §8.

Let \( M \) be a 4-manifold and \( \Sigma \) be a 2-sphere embedded in \( \text{Int}(M) \) which has trivial normal bundle. Choose a framing \( \varphi: N(\Sigma) \to S^2 \times D^2 \) and identify \( S^2 \) with \( \hat{C} \) (the Riemann sphere), and \( D^2 \) with \( U \) (the unit disk in \( C \)). Then, \( \partial N(\Sigma) \) is identified with \( \hat{C} \times \partial U \). Define the map \( \tau: \hat{C} \times \partial U \to \hat{C} \times \partial U \) by \( (x, z) \mapsto (z, x, z, x) \). Put \( \tau(M; \Sigma) = N(\Sigma) \cup (M - \text{Int}(N(\Sigma))) \). We say that \( \tau(M; \Sigma) \) is obtained from \( M \) by Gluck-surgery on \( \Sigma \).

Let \( h \) be a diffeomorphism \( S^2 \times \partial D^2 \to S^2 \times \partial D^2 \) such that \( h_*: H_3(S^2 \times \partial D^2; Z) \to H_3(S^2 \times \partial D^2; Z) \) is identity and that \( h \) cannot be extended to a diffeomorphism \( S^2 \times D^2 \to S^2 \times D^2 \). Then, \( h \) is isotopic to \( \tau \). See [1]. Therefore
the diffeomorphism type of \( \tau(M; \Sigma) \) does not depend on the framing \( \varphi \).

Define \( \tau(M; \Sigma, \cdots, \Sigma_k) \) by \( \tau(\tau(M; \Sigma, \cdots, \Sigma_{k-1}); \Sigma_k) \) if possible.

It is known that
\[
\tau(S^2 \times S^2; \{x\} \times S^2) = S^2 \times S^2
\]
and
\[
\tau(S^3 \times S^3; \{x\} \times S^3, \{y\} \times S^3) = S^3 \times S^3 \quad (* \neq **).
\]

**Lemma 2.11.** Let \( l, r, s \subset \partial(\tau(Tw; R)) \), \( \partial(\tau(Tw; S)) \) be the curves chosen before performing Gluck-surgeries. Then, there exist diffeomorphisms \( h^R, h^S \): \( \tau(Tw; R) \to Tw \) and \( h^S \): \( \tau(Tw; S) \to Tw \) such that
\[
[(h^R_\partial)(l)(h^R_\partial)(r)(h^S_\partial)(s)] = [l \ r \ s]T^{+1}_R,
\]
and
\[
[(h^S_\partial)(l)(h^R_\partial)(r)(h^S_\partial)(s)] = [l \ r \ s]T^{+1}_S.
\]

**Proof.** Regard \( Tw \) as \( S^1 \times (D^3 - \text{Int}(N(K_0))) \cup 2dD^2 \times S^2 \) so that \( \{0\} \times S^2 \subset D^3 \times S^2 \) is \( R \) where \( K_0 \) is an unknot in \( D^3 \). We know that \( S^1 \times (D^3 - \text{Int}(N(K_0))) \cup 2dD^2 \times S^2 \) is \( \tau(Tw; R) \).

Identify \( D^3 \) with the unit ball around the origin \( e \in \mathbb{C} \times R \), and \( K_0 \) with the set \( \{(z_2, t) \in \mathbb{C} \times R \mid z_2^2 = 1/2, t = 0\} \). Let \( N(K_0) \) be the set \( \{x \in \mathbb{C} \times R \mid \text{distance}(x, K_0) \leq 1/4\} \). Define \( h^R_\partial \): \( \tau(Tw; R) \to Tw \) by
\[
h^R_\partial|D^2 \times S^2 = id \quad \text{and} \quad h^R_\partial|S^1 \times (D^3 - \text{Int}(N(K_0)))) : (z_1, z_2, t) \mapsto (z_1, z_1z_2, t) \text{ or } (z_1, z_1^{-1}z_2, t).
\]
The other maps can be constructed similarly. \( \square \)

**Lemma 2.12.** If \( \tau(M_1; \Sigma_i) \) can be defined for \( i = 1, 2 \), then \( \tau(M_1 \# M_2; \Sigma_1 \# \Sigma_2) = \tau(M_1; \Sigma_1) \# \tau(M_2; \Sigma_2) \).

**Proof.** Choose an orientation preserving framing \( \varphi_i: N(\Sigma_i) \to \hat{C} \times U \) for \( i = 1, 2 \).

**Construction of** \( \tau(M_1 \# M_2; \Sigma_1 \# \Sigma_2): \) Remove the interior of the 4-balls \( D^4_1 = \varphi_1^{-1}((z, w_1) \in \hat{C} \times U \mid |z_1| \leq 1) \) and \( D^4_2 = \varphi_2^{-1}((z, w_2) \in \hat{C} \times U \mid |z_2| \geq 1) \) from \( M_1 \) and \( M_2 \) respectively. Then, \( \partial(M_i - \text{Int}(D_i)) = \partial M_i \) is identified with \( \{(z_i, w_i) \in \hat{C} \times U \mid |z_i| = 1 \text{ or } |w_i| = 1\} \). Define \( g: \partial(M_i - \text{Int}(D_i)) - \partial M_i \to \partial(M_i - \text{Int}(D_i)) - \partial M_i \) by \( (z_i, w_i) \mapsto (1/|z_i|, w_i) \). Gluing \( M_i - \text{Int}(D_i) \) and \( M_i - \text{Int}(D_i) \) by \( g \), we have \( M_i \# M_j \) in which \( N(\Sigma_i \# \Sigma_j) \) is identified with \( \hat{C} \times U \) in a natural way. Perform Gluck-surgery on \( \Sigma_i \# \Sigma_j \) by using this framing.

**Construction of** \( \tau(M_1; \Sigma_1) \# \tau(M_2; \Sigma_2): \) By using \( \varphi_i \), construct \( \tau(M_i; \Sigma_i) \) \( (i = 1, 2) \). Remove \( \text{Int}(D_1) \) and \( \text{Int}(D_2) \) defined in the above construction, and paste them by \( g \).
Now it is easy to check the statement. □

We denote the lens space of type \((n, q)\) by \(L(n, q)\). In this paper, we assume that \(L(0, -1)\) represents \(S^1 \times S^2\).
We choose the framing \(\varphi\) of \(N(\{\ast\} \times S^1) \subseteq L(n, -1) \times S^1\) as follows. Let \(i: N(\{\ast\} \times S^1) \rightarrow N(\{\ast\}) \times S^1\) be the natural identification and \(\varphi': N(\{\ast\}) \rightarrow D^1\) be a diffeomorphism. Put \(\varphi = (\varphi' \times \text{id}) \cdot i\). By using this framing \(\varphi\), we define
\[L_n = \mathbb{Z}(L(n, -1) \times S^1; \{\ast\} \times S^1).\]

\(L(n, -1)\) is obtained from two solid tori \(V\) and \(V'\) pasted together by the gluing map which maps the curve \(m\) to \(m + nl\) where \((m, l)\) (resp. \((m', l')\)) is the meridian-longitude pair of \(V\) (resp. \(V'\)). We assume that \(\ast\) is contained in the core \(C\) of \(V\) and that \((N(\{\ast\}), N(\{\ast\}) \cap C)\) is the standard pair \((D^1, D^2)\). In \(L_n = ((V \cup V') - \text{Int}(N(\{\ast\}))) \times S^1 \cup S^2 \times D^2\), the boundary of \(A = (C - \text{Int}(N(\{\ast\}))) \times S^1\) consists of two circles which span disjoint disks \(D_1, D_2\) in \(S^2 \times D^2\). It can be shown easily that the normal bundle of the 2-sphere \(\Sigma = A \cup D_1 \cup D_2\) is trivial. We define \(L_n = \mathbb{Z}(L_n; \Sigma)\).

It is clear that \(\pi_1(L_n) = \pi_1(L_n') = \mathbb{Z}/n\).

The manifolds \(L_n, L_n'\) are the same as the ones in [4], p. 307.

§ 3. Twin singular fibers

In this section we recall the definition of GTF and the GTF structures of fibered neighborhoods of twin singular fibers.

**Definition 3.1.** ([2], §2) Let \(M\) and \(B\) be oriented 4 and 2-dimensional smooth manifolds, \(f: M \rightarrow B\) be a proper, surjective, smooth map. We call \(f: M \rightarrow B\) a good torus fibration (GTF) if it satisfies the following: (i) at each point \(p \in \text{Int}(M)\), the germ \((f, p)\) is smoothly \((+)-\)equivalent (in other words, is conjugate via orientation preserving diffeomorphisms) to the germ at \(0 = (0, 0)\) of the complex valued function \(z_1^n z_2^m \in C^2 \rightarrow C\), where \(m\) and \(n\) are non-negative integers, not necessarily coprime, satisfying \(m + n \geq 2\); (ii) there exists a set of isolated points, \(\sigma\) in \(\text{Int}(B)\) so that \(f|f^{-1}(B - \sigma): f^{-1}(B - \sigma) \rightarrow B - \sigma\) is a smooth \(T^2\)-bundle over \(B - \sigma\).

We call \(f, M\) and \(B\) the projection, the total space and the base space of the GTF, respectively.

Those points \(p \in \text{Int}(M)\), at which the exponents of the germ of \(f\) satisfy the inequality \(m + n \geq 2\), make up a nowhere dense subset \(\Sigma \subset M\). We assume \(f(\Sigma) = \sigma\) and call \(\sigma\) the set of singular values.

The fiber \(F_x = f^{-1}(x)\) is called a general or singular fiber according as \(x \in B - \sigma\) or \(x \in \sigma\).

Since the total space \(M\) and the base space \(B\) are oriented, the general
fibers and each irreducible component of a singular fiber are naturally oriented. In particular, the components $\Theta_1, \ldots, \Theta_k$ of $F_\epsilon$ determines a basis $[\Theta_1], \ldots, [\Theta_k]$ of $H_2(N; \mathbb{Z})$ where $N$ is a fibered regular neighborhood of $F_\epsilon$. If $F$ is a general fiber in $N$, we have $[F] = m_1[\Theta_1] + \cdots + m_k[\Theta_k] \in H_2(N; \mathbb{Z})$ where $m_i \geq 1$. We call $m$, the multiplicity of $\Theta_i$.

A singular fiber $F_\epsilon = \sum_{i=1}^k m_i \Theta_i$ is called a multiple fiber if $\gcd(m_1, \ldots, m_k) > 1$. If $F_\epsilon$ is not multiple, then it is called a simple fiber.

**Definition 3.2.** A twin singular fiber is a singular fiber of a GTF which consists of two embedded 2-spheres which meet at two points transversely with opposite signs.

It is shown that a fibered regular neighborhood of a twin singular fiber is diffeomorphic to a twin. See [2].

In the rest of this section, we construct GTF structures of twins. Let $m, n$ be positive integers, and put $g = \gcd(m, n)$ and $\epsilon = 1/2^{m+n+1}$. The symbol $\epsilon(\alpha)$ means $\exp 2\pi \sqrt{-1}\alpha$.

Put

$$U = \{(u_1, u_2) \in C \times C | |u_1^n u_2^n| \leq \epsilon, |u_1| < 2, |u_2| < 2\}$$
$$V = \{(v_1, v_2) \in C \times C | |v_1^n v_2^n| \leq \epsilon, |v_1| < 2, |v_2| < 2\}.$$

For $i = 1, 2$, put

$$U_i = \{(u_1, u_2) \in U | |u_i| > 1/2\}$$
$$V_i = \{(v_1, v_2) \in V | |v_i| > 1/2\}.$$

Note that $U_1 \cap U_2 = V_1 \cap V_2 = \emptyset$.

Define the diffeomorphism $\phi_{n, \beta, \gamma, \delta}^{m, n} : U_1 + U_2 \rightarrow V_1 + V_2$ by

$$(u_1, u_2) \mapsto \begin{cases} (\epsilon(\alpha)/u_1^n, \epsilon(\beta)u_1 \cdot |u_1|^{m/n}) & \text{if } (u_1, u_2) \in U_1 \\ (\epsilon(\gamma)u_2 \cdot |u_2|^{n/m}, \epsilon(\delta)/u_2^n) & \text{if } (u_1, u_2) \in U_2. \end{cases}$$

This map is orientation reversing. Orient $X_{n, \beta, \gamma, \delta}^{m,n} = U \cup \phi_{n, \beta, \gamma, \delta}^{m,n} V$ so that the inclusion map $U \rightarrow X^{m,n}_{n, \beta, \gamma, \delta}$ is orientation preserving.

**Claim.** $X^{m,n}_{n, \beta, \gamma, \delta}$ is a twin.

**Proof.** $U_1 + U_2$ and $V_1 + V_2$ are 4-balls, and for $i = 1, 2$, the closure of $\phi_{n, \beta, \gamma, \delta}^{m, n}(U_i) = V_i$ is a round 1-handle. We know that $X^{m,n}_{n, \beta, \gamma, \delta} \in \mathcal{Z}$ (see Lemma 2.3).

We examine the intersection form on $H_2(X^{m,n}_{n, \beta, \gamma, \delta}; \mathbb{Z})$. The pair of the elements $x_i, x_2$ where $x_i = \{[u_i = 0] \cup [v_i = 0]\} (i = 1, 2)$ is a basis of $H_2(X^{m,n}_{n, \beta, \gamma, \delta}; \mathbb{Z})$.

Let $\epsilon_i$ be a positive real number which satisfies $\epsilon_i^n < \epsilon$ and $\epsilon_i^n < \epsilon$. Then, the sets

$$\{u_i = \epsilon_i, |u_i| \leq 1\} \cup \{v_i = \epsilon_i, |v_i| \leq 1\}$$
also represent $x_1$ and $x_2$ respectively. Therefore $x_1 \cdot x_2 = x_2 \cdot x_1 = 0$.

Since the map $\phi^{m,n}_{a,b,c,d}$ is orientation reversing, the intersection numbers of $x_1$ and $x_2$ at $(u_1, u_2) = (0, 0)$ and $(v_1, v_2) = (0, 0)$ have opposite signs. Therefore $x_1 \cdot x_2 = 0$. Hence we know that $X^{m,n}_{a,b,c,d}$ is a twin.\[\Box\]

Let $D'(\varepsilon)$ be the 2-disk $\{z \in C | |z| \leq \varepsilon\}$. We define the map $f^{m,n}: U + V \to D'(\varepsilon)$ by $(u_1, u_2) \mapsto u_1^m u_2^n, \ (v_1, v_2) \mapsto v_1^m v_2^n$. This map is compatible with $\phi^{m,n}_{a,b,c,d}$ if and only if $ma + nb \equiv 0, \ m' + nd \equiv 0 \pmod{1}$. Therefore we assume so. Then, there exists a map $f^{m,n}_{a,b,c,d}: X^{m,n}_{a,b,c,d} \to D'(\varepsilon)$ such that $f^{m,n}_{a,b,c,d} \cdot q = f^{m,n}$ where $q: U + V \to X^{m,n}_{a,b,c,d}$ is the natural quotient map. The uniqueness of $f^{m,n}_{a,b,c,d}$ is clear.

Let $x$ be a point of $D'(-\{0\})$. At any point of $(f^{m,n}_{a,b,c,d})^{-1}(x)$, the Jacobian matrix of $f^{m,n}_{a,b,c,d}$ has rank 2. Therefore $(f^{m,n}_{a,b,c,d})^{-1}(x)$ is a submanifold of $X^{m,n}_{a,b,c,d}$.

The set $(f^{m,n}_{a,b,c,d})^{-1}(x) \cap \{u_1 = \text{const.}\}$ is empty or consists of $g$ circles. We have that $(f^{m,n}_{a,b,c,d})^{-1}(x) \cap U$ consists of $g$ annuli. The same argument tells us that $(f^{m,n}_{a,b,c,d})^{-1}(x) \cap V$ consists of $g$ annuli.

Put $m' = ma/g$ and $n' = nb/g$. Then, $\gcd(m', n') = 1$. The map $f^{m,n}_{a,b,c,d}: U + V \to D'(\varepsilon)$ is factored as $f^{m,n}_{a,b,c,d} = ( \ )^{g} f^{m', n'}_{a,b,c,d}$ where $( \ )^{g}: C \to C$ is defined by $z \mapsto z^g$. Let $-\gamma: Z \to Z/g$ be the natural quotient map.

Let $\xi_0$ be a root of $z^g = x$. Put $\xi_j = \xi_0 e(j/g) \ (j = 0, \ldots, g-1)$. Then, $\xi_0, \ldots, \xi_{g-1}$ is the total inverse image of $x$ by $( \ )^g$. We know that $(f^{m,n}_{a,b,c,d})^{-1}(x) = q((f^{m', n'}_{a,b,c,d})^{-1}(\xi_0, \ldots, \xi_{g-1}))$ and that $(f^{m,n}_{a,b,c,d})^{-1}(\xi_j) \cap U$ and $(f^{m,n}_{a,b,c,d})^{-1}(\xi_j) \cap V$ ($j \in Z/g$) are connected.

By the definition of $\phi^{m,n}_{a,b,c,d}$, we have

$$
\phi^{m,n}_{a,b,c,d}(f^{m,n}_{a,b,c,d})^{-1}(\xi_j) \cap U_1 = (f^{m', n'}_{a,b,c,d})^{-1}(e(m' \alpha + n' \beta) \xi_j) \cap V_1,
$$

$$
\phi^{m,n}_{a,b,c,d}(f^{m,n}_{a,b,c,d})^{-1}(\xi_j) \cap U_2 = (f^{m', n'}_{a,b,c,d})^{-1}(e(m' \gamma + n' \delta) \xi_j) \cap V_2.
$$

Put $p = ma + nb$ and $q = m' \gamma + n' \delta$. The numbers $p, q$ are integers. Then, $m' \alpha + n' \beta = p/g$ and $m' \gamma + n' \delta = q/g$. We have $e(m' \alpha + n' \beta) \xi_j = \xi_{j+p}$ and $e(m' \gamma + n' \delta) \xi_j = \xi_{j+q}$. Hence $\phi^{m,n}_{a,b,c,d}|_{U_1}$ maps $(f^{m,n}_{a,b,c,d})^{-1}(\xi_j) \cap U_1$ to $(f^{m', n'}_{a,b,c,d})^{-1}(\xi_{j+p}) \cap V_1$ and $\phi^{m,n}_{a,b,c,d}|_{U_2}$ maps $(f^{m,n}_{a,b,c,d})^{-1}(\xi_j) \cap U_2$ to $(f^{m', n'}_{a,b,c,d})^{-1}(\xi_{j+q}) \cap V_2$. Therefore, we have

**Lemma 3.3.** $f^{m,n}_{a,b,c,d}: X^{m,n}_{a,b,c,d} \to D'(\varepsilon)$ is a GTF if and only if $\gcd(ma + nb - m' \gamma - n' \delta, g) = 1$.

**Proof.** $f^{m,n}_{a,b,c,d}: X^{m,n}_{a,b,c,d} \to D'(\varepsilon)$ is a GTF.

$\Leftrightarrow (f^{m,n}_{a,b,c,d})^{-1}(x)$ is connected for all $x \in D'(\varepsilon) - \{0\}$.

$\Leftrightarrow \bar{p} - \bar{q}$ generates $Z/g$.

$\Leftrightarrow \gcd(p - q, g) = 1$.

$\Leftrightarrow \gcd(ma + nb - m' \gamma - n' \delta, g) = 1. \Box$
Next, we divide \( \{(f^{m,n}_{a,\beta,\tau,\delta}, X^{m,n}_{a,\beta,\tau,\delta} \to D^k)\}_{a,\beta,\tau,\delta} \) into fiber preserving, orientation preserving diffeomorphism classes. Let \( f_i: M_i \to B_i \) be a GTF for \( i=1,2 \). The notation \( f_1 \sim f_2 \) means that there exist orientation preserving diffeomorphisms \( H: M_1 \to M_2 \) and \( h: B_1 \to B_2 \) such that \( f_2 \circ H = h \circ f_1 \).

It is shown that for any GTF \( f: Tw \to D^k \) with only one twin singular fiber, there exist \( m, n; \alpha, \beta, \tau, \delta \) such that \( f \sim f^{m,n}_{a,\beta,\tau,\delta} \).

The following lemma is almost clear.

**Lemma 3.4.** If \( f^{m,n}_{a,\beta,\tau,\delta} \sim f^{m',n'}_{a',\beta',\tau',\delta'} \), then \( \{m, n\} = \{m', n'\} \). \( \square \)

**Lemma 3.5.** \( f^{m,n}_{a,\beta,\tau,\delta} \sim f^{m,n}_{a,\beta,\tau,\delta + g \beta} \) where \( g \) is \( \gcd(m, n) \) and \( i \) is an arbitrary integer. Especially, \( f^{m,n}_{a,\beta,\tau,\delta} \sim f^{m,n}_{a,\beta,\tau,\delta} \sim f^{m,n}_{a,\beta,\tau,\delta} \sim f^{m,n}_{a,\beta,\tau,\delta} \sim f^{m,n}_{a,\beta,\tau,\delta} \).

**Proof.** Put \( \alpha' = (ma + n \beta - m \tau - n \delta + gi)/m \). There exist integers \( x, y \) such that \( gi = mx + ny \).

Let \( \omega: W \to R \) be a smooth function which satisfies

(i) \( \omega = -\beta - y \) on \( W_1 \)

(ii) \( \omega = -\delta \) on \( W_2 \)

(iii) \( V \ni (v_1, v_2) \mapsto \omega(\sqrt{v_1^2 + v_2^2}) \) is a smooth function.

Let \( \psi: W \to R \) be the function which satisfies \( m\psi + n\omega = -m\tau - n\delta \). Therefore \( \psi \) satisfies

(iv) \( \psi = \alpha' - \alpha - x \) on \( W_1 \)

(v) \( \psi = -\tau \) on \( W_2 \)

(vi) \( V \ni (v_1, v_2) \mapsto \psi(\sqrt{v_1^2 + v_2^2}) \) is a smooth function.

Define \( T: U + V \to U + V \) by

\[
\begin{align*}
U \ni (u_1, u_2) &\mapsto (u_1, u_2) \\
V \ni (v_1, v_2) &\mapsto (u_1, \psi(\sqrt{v_1^2 + v_2^2})), v_2, \omega(\sqrt{v_1^2 + v_2^2})).
\end{align*}
\]

The following diagram is commutative for \( i=1,2 \).

\[
\begin{array}{ccc}
U_i & \xrightarrow{T_{a,\beta,\tau,\delta}} & V_i \\ 
\downarrow & & \downarrow \\ 
T\mid U_i & \xrightarrow{T\mid V_i} & T\mid V_i \\
\end{array}
\]

Therefore, \( T \) induces an orientation preserving diffeomorphism \( \tilde{T}: X^{m,n}_{a,\beta,\tau,\delta} \to X^{m,n}_{a',\beta',0,0} \).

The following diagrams are commutative.
Therefore $\tilde{T}$ is fiber preserving. $\square$

From now on, we use the symbols $\phi_a^{m,n}$, $X^{m,n}$, $F^{m,n}$ to represent $\phi_a^{m,n}$, $X^{m,n}$, $F^{m,n}$ respectively. By definition, $\alpha$ satisfies $ma \in \mathbb{Z}$. Therefore there exists an integer $d$ such that $\alpha = d/m$. Lemma 3.3 shows that $f^{m,n}_{d/m} : X^{m,n}_{d/m} \to D(\varepsilon)$ is a GTF if and only if $\gcd(d, m, n) = 1$.

**Corollary 3.6.** If $d \equiv d' \pmod{g}$, then $f^{m,n}_{d/m} \sim f^{m,n}_{d'/m}$, where $g = \gcd(m, n)$. $\square$

**Lemma 3.7.** $f^{m,n}_{d/m} \sim f^{-m,n}_{d/m}$.

**Proof.** Define $T : U + V \to U + V$ by

$$U \ni (u_1, u_2) \rightarrow (u_2, u_1), \quad V \ni (v_1, v_2) \rightarrow (v_2, v_1).$$

By using this map, we can show that $f^{m,n}_{d/m} \sim f^{m,n}_{d'/m}$. Lemma 3.5 tells us that $f^{m,n}_{d/m} \sim f^{m,n}_{d'/m} \sim f^{m,n}_{d'/m'}$. This completes the proof. $\square$

**Lemma 3.8.** If $f^{m,n}_{d/m} \sim f^{m,n}_{d'/m}$, then

$$\begin{cases} d \equiv d' \pmod{g} & \text{if } m \neq n \\ d \equiv \pm d' \pmod{g} & \text{if } m = n \end{cases}$$

where $g = \gcd(m, n)$.

**Proof.** We may assume $0 \leq d < m$, $0 \leq d' < m$. Put

$$\lambda_1 = \{(u_1, u_2) \in U \mid \arg(u_1) = 0, \arg(u_2) = \pi/n, \|u_1^n u_2^n\| = \varepsilon\}$$

$$\lambda_2 = \{(v_1, v_2) \in V \mid \arg(v_1) = 2\pi d/m, \arg(v_2) = \pi/n, \|v_1^n v_2^n\| = \varepsilon, \|v_2^n\| \leq \sqrt{m+n}/\varepsilon\}$$

$$\lambda_3 = \{(v_1, v_2) \in V \mid 0 \leq \arg(v_1) \leq 2\pi d/m, \arg(v_2) = \pi/n, \|v_1^n v_2^n\| = \varepsilon, \|v_2^n\| \leq \sqrt{m+n}/\varepsilon\}$$

$$\lambda_4 = \{(v_1, v_2) \in V \mid \arg(v_1) = 0, \arg(v_2) = \pi/n, \|v_1^n v_2^n\| = \varepsilon, \|v_2^n\| \geq \sqrt{m+n}/\varepsilon\}.$$
same signs. Therefore \(|l_0 \cdot F| = d\) in \(\partial(Tw) = T^3\). Put \(r_0 = \{(u_1, 1) \in U | |u_1| = \sqrt{\varepsilon}\}\) and \(s_0 = \{(1, u_2) \in U | |u_2| = \sqrt{\varepsilon}\}\). Regard them as elements of \(H_1(\partial X_{m,n}^{d/m}; \mathbb{Z})\).

Since \(\langle l_0, r_0, s_0 \rangle\) is a basis of \(H_1(\partial(Tw); \mathbb{Z})\) and \(|r_0 \cdot F| = m, |s_0 \cdot F| = n\), for any element \([l]\) of \(H_1(\partial(Tw); \mathbb{Z})\) such that \([l]\) generates \(H_1(Tw; \mathbb{Z})\), we know that \(|l \cdot F| \equiv d \mod g\).

To complete the proof, we have to show that if \(m \neq n\) and \(d \equiv -d \mod g\), then \(f_{m,n}^{d/m} \neq f_{n,m}^{d/m}\). Therefore we may assume that \(m > 1, n > 1\).

Fix \(m\) and \(n\). We choose a generator of \(H_1(X_{m,n}^{d/m}; \mathbb{Z})\) for \(d \in \mathbb{Z}\) such that \(\gcd (d, m, n) = 1\) as follows. Put

\[
S(a, b) \text{ (resp. } \overline{S}(a, b)) = \left\{ p \in X_{m,n}^{d/m} \mid \text{the germ } (f_{m,n}^{d/m}, p) \text{ is smoothly (+)-equivalent to the germ at } 0 = (0, 0) \text{ of } z_1^a z_2^b \text{ (resp. } \overline{z}_1^a \overline{z}_2^b) \right\}.
\]

Then,

\[
S(m, n) = \{(0, 0) \in U\}
\]
\[
\overline{S}(m, n) = \{(0, 0) \in V\}
\]
\[
S(m, 0) = \{(0, u) \in U | u \neq 0\} \cup \{(0, v) \in V | v \neq 0\}
\]
\[
S(n, 0) = \{(u, 0) \in U | u \neq 0\} \cup \{(v, 0) \in V | v \neq 0\}.
\]

Let \(c_{m,n}^{d/m} : [0, 1] \times X_{m,n}^{d/m} \to \mathbb{C}\) be a continuous map satisfying

\[
c_{d/m}^{m,n}(0) = c_{d/m}^{m,n}(1) \in S(m, n)
\]
\[
c_{d/m}^{m,n}(0, 1/2) \subset S(m, 0)
\]
\[
c_{d/m}^{m,n}(1/2) \in S(m, n)
\]
\[
c_{d/m}^{m,n}(1/2, 1) \subset S(n, 0).
\]

Then, \([c_{d/m}^{m,n}] \in H_1(X_{m,n}^{d/m}; \mathbb{Z})\) is well-defined and it generates \(H_1(X_{m,n}^{d/m}; \mathbb{Z})\). Let \(l_{d/m}^{m,n}\) be a loop in \(\partial(X_{m,n}^{d/m})\) such that \([l_{d/m}^{m,n}] = [c_{d/m}^{m,n}]\) in \(H_1(X_{m,n}^{d/m}; \mathbb{Z})\) and put

\[
Z(m, n, d) = \left[ l_{d/m}^{m,n} \right] \cdot [f_{d/m}^{m,n}]^{-1}(e) \in \mathbb{Z}/g.
\]

CLAIM. \(Z(m, n, -d) = -Z(m, n, d)\) in \(\mathbb{Z}/g\).

PROOF. Define \(T : U \cup V \to U \cup V\) by

\[
U \ni (u_1, u_2) \mapsto (\bar{u}_1, \bar{u}_2)
\]
\[
V \ni (v_1, v_2) \mapsto (\bar{v}_1, \bar{v}_2).
\]

It induces a map \(\tilde{T} : X_{m,n}^{d/m} \to X_{m,n}^{d/m}\) and the following diagram is commutative where \(\text{conj} : D^2(z) \to D^2(\bar{z})\) is defined by \(z \mapsto \bar{z}\).
Since $\tilde{T}$ is orientation preserving and $\text{conj}$ is orientation reversing, $\tilde{T}$ restricted to a general fiber is orientation reversing. See Definition 3.1. Note that $[\tilde{T} \cdot c_m^d, n^d] = [c_m^d, n^d]$ in $H_1(X_{m,n}^d; \mathbb{Z})$. Therefore $Z(m, n, -d) = [l_m^d, n] \cdot (fm,n^d)^{-1}(x) = \text{conj}T \cdot [l_m^d, n] \cdot (fm,n^d)^{-1}(x) = (-1) \text{conj}T \cdot [l_m^d, n] = -Z(m, n, d).

This completes the proof. $\square$

The argument in the proof of Lemma 3.8 shows that we have the following:

**Lemma 3.9.** Let $F_{m,n}^{d/m}$ be a fiber of $fm,n^d/m|\partial$. Then, $F_{m,n}^{d/m} = \partial \cdot l_0 \times r_0 \times s_0 \times l_0 \times n \cdot l_0 \times r_0$ in $H_2(\partial X_{m,n}^d; \mathbb{Z})$. $\square$

Note that for the inclusion map $i : \partial(Tw) \to Tw$, $i_* : H_2(\partial(Tw); \mathbb{Z}) \to H_2(Tw; \mathbb{Z})$ maps $l \times r, r \times s, s \times l$ to $\pm S, 0, \pm R$ respectively. Therefore we have

**Corollary 3.10.** The singular fiber $(fm,n^d)^{-1}(0)$ is simple if and only if $\gcd(m, n) = 1$. $\square$

Now we have

**Proposition 3.11.** (i) Let $N$ be a fibered regular neighborhood of a twin singular fiber of a GTF $f : M^4 \to B^2$. Then, there exist positive integers $m$, $n$ and integer $d$ which satisfy $\gcd(m, n, d) = 1$, such that $f[N] = fm,n^d/m$.

(ii) $fm,n^d/m \sim fm',n'd'/m'$ if and only if both (a) and (b) are satisfied.

(a) $\{m, n\} = \{m', n'\}$.

(b) $d \equiv d' \pmod{\gcd(m, n)}$ if $m \neq n$ and $d \equiv d' \pmod{\gcd(m, n)}$ if $m = n$. $\square$

**Proposition 3.12.** For $i = 1, 2$, let $f_i : \partial(Tw) \to \partial D^2$ be a map which can be decomposed as $f_i = p \cdot \phi_i$, where $p : \partial D^2 \times S^1 \times S^1 \to \partial D^2$ is the projection map to the first coordinate and $\phi_i : \partial(Tw) \to \partial D^2 \times S^1 \times S^1$ is a diffeomorphism. Assume that $(f_i)_* : H_1(\partial(Tw); \mathbb{Z}) \to H_1(\partial D^2; \mathbb{Z})$ maps $l, r, s$ to $a', b', c'$ respectively, where $a, b, c$ are integers and $r$ is a generator of $H_1(\partial D^2; \mathbb{Z})$. Let $f$ be either $f_1$ or $f_2$.

(i) There exists a diffeomorphism $\psi : \partial(Tw) \to \partial(Tw)$ such that $\psi$ is isotopic to identity and that $f_i = f_i \cdot \psi$.

(ii) The map $f$ can be extended to a projection map of a GTF $\tilde{f} : Tw \to D^2$ which has only one twin singular fiber as singular fiber if and only if $b \neq 0$ or $c \neq 0$. 

\[
\begin{array}{ccc}
X_{m,n}^{d/m} & \xrightarrow{T} & X_{m,n}^{d/m} \\
\downarrow f_{m,n}^{d/m} & \quad & \downarrow f_{m,n}^{d/m} \\
D^2 & \xrightarrow{\text{conj}} & D^2
\end{array}
\]
(iii) The twin singular fiber of $\tilde{f}$ is simple if and only if $\gcd(b, c) = 1$.

**Proof.** Regard $\partial D^3 \times S^1 \times S^1 = S^0 \times S^1 \times S^1$ as $(R/Z)^3$, which has the natural affine structure. Introduce the affine structure to $\partial (Tw)$ via $\phi_i$. Put $e = \phi^{-1}_i(0, 0, 0)$.

Let $(\lambda, \rho, \sigma) \in (R/Z)^3$ be an affine coordinate of $\partial (Tw)$ so that the maps $R/Z \ni t \mapsto (t, 0, 0), (0, t, 0), (0, 0, t) \in (R/Z)^3$ represent $l, r, s \in H_1(\partial (Tw); Z)$ respectively, and that $(0, 0, 0)$ corresponds to $e$. Then, $f_i = p \cdot \phi_i$ maps $(\lambda, \rho, \sigma)$ to $\lambda + b\rho + c\sigma$.

Let $\phi'_2: \partial (Tw) \to \partial D^3 \times S^1 \times S^1$ be a diffeomorphism satisfying the following:

(a) $\phi'_2$ is an affine map.

(b) $\phi'_2$ is isotopic to $\phi_2$ under an isotopy $\Phi: \partial (Tw) \times R \to \partial D^3 \times S^1 \times S^1 \times R$ (i.e. $\Phi$ is a diffeomorphism satisfying $\Phi(\partial (Tw) \times \{t\}) = \partial D^3 \times S^1 \times S^1 \times \{t\}$ and $\Phi|\partial (Tw) \times \{t\} = \phi'_2(t \leq 0), \phi_2(t \geq 1)$).

(c) $\phi'_2(e) = (0, 0, 0)$.

Put $\phi = \phi_2^{-1} \cdot \phi'_2$. We know $\phi$ is isotopic to identity under the isotopy $(\phi_2^{-1} \times \text{id}) \cdot \Phi: \partial (Tw) \times R \to \partial D^3 \times R$. Note that $f_i \cdot \phi = p \cdot \phi_2 \cdot \phi'_2 \cdot \phi = p \cdot \phi_2$. Therefore $f_i \cdot \phi$ is an affine map which maps $e$ to $0$. Since $\phi$ is isotopic to identity, $(f_i \cdot \phi)_\# = (f_i)_\#$. Hence $(f_i \cdot \phi)_\#$ maps $l, r, s$ to $a', b', c'$ respectively. Now we know that $f_i \cdot \phi$ maps $(\lambda, \rho, \sigma)$ to $\lambda + b\rho + c\sigma$ and that $f_i = f_i \cdot \phi$.

Before proving (ii) and (iii), note the following fact.

**Fact.** Let $f$ be as in Proposition 3.12. Let $h: \partial (Tw) \to \partial (Tw)$ be a diffeomorphism. Assume that $(f \cdot h)_\#: H_1(\partial (Tw); Z) \to H_1(\partial D^3; Z)$ maps $l, r, s$ to $a', b', c'$ respectively. Then, $[a' b' c'] = [a b c] A^h$ holds. (For the definition of $A^h$, see Proposition 2.4.)

(ii) Assume that $b \neq 0$ or $c \neq 0$. Then, Fact above and Proposition 2.4 shows that there exists a diffeomorphism $\widetilde{h}: Tw \to Tw$ such that $(f \cdot (\tilde{h} | \partial))_\#: H_1(\partial (Tw); Z) \to H_1(\partial D^3; Z)$ maps $l, r, s$ to $a', b', c'$ with $b' \neq 0$ and $c' \neq 0$. Therefore we may assume that $b = 0$ and $c = 0$. Note that $\gcd(a, b, c) = 1$.

Recall that $f \cdot (h | \partial)_\#: H_1(\partial (Tw); Z) \to H_1(\partial D^3; Z)$ maps $l, r, s$ to $a', b', c'$ respectively. Therefore there exists a GTF $\tilde{f}: Tw \to D^3$ such that $(\tilde{f} | \partial)_\#: H_1(\partial (Tw); Z) \to H_1(\partial D^3; Z)$ maps $l, r, s$ to $a', b', c'$ respectively. By (i), there exists a diffeomorphism $\phi: \partial (Tw) \to \partial (Tw)$ such that $\phi$ is isotopic to identity and $f = (\tilde{f} | \partial) \cdot \phi$. Therefore $f$ extends to $\tilde{f}$.

Conversely, assume that $f$ extends to $\tilde{f}$. If $b = 0$ and $c = 0$, then Fact above and Proposition 2.4 shows that $(f \cdot (\tilde{h} | \partial))_\#: H_1(\partial (Tw); Z) \to H_1(\partial D^3; Z)$ maps both $r$ and $s$ to $0$ for any diffeomorphism $\tilde{h}: Tw \to Tw$. Since $\tilde{f} \cdot \tilde{h} \sim f \cdot (\tilde{h} | \partial)_\#$ for some $\tilde{h}$, $m, n, d, d$, this is a contradiction.

(iii) Let $h: Tw \to Tw$ be a diffeomorphism and assume that $(f \cdot (h | \partial))_\#: H_1(\partial (Tw); Z) \to H_1(\partial D^3; Z)$ maps $r, s$ to $b', c'$ respectively. Fact above and
Proposition 2.4 shows that \( \gcd(b, c) = \gcd(b', c') \). Corollary 3.10 completes the proof. □

**Remark 3.13.** As is shown in the proof of Proposition 3.12, there exist countably many \( f \)'s for one \( f \). More precisely, the following holds. (See [2] Lemma 6.1.)

**Corollary 3.14.** Let \( f \) be as in Proposition 3.12 and assume that \( \gcd(b, c) = 1 \). Let \( m, n \) be positive integers such that \( \gcd(m, n) = 1 \). Then, there exists an \( \bar{f} \), an extension of \( f \), such that \( \bar{f} = f_{m,n} \) if and only if \( b + c \equiv m + n \) (mod 2).

**Proof.** There exists a matrix \( A \in H_1 \) such that \( [a \ b \ c] = [0 \ m \ n]A \) if and only if \( b + c \equiv m + n \) (mod 2). This completes the proof. □

**Remark 3.15.** Let \( f: M \to S^1 \) be a GTF with at least one singular fiber. Suppose that each singular fiber is of type \( I^+_1, I^-_1 \) and that \( c(M) = 0 \). As is mentioned in §1, we can deform the projection map \( f \) into \( f' \) so that all the singular fibers of \( f' \) are the twin singular fibers. It is known that all of them are with multiplicities 1,1. Conversely, a twin singular fiber with multiplicities 1,1 can be deformed into \( I^+_1 \) and \( I^-_1 \). See [2].

**Definition 3.16.** Let \( D^2 \) be the unit disk in \( C \), and \( S^1 \) be its boundary. Define \( f: D^2 \times S^1 \times S^1 \to D^2 \) by \( (z, w_1, w_2) \mapsto zw_1w_2, \) where \( a, b, c \) are integers satisfying \( \gcd(a, b, c) = 1 \). If \( |a| > 1 \), then \( f \) is a GTF with the singular fiber \( f^{-1}(0) \). This singular fiber is called a multiple torus. Its multiplicity is \( |a| \). If \( |a| = 1 \), then \( f \) is a trivial \( T^2 \)-bundle.

**Remark 3.17.** Note that \( (f|\partial)_*: H_1(\partial(D^2 \times S^1 \times S^1); Z) \to H_1(\partial D^2; Z) \) maps \( l, r, s \) to \( a\gamma, b\gamma, c\gamma \) respectively where \( \gamma \) is the generator of \( H_1(\partial D^2; Z) \).

§ 4. GTF with one twin singular fiber

In this section, we prove

**Theorem 4.1.** (i) Let \( M' \to S^1 \) be a GTF which has only one twin singular fiber and at most one multiple torus as singular fibers. Then, \( M \) is diffeomorphic to \( L_n \) or \( L'_n \).

(ii) \( L_n \) and \( L'_n \) are diffeomorphic to each other if and only if \( n \equiv 1 \) (mod 2).

(iii) Conversely, \( L_n \) and \( L'_n \) admit such GTF structures. More precisely, the following holds. We restrict our consideration to the case that all the singular fibers are simple. Then, (a) \( L_n \) (b) \( L'_n (n \equiv 0 \mod 2) \) admits the GTF structure over \( S^1 \) with one twin singular fiber with multiplicities \( b, c \) if and only if \( b, c \) are positive integers satisfying \( \gcd(b, c) = 1 \) and (a) \( b + c \equiv 0, 1 \) (b) \( b + c \equiv 1 \) (mod 2).
First, we need a lemma.

**Lemma 4.2.**

\[
\begin{pmatrix}
 n & 1 & 0 \\
 1 & 0 & 0 \\
 0 & 0 & 1 \\
\end{pmatrix}
(n \in N \cup \{0\}), \quad \begin{pmatrix}
 n & 1 & 0 \\
 1 & 0 & 0 \\
 1 & 0 & 1 \\
\end{pmatrix}
(n \in N \cup \{0\}, n \equiv 0 \pmod{2})
\]

is a system of coset representatives of \( H_1/\GL_3 \Z / H_2 \).

**Proof.** In this section, \( A \sim B \) means that \( A \) and \( B \) are in the same coset in \( H_1/\GL_3 \Z / H_2 \), and \( A := B \) means that we rewrite \( A \) by \( B \), and \( x \equiv y \) means that \( x \equiv y \pmod{2} \).

Let \( A \) be an arbitrary matrix of \( \GL_3 \Z \). Clearly, there exists a matrix \( A' \) such that \( A \sim A' \) and the 11-component of \( A' \) is a non-negative integer \( n \).

We have \( A := 
\begin{pmatrix}
 n & x & y \\
 * & * & * \\
 * & * & * \\
\end{pmatrix}
\sim 
\begin{pmatrix}
 * & * & * \\
 * & * & * \\
 * & * & * \\
\end{pmatrix}
\sim 
\begin{pmatrix}
 n & g \gcd(x, y) & 0 \\
 * & * & * \\
 * & * & * \\
\end{pmatrix}
\sim 
\begin{pmatrix}
 n & 1 & 0 \\
 * & * & * \\
 * & * & * \\
\end{pmatrix}
\)

\[
\begin{pmatrix}
 n & 1 & 0 \\
 1 & 0 & 0 \\
 0 & u & v \\
\end{pmatrix}
(n \equiv 0 \pmod{2})
\]

Then \( A \sim A' \) if and only if \( x \equiv y \pmod{2} \).

\[
\begin{pmatrix}
 n & 1 & 0 \\
 1 & 0 & 0 \\
 0 & 0 & 1 \\
\end{pmatrix}
\]

From Proposition 2.4 (iii), \( A \sim A' \) if and only if \( x \equiv y \pmod{2} \).

By Proposition 2.4 (ii) and Lemma 2.6 (ii), we can check the following facts.

(a) \( |11\text{-component}| \) is constant in a coset.

(b) If \( 11\text{-component} \equiv 0 \), then \((21\text{-component})+(31\text{-component}) \pmod{2}\) is constant in a coset.

This completes the proof. □

**Proof of Theorem 4.1.** (i) \( M = D^x \times T^x \cup_h I \times \tilde{T} \) for some diffeomorphism \( h: \partial(D^x \times T^x) \to \partial(I \times \tilde{T}) \). Define \( A^h \in \GL_3 \Z \) by \( h_\alpha[\bar{l} \bar{r} \bar{s}] = [l \ r \ s] A^h \). Put \( M^h = D^x \times T^x \cup_h I \times \tilde{T} \). If \( A^h \sim A'' \), then \( M^h \) and \( M^{A''} \) are diffeomorphic to each other.
If \( A^h \sim A_n \), then \( \mathfrak{z}(M^h; S) = L(n, -1) \times S^1 \). Therefore \( M^h = \mathfrak{z}(L(n, -1) \times S^1; \{\ast\} \times S^1) = L_n \). (There are two framings for \( N(\ast \times S^1) \). We choose the one in the definition of \( L_n \). See also Corollary 4.10.)

If \( A^h \sim A'_n \), then note that \( A'_n = T \theta A_n \). We have that \( M^h = \tau(L_n; R) \) by Lemma 2.11. Since \( (L_n, R) = (L_n, \Sigma_b) \), therefore \( M = L_n \).

(ii) If \( n = 1 \), then \( A_n \sim A'_n \) implies that \( L_n = L'_n \). If \( n = 0 \), the following claim completes the proof.

CLAIM. If \( n \equiv 0 \pmod{2} \), then \( L_n \) is spin, but \( L'_n \) is not spin.

Proof. In this proof, the coefficient groups of homology groups are assumed to be \( \mathbb{Z}/2 \). \( M \) represents \( L_n \) or \( L'_n \). Let \( C_i \) be the surface \( D^i \times \{\ast\} \subset D^i \times T^2 \), and \( C_z \) be a 2-cycle in \( Tw \) such that \( \partial C_z \equiv (\mod 2) \) is \( h(\partial D^i \times \{\ast\}) \). For \( L_n \) (resp. \( L'_n \)), note that \( [h(\partial D^i \times \{\ast\})] = h_*(\bar{\delta}) = r \) (resp. \( r+s \)). The Mayer-Vietoris sequence

\[
H_*(D^i \times T^2) \oplus H_*(Tw) \xrightarrow{j_*} H_*(M) \rightarrow H_*(\partial(D^i \times T^2)) \xrightarrow{i_*} H_*(D^i \times T^2) \oplus H_*(Tw)
\]

shows that \( H_*(M) = \text{Im}(j_*) \oplus \langle [C_i, + C_z] \rangle \). Since \( x^2 = 0 \) for any element \( x \) of \( H_*(D^i \times T^2) \oplus H_*(Tw) \), the self-intersection number on \( \text{Im}(j_*) \) is zero. By Lemma 2.10, we have \( [C_i, + C_z] = 0 \) (resp. \( = 1 \)). This completes the proof. \( \square \)

(iii) Regard \( L_n \) (resp. \( L'_n \)) as \( D^i \times T^2 \cup_h Tw \) such that \( A^h = A_n \) (resp. \( = A'_n \)).

(a) Assume that \( b+c = 0 \). Put \( S^2 = D^i_+ \cup D^i_- \), where \( D^i_+ \) and \( D^i_- \) are 2-disks, and \( D^i_+ \cap D^i_- = \partial D^i_+ = \partial D^i_- \). We construct a GTF \( f: D^i \times T^2 \cup_h Tw \rightarrow S^2 \) as follows. Proposition 3.12 (iii) assures that there exists a GTF \( f': Tw \rightarrow D^i_- \) with only one twin singular fiber such that \( (f'|\delta)_*: H_*(\partial(Tw); Z) \rightarrow H_*(\partial(D^i_-); Z) \) maps \( l, r, s \) to \( 0, r_, r_- \) respectively, where \( r_\cdot \) is the generator of \( H_*(\partial D^i_-; Z) \). Then, \( (f'|\delta)_*: H_*(\partial(D^i_- \times T^2); Z) \rightarrow H_*(\partial(D^i_-); Z) \) maps \( l \) to \( -r_\cdot \), where \( r_\cdot \) is the generator of \( H_*(\partial(D^i_-); Z) \). As is shown in the definition of multiple tori, \( (f'|\delta)_* \cdot h_\cdot: \partial(D^i_- \times T^2) \rightarrow \partial D^i_- \) extends to a GTF without singular fibers. Corollary 3.14 shows that we can replace \( f' \) by \( f_{b,c}^h \) so that \( f'|\delta = f_{b,c}^{h}\delta \). This completes the case \( b+c = 0 \).

If \( b+c = 1 \), then define \( f' \) so that \( f'|\delta \) maps \( l, r, s \) to \( 0, r_\cdot, 2r_- \) respectively. Then, \( ((f'|\delta)_*) \) maps \( l \) to \( -r_\cdot \).

(b) If \( b+c = 1 \), then define \( f' \) so that \( f'|\delta \) maps \( l, r, s \) to \( r_\cdot, -nr_\cdot, r_- \) respectively. Then, \( ((f'|\delta)_*) \) maps \( l \) to \( -r_\cdot \).

Assume that there exists a GTF \( f: L'_n \rightarrow S^2 \) with one twin singular fiber which has multiplicities \( b, c \) with \( b+c = 0 \). Let \( N \) be a fibered regular neighborhood of the twin singular fiber. \( N \) and \( M - \text{Int}(N) \) are diffeomorphic to \( Tw \) and \( D^i \times T^2 \) respectively. By Lemma 4.2 and Theorem 4.1(i), there exist diffeomorphisms \( \phi: N \rightarrow Tw \) and \( \psi: M - \text{Int}(N) \rightarrow D^i \times T^2 \) such that \( A^h(\phi_{\delta}; i \cdot (\phi_{\delta})^{-1} = A'_n \) where \( i: \partial(M - \text{Int}(N)) \rightarrow \partial N \) is the natural identification.
Assume that \( f(N) = D^2 \) and that \((f \circ \partial_\beta \circ \beta^{-1})_*: H_1(\partial(T_w); Z) \to H_1(\partial(D^2); Z)\) maps \( \iota, \rho, \sigma \) to \( \iota', \rho', \sigma' \) respectively. Corollary 3.14 shows us that \( b' + c' = b + c = 0 \). Note that \((f \circ \partial_\beta \circ \beta^{-1})_*: H_1(\partial(D^2 \times T^2); Z) \to H_1(\partial(D^2); Z)\) maps \( \iota \) to \(- (a'n + b' + c') \iota'\). On the other hand, since \( f|\{M - \text{Int}(N)\} \) is a trivial \( T^2 \)-bundle, \((f \circ \partial_\beta \circ \beta^{-1})_*: H_1(\partial(D^2 \times T^2); Z) \to H_1(\partial(D^2); Z)\) maps \( \iota \) to \( \iota' \). Hence \(- (a'n + b' + c') = 1\). By assumption, we have \( 0 \equiv 1 \). This contradiction completes the proof. \( \Box \)

Therefore, the following is proved.

**Corollary 4.3.** Let \( M^4 \to S^2 \) be a GTF which has only one singular fiber of type \( I_+^1 \) and only one singular fiber of type \( I_-^1 \). Then, \( M \) is diffeomorphic to \( L_n \). Conversely, \( L_n \) admits such a GTF structure.

**Remark 4.4.** If \( n \equiv 1 \) (mod 2), then \( L_n (= L'_n) \) is spin, since \( H_2(L_n; Z/2) = 0 \).

**Remark 4.5.** Let \( h: \partial(D^2 \times T^2) \to \partial(T_w) \) be a diffeomorphism. Assume that \( h \) maps the meridian of \( D^2 \times T^2 \) to \( \pm n + a_1 r + a_2 s \subseteq \partial(T_w) \) for \( n \in N \cup \{0\} \). Then, the proofs of Lemma 4.2 and Theorem 4.1 show that \( D^2 \times T^2 \cup \partial(T_w) \) is diffeomorphic to

\[
\begin{cases}
L_n & \text{(if } n \equiv 1 \text{ (mod 2) or if } n \equiv 0 \text{ and } a_1 + a_2 \equiv 1 \text{ (mod 2)}) \\
L'_n & \text{(if } n \equiv 0 \text{ and } a_1 + a_2 \equiv 0 \text{ (mod 2)})
\end{cases}
\]

**Remark 4.6.** \( L_n = S^4 \) (See [3] Corollary 5.6), \( L_n = S^1 \times S^2 \# S^2 \times S^2, L'_n = S^1 \times S^3 \# S^3 \times S^2 \) (See [4] Theorem III.3).

**Remark 4.7.** The universal covering spaces of \( L_n, L'_n (n \neq 0, 1) \) are \( (n-1)(S^2 \# S^2) \). (See [4] Theorem V. 13.)

**Remark 4.8.** For any \( n \), \( -L_n = L_n, -L'_n = L'_n \).

**Corollary 4.9.** Let \( h \) be a diffeomorphism \( \partial(D^2 \times T^2) \to \partial(T_w) \) such that

\[
A^h = \begin{bmatrix}
 n & * \\
 -q & 0 \\
 0 & 0 & *
\end{bmatrix}
\]

Then,

\[
\begin{align*}
(i) & \ D^2 \times T^2 \cup \partial(T_w) = L_n, \\
(ii) & \ \tau(D^2 \times T^2 \cup \partial(T_w); R) = L_n \quad \text{(if } n \equiv 1 \text{ (mod 2)}) \\
(iii) & \ \tau(D^2 \times T^2 \cup \partial(T_w); S) = L'_n.
\end{align*}
\]

**Proof.** Note that if \( n \equiv 0 \), then \( -q \equiv 1 \). Therefore, by Remark 4.5 and Lemma 4.2, (i) \( A^h \sim A_n \), (ii) \( T_n A^h \sim A_n \) (if \( n \equiv 1 \)), \( T_n A^h \sim A'_n \) (if \( n \equiv 0 \)), and (iii) \( T_s A^h \sim A_n \). \( \Box \)
COROLLARY 4.10. $\chi(L(n, q) \times S^1; \{\} \times S^1) = L_n$. Note that the right-hand side does not depend on $q$ (Observation by Y. Matsumoto). Either framing of $N(\{\} \times S^1)$ yields the same manifold.

PROOF. Since $L(n, q) = D^2 \times S^1 \cup D^2 \times S^1$, we have $\chi(L(n, q) \times S^1; \{\} \times S^1) = D^2 \times T^2 \cup T^2$ (See Lemma 2.2), where

$$A^0 = \begin{bmatrix} n & 0 \\ -q & 0 \\ 0 & 1 \end{bmatrix} \quad \text{or} \quad T^0 = \begin{bmatrix} n & 0 \\ -q & 0 \\ 0 & 1 \end{bmatrix}.$$ 

In either case, $A^0 \sim A_n$. □

§ 5. GTF with two twin singular fibers

In this section, we prove

THEOREM 5.1. (i) Let $M^4 \to S^2$ be a GTF which has only two twin singular fibers as singular fibers. Then, $M$ is diffeomorphic to one of the following.

- $L_n \# S^2 \times S^2$ ($n \in N \cup \{0\}$)
- $L_n' \# S^2 \times S^2$ ($n \equiv 0 \pmod{2}$)
- $L_n' \# S^2 \times S^2$ ($n \in N \cup \{0\}$)

(ii) Conversely, the manifolds above admit such GTF structures. More precisely, the following holds. We restrict our consideration to the case that all the singular fibers are simple. Then,

- (a) $L_n \# S^2 \times S^2$ ($n \equiv 1 \pmod{2}$)
- (a') $L_n' \# S^2 \times S^2$ ($n \equiv 0 \pmod{2}$)
- (b) $L_n' \# S^2 \times S^2$ ($n \equiv 0 \pmod{2}$)
- (c) $L_n' \# S^2 \times S^2$

admits the GTF structure over $S^2$ with two twin singular fibers with multiplicity $b_1, c_1$ and $b_2, c_2$ if and only if $b_1, c_1, b_2, c_2$ are positive integers satisfying $\gcd(b_1, c_1) = \gcd(b_2, c_2) = 1$ and $(b_1 + c_1, b_2 + c_2)$ is

- (a) $(0, 0), (0, 1), (1, 0), (1, 1)$
- (a') $(0, 0), (1, 1)$
- (b) $(0, 1), (1, 0), (1, 1)$
- (c) $(0, 1), (1, 0), (1, 1)$

(mod 2).

Put $\Gamma = \begin{bmatrix} * & 0 \\ * & 0 \\ 0 & 1 \end{bmatrix} \in SL_3 \mathbb{Z}$.

LEMMA 5.2. We can choose a system of coset representatives of $H_1 \backslash GL_3 \mathbb{Z} \cap H_3$.
out of $I^\cup T_s^\cup I^\Gamma T_s^\cup T_s^I T_s^\cup I^\Gamma T_r$. 

**Proof.** In this section, $A \sim B$ means that $A$ and $B$ are in the same coset in $H_1 / GL_3 Z / H_1$, and $A \equiv B$ means that we rewrite $A$ by $B$, and $x \equiv y$ means that $x \equiv y \pmod{2}$.

First we show that we can choose a system of coset representatives of $H_0 / GL_3 Z / H_0$ out of $I'$. Let $A$ be an arbitrary matrix of $GL_3 Z$. We have

$$A := \begin{bmatrix} * & x & y \\ * & * & * \\ * & * & * \end{bmatrix} \equiv \begin{bmatrix} * & 0 \\ * & * \\ * & * \end{bmatrix} \equiv \begin{bmatrix} u & * \\ * & 0 \\ * & 1 \end{bmatrix} \equiv \begin{bmatrix} * & 0 \\ * & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$ 

By Proposition 2.4 (iii), we can choose a system of coset representatives of $H_1 / GL_3 Z / H_1$ out of $I'$. 

Let $A$ be an element of $T_{R}^I$ or $T_{R}^\Gamma T_{R}$. It is clear that there exists a matrix $B$ in $I'$ or $\Gamma T_{R}$ such that $A \sim B$.

Let $A$ be an element of $T_{R}^I T_{S}$. Put

$$A = T_{R} \begin{bmatrix} x & y & 0 \\ z & w & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad T_{S} = \begin{bmatrix} x & y & y \\ z & w & w \\ z & w & w+1 \end{bmatrix}.$$

If $y \equiv 0$, then $x \equiv 1$. If $y \equiv 1$, then the formula

$$\begin{array}{l}
\begin{bmatrix} x & y & y \\ z & w & w \\ z & w & w+1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} x+y & y & y \\ z+w & w & w \\ z+w & w & w+1 \end{bmatrix}
\end{array}$$

shows that we may assume $x \equiv 1$. In either case, we may assume $x \equiv 1$.

$$\begin{array}{l}
\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -z & x+1 & -1 \end{bmatrix} \begin{bmatrix} x & y & y \\ z & w & w \\ z & w & w+1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} x & y & y \\ z & w & w \\ 0 & 0 & 1 \end{bmatrix}
\end{array}$$

and the right-hand side is an element of $\Gamma T_{S}$.

Let $A$ be an element of $T_{S}^I T_{R}$. Note that $A^{-1} T_{R}^S A^{-1} T_{S}^I$ and that $T_{S}^S A^{-1} T_{S}^I$ is an element of $T_{R} T_{S}$. Therefore, there exists an element $B$ of $\Gamma T_{S}$ such that $A^{-1} \sim B$. We have that $A \sim B^{-1} T_{S}^S B^{-1}$ and that $T_{S}^S B^{-1}$ is an element of $T_{S}^I$. 

**Proof of Theorem 5.1.** (i) $M = Tw_1 \cup_h Tw_2$ for some diffeomorphism $h \colon \partial(Tw_1) \to \partial(Tw_2)$. Define $A^h \in GL_3 Z$ by $h_* [l_1 r_1 s_1] = [l_2 r_2 s_2] A^h$. Put $M^h = Tw_1$
If $A^h \sim A^{h'}$, then $M^h$ and $M^{h'}$ are diffeomorphic to each other.

Let $A^h$ be an element of $\Gamma$, say 
\[
\left[ \begin{array}{cc}
* & \pm n \\
* & q \\
0 & 0 \\
0 & 1
\end{array} \right] \quad (n \in N \cup \{0\}).
\]
\[
h = \chi(Tw_1; S) \cup h \cdot Tw_2.
\]
Let $h$ be an element of $F$, say $(n \in N \cup \{0\})$. Since $X(M^h; S_1) = X(Tw_1; S_1) \cup h \cdot Tw_2$, by Remark 2.5, $\chi(M^h; S_1)$ is $L_n$. Note that $M^h = \chi(x(M^h; S_1); \lambda)$ for some framing of $\lambda$ where $\lambda$ is a loop which is isotopic to $s_i$. Since $s_i$ is isotopic to $s_i$ and $s_i$ is isotopic to $0$ in $L_n$, $M^h = L_n \# S^2 \times S^2$ or $L_n \# S^2 \times S^2$. The following Claim shows that $M^h = L_n \# S^2 \times S^2$.

**Claim.** $M^h$ is spin.

**Proof.** In this proof, the coefficient groups of homology groups are assumed to be $Z/2$. For $i=1, 2$, let $C_i$ (resp. $C'_i$) be a 2-chain in $Tw_i$ such that $\partial C_i = s_i$ (resp. $\partial C'_i = r_i$) (mod 2). The Mayer-Vietoris sequence
\[
H_2(Tw_i) \oplus H_2(Tw_2) \xrightarrow{i_*} H_2(M^h) \xrightarrow{\partial} H_2(\partial(Tw_i)) \xrightarrow{i_*} H_2(Tw_i) \oplus H_2(Tw_2)
\]
shows that $H_2(M^h) = \text{Im}(j) \oplus \langle [C_i + C'_i] \rangle$ (if $n = 1$) and that $H_2(M^h) = \text{Im}(j) \oplus \langle [C_i + C'_i], [C'_i + C_i] \rangle$ (if $n = 0$). Since $x^2 = 0$ for any element $x$ of $H_2(Tw_i)$, the self-intersection number on $\text{Im}(j)$ is zero. By Lemma 2.10, we have $[C_i + C'_i]^2 = [C'_i + C_i]^2 = 0$. Therefore $M^h$ is spin. $\square$

See Figure 4.

Let $A^h$ be an element of $\Gamma T_s$. Then, $M^h = \tau(L_n \# S^2 \times S^2; S_i)$ for some $n$. Note that $S_i = \{*\} \times S^2 \subset S^2 \times S^2$. See Figure 4. Therefore we have $M^h = L_n \# S^2 \times S^2$.

Let $A^h$ be an element of $T_s \Gamma$. Then, $(A^h)^{-1} = (A^h)^{-1} T_s^2$ and $(A^h)^{-1} T_s^2$ is an element of $\Gamma T_s$. Exchanging $Tw_1$ and $Tw_2$, this case is reduced to the case above.

Let $A^h$ be an element of $T_n \Gamma T_s$, say $T_n^s \left[ \begin{array}{cc}
a_{11} & \pm n \\
a_{21} & a_{22} \\
0 & 0 \\
0 & 1
\end{array} \right] \quad (n \in N \cup \{0\}).$ Note that $M^h = \tau(L_n \# S^2 \times S^2; S_i, S_j)$ and $(L_n \# S^2 \times S^2; S_i, S_j) = (L_n; S) \# (S^2 \times S^2; \{*\} \times S^2, \{**\} \times S^2)$ where $* \neq **$ and $(L_n; S)$ is the pair $(D^2 \times T^2 \cup h \cdot Tw, S)$ with $A^h = \left[ \begin{array}{cc}
a_{21} & - a_{11} \\
0 & 0
\end{array} \right]$. See Figure 4. Lemma 2.12 and Corollary 4.9.

(iii) show that $M^h = \tau(L_n; S \# S^2 \times S^2; \{*\} \times S^2, \{**\} \times S^2) = L_n \# S^2 \times S^2$.

Let $A^h$ be an element of $T_n \Gamma$. Then, $M^h = \tau(L_n \# S^2 \times S^2; R_r)$. By Corollary 4.9 (ii), we have $M^h = L_n \# S^2 \times S^2$ (if $n = 1$), $L_n \# S^2 \times S^2$ (if $n = 0$).

(ii) Regard (a), (a') $L_n \# S^2 \times S^2$ (b) $L_n \# S^2 \times S^2$ (c) $L_n \# S^2 \times S^2$ as $Tw_1 \cup h \cdot Tw_2$ such that $A^h$ is equal to
Put $S^2 = D^2_+ \cup D^2_-$, where $D^2_+$ and $D^2_-$ are 2-disks, and $D^2_+ \cap D^2_- = \partial D^2_+ = \partial D^2_-$. We construct a GTF $f: Tw_1 \cup_h Tw_2 \to S^2$ as follows. First, define a GTF $f': Tw_2 \to D^2_-$ with only one twin singular fiber that $(f'|\partial): H_1(\partial(Tw_2); \mathbb{Z}) \to H_1(\partial D^2_-; \mathbb{Z})$ maps $l_2, r_2, s_2$ to $a_2', b_2', c_2'$ respectively where $a_2', b_2', c_2'$ are as indicated in the table below and $\gamma$ is the generator of $H_1(\partial D^2_-; \mathbb{Z})$. Then,
Good torus fibrations with twin singular fibers

\[(f'|\partial): H_0(\partial(Tw_1); \mathbb{Z}) \to H_0(\partial D^2_+; \mathbb{Z})\] maps \(l_1, r_1, s_1\) to \(a'_1 r^+, b'_1 y^+, c'_1 y^+\) respectively where \(a'_1, b'_1, c'_1\) are as in the table below and \(r^+\) is the generator of \(H_1(\partial D^2_+; \mathbb{Z})\).

<table>
<thead>
<tr>
<th>(\mod 2)</th>
<th>(a'_1)</th>
<th>(b'_1)</th>
<th>(c'_1)</th>
<th>(a'_1)</th>
<th>(b'_1)</th>
<th>(c'_1)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(a)</td>
<td>(0, 0)</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>(0, 1)</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>-1</td>
<td>(n+1)</td>
</tr>
<tr>
<td></td>
<td>(1, 0)</td>
<td>1</td>
<td>2</td>
<td>1</td>
<td>-1</td>
<td>(n+2)</td>
</tr>
<tr>
<td></td>
<td>(1, 1)</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>(a')</td>
<td>(0, 0)</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>(1, 1)</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>(b)</td>
<td>(0, 1)</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>2</td>
</tr>
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<td></td>
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<td>0</td>
<td>2</td>
<td>1</td>
<td>0</td>
<td>3</td>
</tr>
<tr>
<td></td>
<td>(1, 1)</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>0</td>
<td>3</td>
</tr>
<tr>
<td>(c)</td>
<td>(0, 1)</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>1</td>
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<td>0</td>
<td>2</td>
<td>1</td>
<td>0</td>
<td>2</td>
</tr>
</tbody>
</table>

In any case, by Proposition 3.12 (iii), we can extend \((f'|\partial): \partial(Tw_1) \to \partial D^2_+\) to a GTF \(Tw_1 \to D^2_+\) with only one twin singular fiber with required multiplicities \(b, c\). Corollary 3.14 assures us that we can replace \(f'\) by \(f_{b/c}\) such that \(f'|\partial=f_{b/c}|\partial\) where \(b, c\) are required multiplicities.

The "only if" part can be shown by an argument similar to the proof of Theorem 4.1 (iii).

**Remark 5.3.** \(L'_n \# S^2 \times S^2 = L_n \# S^2 \times S^2\) for \(n \equiv 0 \mod 2\). (See [4], p. 313.)

**Proof.** Since \(L'_n \# S^2 \times S^2 = (L_n \# S^2 \times S^2; S, R)\), we have \(L'_n \# S^2 \times S^2 = Tw_1 \cup_h Tw_2\) where \(A^h \in T_R \Gamma T_s\). But Lemma 5.2 shows that there exists an element \(B\) of \(\Gamma T_s\) satisfying \(A^h \sim B\).

**§ 6. GTF with more than two twin singular fibers**

**Lemma 6.1.** Let \(M\) be \(L_n\) or \(L'_n\) and let \(\lambda\) be a simple loop in \(M\). Then, for any framing of \(\lambda\), \(\chi(M; \lambda) = L_{n'} \# S^2 \times S^2\) or \(L_{n'} \# S^2 \times S^2\) or \(L_{n'} \# S^2 \times S^2\) for some \(n' \in \mathbb{N} \cup \{0\}\).
PROOF. $M = D^2 \times T^2 \cup_h Tw$ for some $h$. Note that there exists a basis $\langle a, b \rangle$ of $\pi_1(D^2 \times T^2)$ such that $i_*(a)$ generates $\pi_1(M)$ and $i_*(b)$ vanishes in $\pi_1(M)$ where $i : D^2 \times T^2 \to M$ is the inclusion map. There exists an integer $d$ such that $[\lambda] = d \cdot i_*(a)$ in $\pi_1(M)$. Find a simple loop $\lambda_0$ in $D^2 \times T^2$ which satisfies $[\lambda_0] = da + b$ in $\pi_1(D^2 \times T^2)$. Since $\lambda$ is isotopic to $\lambda_0$ in $M$, $\chi(M; \lambda)$ is diffeomorphic to $\chi(M; \lambda_0)$ for some framing for $\lambda_0$. Since $\lambda_0$ is a primitive element of $\pi_1(D^2 \times T^2)$, $\chi(D^2 \times T^2; \lambda_0)$ is a twin by Lemma 2.2 (ii). Theorem 5.1 completes the proof. \[\boxdot\]

PROOF OF MAIN THEOREM. If $m = 1$ or if $m = 2$ and the condition (A) is satisfied, then Main Theorem is already proved. See Theorem 4.1, 5.1.

Assume that $m \geq 3$, and that the condition (A) is satisfied. Then, $M = (\sum_{i=1}^m Tw_i) \cup_h F_m \times T^2$ for some diffeomorphism $h : \partial(\sum_{i=1}^m Tw_i) \to \partial(F_m \times T^2)$. We may assume that $Tw_i$ $(i = 3, 4, \ldots, m)$ has a simple twin singular fiber and that the projection map $f|F_m \times T^2$ is the projection to the first coordinate. Define $A^h = \oplus_{i=1}^m A_i^h \in \mathbb{GL}_3\mathbb{Z}$ by $h_*[l_i r_i s_i] = [l_i r_i s_i] A_i^h$ in $H_1(\partial(F_m \times T^2); \mathbb{Z})$. Put $M^h = (\sum_{i=1}^m Tw_i) \cup_h F_m \times T^2$. If $A^h$ and $A^h'$ are in the same coset in $H_3(M^h; \mathbb{Z})/\mathbb{GL}_3\mathbb{Z}$, then $M^h$ and $M^{h'}$ are diffeomorphic to each other. Put $A_i^h = \begin{bmatrix} * & * & * \\ b_i & c_i \\ * & * & * \end{bmatrix}$ $(i = 3, 4, \ldots, m)$. By Proposition 3.12 (iii), $\gcd(b_i, c_i) = 1$.

By Proposition 2.4 (i), we may assume that $A_i^h$ $(i = 3, 4, \ldots, m)$ is of the form $\begin{bmatrix} * & 1 & * \\ * & 0 & * \\ * & 0 & * \end{bmatrix}$ without loss of generality. Furthermore, by Lemma 2.9 (i), we may assume that it is of the form $\begin{bmatrix} * & 0 & * \\ 1 & 0 & * \\ * & 0 & * \end{bmatrix}$. Perform surgeries on $S_h, S_4, \ldots, S_m$.

Then, $(\sum_{i=1}^m \chi(Tw_i; S_i)) \cup_h F_m \times T^2$ is $(\sum_{i=1}^m (D^2 \times T^2)) \cup_h F_m \times T^2$. Note that $h$ maps $r_i$, the meridian of $(D^2 \times T^2)$, to $l_i$. By Remark 2.7, we have $(\sum_{i=1}^m (Tw_i; S_i)) \cup_h F_m \times T^2 = F_m \times T^2$. Therefore $\chi(M; S_1, S_3, \ldots, S_m) = Tw_1 \cup F_2 \times T^2 \cup Tw_2 = L_{n_{s_2}} \# S^2 \times S^2$ or $L_{n_{s_2}} \# S^2 \times S^2$ or $L_{n_{s_2}} \# S \times S^2$ for some non-negative integer $n_{s_2}$. $M$ is obtained from $\chi(M; S_1, S_3, \ldots, S_m)$ by surgeries on $(m - 2)$ mutually disjoint simple loops $\lambda_0, \lambda_1, \ldots, \lambda_m$. By Lemma 6.1, we have that $M$ is one of the manifolds in the statement of Main Theorem.

Assume that $m \geq 2$, and that the condition (B) is satisfied. Then, $M = (D^2 \times T^2 + \sum_{i=1}^m Tw_i) \cup_h F_{m+1} \times T^2$ for some diffeomorphism $h : \partial(D^2 \times T^2) + \partial(\sum_{i=1}^m Tw_i) \to \partial(F_{m+1} \times T^2)$. We may assume that $Tw_i$ $(i = 2, 3, \ldots, m)$ has a simple twin singular fiber and that the projection map $f|F_{m+1} \times T^2$ is the projection to the first coordinate. An argument similar to the case above shows
that we may assume that \((\sum_{i=1}^{m} \chi(Tw_i; S_i)) \cup F_{m+1} \times T^2 = F_2 \times T^2\). Therefore \(\chi(M; S_2, S_3, \ldots, S_m) = F_2 \times T^2 \cup F_2 \times T^2 \cup Tw_i = L_n\) or \(L'_n\) for some non-negative integer \(n\). By Lemma 6.1, we have that \(M\) is one of the manifolds in the statement of Main Theorem.

To show the converse, put

\[
(i) \quad A_i = \begin{bmatrix} 1 & 1 & 0 \\ n+1 & n & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad (ii) \quad A_i = \begin{bmatrix} 1 & 1 & 0 \\ n+1 & n & 0 \\ 0 & 1 & 1 \end{bmatrix}, \quad (iii) \quad A_i = \begin{bmatrix} 1 & 1 & -1 \\ n+1 & n & 0 \\ 0 & 0 & 1 \end{bmatrix}
\]

and

\[
A_i = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} (i=2, 3, \ldots, m).
\]

Then we have (i) \(L_n \# (m-1)S^2 \times S^2\), (ii) \(L'_n \# (m-1)S^2 \times S^2\), (iii) \(L_n \# (m-1)S^2 \times S^2\).

**Proof.** We may assume that \(\lambda_i\) is contained in \(\chi(Tw_i; S_i)\) \((i=3, 4, \ldots, m)\). Then, \(\lambda_i\) is isotopic to \(s_i\) in \(\chi(Tw_i; S_i)\). Since \(s_i\) is isotopic to \(s_2\) in \(F_2 \times T^2\) and \(s_2\) is isotopic to 0 in \(Tw_2\), \(M\) is diffeomorphic to \(\chi(M; S_2, S_3, \ldots, S_m) \# (m-2)S^2 \times S^2\) or \(\chi(M; S_2, S_3, \ldots, S_m) \# (m-2)S^2 \times S^2\). Note that \(\chi(M; S_2, S_3, \ldots, S_m)\) is diffeomorphic to (i) \(L_n \# S^2 \times S^2\), (ii) \(L'_n \# S^2 \times S^2\), (iii) \(L_n \# S^2 \times S^2\).

The case (iii) is already proved. To show the cases (i) and (ii), assume that we have (i) \(L_n \# (m-1)S^2 \times S^2\), (ii) \(L'_n \# (m-1)S^2 \times S^2\). Since \(L'_0 \# (m-1)S^2 \times S^2\) is diffeomorphic to \(L'_0 \# (m-1)S^2 \times S^2\), we may assume that \(n>0\) in (ii). Since \(\chi(M; S_2, S_3, \ldots, S_m)\) is of type II and \(M\) is of type I, there exists a number \(k \in \{3, 4, \ldots, m\}\) such that \(\chi(M; S_k, S_{k+1}, \ldots, S_m)\) is of type II and \(\chi(M; S_{k+1}, \ldots, S_m)\) is of type I. Then, \(X = Tw_2 \cup F_2 \times T^2 \cup Tw_k\) is of type I. Let \(C_j\) be a 2-chain in \(Tw_j\) such that \(\partial C_j \pmod{2} = s_j (j=2, k)\) and let \(C\) be a 2-chain in \(F_2 \times T^2\) such that \(\partial C \pmod{2} = h(s_2) + h(s_{k+1}) = s_2 + s_{k+1}\). The Mayer-Vietoris sequence with the coefficient group \(\mathbb{Z}/2\)

\[
H_2(F_2 \times T^2) \oplus H_2(Tw_2 + Tw_k) \xrightarrow{j_s} H_2(X) \xrightarrow{i_0} H_2(F_2 \times T^2 \cap (Tw_2 + Tw_k))
\]

shows that \(H_2(X; \mathbb{Z}/2) = \text{Im}(j_s) \oplus \langle C_2 + C_{k+1} \rangle\). Since the self-intersection number on \(\text{Im}(j_s)\) is zero and \([C_2 + C_{k+1}]^2 = 0\) by Lemma 2.10, \(X\) is spin. This contradiction completes the proof. \(\square\)

By using these constructions, we can show that \(L_n \# (m-1)S^2 \times S^2\), \(L'_n \# (m-1)S^2 \times S^2\) and \(L_n \# (m-1)S^2 \times S^2\) have GTF structures as follows.
We want to construct a GTF map \( f: \sum_{i=1}^{m} Tw_i \cup F_m \rightarrow \sum_{i=1}^{m} D^2_i \cup F_m = S^2 \) where \( D^2_i \) is a 2-disk. Define \( f': F_m \times \mathbb{T}^2 \rightarrow F_m \) by the projection to the first coordinate. Note that \((f' \cdot h)_*: H_1(\partial(\sum_{i=1}^{m} Tw_i); \mathbb{Z}) \rightarrow H_1(\partial(\sum_{i=1}^{m} D^2); \mathbb{Z})\) maps \( l_i, r_i, s_i \) to (i) \(-r_i, -r_i, 0\) (ii) \(-r_i, -r_i, 0\) (iii) \(-T_i, -r_i, r_i\) and \( l_i, r_i, s_i \) to 0, \(-T_i, 0\) respectively where \( r_i \) is the generator of \( H_1(\partial D^2_i; \mathbb{Z}) \). By Proposition 3.12 (iii), \( f' \) can be extended to a GTF map \( f: \sum_{i=1}^{m} Tw_i \cup F_m \times \mathbb{T}^2 \rightarrow \sum_{i=1}^{m} D^2_i \cup F_m = S^2 \). □

**Remark 6.2.** It is shown that \( Tw, D^2 \times \mathbb{T}^2, F_m \times \mathbb{T}^2 \) and all the manifolds in the table of Main Theorem admit effective \( \mathbb{T}^2 \)-actions. See [4]. But our Main Theorem is not a direct corollary to [4]. For example, since an orbit in \( \partial(Tw) \) is null-homologous in \( H_1(Tw; \mathbb{Z}) \), \( Tw_1 \cup Tw_2 \) does not admit an effective \( \mathbb{T}^2 \)-action which leaves \( Tw_1 \cap Tw_2 \) invariant unless \( \pi_1(Tw_1 \cup Tw_2) = \mathbb{Z} \).

**References**


