Irreducible representations of a class of unbounded operator algebras

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Atsushi Inoue [7] has initiated the study of certain algebras of unbounded operators called EC^* -algebras and EW^* -algebras. These algebras occur naturally in the analysis of unbounded generalization of left Hilbert algebras [10]. The purpose of this note is to examine irreducible representations of these algebras in the context of unbounded representation theory [12].

DEFINITIONS. (a) Let $\mathcal{D}$ be a dense subspace of a Hilbert space $\mathcal{H}$. A collection $\mathcal{U}$ of linear operators, not necessarily bounded, all defined on $\mathcal{D}$ is called an EC^* -algebra on $\mathcal{D}$ over $\mathcal{U}_b$ if

- (i) $\mathcal{U}$ is an algebra under the operations of (usual) sum, product and scalar multiplications.
- (ii) $T \rightarrow T^* = T^\#|_{\mathcal{D}}$ defines an involution on $\mathcal{U}$ where $T^\#|_{\mathcal{D}}$ is the restriction to $\mathcal{D}$ of the operator adjoint $T^\#$ of T in $\mathcal{H}$.
- (iii) $\overline{\mathcal{U}}_b = \{\overline{T} | T \in \mathcal{U} \text{ is bounded}\}$ is a C^* -algebra, bar denoting the closure of the operator.
- (iv) For each $T \in \mathcal{U}$, $(1 + T^*T)^{-1} \in \mathcal{U}$.

Further if $\overline{\mathcal{U}}_b$ is a W^* -algebra, then $\mathcal{U}$ is called an EW^* -algebra.

(b) A *representation* $(\pi, \mathcal{D}(\pi), \mathcal{H})$ of a $*$ algebra A is a mapping π of A into linear operators (not necessarily bounded) all defined on a common domain $\mathcal{D}(\pi)$ dense in a Hilbert space $\mathcal{H}$ such that for all $x, y \in A$; $\alpha, \beta \in \mathbb{C}$, $\xi, \eta \in \mathcal{D}(\pi)$, the following hold.

1. $\pi(\alpha x + \beta y)\xi = \alpha\pi(x)\xi + \beta\pi(y)\xi$
2. $\pi(x)\mathcal{D}(\pi) \subset \mathcal{D}(\pi)$ and $\pi(xy)\xi = \pi(x)\pi(y)\xi$.

If A has identity 1, then

3. $\pi(1) = 1$.

Further π is called *hermitian* if

4. $\langle \pi(x)\xi, \eta \rangle = \langle \xi, \pi(x^*)\eta \rangle$.

If $\pi(x)$ is a bounded operator for every x , then π is called a *bounded representation*.

For basic theory, we refer to the references mentioned above. Throughout, all the representations are hermitian.

THEOREM 1. *Let $\mathcal{U}$ be an EC^* -algebra defined on a linear subspace $\mathcal{D}$ dense in a Hilbert space $\mathcal{H}$. Let $(\pi, \mathcal{D}(\pi), \mathcal{H})$ be a closed representation of $\mathcal{U}$ on $\mathcal{H}$. If the only self-adjoint invariant subspaces for π are $\{0\}$ and $\mathcal{D}(\pi)$, then $\mathcal{D}(\pi) = \mathcal{H}$ and π is a bounded representation.*

The proof required the techniques of GB^* -algebra [6]. By [6, Theorem 7.12] an EC^* -algebra $\mathcal{U}$ is a concrete realization of a locally convex GB^* -algebra. Various GB^* -topologies carried by $\mathcal{U}$ are described in [7, (I), § 2]. Further by [6, § 6] $\mathcal{U}$ carries a largest locally convex GB^* -topology (called the Dixon topology and denoted throughout by τ) having the C^* -algebra $\mathcal{U}_b = \{T \in \mathcal{U} \mid T \text{ is bounded}\}$ as the bounded part. The topology τ is known to be barrelled [6, Lemma 6.2] with the result the multiplication in $(\mathcal{U}, \tau)$ is hypocontinuous [6, Lemma 6.3], and hence sequentially jointly continuous. We shall need the following [3].

THEOREM A. *Let $\mathcal{U}$ be an EC^* -algebra and t be any locally convex GB^* -topology on $\mathcal{U}$. Then $\mathcal{U}_b$ is sequentially dense in $\mathcal{U}$.*

In fact, as discussed in [4], for each $T \in \mathcal{U}$, $T_n = T(1 + (1/n)T^*T)^{-1} \in \mathcal{U}_b$ for each $n = 1, 2, \dots$ and $T_n \rightarrow T$ in τ . We shall also need the following version of [10, Theorem 4.1].

THEOREM B. *Let $\mathcal{U}$ be an EC^* -algebra. Let $(\pi, \mathcal{D}(\pi), \mathcal{H})$ be a representation of $\mathcal{U}$. Let σ_π be the weak topology on $\pi(\mathcal{U})$ defined by the semi-norms $\pi(T) \rightarrow \langle \pi(T)\xi, \eta \rangle$ for ξ, η in $\mathcal{D}(\pi)$. Then $\pi: (\mathcal{U}, \tau) \rightarrow (\pi(\mathcal{U}), \sigma_\pi)$ is continuous.*

PROOF OF THE THEOREM. Let $\rho(T) = \overline{\pi(T)}$ ($T \in \mathcal{U}_b$). Then for each such T , the domain of $\rho(T)$ is $\mathcal{D}(\rho(T)) = \mathcal{H}$ and $\rho(T) \in \beta(\mathcal{H})$. We show that ρ is (topologically) irreducible (in the usual sense of C^* -representation theory).

Suppose that $\mathcal{H}_1$ is a norm closed subspace of $\mathcal{H}$ such that $\mathcal{H}_1 \neq (0)$, $\mathcal{H}_1 \neq \mathcal{H}$ and $\rho(\mathcal{U}_b)\mathcal{H}_1 \subset \mathcal{H}_1$. Let $\mathcal{D}_1 = \{\xi \in \mathcal{D}(\pi) \mid \rho(T)\xi \in \mathcal{H}_1 \text{ for all } T \in \mathcal{U}_b\}$. Obviously, $\pi(\mathcal{U}_b)\mathcal{D}_1 \subset \mathcal{D}_1$. We claim that $\pi(\mathcal{U})\mathcal{D}_1 \subset \mathcal{D}_1$. Indeed, let $\xi \in \mathcal{D}_1$, $T \in \mathcal{U}$. Then $\pi(T)\xi \in \mathcal{D}_1$ if for all $S \in \mathcal{U}_b$, $\rho(S)\pi(T)\xi = \pi(ST)\xi \in \mathcal{H}_1$. By Theorem A, $T_n \xrightarrow{\tau} T$ for some sequence (T_n) in $\mathcal{U}_b$. By the sequential joint continuity of multiplication, $(T - T_n)^*S^*S(T - T_n) \rightarrow 0$. Therefore, by Theorem B,

$$\|\pi(ST)\xi - \pi(ST_n)\xi\|^2 = \langle \pi(T - T_n)^*S^*S(T - T_n)\xi, \xi \rangle \rightarrow 0.$$

But $\pi(ST_n)\xi \in \mathcal{H}_1$ and $\mathcal{H}_1$ is norm closed, hence $\pi(ST)\xi \in \mathcal{H}_1$ giving $\pi(\mathcal{U})\mathcal{D}_1 \subset \mathcal{D}_1$. Further, $\mathcal{D}_1$ is a closed subspace of $(\mathcal{D}(\pi), t_u)$ where t_u is the induced topology [11, § 2] on $\mathcal{D}(\pi)$ defined by the semi-norms $\xi \rightarrow \|\pi(A)\xi\|$ ($A \in \mathcal{U}$). For, let $\xi \in \mathcal{D}(\pi)$ be such that for some net (ξ_α) in $\mathcal{D}_1$, $\xi_\alpha \rightarrow \xi$ in t_u . Then for each $T \in \mathcal{U}$, $\|\pi(T)(\xi_\alpha - \xi)\| \rightarrow 0$. But by above, for each $T \in \mathcal{U}$, $\pi(T)\xi_\alpha \in \mathcal{D}_1 \subset \mathcal{H}_1$ and $\mathcal{H}_1$ is norm closed, hence $\pi(T)\xi \in \mathcal{H}_1$. Thus $\xi \in \mathcal{D}_1$. Now π being a closed representation of the symmetric $*$ algebra $\mathcal{U}$, it is self-adjoint by [8, Lemma 3.5]. Hence, as in [11, Theorem 4.7], it follows that $\mathcal{D}_1$ is a self-adjoint π -invariant subspace of $\mathcal{D}(\pi)$. Hence by the hypothesis, $\mathcal{D}_1 = (0)$ or $\mathcal{D}_1 = \mathcal{D}(\pi)$. In the former case, $\mathcal{H}_1 = (0)$; in the latter one, $\mathcal{H}_1 = \mathcal{H}$. In any case, it follows that ρ is topologically irreducible.

Now by the well known result of Kadison [12, Corollary 1.12.17], ρ is algebraically irreducible. Hence, as $\rho(\mathcal{U}_b)\mathcal{D}(\pi) \subset \mathcal{D}(\pi)$, we get $\mathcal{D}(\pi) = \mathcal{H}$. Now the closed graph theorem implies that π is bounded.

Thus if $(\pi, \mathcal{D}(\pi), \mathcal{H})$ is a closed (algebraically) irreducible representation of an EC^* -algebra $\mathcal{U}$, then $\mathcal{D}(\pi) = \mathcal{H}$ and π is bounded; and further, if $\mathcal{U}$ is commutative, then π is one dimensional. Theorem 1 does not hold for arbitrary algebra; e.g. take $\mathcal{H} = L^2(\mathbf{R})$, $\mathcal{D} = \mathcal{S}(\mathbf{R})$, the Schwartz space of rapidly decreasing C^∞ -functions on $\mathbf{R}$. Let $\mathcal{U}$ be the complex algebra with identity generated by the position-momentum operators $p(f) = i(df/dt)$, $q(f)(t) = tf(t)$ ($t \in \mathbf{R}$, $f \in \mathcal{D}$). Let $*$ be the unique involution on $\mathcal{U}$ making p and q self-adjoint. Then the identity representation of $\mathcal{U}$ is known [11] to be self-adjoint having no non-trivial self-adjoint invariant subspaces. It also follows that unlike C^* -algebras, an EC^* -algebra need not admit any irreducible representation at all e.g. consider the EC^* -algebra $\mathcal{U}$ obtained by taking the faithful Dixon representation [6, Theorem 7.1] of the $*$ -algebra $L^w[0, 1] = \bigcap_{1 < p \leq \infty} L^p[0, 1]$ with the pointwise operations [1, Example 3]. Next we discuss several conditions ensuring the existence of irreducible representations.

COROLLARY 2. *Let $\mathcal{U}$ be an EC^* -algebra. The following are equivalent.*

- (a) *There exists a closed irreducible representation of $\mathcal{U}$.*
- (b) *There exists a C^* -semi-norm on $\mathcal{U}$.*
- (c) *There exists a non-zero bounded representation of $\mathcal{U}$.*

PROOF. That (a) $\Rightarrow$ (c) $\Rightarrow$ (b) are immediate in view of Theorem 1. Let p be a C^* -semi-norm on $\mathcal{U}$. Let $\mathbf{I} = \ker p$, a $*$ ideal in $\mathcal{U}$. Let $\tilde{\mathcal{U}} = \mathcal{U}/\mathbf{I}$ and $\tilde{p}$ be the C^* -norm on induced by p . We show that $(\tilde{\mathcal{U}}, \tilde{p})$ is complete (this seems to be of some independent interest). First note that for each $T \in \mathcal{U}$, $p(1 + (1/n)T^*T)^{-1} \leq 1$. (Pass to the quotient, complete it and use the C^* -result.) Now let $\mathbf{J} = \{T \in \mathcal{U}_b | p(T) = 0\} = \mathcal{U}_b \cap \mathbf{I}$. As $\mathcal{U}_b$ is a C^* -algebra, p is norm-continuous on $\mathcal{U}_b$. Hence $\mathbf{J}$ is a norm closed $*$ -ideal of $\mathcal{U}_b$ and $\mathcal{U}_b/\mathbf{J}$ is

a C^* -algebra under the quotient norm $\|\cdot\|$. But $T + \mathbf{J} \rightarrow \tilde{p}(T + \mathbf{J})$ is also a C^* -norm (not necessarily complete a priori) on $\mathcal{U}_b/\mathbf{J}$. Again by the standard C^* -arguments. $p(T) = \tilde{p}(T + \mathbf{J}) = \|T + \mathbf{J}\|$ ($T \in \mathcal{U}_b$). Thus $\phi: T + \mathbf{J} \rightarrow T + \mathbf{I}$ is an isometric embedding of $\mathcal{U}_b/\mathbf{J}$ into $(\tilde{\mathcal{U}}, \tilde{p})$ with the result $\phi(\mathcal{U}_b/\mathbf{J})$ is closed in $(\tilde{\mathcal{U}}, \tilde{p})$. On the other hand, for each $T \in \mathcal{U}$, $T(1 + (1/n)T^*T)^{-1} \in \mathcal{U}_b$ and $\lim_n T(1 + (1/n)T^*T)^{-1} = T$ in τ . Also,

$$\begin{aligned} p\left(T - T\left(1 + \frac{1}{n}T^*T\right)^{-1}\right) &= p\left(T\left(\frac{T^*T}{n}\right)\left(1 + \frac{1}{n}T^*T\right)^{-1}\right) \\ &\leq \frac{1}{n}p(T)p(T^*T) \rightarrow 0 \quad \text{as } n \rightarrow \infty. \end{aligned}$$

Thus $T(1 + (1/n)T^*T)^{-1} + \mathbf{I} \rightarrow T + \mathbf{I}$ in $(\tilde{\mathcal{U}}, \tilde{p})$. Therefore, $\phi(\mathcal{U}_b/\mathbf{J})$ is dense in $(\tilde{\mathcal{U}}, \tilde{p})$. It follows that $(\tilde{\mathcal{U}}, \tilde{p})$ is a C^* -algebra. Now it is easily seen that every irreducible representation of $\tilde{\mathcal{U}}$ lifts to an irreducible representation of $\mathcal{U}$, and (a) follows from (b). This completes the proof.

The existence of a faithful family of irreducible representations greatly simplifies the structure of an EC^* -algebra, namely, makes it a pre b^* -algebra [6, Example 3.2] under a suitable topology. A *weakly unbounded EC^* -algebra* [9] $\mathcal{U}$ is a $*$ -subalgebra of the product $\prod \mathcal{U}_\alpha$ [7 (I), Theorem 3.7] of a family $(\mathcal{U}_\alpha)$ of C^* -algebras such that $\overline{\mathcal{U}}_b = \{\bar{T} \mid T \in \mathcal{U}_b\} = \Sigma^\oplus \mathcal{U}_\alpha$, the usual C^* -direct sum. It follows from Corollary (2) that an EC^* -algebra $\mathcal{U}$ admits a faithfully family of irreducible representations, if and only if $\mathcal{U}$ is ($*$ isomorphic to) a weakly unbounded EC^* -algebra.

PROPOSITION 3. *Let $\mathcal{U}$ be an EW^* -algebra. Let σ be the weak topology on $\mathcal{U}$ defined by the semi-norms $T \rightarrow |\langle T\xi, \eta \rangle|$ for $\xi, \eta \in \mathcal{D}$. The following are equivalent.*

- (a) $\mathcal{U}$ admits a weakly continuous irreducible representation.
- (b) $\mathcal{U}_b$ admits a non-zero minimal projection.

The proof requires the following lemma which is of some independent interest. The proof is a modification of standard W^* -techniques.

LEMMA 4. *Let $\mathcal{U}$ be an EW^* -algebra. A subset $\mathbf{I}$ of $\mathcal{U}$ is a σ -closed left ideal of $\mathcal{U}$ if and only if there is a unique projection P in $\mathcal{U}$ such that $\mathbf{I} = \mathcal{U}P$. Further*

- (a) $\mathbf{I}$ is two sided if and only if P is a central projection.
- (b) $\mathbf{I}$ is maximal if and only if $1 - P$ is a minimal projection.

PROOF OF THE LEMMA. As in [7 (I), § 2], $(\mathcal{U}, \sigma)$ is a locally convex GB^* -algebra with bounded part $\mathcal{U}_b$. By Theorem A, $\mathcal{U}_b$ is σ -dense in $\mathcal{U}$. Now

let $\mathbf{I}$ be a σ -closed left ideal of $\mathcal{U}$. Then $\mathbf{L} = \mathbf{I} \cap \mathcal{U}_b$ is a σ_w -closed left of the W^* -algebra $\mathcal{U}_b$ (Here σ_w is the weak topology on $\mathcal{U}_b$ as the dual of a Banach space). Then by the σ -continuity of the involution on $\mathcal{U}$, $\mathbf{L} \cap \mathbf{L}^* = \mathbf{I} \cap \mathbf{I}^* \cap \mathcal{U}_b$ is a σ_w -closed two sided ideal of $\mathcal{U}_b$. Thus it is a W^* -algebra in its own right admitting an identity P that is a projection in $\mathcal{U}_b$. Next, as in [12, Proposition 1.10], we have $\mathbf{I} \cap \mathcal{U}_b = \mathcal{U}_b P = \{T \in \mathcal{U}_b \mid T = TP\}$. This gives $\mathbf{I} = \mathcal{U}P$. In fact, given $T \in \mathbf{I}$, $T_n = T(1 + (1/n)T^*T)^{-1} \in \mathbf{I} \cap \mathcal{U}_b$ and $T_n P = T_n \xrightarrow{\sigma} T = TP$. The converse, as well as the remaining assertions, are immediate.

PROOF OF THE PROPOSITION. (a) $\Rightarrow$ (b). If $(\pi, \mathcal{D}(\pi), \mathcal{H})$ is a $\sigma - \sigma_\pi$ continuous irreducible representation of $\mathcal{U}$, then for each $\xi \neq 0$ in $\mathcal{D}(\pi)$, $\mathbf{L} = \{T \in \mathcal{U} \mid \pi(T)\xi = 0\}$ is a maximal left ideal that is σ -closed. The lemma gives (b).

Conversely, let P be a minimal projection in $\mathcal{U}_b$. Then $\mathbf{I} = \mathcal{U}(1 - P)$ is a σ -closed maximal left ideal, and $P\mathcal{U}_b P \simeq C$. By theorem A, $P\mathcal{U}P \simeq C$ and $f: T \rightarrow PTP$ define a linear functional on $\mathcal{U}$ that is easily seen to be positive using [7 (1), Proposition 2.11]. Let $\mathbf{L}' = \{T \mid f(T^*T) = 0\}$. Then, if $T = S(1 - P) \in \mathbf{L}$, $f(T^*T) = P(1 - P)S^*S(1 - P)P = 0$ and so $\mathbf{L} \subset \mathbf{L}'$. Since $\mathbf{L}$ is maximal, $\mathbf{L} = \mathbf{L}'$. Let $(\pi, \mathcal{D}(\pi), \mathcal{H})$ be the closed strongly cyclic representation of $\mathcal{U}$ constructed from f by the GNS construction. Then, as f is σ -continuous, π is $\sigma - \sigma_\pi$ continuous; and by the standard arguments, it is irreducible. Thus (b) $\Rightarrow$ (a) and the proposition follows.

Thus an EW^* -algebra $\mathcal{U}$ admits a faithfully family of weakly continuous irreducible representations, if and only if $\mathcal{U}_b$ is an atomic W^* -algebra.

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