Linear partial differential equations with regular singularities in all variables

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§ 1. Introduction and notations

The purpose of this paper is to extend the well-known theory of regular singularities of ordinary differential equations to the case of partial differential equations.

We consider an equation of the form

$$\sum_{|\alpha| \leq m} a_{\alpha_1 \alpha_2 \dots \alpha_n}(x) \left(x_1 \frac{\partial}{\partial x_1} \right)^{\alpha_1} \left(x_2 \frac{\partial}{\partial x_2} \right)^{\alpha_2} \cdots \left(x_n \frac{\partial}{\partial x_n} \right)^{\alpha_n} u(x) = 0$$

and obtain solutions of the form

$$x_1^{\rho_1} \cdots x_n^{\rho_n} \cdot \sum p_{\nu_1 \dots \nu_n} (\log x_1)^{\nu_1} \cdots (\log x_n)^{\nu_n}$$

which are called regular type solutions.

Kashiwara-Kawai-Sjöstrand [4] and Bengle-Gérard [1] treated a class of more general equations but they studied only power series solutions.

Throughout this paper we use the following notations:

$$Z_0 = \{0, 1, 2, \dots\}, \quad \alpha = (\alpha_1, \dots, \alpha_n) \in Z_0^n, \quad |\alpha| = \alpha_1 + \dots + \alpha_n,$$

$$\gamma \leq \delta \iff \gamma_i \leq \delta_i \quad (\gamma = (\gamma_1, \dots, \gamma_n), \quad \delta = (\delta_1, \dots, \delta_n) \in Z_0^n),$$

$$\gamma < \delta \iff \gamma \leq \delta \quad \text{and} \quad \gamma \neq \delta,$$

$$x = (x_1, \dots, x_n) \in C^n, \quad |x| = |x_1| + \dots + |x_n|, \quad x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n},$$

$$\frac{\partial^{|\alpha|}}{\partial x^\alpha} = \frac{\partial^{\alpha_1 + \dots + \alpha_n}}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}.$$

§ 2. Preliminaries

We denote by $\mathcal{O}_0$ the ring of germs of holomorphic functions at the origin $x=0$ of C^n , and by $\mathcal{D}_j$ ($j=1, \dots, n$) the differential operator $x_j \partial / \partial x_j$. Let us consider a differential operator

$$A(x, \vartheta) = \sum_{|\alpha| \leq m} a_\alpha(x) \vartheta^\alpha$$

with $a_\alpha \in \mathcal{O}_0$. The polynomial in ρ

$$f(\rho) = A(0, \rho)$$

is called the *indicial polynomial* of $A(x, \vartheta)$. Assume that the degree of f is m and denote by $f_m(\rho)$ the homogeneous part of degree m of f .

We suppose that there exists a positive number K such that for all $\gamma \in \mathbf{Z}_0^n$

$$(1) \quad |f_m(\gamma)| \geq K |\gamma|^m.$$

The condition (1) is called the *Poincaré condition*. We decompose f into irreducible polynomials:

$$f = f_1^{d_1} \cdots f_k^{d_k}$$

and put

$$\begin{aligned} V &= \{\rho \in \mathbf{C}^n; f(\rho) = 0\}, \\ V_j &= \{\rho \in \mathbf{C}^n; f_j(\rho) = 0\} \quad (j = 1, \dots, k). \end{aligned}$$

Then

$$V = V_1 \cup \cdots \cup V_k.$$

We define

$$\Sigma = \{\rho \in V; f(\rho + \gamma) = 0 \text{ for some } \gamma \in \mathbf{Z}_0^n, \gamma \neq 0\}.$$

LEMMA 1. For any point $p \in \mathbf{C}^n$, there exist a neighborhood U_p of p and a point $\delta \in \mathbf{Z}_0^n$ such that

$$\Sigma \cap U_p = \bigcup_{0 < \gamma \leq \delta} \{\rho \in V; f(\rho + \gamma) = 0\}.$$

PROOF. If we consider $f(\rho + \gamma)$ as a polynomial in γ , then its homogeneous part of degree m is equal to $f_m(\gamma)$. Hence $f(\rho + \gamma)$ can be written as

$$f(\rho + \gamma) = f_m(\gamma) + \sum_{|\alpha| \leq m-1} f_\alpha(\rho) \gamma^\alpha$$

where $f_\alpha(\rho)$ are polynomials. The Poincaré condition leads us to the estimate

$$|f(\rho + \gamma)| / |\gamma|^m \geq K - \sum_{|\alpha| \leq m-1} |f_\alpha(\rho)| \cdot |\gamma|^{-1} \quad (\gamma \neq 0).$$

Since there exist a positive number $B > 0$ such that

$$\sum_{|\alpha| \leq m-1} |f_\alpha(\rho)| < B |\rho|^m,$$

we have for $|\gamma| > B|\rho|^m/K$

$$|f(\rho + \gamma)| > 0.$$

Choose a neighborhood U_p of p such that $|\rho| \leq L$ for $\rho \in U_p$. Then clearly

$$\Sigma \cap U_p = \bigcup_{0 < |\gamma| \leq BL^m/K} \{\rho \in V; f(\rho + \gamma) = 0\}.$$

Let k be the least integer $\geq BL^m/K$. Then we can take $(k, \dots, k)$ as δ . Thus the lemma is established.

Modifying the proof of this lemma, we obtain easily the following lemma.

LEMMA 2. *For any point $p \in V$, there exist a point $\delta' \in Z_0^n$, a neighborhood U_p of p and a positive number K' such that*

$$(2) \quad |f(\rho + \gamma)| \geq K'(|\rho| + |\gamma| + m)^m$$

for all $\rho \in U_p$ unless $\gamma \leq \delta'$.

We see from lemma 1 that $\dim \Sigma \geq n-2$, but it happens that Σ contains components of dimension $n-1$. We decompose $\Sigma = \Sigma_{n-1} \cup \Sigma_{n-2}$ where Σ_i ($i = n-1, n-2$) is a variety of dimension i . We define

$$\text{Reg } V = \left\{ p \in V; \left(\frac{\partial}{\partial \rho_1}, \dots, \frac{\partial}{\partial \rho_n} \right) (f_1 \cdots f_k)|_{\rho=p} \neq 0 \right\}.$$

For $\rho \in \text{Reg } V - \Sigma_{n-2}$, there exists $j_\rho \in \{1, \dots, k\}$ such that $\rho \in V_{j_\rho}$. Let $\{j_{1\rho}, \dots, j_{s\rho}\} \subset \{1, \dots, k\}$ be the subset of indices such that we can write $V_{j_\rho} = V_{j_\rho} + \gamma_{j_\rho}$ for some $\gamma_{j_\rho} \in Z_0^n - \{0\}$. Define $d(\rho)$ and $m(\rho)$ by

$$d(\rho) = d_{j_\rho}, \quad m(\rho) = d_{j_\rho} + d_{j_{1\rho}} + \dots + d_{j_{s\rho}}.$$

§ 3. Existence of simple regular type solutions

We consider the partial differential equation

$$(3) \quad A(x, \mathcal{D})u(x) = 0.$$

First we will prove the existence of solutions of the form

$$u(x) = x^\rho \phi(x)$$

with $\phi \in \mathcal{O}_0$. We call a solution of this form *simple regular type solution*.

THEOREM 1. *Suppose that the Poincaré condition is satisfied.*

i) *For any point $\rho \in V - \Sigma$, there exists a unique simple regular type solution*

$$u(x, \rho) = x^\rho \phi(x, \rho)$$

such that $\phi(0, \rho) = 1$.

ii). $\phi(x, \rho)$ is holomorphic with respect to ρ on $\text{Reg } V - \Sigma$.

PROOF. First we note the relation

$$(4) \quad A(x, \mathcal{G})x^\nu = A(x, \nu)x^\nu$$

for any $\nu \in \mathbb{C}^n$. The function $A(x, \nu)$ is expressed as

$$(5) \quad A(x, \nu) = \sum_{|\beta| \geq 0} x^\beta f_\beta(\nu),$$

where $f_\beta(\nu)$ are polynomials in ν with constant coefficients of degree at most m , and $f_0(\nu)$ coincides with the indicial polynomial $f(\nu)$.

We consider a formal solution

$$(6) \quad u(x, \rho) = x^\rho \sum_{r \in \mathbb{Z}_0^n} u_r(\rho) x^r,$$

then we have

$$(7) \quad \begin{aligned} A(x, \mathcal{G})u(x, \rho) &= \sum_{r \in \mathbb{Z}_0^n} u_r(\rho) A(x, \rho + r) x^{\rho+r} \\ &= \sum_{r \in \mathbb{Z}_0^n} u_r(\rho) x^{\rho+r} \sum_{\beta \in \mathbb{Z}_0^n} f_\beta(\rho) x^\beta \\ &= x^\rho \sum_{r \in \mathbb{Z}_0^n} \sum_{\beta + r' = r} u_{r'}(\rho) f_\beta(\rho + r') x^r \end{aligned}$$

from which we obtain the following recurrence formula:

$$(8) \quad u_0(\rho) f(\rho) = 0,$$

$$(9) \quad u_r(\rho) f(\rho + r) + \sum_{\substack{\beta + r' = r \\ |\beta| > 0}} u_{r'}(\rho) f_\beta(\rho + r') = 0.$$

The condition $\phi(0, \rho) = 1$ implies $u_0(\rho) = 1$.

Since the series in x

$$A(x, \rho) = \sum_{\beta \in \mathbb{Z}_0^n} f_\beta(\rho) x^\beta$$

is convergent for $|\rho| < 1$, we have for $|\rho| < 1$

$$|f_\beta(\rho)| \leq \frac{M}{R^\beta}$$

with $M > 0$ and $R = (R_1, \dots, R_n)$. It follows from the fact that $f_\beta(\rho)$ are polynomials in ρ of degree at most m that we get

$$(10) \quad |f_\beta(\rho)| \leq \frac{A}{R^\beta} |\rho|^m,$$

where A is a positive constant independent of β .

From (9), (10), and lemma 2, we can derive the estimate

$$(11) \quad |u_r(\rho)| \leq \tilde{K} \sum_{\substack{\beta+r'=r \\ |\beta|>0}} \frac{|u_{r'}(\rho)|}{R^\beta} \quad (r \leq \delta')$$

where $\tilde{K}$ is a positive number.

It is clear that the coefficients $u_r(\rho)$, $r \leq \delta'$, are all holomorphic in a neighborhood U_p of p . Hence we can suppose that $u_r(\rho)$ ($r \leq \delta'$) are majorized by constants $d_r > 0$;

$$|u_r(\rho)| \leq d_r$$

in U_p .

We define inductively d_r ($r \leq \delta'$) by the formula

$$(12) \quad d_r = \tilde{K} \sum_{r' < r} \frac{d_{r'}}{R^{r-r'}}.$$

Then we have

$$|u_r(\rho)| \leq d_r$$

for all $r \in \mathbb{Z}_0^n$, $\rho \in U_p$.

To prove the theorem, it is sufficient to show that $\sum_{r \in \mathbb{Z}_0^n} d_r x^r$ is convergent. We note that we can take d_r so that

$$(13) \quad b_0 := d_0 > 0$$

$$(14) \quad b_r := d_r - \tilde{K} \sum_{r' < r} \frac{d_{r'}}{R^{r-r'}} > 0 \quad (0 < r \leq \delta').$$

Now let us consider the function

$$(15) \quad E(x) = \sum_{r \leq \delta'} b_r x^r / (1 - \tilde{K} \sum_{r \in \mathbb{Z}_0^n - \{0\}} x^r / R^r).$$

It is clear that $E(x)$ is holomorphic at the origin and that we can expand $E(x)$ into a convergent power series:

$$(16) \quad E(x) = \sum_{r \in \mathbb{Z}_0^n} e_r x^r.$$

Comparing the coefficients of (15) and (16), we obtain equalities

$$(17) \quad b_0 = e_0,$$

$$(18) \quad b_r = e_r - \tilde{K} \sum_{r' < r} \frac{e_{r'}}{R^{r-r'}} \quad (r \leq \delta'),$$

and

$$(19) \quad 0 = e_r - \tilde{K} \sum_{r' < r} \frac{c_{r'}}{R^{r-r'}} \quad (r \leq \delta').$$

Comparing (17) with (13), (18) with (14), and (19) with (12), we conclude that for all, $r \in \mathbb{Z}_0^n$, we have $d_r = e_r$. Therefore the series $\sum_{r \in \mathbb{Z}_0^n} d_r x^r$ converges, and this implies the convergence of the formal power series (6). Thus the theorem is proved.

§ 4. Regular type solutions

We say that a solution $u(x)$ of (3) is a *regular type solution of exponent ρ* if $u(x)$ is represented as

$$u(x) = x^\rho F(x, \log x),$$

where F is a polynomial in $\log x_1, \dots, \log x_n$ with coefficients in $\mathcal{O}_0$, namely

$$F(x, \log x) \in \mathcal{O}_0 [\log x_1, \dots, \log x_n].$$

We denote by $\mathcal{A}^\rho$ the set of regular type solutions of exponent ρ .

THEOREM 2. *Suppose that $\rho \in \text{Reg } V - \Sigma$. Then the linear space $\mathcal{A}^\rho$ is isomorphic to $C[X_1, \dots, X_{n-1}]^{d(\rho)}$.*

PROOF. We regard $\tilde{\rho}$ as a parameter and consider the series

$$(20) \quad u(x, \tilde{\rho}) = x^{\tilde{\rho}} \sum_{r \in \mathbb{Z}_0^n} u_r(\tilde{\rho}) x^r,$$

where $u_0(\tilde{\rho}) \equiv 1$ and $u_r(\tilde{\rho})$ ($r > 0$) are determined inductively by the formula (9). This series (20) is convergent by the proof of Theorem 1 and give a holomorphic function of $2n$ variables $(x, \tilde{\rho})$. It is easy to see that

$$(21) \quad A(x, \mathcal{J})u(x, \tilde{\rho}) = f(\tilde{\rho})x^{\tilde{\rho}}.$$

By a change of the order of the variables $x_1, \dots, x_n$, we may suppose that $(\lambda_1, \dots, \lambda_{n-1}) = (\tilde{\rho}_1 - \rho_1, \dots, \tilde{\rho}_{n-1} - \rho_{n-1})$ is a local coordinate system of $\text{Reg } V - \Sigma$ in a neighborhood of ρ . Then $\text{Reg } V - \Sigma$ can be expressed locally by

$$\tilde{\rho}_n - \rho_n - \psi(\lambda_1, \dots, \lambda_{n-1}) = 0$$

with a function ψ holomorphic in a neighborhood of 0. Let $\lambda_n = \tilde{\rho}_n - \rho_n - \psi(\lambda_1, \dots, \lambda_{n-1})$. Then $(\lambda_1, \dots, \lambda_{n-1}, \lambda_n)$ is a local coordinate system around ρ in C^n . It is clear that $f(\rho)$, as a function of $(\lambda_1, \dots, \lambda_{n-1}, \lambda_n)$, can be expressed as

$$(22) \quad f(\tilde{\rho}(\lambda)) = \lambda_n^{d(\rho)} \times (\text{a unit in } \mathcal{O}_0).$$

We define a mapping $\Phi_\rho: C[X_1, \dots, X_{n-1}]^{d(\rho)} \rightarrow \mathcal{A}^\rho$ by

$$(23) \quad \begin{aligned} & \Phi_\rho((P_j(X_1, \dots, X_{n-1}))_{j=1, \dots, d(\rho)}) \\ &= \sum_{j=1}^{d(\rho)} P_j \left(\frac{\partial}{\partial \lambda_1}, \dots, \frac{\partial}{\partial \lambda_{n-1}} \right) \frac{\partial^{j-1}}{\partial \lambda_n^{j-1}} u(x, \tilde{\rho}(\lambda))|_{\lambda=0}. \end{aligned}$$

It can be easily seen from (21) and (22) that the mapping Φ_ρ is well-defined.

Suppose that

$$(24) \quad \sum_{j=1}^{d(\rho)} P_j \left(\frac{\partial}{\partial \lambda_1}, \dots, \frac{\partial}{\partial \lambda_{n-1}} \right) \frac{\partial^{j-1}}{\partial \lambda_n^{j-1}} u(x, \tilde{\rho}(\lambda))|_{\lambda=0} = 0.$$

The operator

$$(25) \quad \sum_{j=1}^{d(\rho)} P_j \left(\frac{\partial}{\partial \lambda_1}, \dots, \frac{\partial}{\partial \lambda_{n-1}} \right) \frac{\partial^{j-1}}{\partial \lambda_n^{j-1}}$$

can be written as

$$\sum_{|\alpha| \leq J} P_\alpha \frac{\partial^{|\alpha|}}{\partial \lambda^\alpha}$$

and x^ρ is written as

$$x_1^{\rho_1 + \lambda_1} \dots x_{n-1}^{\rho_{n-1} + \lambda_{n-1}} x_n^{\rho_n + \lambda_n + \psi(\lambda_1, \dots, \lambda_{n-1})}.$$

From (24), we have

$$\begin{aligned} & \sum_{|\alpha| \leq J} P_\alpha \frac{\partial^{|\alpha|}}{\partial \lambda^\alpha} (x_1^{\rho_1 + \lambda_1} \dots x_{n-1}^{\rho_{n-1} + \lambda_{n-1}} x_n^{\rho_n + \lambda_n + \psi(\lambda_1, \dots, \lambda_{n-1})}) \sum_{r \in \mathbb{Z}_0^n} u_r(\tilde{\rho}(\lambda)) x^r|_{\lambda=0} \\ &= \sum_{|\alpha| \leq J} P_\alpha \prod_{j=1}^{n-1} \left(\log x_j + \frac{\partial \psi}{\partial \lambda_j}(0) \log x_n \right)^{\alpha_j} \cdot (\log x_n)^{\alpha_n} \cdot x^\rho \sum_{r \in \mathbb{Z}_0^n} u_r(\rho) x^r + \dots \\ &= 0 \end{aligned}$$

where $\dots$ means a polynomial of $\log x_1, \dots, \log x_n$ of degree $\leq J-1$. It follows that $p_\alpha = 0$ if $|\alpha| = J$. This fact enables us to conclude that

$$\sum_{j=1}^{d(\rho)} P_j \left(\frac{\partial}{\partial \lambda_1}, \dots, \frac{\partial}{\partial \lambda_{n-1}} \right) \frac{\partial^{j-1}}{\partial \lambda_n^{j-1}} = 0.$$

This means that Φ_ρ is injective.

Now, take any regular type solution of exponent ρ

$$(26) \quad w(x) = x^\rho F(x, \log x) \in \mathcal{A}^\rho$$

We will prove by induction on $\deg F$ that there exists an operator of the form (25) such that

$$(27) \quad x^\rho F(x, \log x) = \sum_{j=1}^{d(\rho)} P_j \left(\frac{\partial}{\partial \lambda_1}, \dots, \frac{\partial}{\partial \lambda_{n-1}} \right) \frac{\partial^{j-1}}{\partial \lambda^{j-1}} \cdot (x_1^{\rho_1 + \lambda_1} \dots x_{n-1}^{\rho_{n-1} + \lambda_{n-1}} x_n^{\rho_n + \lambda_n + \psi(\lambda_1, \dots, \lambda_{n-1})} \sum_{r \in \mathbb{Z}_0^n} u_r(\tilde{\rho}(\lambda)) x^r) |_{\lambda=0}$$

where $\deg F$ stands for the degree of the polynomial F with respect to $\log x_1, \dots, \log x_n$.

If $\deg F = 0$, we can write $w(x) = x^\rho \tilde{w}(x)$. It is obvious from Theorem 1 that

$$\begin{aligned} w(x) &= x^\rho \tilde{w}(0) \sum_{r \in \mathbb{Z}_0^n} u_r(\rho) x^r \\ &= \tilde{w}(0) u(x, \rho). \end{aligned}$$

Assume that the assertion is true for $\deg F \leq l-1$, and proceed to the case $\deg F = l$. We can write

$$(28) \quad w(x) = x^\rho \left\{ \sum_{|a|=l} \tilde{c}_a(x) \prod_{j=1}^{n-1} \left(\log x_j + \frac{\partial \psi}{\partial \lambda_j}(0) \log x_n \right)^{\alpha_j} \cdot (\log x_n)^{\alpha_n} + \dots \right\}$$

where $\dots$ is a polynomial of degree $\leq l-1$.

Then the function

$$(29) \quad \begin{aligned} w(x) - \sum_{\substack{\alpha_1 + \dots + \alpha_{n-1} + \alpha_n = l \\ \alpha_n \leq (\rho) - 1}} \tilde{c}_{\alpha_1, \dots, \alpha_{n-1}, \alpha_n}(0) \frac{\partial^{\alpha_1 + \dots + \alpha_n}}{\partial \lambda_1^{\alpha_1} \dots \partial \lambda_{n-1}^{\alpha_{n-1}} \partial \lambda_n^{\alpha_n}} \\ \cdot [x_1^{\rho_1 + \lambda_1} \dots x_{n-1}^{\rho_{n-1} + \lambda_{n-1}} x_n^{\rho_n + \lambda_n + \psi(\lambda_1, \dots, \lambda_{n-1})} \sum_{r \in \mathbb{Z}_0^n} u_r(\tilde{\rho}(\lambda)) x^r] |_{\lambda=0} \\ = x^\rho \left\{ \sum_{\alpha_1 + \dots + \alpha_{n-1} + \alpha_n = l} d_\alpha(x) \prod_{j=1}^{n-1} \left(\log x_j + \frac{\partial \psi}{\partial \lambda_j}(0) \log x_n \right)^{\alpha_j} \cdot (\log x_n)^{\alpha_n} + \dots \right\} \end{aligned}$$

belongs to $\mathcal{A}^\rho$.

We will show by induction on α_n that $d_\alpha(x) = 0$ ($|\alpha| = l$). Note that the following Leibniz rule holds:

$$(30) \quad A(x, \vartheta)(g(x)h(x)) = \sum_{|\alpha| \leq m} \frac{1}{\alpha!} (A^{(\alpha)}(x, \vartheta)g(x))(\vartheta^\alpha h(x))$$

where

$$A^{(\alpha)}(x, \xi) = \frac{\partial^{|\alpha|}}{\partial \xi^\alpha} A(x, \xi).$$

If $\alpha_n \leq d(\rho) - 1$, then we have from (29) $d_\alpha(0) = 0$. On the other hand, $x^\rho d_\alpha(x)$ belongs to $\mathcal{A}^\rho$, and we see that $d_\alpha(x) \equiv 0$.

Suppose that $d_\alpha(x) = 0$ for $\alpha_n \leq k - 1$. Then it follows from the fact that (29) belongs to $\mathcal{A}^\rho$ that

$$(31) \quad A(x, \mathcal{V})x^\rho \left[\sum_{\substack{\alpha_1 + \dots + \alpha_{n-1} + \alpha_n = l \\ \alpha_n \geq k (\geq d(\rho))}} d_\alpha(x) \prod_{j=1}^{n-1} \left(\log x_j + \frac{\partial \psi}{\partial \lambda_j}(0) \right)^{\alpha_j} \cdot (\log x_n)^{\alpha_n} \right. \\ \left. + \sum_{\alpha_1 + \dots + \alpha_{n-1} + \alpha_n \leq l-1} e_\alpha(x) \prod_{j=1}^{n-1} \left(\log x_j + \frac{\partial \psi}{\partial \lambda_j}(0) \right)^{\alpha_j} \cdot (\log x_n)^{\alpha_n} \right] = 0.$$

Equating the coefficients of $\prod_{j=1}^{n-1} (\log x_j + (\partial \psi / \partial \lambda_j)(0) \log x_n)^{\alpha_j} \cdot (\log x_n)^k$ ($\alpha_1 + \dots + \alpha_{n-1} + k = l$) with zero, we obtain

$$(32) \quad A(x, \mathcal{V})x^\rho d_{\alpha_1 \dots \alpha_{n-1} k}(x) = 0.$$

If we equate the coefficients of $\prod_{j=1}^{n-1} (\log x_j + (\partial \psi / \partial \lambda_j)(0) \log x_n)^{\alpha_j} \cdot (\log x_n)^{k-d(\rho)}$ ($\alpha_1 + \dots + \alpha_{n-1} + k = l$) with zero, we get by the use of (30)

$$(33) \quad k(k-1) \dots (k-d(\rho)+1) A^{(0, \dots, 0, d(\rho))}(x, \mathcal{V})x^\rho d_{\alpha_1 \dots \alpha_{n-1} k}(x) \\ + k(k-1) \dots (k-d(\rho)+2) A^{(0, \dots, 0, d(\rho)-1)}(x, \mathcal{V})x^\rho e_{\alpha_1 \dots \alpha_{n-1} (k-1)}(x) \\ + \dots \\ + A(x, \mathcal{V})x^\rho e_{\alpha_1 \dots \alpha_{n-1} (k-d(\rho))}(x) = 0.$$

If $d_{\alpha_1 \dots \alpha_{n-1} k}(x) \equiv 0$, we obtain, from (32) and Theorem 1,

$$d_{\alpha_1 \dots \alpha_{n-1} k}(0) \neq 0.$$

Then it is easy to see that

$$x^{-\rho} A^{(0, \dots, 0, d(\rho))}(x, \mathcal{V})x^\rho d_{\alpha_1 \dots \alpha_{n-1} k}(x) \Big|_{x=0} \\ = \frac{\partial^{d(\rho)} f}{\partial \lambda_n^{d(\rho)}}(0) d_{\alpha_1 \dots \alpha_{n-1} k}(0) \neq 0.$$

This contradicts to the equality (33). Hence $d_{\alpha_1 \dots \alpha_{n-1} k}(x) \equiv 0$.

Thus it has been shown by induction on α_n that $d_\alpha(x) \equiv 0$ for all satisfy $\alpha_1 + \dots + \alpha_n = l$.

Therefore

$$(34) \quad w(x) - \sum_{\substack{\alpha_1 + \dots + \alpha_{n-1} + \alpha_n = l \\ \alpha_n \leq d(\rho) - 1}} \tilde{c}_{\alpha_1 \dots \alpha_{n-1} \alpha_n}(0) \frac{\partial^{\alpha_1 + \dots + \alpha_n}}{\partial \lambda_1^{\alpha_1} \dots \partial \lambda_{n-1}^{\alpha_{n-1}} \partial \lambda_n^{\alpha_n}} \\ \cdot [x_1^{\rho_1 + \lambda_1} \dots x_{n-1}^{\rho_{n-1} + \lambda_{n-1}} x_n^{\rho_n + \lambda_n + \psi(\lambda_1, \dots, \lambda_{n-1})} \sum_{\gamma \in \mathbb{Z}_0^n} u_\gamma(\tilde{\rho}(\lambda)) x^\gamma] \Big|_{x=0}$$

is reduced to a polynomial of $\log x_1, \dots, \log x_n$ of degree $\leq l-1$, which belongs to $\mathcal{A}^\rho$. Thus it is proved by induction on $\deg F$ that the mapping Φ_ρ is surjective.

§ 5. Generalization of Theorem 2

In Theorem 2, ρ was supposed to be an element of $\text{Reg } V - \Sigma$. In this section, we will show that this restriction can be weakened.

Let $\rho \in \text{Reg } V - \Sigma_{n-2}$ and assume that

$$\{\gamma \in \mathbb{Z}_0^n - \{0\}; \{V_{j_\rho} + \gamma\} \subset V\} = \{\gamma_{1,\rho}, \dots, \gamma_{s,\rho}\}.$$

Consider a power series

$$(35) \quad u(x, \tilde{\rho}) = x^\delta \sum_{\gamma \in \mathbb{Z}_0^n} u_\gamma(\tilde{\rho}) x^\gamma.$$

We suppose that

$$(36) \quad u_0(\tilde{\rho}) = \prod_{0 < \gamma \leq \delta} f(\tilde{\rho} + \gamma)$$

and $u_\gamma(\rho)$ are determined by the recurrence formula (9), where $\delta \in \mathbb{Z}_0^n$ is such that $\gamma_{r,\rho} \leq \delta$ ($r=1, 2, \dots, s$). Then it follows from (9) that

$$(37) \quad A(x, \mathcal{D})u(x, \tilde{\rho}) = f(\tilde{\rho})u_0(\rho)x^\delta,$$

and that

$$(38) \quad u_\gamma(\tilde{\rho}) = \frac{u_0(\tilde{\rho}) \cdot (\text{a polynomial in } \rho)}{f(\tilde{\rho} + \gamma) \prod_{0 < \alpha < \gamma} f(\tilde{\rho} + \alpha)}.$$

Therefore $u_\gamma(\rho)$ ($\gamma \in \mathbb{Z}_0^n$) are holomorphic in a neighborhood of $\tilde{\rho} = \rho$. It can be shown by a proof similar to that of Theorem 1 that the series (35) is convergent, and so $u(x, \tilde{\rho})$ is a holomorphic function of $(x, \tilde{\rho})$.

We choose a local coordinate system $(\lambda_1, \dots, \lambda_{n-1}, \lambda_n)$ around $\tilde{\rho} = \rho$ as in the proof of Theorem 2. Then, $f(\rho)$ and $u_0(\rho)$, as a function of $(\lambda_1, \dots, \lambda_{n-1}, \lambda_n)$, can be expressed as

$$(39) \quad f(\tilde{\rho}(\lambda)) = \lambda_n^{d(\rho)} \times (\text{a unit in } \mathcal{O}_0)$$

$$(40) \quad u_0(\tilde{\rho}(\lambda)) = \lambda_n^{m(\rho) - d(\rho)} \times (\text{a unit in } \mathcal{O}_0).$$

Now we define the mapping $\Phi_\rho: C[X_1, \dots, X_{n-1}]^{m(\rho)} \rightarrow \mathcal{A}^\rho$ by

$$(41) \quad \begin{aligned} & \Phi_\rho((P_j(X_1, \dots, X_{n-1}))_{j=1, \dots, m(\rho)}) \\ &= \sum_{j=1}^{m(\rho)} P_j \left(\frac{\partial}{\partial \lambda_1}, \dots, \frac{\partial}{\partial \lambda_{n-1}} \right) \frac{\partial^{j-1}}{\partial \lambda_n^{j-1}} u(x, \tilde{\rho}(\lambda))|_{\lambda=0}. \end{aligned}$$

Φ_ρ is well-defined by (37), (39) and (40).

We can prove that Φ_ρ is surjective, although its proof is more complicated than that of Theorem 2.

Thus we arrive at the following theorem

THEOREM 3. *Suppose that $\rho \in \text{Reg } V - \Sigma_{n-2}$. Then $\mathcal{A}^\rho$ is isomorphic to $C[x_1, \dots, X_{n-1}]^{m(\rho)}$.*

§ 6. Existence of local holomorphic solution and application to the Goursat problem

Suppose $A(x, \vartheta)$ satisfies the Poincaré condition. Then the following theorem holds, which can be proved by the method of majorants as in the proof of Theorem 1.

THEOREM 4 (cf. Bengel-Gerard [1]). i) *If $V \cap Z_0^n = \phi$, there exists a polydisc $\Delta(0; r_0)$ such that*

$$(42) \quad A(x, \vartheta): \mathcal{O}(\Delta(0; r)) \xrightarrow{\sim} \mathcal{O}(\Delta(0; r))$$

is a topological isomorphism for any $\Delta(0; r) \subset \Delta(0; r_0)$.

ii) *If $V \cap Z_0^n \neq \phi$, for k_0 being the maximum of the length of the points in $V \cap Z_0^n$, there exists a polydisc $\Delta(0; r_0)$ such that $A(x, \vartheta)$ induces a topological isomorphism*

$$(43) \quad A(x, \vartheta)|_{m^k(\Delta(0; r))}: m^k(\Delta(0; r)) \xrightarrow{\sim} m^k(\Delta(0; r))$$

for any $k \geq k_0 + 1$ and $\Delta(0; r) \subset \Delta(0; r_0)$, where $m \subset \mathcal{O}_0$ is the maximal ideal and $m^k(\Delta(0; r)) = m^k \cap \mathcal{O}(\Delta(0; r))$.

We will apply this result to the Goursat problem. Consider a differential operator

$$(44) \quad P = \frac{\partial^{|\beta|}}{\partial x^\beta} + \sum_{\gamma < \beta} a_\gamma(x) \frac{\partial^{|\gamma|}}{\partial x^\gamma}$$

with $a_\gamma(x) \in \mathcal{O}_0$. Suppose $\beta \geq (1, \dots, 1)$.

We note that the differential operator Px^β satisfies assumptions of Theorem 4, i).

Let $\mathcal{S}_r^\beta = \{(f_{i,r_i})_{r_i=0,1,\dots,\beta_i-1, i=1,\dots,n} \in \bigoplus_{i=1}^n \bigoplus_{r_i=0}^{\beta_i-1} (\Delta(0; r_1, \dots, r_{i-1}, r_{i+1}, \dots, r^n));$
 $[\partial^{\gamma_j} f_{i,r_i} / \partial x_j^{\gamma_j}]_{x_j=0} = [\partial^{\gamma_j} f_{j,r_j} / \partial x_i^{\gamma_j}]_{x_i=0}\}.$

Then the following theorem is obtained as an application of Theorem 4 to the Goursat problem.

THEOREM 5. *There exists a polydisc $\Delta(0; r_0)$ such that for any polydisc $\Delta(0; r) \subset \Delta(0; r_0)$ the mapping*

$$(45) \quad \mathcal{O}(\Delta(0; r)) \longrightarrow \mathcal{O}(\Delta(0; r)) \oplus \mathcal{I}_r^\beta$$

$$u \longrightarrow \left(Pu, \left. \frac{\partial^{r_i} u}{\partial x_i^{r_i}} \right|_{x_i=0} \right)_{\gamma_i=0, 1, \dots, \beta_i-1, i=1, \dots, n}$$

is a topological isomorphism.

This theorem is already known (cf. Hörmander [2]), but the method used here may be of some interest. Oshima [6] applied some type of singular differential equations to prove the Cauchy-Kowalevsky theorem for linear equations including the fact that there is no loss of the domain of convergence. Our method shows a close relation between singular differential operators treated here in this paper and the Goursat problem.

More generally, consider a differential operator

$$(46) \quad Q = \frac{\partial^{|\beta|}}{\partial x^\beta} + \sum_{|r| < |\beta|} a_r(x) \frac{\partial^{|r|}}{\partial x^r}.$$

To obtain a result similar to Theorem 5 for this operator, we use Theorem 2.2 of Kashiwara-Kawai-Sjöstrand [4]. The same argument as above for the operator Qx^β leads us to the following theorem.

THEOREM 6. *There exists a polydisc $\Delta(0; r)$ such that for any polydisc $\Delta(0; r) \subset \Delta(0; r_0)$ the mapping*

$$(47) \quad (\Delta(0; r)) \longrightarrow (\Delta(0; r)) \oplus \mathcal{I}_r^\beta$$

$$u \longrightarrow \left(Qu, \left. \frac{\partial^{r_i} u}{\partial x_i^{r_i}} \right|_{x_i=0} \right)_{\gamma_i=0, 1, \dots, \beta_i-1, i=1, \dots, n}$$

is a topological isomorphism.

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