Non-integrated defect relation for meromorphic maps of complete Kähler manifolds into $P^{N_1}(C) \times \cdots \times P^{N_k}(C)$

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§ 1. Introduction

Relating to the study of value distribution of the Gauss maps of a complete minimal surfaces in $R^m$, the author introduced the new notion of the non-integrated defect for a holomorphic map of an open Riemann surface into $P^N(C)$ and obtained some results analogous to the classical defect relation in the previous paper [6]. The purposes of this paper are to generalize these results to the case of a meromorphic map of a complete Kähler manifold into $P^{N_1} \cdots N_k(C):=P^{N_1}(C) \times \cdots \times P^{N_k}(C)$, and to give an application to the study of the Gauss map of a closed regular submanifold of $C^m$.

Let $f$ be a meromorphic map of an $n$-dimensional connected complex manifold $M$ into $P^N(C)$, $\mu_0$ be a positive integer and $H$ be a hyperplane in $P^N(C)$ with $f(M) \not\subset H$. We denote the intersection multiplicity of the image of $f$ and $H$ at $f(p)$ by $v^f(H)(p)$ and the pull-back of the normalized Fubini-Study metric form $\Omega$ on $P^N(C)$ by $\Omega_f$. The non-integrated defect of $H$ cut by $\mu_0$ is defined by

$$\delta^f_\mu(H):=1-\inf\{\gamma \geq 0: \gamma \text{ satisfies condition } (*)\}.$$  

Here, condition $(*)$ means that there exists a bounded nonnegative function $h$ on $M$ with zeros of order not less than $\min (v^f(H), \mu_0)$ such that

$$\gamma \Omega_f + dd^c \log h \leq [\min (v^f(H), \mu_0)]$$

where $d^c = (\sqrt{-1}/4\pi)(\bar{\partial} - \partial)$ and we mean by $[\nu]$ the (1,1)-current associated with a divisor $\nu$.

Similarly to the classical defect, we have always $0 \leq \delta^f_\mu(H) \leq 1$, and $\delta^f_\mu(H) = 1$ if $f(M) \cap H = \emptyset$. Moreover, if $v^f(H) \geq \mu$ for every $p \in f^{-1}(H)$, then

$$\delta^f_\mu(H) \geq 1 - \frac{\mu_0}{\mu}.$$  

We shall study in the following sections value distributions of
meromorphic maps of a Kähler manifold into $\mathbb{P}^n(C)$ and give some non-integrated defect relations under suitable conditions. We state here only the main result restricted to the case of nondegenerate meromorphic maps into $\mathbb{P}^n(C)$.

For a Kähler manifold $M$ with Kähler form $\omega=(\sqrt{-1}/2)\sum_i h_{ij} dz_i \wedge d\bar{z}_j$, we define
\[ \text{Ric } \omega = dd^c \log (\det (h_{ij})). \]

For a nondegenerate meromorphic map $f: M \to \mathbb{P}^n(C)$, we shall give the following non-integrated defect relation.

**Theorem 1.1.** Assume that $M$ is complete and the universal covering of $M$ is biholomorphically isomorphic to a ball in $\mathbb{C}^n$. For some $\rho \geq 0$, if there exists a bounded continuous function $h \geq 0$ on $M$ such that
\[ \rho \Omega + dd^c \log h \geq \text{Ric } \omega, \]

then
\[ \sum_{j=1}^q \delta_\rho(H_j) \leq N+1+\rho N(N+1) \]

for arbitrary hyperplanes $H_1, \ldots, H_q$ in $\mathbb{P}^n(C)$ in general position.

We next study a regular submanifold $M$ of $\mathbb{C}^m$ immersed by a holomorphic map $f$. By definition, the Gauss map of $M$ is the map $G$ of $M$ into the Grassmannian manifold $G(n, m)$ of all $n$-dimensional linear subspaces of $\mathbb{C}^m$ such that for each $p \in M$ $G(p)$ is the point in $G(n, m)$ corresponding to the tangent space of $M$ at $p$. We may consider $G(n, m)$ as a submanifold of $\mathbb{P}(\wedge^n \mathbb{C}^m)$ and so $G$ as a map into $\mathbb{P}^n(C)$, where $N=\binom{m}{n}-1$. As an application of Theorem 1.1, we shall prove

**Theorem 1.2.** Let $M$ be a closed regular submanifold of $\mathbb{C}^m$ whose universal covering is biholomorphically isomorphic to a ball in $\mathbb{C}^n$. If the Gauss map $G$ of $M$ is nondegenerate, then
\[ \sum_{j=1}^q \delta_\rho(H_j) \leq \binom{m}{n}^2 \]

for arbitrary hyperplanes $H_1, \ldots, H_q$ in general position.

Taking holomorphic local coordinates $z=(z_1, \ldots, z_n)$ arbitrarily, we define the generalized absolute Jacobian of a holomorphic map $f=(f_1, \ldots, f_m): M \to \mathbb{C}^m$ by
We shall give another type of non-integrated defect relation for the Gauss map as follows.

**Theorem 1.3.** In the same situation as in Theorem 1.2, if $J_1 f \neq 0$ for some holomorphic local coordinates $z$, then

$$
\sum_{j=1}^{q} \Omega_{m-n}(H_j) \leq \binom{m}{n} + 2n \binom{m}{n+1}
$$

for arbitrary hyperplanes $H_1, \ldots, H_q$ in general position.

We shall first recall some properties of characteristic functions and counting functions in §2 and §3 give a new type of the lemma of the logarithmic derivative in §3. We shall next discuss some types of degeneracy of meromorphic maps in §4. The main theorem will be stated in §5 and proven in §6. In the last section, we shall give applications to the study of the Gauss maps of regular submanifolds of $\mathbb{C}^{\mu}$.

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### §2. Preliminaries

We set $|z| = (|z_1|^2 + \cdots + |z_n|^2)^{1/2}$ for $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$ and define $B(r) := \{z \in \mathbb{C}^n : |z| < r\}$, $S(r) := \{z \in \mathbb{C}^n : |z| = r\}$ for $0 < r < +\infty$, where we mean $B(\infty) = \mathbb{C}^n$ and $S(\infty) = \emptyset$. Define

$$
\sigma_n := d^n \log |z|^2 \wedge (dd^c \log |z|^2)^{n-1} \quad \text{on } \mathbb{C}^n - \{0\},
$$

$$
u_1 := (dd^c |z|^2)^{1/2}.
$$

Let $g$ be a nonzero holomorphic function on a domain $D$ in $\mathbb{C}^n$. For each $a \in D$, expanding $g$ as

$$
g = \sum_m P_m(z-a)
$$

with homogeneous polynomials $P_m$ of degree $m$ around $a$, we define

$$
\nu_g(a) = \min \{m : P_m \neq 0\}.
$$

By a divisor on a domain $D$ we mean an integer-valued function $\nu$ on $D$ such that it can be locally written as $\nu = \nu_g - \nu_h$ with nonzero holomorphic functions $g, h$. We set

$$
|\nu| = \{z \in D : \nu(z) \neq 0\} \cap D,
$$

which is a purely $(n-1)$-dimensional analytic set if $|\nu| \neq \emptyset$. 
DEFINITION 2.1. The counting function of a divisor $\nu$ on $B(R_0)$ ($0 < R_0 \leq +\infty$) is defined by

$$n_\nu(r) = \begin{cases} \frac{1}{r^{2n-2}} \int_{|z| \leq r} \nu v_{n-1} & \text{if } n > 1 \\ \sum_{z \in \partial B(r)} \nu(z) & \text{if } n = 1 \end{cases}$$

and the valence function of $\nu$ is defined by

$$N_\nu(r, r_0) = \int_{r_0}^r \frac{n_\nu(t)}{t} \, dt$$

for $0 < r_0 < r < R_0$.

Let $\varphi$ (\$ \neq \text{const} \$) be a meromorphic function on $B(R_0)$. Representing $\varphi$ as $\varphi = g/h$ with nonzero holomorphic functions $g$, $h$ such that $\operatorname{codim} \{g = h = 0\} \geq 2$, we define $\nu_\varphi = \nu_\varphi$, $\nu_\varphi^a = \nu_\varphi^a - \nu_\varphi^a$ for $a \in \mathbb{C}$ and $\nu_\varphi = \nu_\varphi - \nu_\varphi$. For brevity, set $N_\varphi^a(r, r_0) := N_\varphi^a(r, r_0)$ for each $a \in P^1(\mathbb{C}) = \mathbb{C} \cup \{\infty\}$ and $N_\varphi(r, r_0) = N_\varphi(r, r_0)$.

(2.2) (Jensen’s formula) For $0 < r_0 < r < R_0$, we have

$$N_\varphi(r, r_0) = \int_{s(r_0)} \log |\varphi| \, \sigma_n - \int_{s(r_0)} \log |\varphi| \, \sigma_n.$$ 

For the proof, see [10], p. 248.

Let $f$ be a meromorphic map of $B(R_0)$ ($0 < R_0 \leq +\infty$) into $P^N(\mathbb{C})$. For arbitrarily fixed homogeneous coordinates $(w_1 : \cdots : w_{N+1})$ on $P^N(\mathbb{C})$, we take a reduced representation $f = (f_1 : \cdots : f_{N+1})$ on $B(R_0)$, which means that each $f_i$ is a holomorphic function on $B(R_0)$ and

$$f(z) = (f_1(z) : \cdots : f_{N+1}(z))$$

outside the analytic set $\{f_1 = \cdots = f_{N+1} = 0\}$ of codimension $\geq 2$. Set

$$||f|| = (|f_1|^2 + \cdots + |f_{N+1}|^2)^{1/2}.$$ 

Then, the pull-back of the normalized Fubini-Study metric form $\Omega$ on $P^N(\mathbb{C})$ by $f$ is given by

$$\Omega_f = dd^c \log ||f||^2.$$ 

DEFINITION 2.3. The characteristic function of $f$ is defined by

$$T_f(r, r_0) = \int_{r_0}^r \frac{dt}{t^{2n-1}} \int_{B(t)} \Omega_f \wedge v_{n-1} \quad (0 < r_0 < r < R_0).$$

We then have
For the proof, see [10], pp. 251-255.

In the following, we denote some constant not depending on each \( r \) with \( r_0 < r < R_0 \) by the common letter \( K \) even when it should be replaced by a new constant.

For \( (a_1, \ldots, a_{N+1}) \in C^{N+1} \) (\( i = 1, 2 \)), assume that \( F_i = a_1 f_1 + \cdots + a_{N+1} f_{N+1} \neq 0 \) and set \( \varphi = F_1 / F_2 \). Then, considering \( \varphi \) a meromorphic map into \( P^1(C) \), we see

\[
T_\varphi (r, r_0) \leq T_f (r, r_0) + K.
\]

In fact, for a reduced representation \( \varphi = (g: h) \) on \( B(R_0) \), \( k = F_1 / g = F_2 / h \) is a nonzero holomorphic function. Then,

\[
||\varphi||k^2 = (|g|^2 + |h|^2) k^2 = |F_1|^2 + |F_2|^2 \leq K||f||^2
\]

and so

\[
\int_{S(r)} \log ||\varphi|| \sigma + \int_{S(r)} \log |k| \sigma + \int_{S(r_0)} \log ||f|| \sigma + K.
\]

Since

\[
\int_{S(r)} \log |k| \sigma = N_\varphi (r, r_0) + \int_{S(r_0)} \log |k| \sigma \geq \int_{S(r_0)} \log |k| \sigma,
\]

we see (2.5).

(2.6) Let \( \varphi_1, \varphi_2 \) be nonzero meromorphic functions on \( B(R_0) \). Then,

\[
T_{\varphi_1 \varphi_2} (r, r_0) \leq T_{\varphi_1} (r, r_0) + T_{\varphi_2} (r, r_0) + K.
\]

In fact, we take reduced representations \( \varphi_i = (g_i: h_i) \) (\( i = 1, 2 \)), and \( \varphi_1 \varphi_2 = (g_3: h_3) \). Then, \( k = g_1 g_2 / h_1 h_2 / h_3 \) is holomorphic. Since

\[
\log (|g_1|^2 + |h_1|^2)^{1/2} + \log |k| \leq \log (|g_1|^2 + |h_1|^2)^{1/2} + \log (|g_3|^2 + |h_3|^2)^{1/2},
\]

\[
\int_{S(r)} \log ||\varphi_1 \varphi_2|| \sigma \leq \int_{S(r)} \log ||\varphi_1|| \sigma + \int_{S(r)} \log ||\varphi_2|| \sigma + \int_{S(r)} \log |k| \sigma \leq N_\varphi (r, r_0) + K.
\]

This gives (2.6).

Let \( \varphi \) be a nonzero meromorphic function on \( B(R_0) \) with a reduced representation \( \varphi = g / h \) (\( = (g: h) \)).

(2.7)

\[
\int_{S(r)} |\log |\varphi|| \sigma \leq 2T_f (r, r_0) + K.
\]
In fact, since
\[ |\log |\varphi|| = 2 \max (\log |g|, \log |h|) - \log |g| - \log |h| \]
\[ \leq 2 \log \|\varphi\| - \log |g| - \log |h|, \]
we have
\[ \int_{S(r)} |\log |\varphi|| \sigma_n \]
\[ \leq 2 \int_{S(r)} \log \|\varphi\| \sigma_n - \int_{S(r)} \log |g| \sigma_n - \int_{S(r)} \log |h| \sigma_n \]
\[ \leq 2T_\varphi(r, r_0) - N_\varphi(r, r_0) - N_\varphi(r, r_0) + K \]
\[ \leq 2T_\varphi(r, r_0) + K. \]

(2.8) \( N_\varphi(r, r_0) \leq T_\varphi(r, r_0) + K \) for every \( a \in P^1(C) \).

In fact, for \( a = (a_1 : a_2) \in P^1(C) \),
\[ N_\varphi(a, r_0) = \int_{S(r)} \log |a_2g - a_1h| \sigma_n - \int_{S(r_0)} \log |a_2g - a_1h| \sigma_n \]
\[ \leq \int_{S(r)} \log \|\varphi\| \sigma_n + K \]
\[ \leq T_\varphi(r, r_0) + K. \]

(2.9) \[ \left| T_\varphi(r, r_0) - \left( \int_{S(r)} \log^+ |\varphi| \sigma_n + N_\varphi^\alpha(r, r_0) \right) \right| \leq K, \]

where \( \log^+ x = \max (\log x, 0) \) for \( x \geq 0 \).

In fact, this is a direct result of the inequality
\[ \log \|\varphi\| \leq \log (2 \max (|g|, |h|)) = \log^+ \left| \frac{g}{h} \right| + \log |h| + \log 2 \]
\[ \leq \log \|\varphi\| + K \]
and the monotonicity of the integral over \( S(r) \).

(2.10) Let \( 0 < r_0 < r < R < R_0 \) and set \( \rho := (r + R)/2 \). Then
\[ n_{\varphi}(\rho) \leq \frac{2R}{R-r} (T_\varphi(R, r_0) + K). \]

In fact, by Definition 2.1,
\[ N_\varphi^\alpha(R, r_0) = \int_{r_0}^\rho n_\varphi(t) \frac{dt}{t} \geq \int_{r_0}^\rho n_\varphi(t) \frac{dt}{t} \geq n_\varphi(\rho) \frac{R - \rho}{R} \]
and so
§ 3. The lemma of the logarithmic derivative

Let \( \varphi(z_1, \ldots, z_n) \) be a nonzero meromorphic function on \( B(R_0) (0 < R_0 \leq +\infty) \). For a set \( \alpha=(\alpha_1, \ldots, \alpha_n) \) of integers \( \alpha_i \geq 0 \) and \( z=(z_1, \ldots, z_n) \in \mathbb{C}^n \), we set \( \alpha!:=\alpha_1! \cdots \alpha_n! \), \( |\alpha|:=\alpha_1 + \cdots + \alpha_n \), \( z^\alpha:=z_1^{\alpha_1} \cdots z_n^{\alpha_n} \) and \( D^\alpha \varphi = D_1^{\alpha_1} \cdots D_n^{\alpha_n} \varphi \), where \( D_i \varphi := (\partial / \partial z_i) \varphi \). The purpose of this section is to prove the following lemma of the logarithmic derivative.

**Theorem 3.1.** Let \( \alpha=(\alpha_1, \ldots, \alpha_n) \neq (0, \ldots, 0) \), \( 0 < r_0 < R_0 \) and take positive numbers \( p, p' \) such that \( 0 < p|\alpha| < p' < 1 \). Then, for \( r_0 < r < R < R_0 \),

\[
\int_{S(r)} |z^\alpha (D^\alpha \varphi / \varphi)(z)|^p \sigma_n(z) \lesssim K \left( \frac{R^{2n-1}}{R-r} T_p(R, r_0) \right)^{p'},
\]

where \( K \) is a constant not depending on each \( r \) and \( R \).

Before proving this, we give the following Corollary to Theorem 3.1, which is essentially the same as the lemma of the logarithmic derivative in several complex variables given by Vitter (\[12\]).

**Corollary 3.2.** Let \( \alpha=(\alpha_1, \ldots, \alpha_n) \neq (0, \ldots, 0) \) and \( 0 < r_0 < R_0 \). For \( r_0 < r < R < R_0 \), we have

\[
\int_{S(r)} \log^+ |(D^\alpha \varphi / \varphi)(z)| \sigma_n(z) \lesssim K \log^+ \left( \frac{R^{2n-1}}{R-r} T_p(R, r_0) \right).
\]

To conclude this from Theorem 3.1, we use the following result by Biancofiore and Stoll (\[2\], Lemma 3.5).

(3.3) Let \( h \geq 0 \) be an integrable function on \( S(r) \). Then

\[
\int_{S(r)} \log^+ h \sigma_n \leq \log^+ \int_{S(r)} h \sigma_n + \log 2.
\]

To apply Theorem 3.1, take \( p, p' \) such that \( 0 < p|\alpha| < p' < 1 \). By (3.3) and Theorem 3.1 we have

\[
\int_{S(r)} \log^+ |D^\alpha \varphi / \varphi| \sigma_n \leq \frac{1}{p} \int_{S(r)} \log^+ |z^\alpha (D^\alpha \varphi / \varphi)(z)|^p \sigma_n(z) + K
\]
Thus, we conclude Corollary 3.2.

For the proof of Theorem 3.1, we recall some known facts.

**Lemma 3.4** ([6], Lemma 2.5). Let \( r > 0 \) and \( 0 < p < 1 \). For every \( a \in C \), we have

\[
\frac{1}{2\pi} \int_{0}^{2\pi} \frac{r^{p}}{|re^{i\theta} - a|^{p}} \, d\theta \leq \frac{2 - p}{2(1 - p)}.
\]

**Proposition 3.5** ([7], p. 1). Let \( \varphi \) be a nonzero meromorphic function on \( \{u \in C : |u| < R_{0}\} \) and \( 0 < r < R < R_{0} \). Take \( z \in C \) such that \( |z| = r \) and \( \varphi(z) \neq 0, \infty \). Then,

\[
\log|\varphi(z)| = \frac{1}{2\pi} \int_{0}^{2\pi} \log|\varphi(Re^{i\theta})| Re\left(\frac{Re^{i\theta} + z}{Re^{i\theta} - z}\right) d\phi + \sum_{u \in R} \nu_{\varphi}(u) \log \left| \frac{R(z - u)}{R^{2} - \bar{u}z} \right|.
\]

For \( z = (z_{1}, \ldots, z_{n}) \in C^{n} \) set \( \eta = (z_{1}, \ldots, z_{n-1}) \), \( \zeta = z_{n} \), \( z = (\eta, \zeta) \) and \( |\eta| = (|z_{1}|^{2} + \cdots + |z_{n-1}|^{2})^{1/2} \).

**Lemma 3.6** ([2], p. 35). Let \( h \) be an integrable function on \( S(r) \) \((r > 0)\). Then

\[
\int_{S(r)} h_{\sigma} = \frac{1}{r^{2n-2}} \int_{B(r)} v_{n-1}(\eta) \int_{|\zeta| = \sqrt{r^{2} - |\eta|^{2}}} h(\eta, \zeta) \sigma(\zeta),
\]

where \( \bar{B}(r) := \{\eta \in C^{n-1} : |\eta| < r\} \).

For a nonzero meromorphic function \( \varphi \) on \( B(R_{0}) \) there exists a subset \( E \) of \( \bar{B}(R_{0}) \) of measure zero such that for each \( \eta \in \bar{B}(R_{0}) \setminus E \) a meromorphic function \( (\varphi_{\eta})(\zeta) = \varphi(\eta, \zeta) \) is well-defined on \( \{\zeta \in C : |\zeta| < \sqrt{R_{0}^{2} - |\eta|^{2}}\} \).

**Lemma 3.7** ([2], p. 37). For each \( a \in P(C) \) and \( 0 < r < R_{0} \), we have

\[
\frac{1}{r^{2n-2}} \int_{\bar{B}(r) \setminus E} v_{n-1}(\eta) n_{\varphi_{\eta}}(\sqrt{r^{2} - |\eta|^{2}}) \varphi_{\eta}(\eta) \leq n_{\varphi}(r).
\]

We now prove the following

**Lemma 3.8.** Let \( 0 < \rho < 1 \) and \( 0 < r < \rho < R_{0} \). For every \( \eta \in \bar{B}(r) \setminus E \), we have
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\[
\int_{|\zeta|<\sqrt{r^2-|\eta|^2}} \left| \zeta \left( \frac{\partial \varphi}{\partial \zeta} / \varphi \right)(\eta, \zeta) \right|^\beta \sigma_i(\zeta)
\]
\[\leq \left( \frac{\rho}{\rho - r} \right) \int_{|\zeta|<\sqrt{r^2-|\eta|^2}} \left| \log |\varphi(\eta, \zeta)| \right| \sigma_i(\zeta) \right|^\beta
\]
\[+ K(n, \rho(\sqrt{\rho^2-|\eta|^2} + n, \rho(\sqrt{\rho^2-|\eta|^2})).
\]

**Proof.** We may assume that \( \varphi(\zeta) \neq 0 \) on \( \{ \zeta : |\zeta| = \sqrt{\rho^2-|\eta|^2} \} \), because each term is continuous on \( \rho \). By differentiating the equation in Proposition 3.5 applied to the function \( \varphi|\eta \) and \( R = \bar{\rho} = \sqrt{\rho^2-|\eta|^2} \), we obtain
\[
\left( \frac{\partial \varphi}{\partial \zeta} / \varphi \right)(\eta, \zeta) = \frac{\bar{\rho}}{\pi} \int_0^{2\pi} \frac{\log |\varphi(\eta, \bar{\rho} e^{i \theta})| e^{i \theta}}{(\bar{\rho} e^{i \theta} - \zeta)^2} d\theta
\]
\[+ \sum_{|u| \leq \bar{\rho}} \nu_{\psi}^o(u) \left\{ \frac{1}{\zeta - \bar{u}} \frac{u - \zeta}{\bar{\rho}^2 - \bar{u} \bar{\zeta}} \right\}.
\]

Therefore,
\[
\left| \zeta \left( \frac{\partial \varphi}{\partial \zeta} / \varphi \right)(\eta, \zeta) \right|^\beta \leq \left( 2\bar{\rho} |\zeta| \right) \int_{|u| \leq \bar{\rho}} \frac{\log |\varphi(\eta, \zeta)|}{|u - \zeta|^2} \sigma_i(u) \left| \sigma_i(u) \right|^\beta
\]
\[+ \sum_{|u| \leq \bar{\rho}} (\nu_{\psi}^o(u) + \nu_{\psi}^m(u)) \left( \frac{|\zeta|^\beta}{|u - \zeta|^2} + \frac{|\zeta|^\beta}{(\bar{\rho}^2 - |\eta|^2)} \right) \sigma_i(\zeta).
\]

Integrating this and using Lemma 3.4, we see
\[
\int_{|\zeta|<\sqrt{r^2-|\eta|^2}} \left| \zeta \left( \frac{\partial \varphi}{\partial \zeta} / \varphi \right)(\eta, \zeta) \right|^\beta \sigma_i(\zeta)
\]
\[\leq \left( 2\bar{\rho} \int_{|\zeta|<\sqrt{r^2-|\eta|^2}} |\zeta| \sigma_i(\zeta) \int_{|u| \leq \bar{\rho}} \frac{\log |\varphi(\eta, \zeta)|}{|u - \zeta|^2} \sigma_i(u) \right) \sigma_i(\zeta)
\]
\[+ \sum_{|u| \leq \bar{\rho}} (\nu_{\psi}^o(u) + \nu_{\psi}^m(u)) \int_{|\zeta|<\sqrt{r^2-|\eta|^2}} \left( \frac{|\zeta|^\beta}{|u - \zeta|^2} + \frac{|\zeta|^\beta}{(\bar{\rho}^2 - |\eta|^2)} \right) \sigma_i(\zeta)
\]
\[\leq \left( 2\bar{\rho} \int_{|u| \leq \bar{\rho}} \log |\varphi(\eta, \zeta)| \sigma_i(u) \right) \bigg|_{|\zeta|<\sqrt{r^2-|\eta|^2}} \frac{1}{|u - \zeta|^2} \sigma_i(\zeta)
\]
\[+ K \left( \sum_{|u| \leq \bar{\rho}} (\nu_{\psi}^o(u) + \nu_{\psi}^m(u)) \right).
\]

On the other hand, we know
\[
\int_{|\zeta|<\sqrt{r^2-|\eta|^2}} \frac{1}{|u - \zeta|^2} \sigma_i(\zeta) = \frac{1}{\bar{\rho}^2 - (\bar{\rho}^2 - |\eta|^2)} = \frac{1}{\rho^2 - r^2}
\]
for every \( u \) with \( |u| = \bar{\rho} \). From this we can conclude that
This gives Lemma 3.8.

**Proof of Theorem 3.1 for the case $|\alpha|=1$.** We prove Theorem 3.1 by induction on $|\alpha|$. We first consider the case $|\alpha|=1$. Without loss of generality, we may assume $D^* = D_n$. Let $r < r < R < R_0$, $0 < p < p' < 1$ and set $\tilde{\rho} = p/p'$, $\rho = (r + R)/2$. Since each pole of $D_n\varphi/\varphi$ is of order $\leq 1$, $z_n(D_n\varphi/\varphi)(z)\tilde{\rho}$ is integrable on $S(r)$. By Lemmas 3.1, 3.8, 3.7 and the Hölder's inequality, we get

\[
\int_{S(r)} |z_n(D_n\varphi/\varphi)(z)^\tilde{\rho} \sigma_n(z)\\
\leq \frac{1}{r^{2n-2}} \left( \frac{\rho}{\rho - r} \right)^\delta \left( \int_{|\eta| \leq r} v_{n-1}(\eta) \right)^{1-\delta} \\
\times \left( \int_{|\eta| \leq r} v_{n-1}(\eta) \int_{|\xi| \leq \sqrt{\rho^2 - |\eta|^2}} |\log |\varphi(\eta, \xi)||\sigma_n(\xi)\right)^\delta \\
+ \frac{K}{r^{2n-2}} \left( n_{\rho,1}(\sqrt{\rho^2 - |\eta|^2}) + n_{\rho,1}(\sqrt{\rho^2 - |\eta|^2})v_{n-1}(\eta) \right) \\
\leq \left( \frac{\rho}{\rho - r} \int_{S(r)} |\log |\varphi||\sigma_n\right)^\delta + K \left( \frac{\rho}{r} \right)^{2n-2} (n_{\rho,1}(\rho) + n_{\rho,1}(\rho)) \\
\]  

Moreover, using (2.7) and (2.10), we conclude

\[
\int_{S(r)} |z_n(D_n\varphi/\varphi)(z)^\tilde{\rho} \sigma_n(z)\\
\leq \left( \int_{S(r)} |z_n(D_n\varphi/\varphi)(z)^\tilde{\rho} \sigma_n(z) \right)^{\tilde{\rho}'} \\
\leq \left( \frac{\rho}{\rho - r} \int_{S(r)} |\log |\varphi||\sigma_n\right)^{\tilde{\rho}'} + K \left( \frac{\rho}{r} \right)^{2n-2} (n_{\rho,1}(\rho)^{\tilde{\rho}'} + n_{\rho,1}(\rho)^{\tilde{\rho}'}) \\
\leq \left( \frac{2R}{R - r} \int_{S(r)} |\log |\varphi||\sigma_n\right)^{\tilde{\rho}'} + K \left( \frac{4R^{2n-1}}{R - r} (T_s(R, r_0) + K)\right)^{\tilde{\rho}'} \\
\leq K \left( \frac{R^{2n-1}}{R - r} T_s(r, r_0) \right)^{\tilde{\rho}'}.
\]

This shows Theorem 3.1 for the case $D^* = D_n$. 
To complete the proof of Theorem 3.1, we need

**Lemma 3.9.** Let \( \varphi \) be a nonzero meromorphic function on \( B(R_0) \) and \( 0 < r_0 < r < R < R_0 \). Then

\[
T_{D_{\varphi}^n}(r, r_0) \leq 3T_\varphi(r, r_0) + K \log^+ \left( \frac{R^{2n-1}}{R-r} T_\varphi(R, r_0) \right) \quad \text{for } i = 1, 2, \ldots, n.
\]

**Proof.** By (2.6) and (2.9), we see

\[
T_{D_{\varphi}^n}(r, r_0) \leq T_{D_{\varphi}^n/p}(r, r_0) + T_\varphi(r, r_0) + K
\]

\[
\leq \int_{S(r)} \log^+ |D_{\varphi}^n| \sigma_n + N_{D_{\varphi}^n/p}(r, r_0) + T_\varphi(r, r_0) + K.
\]

On the other hand, (2.8) gives

\[
N_{D_{\varphi}^n/p}(r, r_0) \leq N_{\varphi}(r, r_0) + N_\varphi(r, r_0)
\]

\[
\leq 2T_\varphi(r, r_0) + K.
\]

Since we have proved Theorem 3.1 for the case \(|\alpha| = 1\), we may use Corollary 3.2 in this case. We have thus

\[
\int_{S(r)} \log^+ |D_{\varphi}^n| \sigma_n \leq K \log^+ \left( \frac{R^{2n-1}}{R-r} T_\varphi(R, r_0) \right).
\]

From these facts, we have easily Lemma 3.9.

**Proof of Theorem 3.1 for the general case.** Assume that Theorem 3.1 holds for the case \(|\alpha| \leq \kappa\). Take an arbitrary \( \alpha \) with \(|\alpha| = \kappa + 1\) and write

\[
D^n = D^nD_\alpha, \quad \text{where } 1 \leq i \leq n \quad \text{and } |\alpha'| = \kappa.
\]

Then \( D^n\varphi/\varphi = (D_i\varphi/\varphi)(D^n(D_i\varphi)/D_i\varphi) \),

\[
z^n = z_i z_i' \quad \text{and } |z| p = (|\alpha'| + 1)p < p' < 1.
\]

Set \( p_1 := 1/(|\alpha'| + 1) \) and \( p_2 := |\alpha'|/(|\alpha'| + 1) \).

By the Hölder's inequality and the induction hypothesis, we have

\[
\int_{S(r)} |z^n(D^n\varphi/\varphi)(z)|^p \sigma_n(z)
\]

\[
\leq \left( \int_{S(r)} |z_i(D_i\varphi/\varphi)(z)|^{p/p_1} \sigma_n(z) \right)^{p_1} \left( \int_{S(r)} |z^n(D^n(D_i\varphi)/D_i\varphi)(z)|^{p/p_2} \sigma_n(z) \right)^{p_2}
\]

\[
\leq K \left( \frac{R^{2n-1}}{R-r} T_\varphi(R, r_0) \right)^{p_1} \left( \frac{R^{2n-1}}{R-r} T_{D_{\varphi}^n}(R, r_0) \right)^{p_2}.
\]

For any \( \varepsilon > 0 \) there exists a positive constant \( K_\varepsilon \) such that

\[
\log^+ \left( \frac{R^{2n-1}}{R-r} T_\varphi(R, r_0) \right) \leq K_\varepsilon \left( \frac{R^{2n-1}}{R-r} T_{D_{\varphi}^n}(R, r_0) \right)^\varepsilon.
\]

Apply the above argument for slightly smaller \( p' \) if necessary. We can conclude
\[ \int_{S(r)} |z^2(D^r\varphi)(z)|^p |\varphi(z)|^p \leq K \left( \frac{R^{2n-1}}{R-r} T_s(R, r) \right)^p \]

by the help of Lemma 3.9. This completes the proof of Theorem 3.1.

§ 4. Nondegeneracy of meromorphic maps and generalized Wronskians

Let \( f \) be a meromorphic map of a connected \( n \)-dimensional complex manifold \( M \) into \( P^N(\mathbb{C}) \) and choose a Cousin II domain \( U \) where holomorphic local coordinates \( z_1, \ldots, z_n \) are defined. Taking a reduced representation \( f=(f_1: \cdots : f_{N+1}) \) on \( U \), we consider the system of \( N+1 \) holomorphic functions \((f_1, \ldots, f_{N+1})\), which we denote by the same letter \( f \).

We denote by \( \mathcal{M}_p \) the field of all germs of meromorphic functions at a point \( p \in M \) and \( \mathcal{F}^r \) the \( \mathcal{M}_p \)-submodule of \( \mathcal{M}_p^{N+1} \) generated by \( \{D^a f: |a| \leq \kappa\} \), where \( D^a f=(D^a f_1, \ldots, D^a f_{N+1}) \in \mathcal{M}_p^{N+1} \) and \( D^a f \) has \( \kappa \) distinct factors in each factor. Set \( \mathcal{F}=\bigcup \mathcal{F}^r \), which is the \( \mathcal{M}_p \)-submodule of \( \mathcal{M}_p^{N+1} \) generated by \( \{D^r f\} \).

**Proposition 4.1.** The set \( \mathcal{F}^r \) does not depend on the choices of holomorphic local coordinates \((z_1, \ldots, z_n)\) and a reduced representation \( f=(f_1: \cdots : f_{N+1}) \).

**Proof.** This will be shown by induction on \( \kappa \). For the case \( \kappa=0 \) we have nothing to prove. Assume that it holds for the case \( |\alpha| \leq \kappa \). Take another system of holomorphic local coordinates \( u=(u_1, \ldots, u_n) \). We use notations

\[ \dfrac{\partial^{|\alpha|}}{\partial u_1^{i_1} \cdots \partial u_n^{i_n}} \]

for \( \alpha=(\alpha_1, \ldots, \alpha_n) \). For an arbitrary \( \alpha \) with \( |\alpha|=\kappa+1 \) we write \( D^r=D_i D^\alpha \), where \( 1 \leq i \leq n \) and \( |\alpha'|=\kappa \). By the induction hypothesis, we can write

\[ D^r f=\sum_{|\beta| \leq \kappa} h_\beta D^\beta f \]

with some \( h_\beta \in \mathcal{M}_p \). Then,

\[ D^r f=D_i(D^r f)=\sum_{|\beta| \leq \kappa} h_\beta D_i D^\beta f+\sum_{i=1}^n h_\beta \frac{\partial z_i}{\partial u_i} D_i D^\alpha f \in \mathcal{F}^{r+1}. \]

This implies that \( \mathcal{F}^{r+1} \) does not depend on a particular choice of systems of holomorphic local coordinates.

We next take another reduced representation \( f=(\tilde{f}_1: \cdots : \tilde{f}_{N+1}) \) and set \( \tilde{f}:=\tilde{f}_1/i \cdots, \tilde{f}_{N+1} \). Then \( h=\tilde{f}_i/f_i \) (\( i=1, 2, \ldots, N+1 \)) is a nowhere zero hol-
morphic function. Let $D^\alpha = D_{\alpha} D^\gamma$ for $|\alpha'| = \kappa$. By the induction hypothesis, we can write

$$D^\alpha \hat{f} = \sum_{|\beta| \leq \kappa} g_\beta D^\beta f$$

with some $g_\beta \in \mathcal{M}_p$. Then,

$$D^\alpha \hat{f} = D_{\alpha} D^\gamma f = \sum_{|\beta| \leq \kappa} D_{\alpha} g_\beta D^\beta f + \sum_{|\beta| \leq \kappa} g_\beta D_{\alpha} D^\beta f \in \mathcal{F}_{\kappa+1}.$$ 

This gives that $\mathcal{F}_{\kappa+1}$ does not depend on a particular choice of reduced representations.

**Definition 4.2.** We shall say that $f$ is nondegenerate if $f(M) \not\subset H$ for every hyperplane $H$ in $P^n(C)$, or equivalently $f_1, \ldots, f_{N+1}$ on $U$ are linearly independent over $C$.

Set

$$l(\kappa) = \dim_{\mathcal{F}_p} \mathcal{F}$$

which does not depend on a point $p$ because $M$ is connected.

**Proposition 4.3** (cf., [12], p. 120). The map $f$ is nondegenerate if and only if $\mathcal{F} = \mathcal{M}_p^{N+1}$ for some point $p$, or equivalently $l(\kappa_0) = N+1$ for some $\kappa_0$.

**Proof.** Assume that $f$ is degenerate and so

$$a_1 f_1 + \cdots + a_{N+1} f_{N+1} = 0$$

for some $(a_1, \ldots, a_{N+1}) \neq (0, \ldots, 0)$. We then have

$$a_1 D^\alpha f_1 + \cdots + a_{N+1} D^\alpha f_{N+1} = 0$$

for all $\alpha$. Therefore, $l(\kappa) = \text{rank}_{\mathcal{F}_p} (D^\alpha f: |\alpha| \leq \kappa) < N+1$ for every $\kappa = 1, 2, \ldots$.

Conversely, assume that $\max_l l(\kappa) < N+1$. Then, there exists some $(\varphi_1, \ldots, \varphi_{N+1}) \neq (0, \ldots, 0)$ in $\mathcal{M}_p^{N+1}$ such that

$$\varphi_1 D^\alpha f_1 + \cdots + \varphi_{N+1} D^\alpha f_{N+1} = 0$$

for all $\alpha$. Take a point $q \in M$ sufficiently near $p$ such that $\varphi_1, \ldots, \varphi_{N+1}$ are holomorphic in a neighborhood of $q$ and $(\varphi_1(q), \ldots, \varphi_{N+1}(q)) \neq (0, \ldots, 0)$. Set

$$\mathcal{F}(z) = \varphi_1(q) f_1(z) + \cdots + \varphi_{N+1}(q) f_{N+1}(z)$$

on $U$. Then,

$$(D^\gamma \mathcal{F})(q) = \varphi_1(q) D^\gamma f_1(q) + \cdots + \varphi_{N+1}(q) D^\gamma f_{N+1}(q) = 0$$
for all $\alpha$. By the theorem of identity, we see $\psi = 0$. This shows that $f$ is degenerate.

**Definition 4.4.** A meromorphic map $f: M \to P^N(C)$ is said to be non-degenerate of type $(\kappa_0, l_0)$ if it satisfies the conditions

(i) $l(\kappa_0) = N + 1$,

(ii) $\sum_{\kappa = 1}^{\kappa_0} \kappa (l(\kappa) - l(\kappa - 1)) \leq l_0$.

**Remark.** If $f$ is nondegenerate of type $(\kappa_0, l_0)$ and $\kappa_0 \leq \kappa', l_0 \leq l'$, then $f$ is nondegenerate of type $(\kappa', l')$.

**Proposition 4.5.** Every nondegenerate meromorphic map $f: M \to P^N(C)$ is nondegenerate of type $(N, N(N + 1)/2)$.

To prove this, we first show

**Lemma 4.6.** If $F^\kappa = F^{\kappa+1}$ for a positive integer $\kappa$, then $F = F^\kappa$.

**Proof.** We show $F^{\kappa} \subseteq F^{\kappa+1}$ for every $\kappa' (> \kappa)$ by induction on $\kappa'$. There is nothing to prove for the case $\kappa' = \kappa + 1$. Assume that $F^{\kappa} \subseteq F^{\kappa+1}$ for $\kappa' > \kappa$. Take $\alpha$ with $|\alpha| = \kappa' + 1$ and write $D^\kappa = D_i D^\kappa$ for some $i$ and $\alpha'$ with $|\alpha'| = \kappa'$. By the induction hypothesis, we can write

$$D^\kappa f = \sum_{\beta \in \mathcal{M}_p} D_i D^\kappa f$$

with some $\varphi_{\alpha', \beta} \in \mathcal{M}_p$. Then,

$$D^\kappa f = D_i D^\kappa f$$

$$= \sum_{|\beta| \leq \kappa} D_i \varphi_{\alpha', \beta} D^\kappa f + \sum_{|\beta| \leq \kappa} \varphi_{\alpha', \beta} D_i D^\kappa f,$$

which is contained in $F^{\kappa+1} = F^{\kappa}$. We have thus Lemma 4.6.

**Proof of Proposition 4.5.** By Proposition 4.3 there is at least one $\kappa$ with $l(\kappa) = N + 1$. Set

$$\kappa_0 = \min \{ \kappa : l(\kappa) = N + 1 \}.$$

Lemma 4.6 implies that

$$1 = l(0) < l(1) < \cdots < l(\kappa_0) = N + 1.$$

Therefore,

$$N + 1 = l(\kappa_0) = \sum_{\kappa = 1}^{\kappa_0} (l(\kappa) - l(\kappa - 1)) + l(0) \geq \kappa_0 + 1.$$

This gives $\kappa_0 \leq N$ and so $l(N) = l(\kappa_0) = N + 1$. On the other hand, since
\( \kappa + 1 \leq l(\kappa) \) for every \( \kappa = 1, 2, \ldots, \kappa_0 \), we have

\[
\sum_{\kappa=1}^{\kappa_0} \kappa (l(\kappa) - l(\kappa - 1)) = \sum_{\kappa=1}^{\kappa_0} \kappa (l(\kappa) - l(\kappa - 1)) = \kappa_0 (l(\kappa_0) - (l(0) + \cdots + l(\kappa_0 - 1)) \leq \kappa_0 (N + 1) - (1 + \cdots + \kappa_0) = N(N + 1) - (N - \kappa_0)^2 - (N - \kappa_0) 2 \leq N(N + 1). \]

This completes the proof of Proposition 4.5.

**Definition 4.7.** Take arbitrarily \( N + 1 \) sets \( \alpha_i = (\alpha_{i1}, \ldots, \alpha_{in}) \) \( (1 \leq i \leq N + 1) \) of \( n \) nonnegative integers. Following Al Vitter [12], we define the generalized Wronskian of \( f \) by

\[ W_{\alpha_1, \ldots, \alpha_{N+1}}(f) = \det (D^{\alpha_i} f: 1 \leq i \leq N + 1). \]

**Remark.** By definition, \( W_{\alpha_1, \ldots, \alpha_{N+1}}(f) \neq 0 \) if and only if \( \{D^{\alpha_1} f, \ldots, D^{\alpha_{N+1}} f\} \) is a basis of \( \mathcal{F} \) as an \( \mathcal{M}_p \)-module.

**Definition 3.8.** Assume that \( f \) is nondegenerate. We say \( \{D^{\alpha_1} f, \ldots, D^{\alpha_{N+1}} f\} \) an admissible basis of \( \mathcal{F} \) if \( \{D^{\alpha_1} f, \ldots, D^{\alpha_{N+1}} f\} \) is a basis of \( \mathcal{F} \) for each \( \kappa = 1, 2, \ldots, \kappa_0 := \min \{\kappa: l(\kappa) = N + 1\} \).

**Proposition 4.9.** Assume that \( f \) is nondegenerate and \( \{D^{\alpha_1} f, \ldots, D^{\alpha_{N+1}} f\} \) is an admissible basis of \( \mathcal{F} \). Then,

\[ W_{\alpha_1, \ldots, \alpha_{N+1}}(gf) = g^{N+1} W_{\alpha_1, \ldots, \alpha_{N+1}}(f) \]

for any nonzero holomorphic function \( g \), where \( gf = (gf_1, \ldots, gf_{N+1}) \).

**Proof.** As is easily seen by induction on \( |\alpha| \), for each \( \alpha \) we can write

\[ D^\alpha(gf) = g D^\alpha f + \sum_{|\beta| < |\alpha|} a_{\alpha \beta} D^{\alpha - \beta} g D^\beta f \]

with some constants \( a_{\alpha \beta} \). We replace each \( D^\alpha(gf) \) in \( W_{\alpha_1, \ldots, \alpha_{N+1}}(gf) = \det (D^\alpha(gf): 1 \leq i \leq N + 1) \) by the right hand side of the above equation for \( \alpha = \alpha_i \) and repeat the additions of a multiple of one row to another so that

\[ \det (D^\alpha(gf)) = \det (g D^\alpha f). \]

We have thus Proposition 4.9.

Now, we consider \( q (q \geq N + 1) \) hyperplanes
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\[ H_j : a_1^j w_1 + \cdots + a_{N+1}^j w_{N+1} = 0 \]

in \( \mathbb{P}^N(\mathbb{C}) \) in general position. For a nondegenerate meromorphic map \( f : M \to \mathbb{P}^N(\mathbb{C}) \) with a reduced representation \( f = (f_1 : \cdots : f_{N+1}) \) on \( U (\subseteq M) \), we set

\[ F_j = a_1^j f_1 + \cdots + a_{N+1}^j f_{N+1} \]
on \( U \) and take an admissible basis \( \{ D^1 f, \cdots, D^{N+1} f \} \) for \( f = (f_1, \cdots, f_{N+1}) \).

**PROPOSITION 4.10.** Set \( \kappa_0 := \min \{ k : l(k) = N+1 \} \) and

\[ \varphi := \frac{W_{a_1 \cdots a_{N+1}}(f)}{F_1 F_2 \cdots F_{q}} \]

Then

\[ \nu^\varphi \leq \sum_{i=1}^{q} \min (\nu^\varphi_i, \kappa_0) \]

outside an analytic set of codimension \( \geq 2 \).

For the proof, we first show

(4.11) Let \( F \) be a nonzero holomorphic function such that \( \psi := D^s F | F \neq 0 \).

Then,

\[ \nu^\psi \leq \min (\nu^\psi, |\alpha|) \]

outside an analytic set of codimension \( \geq 2 \).

To see this, we consider the set \( E \) of all singularities of the analytic set \( \{ F = 0 \} \), which is an analytic set of codimension \( \geq 2 \). For each \( z_0 \in E \) we can find a system of holomorphic local coordinates \( (z_1, \cdots, z_n) \) in a neighborhood of \( z_0 \) such that \( z_0 = (0) \) and \( F \) can be written as \( F = z_1^m h \), where \( m = \nu^\psi(z_0) \) and \( h \) has no zero in a neighborhood of \( z_0 \). From this representation of \( F \), we can easily conclude (4.11).

**PROOF OF PROPOSITION 4.10.** It suffices to prove the desired inequality at a point \( z_0 \) such that \( (f_1(z_0), \cdots, f_{N+1}(z_0)) \neq (0, \cdots, 0) \) and

\[ \nu_{D^{s+1} f \cup F}(z_0) \leq \min (\nu^\psi(z_0), |\alpha|) \]

for every \( i, j \). Changing indices if necessary, we may assume

\[ F_i(z_0) = \cdots = F_s(z_0) = 0, \quad F_{s+1}(z_0) \neq 0, \cdots, F_{N+1}(z_0) \neq 0. \]

Here, \( k \leq N \). In fact, if \( k > N \), we have necessarily \( f_i(z_0) = \cdots = f_{N+1}(z_0) = 0 \) because \( f_1, \cdots, f_{N+1} \) can be represented as linear combinations of \( F_1, \cdots, F_{N+1} \).
Set

$$\chi := \frac{W_{a_1 \ldots a_{N+1}}(F_1, \ldots, F_{N+1})}{F_1 \cdots F_{N+1}}. $$

Then, we see $v^\omega(z_0) = v^\omega(x_0)$. For, $1/F_{N+2} \cdots F_0$ is holomorphic in a neighborhood of $z_0$ and

$$W_{a_1 \ldots a_{N+1}}(F_1, \ldots, F_{N+1}) = cW_{a_1 \ldots a_{N+1}}(f)$$

for some nonzero constant $c$. On the other hand, we can rewrite $\chi$ as

$$\chi = \det (D^{i_j}F_i/F_j: 1 \leq i, j \leq N+1)$$

$$= \sum_{\{i_1, \ldots, i_{N+1}\}} \text{sgn} \left( \frac{1, 2, \ldots, N+1}{i_1, i_2, \ldots, i_{N+1}} \right) \frac{D^{i_1}F_{i_1} \cdots D^{i_{N+1}}F_{i_{N+1}}}{F_{i_{N+1}}}. $$

By (4.11), we obtain

$$v^\omega(z_0) \leq \sum_{i=1}^{N+1} \min (v^\omega_{F_i}(z_0), \kappa_0)$$

$$= \sum_{i=1}^{N+1} \min (v^\omega_{F_i}(z_0), \kappa_0).$$

This gives Proposition 4.10.

§ 5. Statement of the main theorem

Let $f$ be a nondegenerate meromorphic map of an $n$-dimensional connected complex manifold $M$ into $\mathbb{P}^N(\mathbb{C})$ and

$$H: a^i w_i + \cdots + a^{N+1} w_{N+1} = 0$$

be a hyperplane in $\mathbb{P}^N(\mathbb{C})$. For each point $p \in M$ taking a reduced representation $f=(f_1: \cdots: f_{N+1})$ on a neighborhood of $p$, we set

$$F:= a^i f_i + \cdots + a^{N+1} f_{N+1}. $$

The intersection multiplicity of the image of $f$ and $H$ at $f(p)$ is defined by

$$v'(H)(p) = v_p(p).$$

DEFINITION 5.1. For a positive integer $\mu_0$, we define the non-integrated defect of $H$ cut by $\mu_0$ as

$$\delta_{\mu_0}(H) = 1 - \inf \{ \eta \geq 0: \eta \text{satisfies condition } (*) \}. $$

Here, condition (*) means that there exists a bounded nonnegative function
$h$ such that, for each holomorphic function $\varphi (\not \equiv 0)$ on an open subset $U$ of $M$ with $\nu = \min (\nu^{\prime} (H), \mu_0)$ outside an analytic set codimension $\geq 2$, the function $u := \log (h^2 |f|^2 |\varphi|^2)$ is continuous and plurisubharmonic on $U$. We can rewrite this

$$\gamma \Omega_f + dd^c \log h^2 \geq [\min (\nu^{\prime} (H), \mu_0)]$$

as an inequality of currents, where we denote by $[\nu]$ the (1.1)-current associated with a divisor $\nu$.

**Remark.** In the case where $f$ has a reduced representation $f= (f_1 : \cdots : f_{n+1})$ and there exists a nonzero holomorphic function $\varphi$ with $\nu_\varphi = \min (\nu^{\prime} (H), \mu_0)$ on the totality of $M$, e.g., $M$ is a ball in $C^n$, condition (*) is satisfied if and only if there exists a continuous plurisubharmonic function $u (\not \equiv - \infty)$ such that $e^u |\varphi| \leq ||f||^\gamma$.

We have always

$$0 \leq \delta_\nu^{\prime} (H) \leq 1.$$  

Obviously, $\delta_\nu^{\prime} (H) \leq 1$. To see $0 \leq \delta_\nu^{\prime} (H)$, it suffices to take $\gamma = 1$ which satisfies condition (*) for the function $h = ||F||/||f||$.

(5.4) **If there exists a bounded nonzero holomorphic function $g$ on $M$ such that $\nu_\varphi \leq \min (\nu^{\prime} (H), \mu_0)$ on $M$, in particular if $f (M) \cap H = \emptyset$, then $\delta_\nu^{\prime} (H) = 1$.**

In fact, in this case, $\gamma = 0$ satisfies condition (*) for the function $h = ||g||$. For the case $f (M) \cap H = \emptyset$, take $g = 1$.

(5.5) **If there is a positive integer $\mu > \mu_0$ such that $\nu^{\prime} (H) (p) \geq \mu$ for each point $p \in f^{-1} (H)$, then $\delta_\nu^{\prime} (H) \geq 1 - \frac{\mu_0}{\mu}$.

To see this, set $h = (||F||/||f||)^{\mu_0/\mu}$. Then

$$\frac{\mu_0}{\mu} dd^c \log ||f||^\gamma + dd^c \log h^2 = \frac{\mu_0}{\mu} [\nu^{\prime} (H)]$$

$$\geq [\min (\nu^{\prime} (H), \mu_0)]$$

by the assumption. This shows that $\frac{\mu_0}{\mu}$ satisfies condition (*).

We shall discuss here the relation between the non-integrated defect and the classical defect in the case of $M = B(R_0)$ $(0 < R_0 \leq + \infty)$. Set

$$N^H_f (r, r_0) = N_{\min (\nu^{\prime} (H), \mu_0)} (r, r_0).$$

By definition, the classical defect of $H$ cut by $\mu_0$ is given by
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\[ \delta^*_p(H) := 1 - \lim_{r \to R_0} \sup_{r \to R_0} \frac{N^H_H(r, r_0)^{[p_o]}}{T_f(r, r_0)} . \]

**PROPOSITION 5.6.** If \( \lim_{r \to R_0} T_f(r, r_0) = +\infty \), then
\[
0 \leq \delta^*_p(H) \leq \delta^*_p(H) \leq 1.
\]

**PROOF.** Take \( \gamma \) satisfying condition (\( \ast \)). The function
\[
v := \gamma \log \|f\| + \log h - \log |\varphi|
\]
is plurisubharmonic for the function \( h \) and \( \varphi \) as in Definition 5.1. Therefore,
\[
0 \leq \int_{\mathcal{S}(r)} v \sigma_n - \int_{\mathcal{S}(r_0)} v \sigma_n
\]
\[
= \gamma \int_{\mathcal{S}(r)} \log \|f\| \sigma_n + \int_{\mathcal{S}(r)} \log h \sigma_n - \int_{\mathcal{S}(r)} \log |\varphi| \sigma_n + K
\]
\[
\leq \gamma T_f(r, r_0) - N^H_f(r, r_0)^{[p_o]} + K
\]
because \( h \) is bounded from above. This implies that
\[
\frac{N^H_f(r, r_0)^{[p_o]}}{T_f(r, r_0)} \leq \gamma + \frac{K}{T_f(r, r_0)}.
\]

As \( r \to R_0 \), we obtain \( \delta^*_p(H) \geq 1 - \gamma \), and hence Proposition 5.6.

Now, we consider a meromorphic map \( f \) of an \( n \)-dimensional Kähler manifold \( M \) into \( P^{N_1 \cdots N_k}(C) := P^{N_1}(\mathbb{C}) \times \cdots \times P^{N_k}(\mathbb{C}) \).

**DEFINITION 5.7.** We shall say a meromorphic map \( f = (f_1, \ldots, f_k) : M \to P^{N_1 \cdots N_k}(\mathbb{C}) \) is nondegenerate if each map \( f_i : M \to P^{N_i}(\mathbb{C}) \) is nondegenerate, and is nondegenerate of type \((\kappa_i, l_i ; \cdots ; \kappa_k, l_k)\) if each \( f_i \) is nondegenerate of type \((\kappa_i, l_i)\).

Let \( \omega \) be the Kähler form of \( M \), which we represent as
\[
(5.8) \quad \omega = \frac{\sqrt{-1}}{2} \sum_{i,j} h_{i\bar{j}} dz_i \wedge d\bar{z}_j
\]
locally. We define the Ricci form of \( \omega \) by
\[
\text{Ric} \omega = dd^c \log (\det (h_{i\bar{j}})).
\]

**DEFINITION 5.9.** For nonnegative numbers \( \rho_1, \ldots, \rho_k \) we shall say \( f = (f_1, \ldots, f_k) : M \to P^{N_1 \cdots N_k}(\mathbb{C}) \) satisfies condition \((C_{\rho_1, \ldots, \rho_k})\) if there exists a bounded function \( h (\neq 0) \) on \( M \) such that the function \( u := \log (h^2 \|f_1\|^{\rho_1} \cdots \|f_k\|^{\rho_k} / \det (h_{i\bar{j}})) \) is continuous and plurisubharmonic locally. We can rewrite this as
\[
\text{Ric} \omega \leq \rho_1 \Omega_{f_1} + \cdots + \rho_k \Omega_{f_k} + dd^c \log h^2,
\]
where $\Omega_{f_1}$ denotes the pull-back of the normalized Fubini-Study metric form on $P^N(C)$.

**Remark.** In the case where $M$ is a Cousin II domain in $C^n$, each $f_i$ has a reduced representation $f_i = (f_{i1} : \cdots : f_{i,n+1})$ and $\omega$ has a representation (5.8) on the totality of $M$. Then, condition $(C_{\rho_1,\ldots,\rho_k})$ is satisfied if and only if there exists a continuous plurisubharmonic function $u^\omega \geq -\infty$ on $M$ such that

$$e^u dV \leq ||f_1||^p \cdots ||f_k||^p dV_n,$$

where $dV$ denotes the volume form of $M$ and $||f_i|| = (|f_{i1}|^p + \cdots + |f_{i,n+1}|^p)^{1/p}$.

The main theorem is stated as follows.

**Theorem 5.10.** Let $M$ be an $n$-dimensional complete Kähler manifold and $f = (f_1, \cdots, f_k): M \to P^N(C)$ be a meromorphic map which is nondegenerate of type $((\kappa_1, l_1; \cdots ; \kappa_k, l_k))$. Assume that $f$ satisfies condition $(C_{\rho_1,\ldots,\rho_k})$ and the universal covering $\tilde{M}$ of $M$ is biholomorphically isomorphic to a ball $B(R_0)$ ($0 < R_0 \leq +\infty$) in $C^n$. For each $i$ ($1 \leq i \leq k$) take hyperplanes $H_{i1}, \ldots, H_{iq}$ in $P^N(C)$ in general position. If

$$\sum_{j=1}^{q_i} \delta_{i1}^j(H_{ij}) > N_i + 1$$

for every $i$, then

$$\sum_{i=1}^{k} \delta_{i1}^j(H_{i1}) + \cdots + \delta_{i1}^j(H_{iq}) - N_i - 1 \geq 1.$$

**Remark.** (i) This is a generalization of Theorem 4.10 in [6] to the case of several complex variables.

(ii) Theorem 1.1 stated in § 1 is a direct result of Theorem 5.10 by virtue of Proposition 4.5.

The proof of Theorem 5.10 will be given in the next section. Here, we note that, for our purpose we may assume $\tilde{M} = M$ and so $M$ itself is biholomorphically isomorphic to $B(R_0)$. In fact, for the universal covering $\varphi: \tilde{M} \to M$, $\tilde{f} = f \circ \varphi: \tilde{M} \to P^N(C)$ is also nondegenerate of type $((\kappa_1, l_1; \cdots ; \kappa_k, l_k))$ and satisfies condition $(C_{\rho_1,\ldots,\rho_k})$. Moreover, it holds that $\delta_{i1}^j(H_{i1}) \leq \delta_{i1}^j(H_{i1})$. If Theorem 5.10 is true for $\tilde{f}$, then it is also true for $f$.

**§ 6. Proof of the main theorem**

We first consider a nondegenerate meromorphic map $f: B(R_0) \to P^N(C)$ with a reduced representation $f = (f_1 : \cdots : f_{N+1})$ and $q$ hyperplanes

$$H_j: a_j^1w_1 + \cdots + a_j^{N+1}w_{N+1} = 0 \quad (1 \leq j \leq q)$$
in general position. As in the previous sections, we set \( \|f\| = (|f_1|^2 + \cdots + |f_{n+1}|^2)^{1/2} \) and \( F_j = a_j^1 f_1 + \cdots + a_j^{N+1} f_{N+1} \) and take an admissible basis \( \{D^n f, \cdots, D^n x_1 f\} \).

**Proposition 6.1.** In the above situation, set \( l_0 = |\alpha_1| + \cdots + |\alpha_{N+1}| \) and take \( t, p' \) with \( 0 < l_0 < p' < 1 \). Then, for \( 0 < r_0 < R_0 \) there exists a positive constant \( K \) such that for \( r_0 < r < R < R_0 \)

\[
\int_{S(r)} \left| \frac{W_{a_1 \cdots a_{N+1}}(f)}{F_1 \cdots F_q} \right|^t \|f\|^{(q-N-1)} \sigma_n^t \leq K \left( \frac{R_0^{2n-1}}{R-r} T_{ij}(R, r_0) \right)^{p'}.
\]

**Proof.** This will be proven by the argument similar to the proof of Proposition 3.3 in [6]. We describe here only the outline of the proof. As in [6],

\[
\frac{W_{a_1 \cdots a_{N+1}}(f)}{F_1 \cdots F_q} \leq K \left( \sum_{1 \leq j_1 < \cdots < j_{N+1} \leq q} \frac{W_{a_1 \cdots a_{N+1}}(F_{j_1}, \cdots, F_{j_{N+1}})}{F_{j_1} \cdots F_{j_{N+1}}} \right)
\]

for some constant \( K \). To obtain the desired inequality, we shall estimate

\[
I_1 := \int_{S(r)} \left| \frac{W_{a_1 \cdots a_{N+1}}(F_j)}{F_1 \cdots F_{j_{N+1}}} \right|^t \sigma_n.
\]

By the same argument as in the proof of Lemma 3.6 in [6] the integrand can be estimated from above by a positive constant multiple of the sum of some functions of the type

\[
\psi_{i_1, \cdots, i_{N+1}} := \left| \frac{D^n \varphi_{i_1} \cdots D^n \varphi_{i_{N+1}}}{\varphi_{i_1} \cdots \varphi_{i_{N+1}}} \right|^t,
\]

where \( \varphi_i = F_i/F_j(1 \leq i \leq q) \) and \( 1 \leq i_1, \cdots, i_{N+1} \leq q \). Set \( p_j = |\alpha_j|/(|\alpha_1| + \cdots + |\alpha_{N+1}|) \) for \( 1 \leq j \leq N+1 \). By Hölder's inequality, we obtain

\[
\int_{S(r)} \psi_{i_1, \cdots, i_{N+1}} \sigma_n \leq \prod_{j=1}^{N+1} \left( \int_{S(r)} \left| \frac{D^n \varphi_{i_j}}{\varphi_{i_j}} \right|^{p_j} \sigma_n \right)^{p_j}.
\]

Since \( t|p_j| \alpha_j = (|\alpha_1| + \cdots + |\alpha_{N+1}|) t < p' < 1 \) for \( 1 \leq j \leq N+1 \), we can apply Theorem 3.1 to show

\[
\int_{S(r)} \psi_{i_1, \cdots, i_{N+1}} \sigma_n \leq K \prod_{j=1}^{N+1} \left( \frac{R_0^{2n-1}}{R-r} T_{ij}(R, r_0) \right)^{p_j}.
\]

On the other hand, by (2.5)
for every $i=1, 2, \ldots, q$. Therefore, we conclude

$$I \leq K \left( \frac{R^{2n-1}}{R-r} T_f(R, r_0) \right)^p.$$ 

This completes the proof of Proposition 6.1.

As a result of Proposition 6.1, we have the following refinement of the well-known second main theorem for meromorphic maps into $P^N(C)$ (cf., [11], [12]).

**PROPOSITION 6.2.** Let $f: B(R_0) \rightarrow P^N(C)$ be a meromorphic map which is nondegenerate of type $(\kappa_0, l_0)$ and $H_1, \ldots, H_q$ be hyperplanes in $P^N(C)$ located in general position. Then,

$$(q-N-1)T_f(r, r_0) \leq \sum_{i=1}^{q} N_{f,i}^{(r_0)} + S(r),$$

where $S(r)$ is evaluated as follows.

(i) In the case $R_0 < \infty$,

$$S(r) \leq K \left( \log^+ \frac{1}{R_0-r} + \log^+ T_f(r, r_0) \right)$$

for every $r \in [0, R)$ excluding a set $E$ with $\int_E 1/(R_0-t)dt < \infty$.

(ii) In the case $R_0 = \infty$,

$$S(r) \leq K (\log^+ T_f(r, r_0) + \log r)$$

for every $r \in [0, \infty)$ excluding a set $E'$ with $\int_{E'} dt < \infty$.

**PROOF.** By virtue of the concavity of logarithm, Proposition 6.1 implies that

$$t \int_{S(r)} \log |z^{\alpha_1} \cdots \alpha_\kappa \cdots \alpha_N |^p |\sigma_n| + t \int_{S(r)} \log \left| \frac{W_{\alpha_1} \cdots \alpha_N (f)}{F_1 \cdots F_q} \right| |\sigma_n|$$

$$+ t(q-N-1) \int_{S(r)} \log \|f\| |\sigma_n|$$

$$\leq \log \int_{S(r)} \left| z^{\alpha_1} \cdots \alpha_N W_{\alpha_1} \cdots \alpha_N (f) \right|^p \|f\|^{(q-N-1)} |\sigma_n|$$

$$\leq K \left( \log^+ \frac{R}{R-r} + \log^+ T_f(R, f) \right).$$
On the other hand, by (2.2) and Proposition 4.10,

$$- \sum_{j=1}^{q} N_f^j(r, r_0) \leq \int_{\mathbb{R}(r)} \log \left| \frac{W_{q+1} f}{F_1 \cdots F_q} \right| \sigma + K.$$

Therefore,

$$
(6.3) \quad (q-N-1) T_f(r, r_0) \\
\leq \sum_{j=1}^{q} N_f^j(r, r_0) + K \left( \log r + \log^+ \frac{R}{R-r} + \log^+ T_f(R, r_0) \right).
$$

Since $T_f(r, r_0)$ is continuous, increasing and we may assume $T_f(r, r_0) \geq 1$, we can apply Lemma 2.4 in [7] to show

$$T_f \left( r + \frac{R_0-r}{e T_f(r, r_0)} , r_0 \right) \leq 2 T_f(r, r_0)$$

outside a set $E$ of $r$ such that $\int_{E} 1/(R_0-r)dr < \infty$ in the case $R_0 < \infty$ and

$$T_f \left( r + \frac{1}{T_f(r, r_0)} , r_0 \right) \leq 2 T_f(r, r_0)$$

outside a set $E'$ of $r$ such that $\int_{E'} dr < \infty$ in the case $R_0 = \infty$. For (6.3), substitute $R = r + (R_0-r)/e T_f(r, r_0)$ if $R_0 < \infty$ and $R = r + 1/T_f(r, r_0)$ if $R_0 = \infty$. We have easily Proposition 6.2.

**Corollary 6.4.** In the same situation as in Theorem 6.2, if (i) $R_0 < \infty$ and

$$\limsup_{r \to R_0} \frac{T_f(r, r_0)}{\log 1/(R_0-r)} = \infty$$

or (ii) $R_0 = \infty$, then

$$\sum_{j=1}^{q} \Delta^*(H_j) \leq N + 1.$$

**Proof.** Proposition 6.2 implies that

$$
(6.5) \quad \sum_{j=1}^{q} \left( 1 - \frac{N_f^j(r, r_0)}{T_f(r, r_0)} \right) \leq N + 1 + \frac{S(r)}{T_f(r, r_0)}.
$$

For the case (i), we note that $\limsup_{r \to R_0} T_f(r, r_0)/-\log (R_0-r) = \infty$ is equivalent to $\limsup_{r \to R_0, r \in E} T_f(r, r_0)/-\log (R_0-r) = \infty$ for a set $E$ with $\int_{E} 1/(R_0-r)dr$
In fact, this is easily seen by the same argument as in the proof of Proposition 5.5 in [6]. In any case, as $r \to R_0$ in (6.5), we have Corollary 6.4.

We now start to prove Theorem 5.10. Let $M, f = (f_1, \ldots, f_k): M \to \mathbb{P}^{N_1} \cdots \mathbb{P}^{N_k}(\mathbb{C})$ and $H_{i,j} (1 \leq i \leq k, 1 \leq j \leq q_i)$ be a Kähler manifold, a meromorphic map and hyperplanes respectively which satisfy all assumptions in Theorem 5.10. As stated in the last paragraph in § 5, we may assume $M = B(R_0)$ for some $R_0 (0 < R_0 \leq +\infty)$. Let

$$H_{i,j}: a_{ij} w_1 + \cdots + a_{ij}^{N_{j+1}} w_{N_{j+1}} = 0$$

and, taking reduced representations $f_i = (f_{i1}, \ldots, f_{iN_{i+1}})$, we set $\|f_i\| = (|f_{i1}| + \cdots + |f_{iN_{i+1}}|)^{1/\gamma}$. Moreover, we choose an admissible basis $\{D^i f_r, \ldots, D^{iN_{i+1}} f_r\}$ for each $f_i$. By the assumption,

$$\left|\alpha_{i1}\right| + \cdots + \left|\alpha_{iN_{i+1}}\right| \leq l_i$$

and

$$\max(\left|\alpha_{i1}\right|, \ldots, \left|\alpha_{iN_{i+1}}\right|) \leq \varepsilon_i.$$  

According to Corollary 6.4 and Proposition 5.6, we may assume that $R_0 = 1$ and

$$\limsup_{r \to -1} \frac{T_{f_i}(r, r_0)}{\log 1/(1-r)} < \infty$$

for every $i$.

The proof of Theorem 5.10 is given by reduction to absurdity. We assume that

$$\sum_{f=1}^k \beta'_{f} (H_{i,j}) > N_{i} + 1$$

for every $i$ and

$$\sum_{i=1}^k \left( \frac{2\rho_i l_i}{\beta'_{i} (H_{i}) + \cdots + \beta'_{i} (H_{iN_i}) - N_i - 1} < 1. \right.$$  

Then, by Remark to Definition 5.1, there exist some $\eta_{i,j} \geq 0$ and continuous plurisubharmonic functions $u_{i,j} (\neq -\infty)$ such that

$$\sum_{i=1}^k \left( \frac{2\rho_i l_i}{(1-\eta_{i1}) + \cdots + (1-\eta_{iN_i}) - N_i - 1} < 1, \right.$$  

and $u_{i,j} - \log |\varphi_{i,j}|$ is plurisubharmonic, where $\varphi_{i,j}$ is a nonzero holomorphic
function with $\nu_{ij} = \min(\nu^f(H_{ij}), \kappa_i)$. Set

$$u_i := \log \left| z_1^{a_{i1}+\cdots+a_{iN}+1} \frac{W_{a_{i1}\cdots a_{iN}+1}}{F_{i1}\cdots F_{iN}} \right| + \sum_{j=1}^{g_i} u_{ij},$$

which are plurisubharmonic on $M = B(1)$ by virtue of Proposition 4.10. On the other hand, by the assumption there exists a continuous plurisubharmonic function $w \equiv -\infty$ on $B(1)$ such that

$$e^w dV \leq |f_1|^{p_1} \cdots |f_k|^{p_k} u_n.$$

Set

$$t_i := \frac{2\rho_i}{q_i^2 - N_i - 1 - (\eta_{i1} + \cdots + \eta_{iN_i})},$$

$$\chi_i := z_1^{a_{i1}+\cdots+a_{iN}+1} \frac{W_{a_{i1}\cdots a_{iN}+1}(f_i)}{F_{i1}\cdots F_{iN_i}},$$

and define

$$u := w + t_1v_1 + \cdots + t_kv_k.$$

Obviously, $u$ is plurisubharmonic and so subharmonic on a Kähler manifold $M$. We then have

$$e^u dV \leq e^{t_1v_1 + \cdots + t_kv_k} |f_1|^{p_1} \cdots |f_k|^{p_k} u_n \leq \prod_{i=1}^k |\chi_i|^{t_i} |f_i|^{t_i(q_i^2 - N_i - 1)} u_n$$

for each $i$ by the same argument as in the proof of Theorem 4.10 in [6], p. 676. Therefore, if we set $s_i = t_i l_i$, $p_i = s_i/(s_1 + \cdots + s_k)$ and $t_i' = t_i/p_i$, then we obtain

$$\int_{B(1)} e^u dV \leq \int_{B(1)} \prod_{i=1}^k |\chi_i|^{t_i'} |f_i|^{t_i'(q_i^2 - N_i - 1)} u_n$$

$$\leq 2n \int_0^{\sigma_1} \left( \int_{B(1)} |\chi_i|^{t_i'} |f_i|^{t_i'(q_i^2 - N_i - 1)} u_n \right)^{p_i'} dt$$

by the help of the Hölder’s inequality and the identity

$$u_n = (dd^c |z|^p)^n = 2n |z|^{2n-1} \sigma_n \wedge d|z|$$

(cf., [10], p. 226).

Since $t_1 := s_1 + \cdots + s_k < 1$, we can choose $p'$ such that $t_o < p' < 1$. Then,

$$0 < (|\alpha_{i1}| + \cdots + |\alpha_{iN+i}|) t_i' \leq t_i' < p' < 1$$
by (6.6). Therefore, we can apply Proposition 6.1 to show

\[ \int_{B(r)} |\mathcal{L}^i|^{1/n} \|f_i\|^{1/(q_1-q_i-1)} \sigma_n \leq K \left( \frac{R_i^{n-1}}{R_i-r} T_{f_i}(R_i, r_0) \right)^{\sigma_n} \]

for \( r_0 < r < R_i < 1 \). According to Lemma 2.4 in [7], if we choose \( R_i := r + (1-r)/eT_{f_i}(r, r_0) \), then

\[ T_{f_i}(R_i, r_0) \leq 2T_{f_i}(r, r_0) \]

outside a set \( E \) with \( \int_E 1/(1-r)dr < \infty \). By the assumption (6.8), we have

\[ \int_{B(r)} |\mathcal{L}^i|^{1/n} \|f_i\|^{1/(q_1-q_i-1)} \sigma_n \leq \left( \frac{1}{1-r} T_{f_i}(R_i, r_0) \right)^{\sigma_n} \]

\[ \leq K \left( \frac{1}{1-r} \right)^{\sigma_n} \left( \log \frac{1}{1-r} \right)^{\sigma_n} \]

for all \( r \in [0, 1) \). Varying a constant \( K \) slightly, we may assume that the above inequality holds for all \( r \in [0, 1) \) because of Proposition 5.5 in [6]. From these facts, we conclude

\[ \int_{B(1)} e^s dV \leq K \int_0 ^{r_0} t^{n-1} \left[ \prod_{i=1}^k \left( \frac{1}{(1-t)^{\sigma_n}} \left( \log \frac{1}{1-t} \right)^{\sigma_n} \right) dt \right] \]

\[ \leq K \int_0 ^{r_0} t^{n-1} \left( \log \frac{1}{1-t} \right)^{\sigma_n} dt < \infty. \]

On the other hand, by the result of S. T. Yau ([13]) and L. Karp ([8]), we have necessarily

\[ \int_{B(1)} e^s dV = \infty, \]

because \( B(1) \) has infinite volume with respect to the given complete Kähler metric (cf. [8], Theorem B). This is a contradiction. So, the proof of Theorem 5.10 is completed.

**§ 7. Value distribution of the Gauss map of a complete regular submanifold of \( \mathbb{C}^m \)**

Let \( f=(f_1, \ldots, f_m): M \to \mathbb{C}^m \) be a regular submanifold of \( \mathbb{C}^m \), namely, \( M \) be a connected complex manifold and \( f \) be a holomorphic map of \( M \) into \( \mathbb{C}^m \) such that rank \( d_f = \dim M \) for every point \( p \in M \).

To each point \( p \in M \), we assign the tangent space \( T_p(M) \) of \( M \) at \( p \) which may be regarded as an \( n \)-dimensional linear subspace of \( T_{f(p)} \mathbb{C}^m \). On the
other hand, each \( T_p(C^n) \) is identified with \( T_0(C^n)=C^n \) by a parallel translation. Therefore, to each \( T_p(M) \) corresponds a point \( G(p) \) in the complex Grassmannian manifold \( G(n, m) \) of all \( n \)-dimensional linear subspaces of \( C^m \), where \( n=\dim M \).

**Definition 7.1.** We call the map \( G: M \to G(n, m) \) the Gauss map of \( f: M \to C^n \).

The space \( G(n, m) \) is canonically imbedded in \( P^N(C)=P(\wedge^n C^m) \), where \( N=\binom{m}{n}-1 \). The Gauss map \( G \) may be identified with the holomorphic map of \( M \) into \( P^N(C) \) given as follows.

Taking holomorphic local coordinates \( (z_1, \cdots, z_n) \) defined on an open set \( U \), we consider the map

\[
\Lambda:=D_1f \wedge \cdots \wedge D_nf: U \to \wedge^n C^m-{0}=C^{n+1}-{0},
\]

where \( D_if=\left(\frac{\partial}{\partial z_i}f_1, \cdots, \frac{\partial}{\partial z_i}f_{n+1}\right) \). Then,

\[
G=\pi \cdot \Lambda
\]

locally, where \( \pi: C^{n+1}-{0} \to P^N(C) \) is the canonical projection map.

A regular submanifold \( M \) of \( C^n \) is considered a Kähler manifold with the metric \( \omega \) induced from the standard flat metric on \( C^m \). By \( dV \) we denote the volume form on \( M \).

We see easily

(7.2) For arbitrary holomorphic local coordinates \( z_1, \cdots, z_n \),

\[
dV=|A|^2\left(\frac{\sqrt{-1}}{2}\right)^n dz_1 \wedge d\bar{z}_1 \wedge \cdots \wedge dz_n \wedge d\bar{z}_n,
\]

where

\[
|A|^2=\sum_{1 \leq i_1 < \cdots < i_n \leq m} \left| \frac{\partial (f_{i_1}, \cdots, f_{i_n})}{\partial (z_{i_1}, \cdots, z_{i_n})} \right|^2.
\]

(7.3) For a regular submanifold \( f: M \to C^n \) the Gauss map \( G: M \to P^N(C) \) satisfies condition \((C_i)\).

For, if we take \( h=1 \), then

\[
\Omega_0 + dd^c \log h^2 = dd^c \log |A|^2 = \text{Ric}(\omega).
\]

As a direct result of Theorem 1.1, we have

**Theorem 7.4.** Let \( f: M \to C^n \) be a complete regular submanifold such that the universal covering of \( M \) is biholomorphically isomorphic to \( B(R_0) \)
If the Gauss map $G : M \to P^N(C)$ is nondegenerate, then
\[ \sum_{j=1}^{q} \delta^N_j(H_j) \leq (N+1)^2 = \binom{m}{n} \]
for every hyperplanes $H_1, \ldots, H_q$ in general position.

If $M$ is a closed regular submanifold of $C^n$, then $M$ is necessarily complete. Theorem 1.2 stated in §1 is a particular case of Theorem 7.4. In the case $\dim M = 1$, we have the same conclusion as in Theorem 7.4 under the only assumption that $M$ is complete. In fact, in this case, the universal covering of $M$ is always biholomorphically isomorphic to $A(R_0) = \{ |z| < R_0 \}$ ($0 < R_0 \leq +\infty$).

To give another type of non-integrated defect relation, we shall introduce a new notion which connects the notions of Wronskians and Jacobians. Let $f = (f_1, \ldots, f_m) : M \to C^n$ be a holomorphic map. Taking holomorphic local coordinates $z = (z_1, \ldots, z_n)$ on $M$, we consider the Jacobian
\[ J_m = \frac{\partial (f_{i_1}, \ldots, f_{i_n})}{\partial (z_{i_1}, \ldots, z_{i_n})} \]
for each combination $I = (i_1, \ldots, i_n)$ with $1 \leq i_1 < \cdots < i_n \leq m$. We label all combinations $(i_1, \ldots, i_n)$ of the letters $1, 2, \ldots, m$ as
\[ I_1, I_2, \ldots, I_{N+1}, \]
where $N = \binom{m}{n} - 1$. On the other hand, the number of all sets $\alpha = (\alpha_1, \ldots, \alpha_n)$ of nonnegative integers with $|\alpha| = \alpha_1 + \cdots + \alpha_n \leq m - n$ is $\binom{m}{n}$. We label them as
\[ \alpha_1, \alpha_2, \ldots, \alpha_{N+1}. \]

**DEFINITION 7.5.** We define the **generalized absolute Jacobian** of $f$ by
\[ J_f = |\det (D^n(I_1,f)) : 1 \leq i, j \leq N+1)|. \]

**REMARK.** For the case $n = 1$, $J_f$ gives the absolute value of the Wronskian of the map $Df = \frac{df_1}{dz}, \ldots, \frac{df_m}{dz}$, and for the case $m = n$, $J_f$ means the absolute value of the Jacobian of $f$.

**DEFINITION 7.6.** We shall say a holomorphic map $f : M \to C^n$ is strictly nondegenerate if $J_f \neq 0$ for a system of holomorphic local coordinates $z = (z_1, \ldots, z_n)$.

(7.7) If for a regular submanifold $f : M \to C^n$ $f$ is strictly nondegenerate,
then the Gauss map $G: M \to \mathbb{P}^n(C)$ is nondegenerate of type $(m-n, n\binom{m}{n+1})$.

In fact, in this case, we see

$$l(k) = \binom{n+k}{n} \quad (0 \leq k \leq m - n)$$

and so $l(m-n) = \binom{m}{n}$ and

$$\sum_{k=0}^{m-n} k \left( \binom{n+k}{n} - \binom{n+k-1}{n} \right) = n \binom{m}{n+1}.$$

As a consequence of Theorem 5.10 and (7.7), we have

**Theorem 7.8.** Let $f: M \to \mathbb{C}^m$ be a complete regular submanifold of $\mathbb{C}^m$ such that the universal covering of $M$ is biholomorphically isomorphic to $\mathbb{B}(R_0)$. If $f$ is strictly nondegenerate, then

$$\sum_{j=1}^{q} \partial^2_{x_{-k}}(H_j) \leq \binom{m}{n} + 2n \binom{m}{n+1}$$

for the Gauss map $G$ and arbitrary hyperplanes $H_1, \ldots, H_q$ in general position.

Theorem 1.3 stated in §1 is a particular case of Theorem 7.8.

We finally give a basic property of generalized absolute Jacobians.

**Proposition 7.9.** (i) If $g = Af$ for a nonsingular constant matrix $A$, then

$$J_ug = |\det A|^{m-1} J uf.$$

(ii) For another system of holomorphic local coordinates $u = (u_1, \ldots, u_n)$, we have

$$J uf = \left| \frac{\partial (z_1, \ldots, z_n)}{\partial (u_1, \ldots, u_n)} \right|^{m+1} J uf.$$

**Proof.** (i) Let $A = (a_{ij})$ and so $g_i = \sum_{j=1}^{m} a_{ij} f_j$. Then, it is easily seen that

$$D^\alpha \left( \frac{\partial (g_1, \ldots, g_m)}{\partial (z_1, \ldots, z_n)} \right) = \sum_{1 \leq j_1 < \ldots < j_k \leq m} \det (a_{ij}) D^\alpha \left( \frac{\partial (f_1, \ldots, f_m)}{\partial (z_1, \ldots, z_n)} \right)$$

for every $\alpha = (\alpha_1, \ldots, \alpha_n)$ and combination $(i_1, \ldots, i_n)$. According to the classical theorem of Sylvester and Franke ([9], p. 94), we conclude
(ii) Take a point \( p \) in the common domain of definition of \( z \) and \( u \). Since \( J \cdot f \) does not change by parallel translations of holomorphic local coordinates, we may assume \( z_i(p) = u_j(p) = 0 \). We denote all \( n \) combinations of \( m \) letters \( 1, 2, \ldots, m \) by

\[
I_l = (i_{l1}, \ldots, i_{ln}) \quad 1 \leq l \leq \binom{m}{n}
\]

and set

\[
g_i(z) := \frac{\partial (f_{i_{l1}}, \ldots, f_{i_{ln}})}{\partial (z_{i_1}, \ldots, z_n)}.
\]

Each \( g_i \) is considered a function of \( u_1, \ldots, u_n \), which we denote by \( h_i \). We expand them in the Taylor series as

\[
g_i(z) = \sum a_{i\alpha} z^\alpha
\]

and

\[
h_i(u) = \sum b_{i\beta} u^\beta
\]

around the origin. Moreover, considering each \( z_i \) as a function of \( u_1, \ldots, u_n \), we expand it as

\[
z_i = \sum c_i u_j + \text{the terms of higher degree},
\]

where \( c_i = \partial z_i / \partial u_j \) and so \( \partial (z_1, \ldots, z_n) / \partial (u_1, \ldots, u_n) = \det (c_i) \). For our purpose, it suffices to show that

\[
\det \left( \frac{1}{\alpha!} \left( D^\alpha f_i \right)(0) \right) \left( \frac{1}{\alpha!} \left( D^\alpha g_i \right)(0) \right) = \det \left( \frac{1}{\beta!} \left( D^\beta f_i \right)(0) \right) \left( \frac{1}{\beta!} \left( D^\beta g_i \right)(0) \right).
\]

In fact, since \( a_{i\alpha} = (1/\alpha!) \left( D^\alpha f_i \right)(0) \) and \( b_{i\beta} = (1/\beta!) \left( D^\beta g_i \right)(0) \), (7.11) implies that

\[
\det \left( \frac{1}{\alpha!} \left( D^\alpha f_i \right) \right) \left( \frac{1}{\alpha!} \left( D^\alpha g_i \right) \right) = \det \left( \frac{1}{\beta!} \left( D^\beta f_i \right) \right) \left( \frac{1}{\beta!} \left( D^\beta g_i \right) \right).
\]

On the other hand, since
we obtain the desired conclusion by the help of Proposition 4.9 and the identity \( \binom{m}{n} + \binom{m}{n+1} = \binom{m+1}{n+1} \).

To show (7.11), consider the functions

\[
\sum_{\beta} b_{\beta}^* = \sum_{\beta} \left( \sum_{j=1}^{m} c_j u_j \right)^{a_j} \left( \sum_{j=1}^{n} c_j u_j \right)^{a_n} = \sum b_{\beta}^* u_\beta.
\]

Then, we can prove

\[
| \det \left( \begin{array}{c}
1 \leq l \leq N+1 \\
|\beta| \leq m-1
\end{array} \right) | = | \det \left( \begin{array}{c}
1 \leq l \leq N+1 \\
|\beta| \leq m-1
\end{array} \right) |.
\]

In fact, as is easily seen by induction on \(|\beta|\), we can write

\[
b_{\beta} = b_{\beta}^* + \sum_{|\gamma|<|\beta|} q_{\gamma} a_{\gamma},
\]

where each \( q_{\gamma} \) is a function of \( \partial z_i/\partial u_j, \partial^2 z_i/\partial u_j \partial u_k, \cdots \) and does not depend on \( l \). On the other hand, representing each \( u_i \) as a power series of \( z_1, \cdots, z_n \) and substituting it into (7.10), we compare the coefficients of \( z^\alpha \) in the two expansions of \( g_i \). Then, we see easily that each \( a_{\gamma} \) is a linear function of \( b_{\beta}^* \)'s with \(|\gamma|\leq|\alpha|\) whose coefficients are functions of \( \partial z_i/\partial u_j, \partial^2 z_i/\partial u_j \partial u_k, \cdots \) and does not depend on \( l \). From these facts, we can conclude (7.12) by repeating the addition of a constant multiple of one row to another.

In our arguments, we may disregard the terms of degree \( > m-n \) and therefore we may replace each \( g_i \) by a polynomial

\[
P_i(z_1, \cdots, z_n) = \sum_{|\alpha| \leq m-n} a_{i\alpha} z^\alpha.
\]

Moreover, instead of \( P_i \), we may consider a homogeneous polynomial \( Q_i(z_0, z_1, \cdots, z_n) \) of degree \( m-n \) such that \( Q_i(1, z_1, \cdots, z_n) = P_i(z_1, \cdots, z_n) \).

To complete the proof of (7.11), it suffices to prove the following

**Lemma 7.13.** Let \( H^{m-n} \) be the vector space of all homogeneous polynomials in \( z_0, z_1, \cdots, z_n \) of degree \( m-n \). For a regular matrix \( C=(c_{ij}: 0 \leq i, j \leq n) \) define the linear map \( L_C: H^{m-n} \to H^{m-n} \) by

\[
L_C(Q)(z_0, \cdots, z_n) = Q(\sum_{j=1}^{m} c_{0j} z_j, \cdots, \sum_{j=1}^{n} c_{nj} z_j).
\]

Then, \( \det L_C = |C|^{m-n} \).

**Proof.** The map \( D: GL(n, C) \to C^* = C - \{0\} \) defined by \( D(C) = \det (L_C) \)
(\(C \in GL(n, \mathbb{C})\)) is a rational abelian character for the rational representation \(L\) of \(GL(n, \mathbb{C})\) in the space \(H^{m-n}\). By the well-known theorem of rational abelian characters of \(GL(n, \mathbb{C})\), \(D\) is of the form

\[
D(C) = \text{det}(C)^t
\]

for some integer \(t\) (cf., [4], p. 21). Comparing degrees of both sides, we see easily \(t = \binom{m}{n+1}\).

References