The non-existence of elliptic curves with everywhere good reduction over certain quadratic fields

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Introduction

The main purpose of the present paper is to show the non-existence of elliptic curves with everywhere good reduction defined over certain *real* quadratic fields. (See Theorem 2.3.) For this purpose we shall study k -rational point of order 2 of such an elliptic curve defined over a real quadratic field k . Then the problem is reduced to a certain Diophantine problem over k .

As for elliptic curves defined over *imaginary* quadratic field k , Stroeker has proved that any elliptic curve with everywhere good reduction defined over k does not have a global minimal equation by a Diophantine method in [9]. By using Stroeker's result, Setzer has shown that there exists no elliptic curve with everywhere good reduction defined over k if the class number of k is prime to 6 in [11]. (See also [10, p. 196].) Although the result is weaker than Setzer's, the present author has independently shown a result of the same type by a quite different method (non Diophantine method) in [2]. In § 3, we shall give a mild improvement of the Theorem of [2]. (See Cor. 3.3.) In § 5, we shall show some examples of elliptic curves with everywhere good reduction. In Appendix we shall give a table of elliptic curves of prime power conductor with a rational point of order 3 defined over imaginary quadratic fields with the class number one.

We should note that the problem of determining elliptic curves with everywhere good reduction defined over quadratic fields having $\mathbb{Q}$ -rational j -invariant has been discussed by Setzer in [12].

§ 1. Let k be an algebraic number field of finite degree, $\mathfrak{o}_k$ the maximal order of k and $\mathfrak{o}_k^\times$ the group of units of k . Any elliptic curve defined over k has a plane cubic model of the form:

$$(1) \quad y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6,$$

where the coefficients a_i are in $\mathfrak{o}_k$. Following Tate [10], we define associated

quantities b_i , c_i , Δ , and j by the following formulas:

$$\begin{aligned} b_2 &= a_1^2 + 4a_2, & b_4 &= a_1a_3 + 2a_4, & b_6 &= a_3^2 + 4a_6, \\ b_8 &= a_1^2a_6 - a_1a_3a_4 + 4a_2a_6 + a_2a_3^2 - a_4^2, \\ c_4 &= b_2^2 - 24b_6, & c_6 &= 36b_2b_4 - b_2^3 - 216b_6, \\ \Delta &= b_4^3 - 27b_6^2 + b_6(36b_4 - b_2^2), & j &= c_4^3/\Delta. \end{aligned}$$

We call the equation (1) global minimal if $\text{ord}_{\mathfrak{p}}(\Delta)$ is minimal for any prime ideal $\mathfrak{p}$ of k , where $\text{ord}_{\mathfrak{p}}(\ )$ is the additive valuation of k at $\mathfrak{p}$. There exists such a global minimal equation if the class number of k is one. In such a case the discriminant Δ of a global minimal equation of an elliptic curve with everywhere good reduction defined over k is a unit of k .

§ 2. Let k be a real quadratic field and σ be a generator of $\text{Gal}(k/\mathbb{Q})$. We consider the following Diophantine equation:

$$(2) \quad 2^6\varepsilon_1 + \varepsilon_2 = a^2,$$

where ε_1 and ε_2 are units of k and a is an integer of k . Then we can prove the following.

LEMMA 2.1. *Let m be the discriminant of k and suppose that $m \equiv 1 \pmod{4}$. Let u_0 ($u_0 > 1$) be a fundamental unit and suppose that $N_{k/\mathbb{Q}}(u_0) = -1$.*

- (i) *If $m \neq 41, 65$, the equation (2) has no solution.*
- (ii) *If $m = 41$, the solutions of (2) are given by $(\varepsilon_1, \varepsilon_2, a) = (\varepsilon^2 u_0, \varepsilon^2, \varepsilon u_0)$, $(\varepsilon^2 u_0^\sigma, \varepsilon^2, \varepsilon u_0^\sigma)$, where ε is any unit of k .*
- (iii) *If $m = 65$, the solutions of (2) are given by $(\varepsilon_1, \varepsilon_2, a) = (\varepsilon^2 u_0^3, \varepsilon^2, \varepsilon(2u_0^2 - 1))$, $(\varepsilon^2 u_0^{3\sigma}, \varepsilon^2, \varepsilon(2u_0^2 - 1)^\sigma)$, $(\varepsilon^2, \varepsilon^2, \varepsilon\sqrt{65})$, where ε is any unit of k .*

PROOF. Let $(\varepsilon_1, \varepsilon_2, a)$ be a solution of (2). Then $(\varepsilon^2\varepsilon_1, \varepsilon^2\varepsilon_2, \varepsilon a)$ is also a solution for any unit ε . Therefore we may assume that $\varepsilon_2 = \pm 1, \pm u_0$. Since $\varepsilon_2 \equiv a^2 \pmod{4}$, we have $N_{k/\mathbb{Q}}(\varepsilon_2) = 1$. Therefore $\varepsilon_2 = \pm 1$ by assumption. Furthermore it is easy to prove that $\varepsilon_2 = 1$. Put $\alpha = (a-1)/2$. By (2), we have $\alpha(\alpha+1) = 2^4\varepsilon_1$.

Case (I). Assume that either α or $\alpha+1$ is a unit. Then we may assume that $\alpha+1$ is a unit by changing a by $-a$ if necessary. Put $\alpha = 2^4u$ and $\alpha+1 = v$, then u and v are units of k and $2^4u+1=v$. By considering the norms of the both sides, we have $N_{k/\mathbb{Q}}(v)=1$ and $2^4N_{k/\mathbb{Q}}(u)+\text{Tr}(u)=0$, where $\text{Tr}(u)$ is the trace of u . Thus we have $u=8\pm\sqrt{65}$, $\alpha=2u^2-1$ and $\varepsilon_1=u^3$.

Case (II). Assume that neither α nor $\alpha+1$ is a unit. In this case 2 must be decomposed into two prime factors $\mathfrak{p}$ and $\mathfrak{p}^\sigma$ in k , and $\alpha\mathfrak{o}_k = \mathfrak{p}^4$, $(\alpha+1)\mathfrak{o}_k = \mathfrak{p}^{\sigma^4}$. Put $\alpha+1 = u\alpha^\sigma$, where u is a unit of k . Then $N_{k/\mathbb{Q}}(\alpha) + \text{Tr}(\alpha) + 1 = N_{k/\mathbb{Q}}(u)N_{k/\mathbb{Q}}(\alpha)$.

If $N_{k/Q}(\alpha) = 2^4$, we have $N_{k/Q}(u) = -1$ and $\text{Tr}(\alpha) = -33$. Then we have $\alpha = (-33 \pm 5\sqrt{41})/2$. Hence $a = -u_0$ or $-u_0^\sigma$, and we have $\varepsilon_1 = u_0$ or u_0^σ . On the other hand, if $N_{k/Q}(\alpha) = -2^4$, we have solutions $(\varepsilon_1, a) = (1, \pm\sqrt{65})$, (u_0, u_0) , (u_0^σ, u_0^σ) similarly. This completes the proof of Lemma 2.1.

Let p be a prime number such that $p \equiv 5 \pmod{8}$. Put $k = \mathbb{Q}(\sqrt{p})$ and assume that the class number of k is one. Let E be an elliptic curve with everywhere good reduction defined over k . By the remark in §1, we can take an equation (1) of E with $\Delta \in \mathfrak{o}_k^\times$. Let $P = (r, t)$ be a rational point of order 2 of E . Under a coordinate-transformation $x = x' + r$, $y = y' + t$, (1) is transformed to $y'^2 + a'_1xy + a'_3y = x'^3 + a'_2x'^2 + a'_4x + a'_6$, where the coefficients are determined by the following;

$$a'_1 = a_1, \quad a'_2 = a_2 + 3r, \quad a'_3 = a_3 + ra_1 + 2t = 0.$$

$$a'_4 = a_4 - 2ra_2 - ta_1 + 3r^2, \quad a'_6 = 0.$$

The associated b'_i are $b'_2 = b_2 + 12r$, $b'_4 = b_4 + rb_2 + 6r^2$ and $b'_6 = b_6 + 2rb_4 + r^2b_2 + 4r^3 = 0$. (cf. Ogg[4], Tate [10]). Then $4r \in \mathfrak{o}_k$, so $8b'_4 \in \mathfrak{o}_k$ and $b'_2 \in \mathfrak{o}_k$. Put $b = 8b_4$, then we have $2^8\Delta = b^2(b'^2_2 - 4b)$. If $N_{k/Q}(b)$ is odd, then $b \in \mathfrak{o}_k^\times$ and $b'_2 \equiv 0 \pmod{2}$. Put $a = b'_2/2$, then we have $2^8\Delta = b^2(a^2 - b)$. This is a contradiction by Lemma 2.1. Therefore $N_{k/Q}(b)$ is even. Since $2\mathfrak{o}_k$ is a prime ideal of k , $b \equiv 0 \pmod{2}$. Then we have $4b'_4 \in \mathfrak{o}_k$ and $\text{ord}_2(r) \geq -1$. Hence $r \in \mathfrak{o}_k$ and $s \in \mathfrak{o}_k$. Thus we may assume that $(r, s) = (0, 0)$. Then we have $a_3 = a_6 = 0$ and $\Delta = a_4^2(b_2^2 - 2^6a_4)$. Since $\Delta \in \mathfrak{o}_k^\times$, we have $a_4 \in \mathfrak{o}_k^\times$. This is a contradiction by Lemma 2.1. Thus we have;

PROPOSITION 2.2. *Let p be a prime number such that $p \equiv 5 \pmod{8}$, and put $k = \mathbb{Q}(\sqrt{p})$. Let E be an elliptic curve with everywhere good reduction defined over k . Assume that the class number of k is one. Then E has no rational point of order 2.*

The main theorem of this section is the following;

THEOREM 2.3. *Let p be a prime number such that $p \equiv 5 \pmod{8}$ and u_0 be a fundamental unit of $k = \mathbb{Q}(\sqrt{p})$. Assume the following conditions (i), (ii) and (iii).*

- (i) *The class number of k is one.*
- (ii) *$\text{Tr}(u_0)$ is odd.*
- (iii) *The class numbers of $\mathbb{Q}(\sqrt{-p})$ and $k(\sqrt{u_0})$ are prime to 3. Then there exists no elliptic curve with everywhere good reduction defined over k .*

PROOF. Let E be an elliptic curve with everywhere good reduction defined over k . Take a global minimal equation of E . We denote its discriminant

inant by Δ . Then $\Delta \in \mathfrak{o}_k^\times$. By the assumption (iii) the class number of $K = k(\sqrt{\Delta})$ is prime to 3. Put $L = k({}_3E)$, where ${}_3E$ is the kernel of 2-multiplication of E . Then L is a cyclic extension of K of degree 3 by Proposition 2.2. If $E \bmod 2$ is ordinary, L is unramified over K by [6, p. 274, Cor.]. Since the class number of K is prime to 3, this is a contradiction. Thus $E \bmod 2$ is supersingular. Then K is unramified over k by [6, p. 275, Prop. 12]. Since the class number of k in narrow sense is one, we have $K = k$. Hence L is a cyclic extension of k of degree 2 which is unramified outside $\{2\}$. (cf. [8].) Since $\text{Tr}(u_0)$ is odd, there exists no such extension of k . This completes the proof of Theorem 2.3.

EXAMPLE. Let $k = \mathbb{Q}(\sqrt{5})$ or $\mathbb{Q}(\sqrt{13})$ and u_0 a fundamental unit of k . Here we note that the class number of $k(\sqrt{u_0})$ is one. In this case the assumptions (i), (ii) and (iii) of Theorem 2.3 are satisfied.

§ 3. In this section we consider elliptic curves defined over certain imaginary quadratic fields. The following lemma is easy to prove.

LEMMA 3.1. *Let k be an imaginary quadratic field. Assume that the class number of k is odd. Then there exists at most one quadratic extension of k which is unramified outside $\{3\}$.*

REMARK. More precisely, such an extension is $k(\sqrt{-3})$ unless $k = \mathbb{Q}(\sqrt{-3})$. In case $k = \mathbb{Q}(\sqrt{-3})$, there exists no such extension.

THEOREM 3.2. *Let k be an imaginary quadratic field. Assume that the class number of k is odd. Let E be a semi-stable elliptic curve defined over k such that the discriminant Δ is a cube in k . Then, ${}_3E = \mu_3 \oplus \mathbb{Z}/3\mathbb{Z}$ as $\text{Gal}(\bar{k}/k)$ -modules. In particular, E has a k -rational point of order 3.*

PROOF. Put $L = k({}_3E)$. Since Δ is a cube, L is a 2-extension of k which is unramified outside $\{3\}$. In case $k = \mathbb{Q}(\sqrt{-3})$, we have $L = k$. Hence we have ${}_3E = \mu_3 \oplus \mathbb{Z}/3\mathbb{Z}$ in this case. In case $k \neq \mathbb{Q}(\sqrt{-3})$, $\text{Gal}(L/k)$ is cyclic because it has unique subgroup of index 2 by Lemma 3.1. Then a simple calculation of the class number mod 3 implies that $[L:k] = 2$ or 4, $\text{Gal}(L/k)$ is a cyclic subgroup of 2-Sylow subgroup of $\text{GL}_2(F_3)$ which is not included in $\text{SL}_2(F_3)$ because $k \neq \mathbb{Q}(\sqrt{-3})$. It is easy to show that $\text{Gal}(L/k)$ is conjugate to the subgroup generated by $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ in $\text{GL}_2(F_3)$. This completes the proof.

REMARK. We should note that similar argument is discussed in [1].

COROLLARY 3.3. *Let k be an imaginary quadratic field. Assume that the*

class number of k is prime to 6. Then there exists no elliptic curve with everywhere good reduction defined over k .

PROOF. When the class number of k is one, the assertion is proved by Stroeker [9]. So we may assume that $k \neq \mathbb{Q}(\sqrt{-3})$. Let E be an elliptic curve with everywhere good reduction defined over k . By [2, Lemma 4], Δ is a cube in k . Then E has a rational point of order 3 by Theorem 3.2. This contradicts [2, Lemma 3]. This completes the proof.

§ 4. In this section we consider rational points of order 5 of elliptic curves defined over certain real quadratic fields.

PROPOSITION 4.1. Let p be a prime number such that $p \equiv 1 \pmod{8}$. Put $k = \mathbb{Q}(\sqrt{p})$. Assume that the class number of k is one. Let E be an elliptic curve with everywhere good reduction defined over k . Then E has no rational point of order 5.

PROOF. Let (1) (in § 1) be a global minimal equation of E and $P = (r, s)$ a rational point of order 5. Since 5 is unramified in k , r and s are integers of k . Then we may assume that $P = (0, 0)$. Similarly as in [3], we can transform (1) to the following;

$$y^2 + ab^2y = x^3 + \frac{1}{4}(a^2 + 6ab + b^2)x^2 + \frac{1}{2}ab^2(a + b)x,$$

where $a, b \in \mathcal{O}_k$ with $a \equiv b \equiv 1 \pmod{2}$. Then the discriminant $\Delta = -a^5b^5(a^2 + 11ab - b^2)$. Since Δ is a unit of k , a, b and $a^2 + 11ab - b^2$ are units of k . But it is easy to show that there exists no such pair of units. This completes the proof.

REMARK. As a byproduct of this argument, we obtain an elliptic curve 5.4 in § 5.

§ 5. Examples

In this section we consider some examples of elliptic curves with everywhere good reduction.

5.1. Let n be a positive integer which is prime to 6. Let j be a root of the equation $j(j - 1728) = \pm n^{12}$. In [9], Stroeker has proved that the following elliptic curve E has good reduction everywhere over $k = \mathbb{Q}(\sqrt{2^{10}3^8 \pm n^{12}})$.

$$E: y^2 + xy = x^3 - \frac{36}{j - 1728}x - \frac{1}{j - 1728}.$$

For example in case $n=5$, the class number of k is $2^3 \cdot 3 \cdot 7$. (This is calculated by M. Kamei.)

5.2. Let $k = \mathbb{Q}(\sqrt{733})$. The class number of k is 3.

$$E: y^2 + xy - u_0 y = x^3; \quad \Delta = -u_0^5,$$

where $u_0 = (27 + \sqrt{733})/2$. Then E has good reduction everywhere over k .

5.3. Let $k = \mathbb{Q}(\sqrt{7})$ and $u_0 = 8 + 3\sqrt{7}$.

$$E: y^2 + xy = x^3 - 2u_0 x^2 + u_0^2 x; \quad \Delta = -u_0^6; \quad j = -15^3.$$

This example is due to Stroeker [9]. It is easy to see that $\text{End}(E) \otimes \mathbb{Q} = \mathbb{Q}(\sqrt{-7})$.

5.4. Let $k = \mathbb{Q}(\sqrt{37})$ and $u_0 = 6 + \sqrt{37}$.

$$E: y^2 - u_0 y = x^3 + ax^2 + bx; \quad \Delta = u_0^6; \quad j = 2^{12},$$

where $a = (3u_0 + 1)/2$ and $b = (11u_0 + 1)/2$. (cf. [12, P. 245].) $(0, 0)$ is a rational point of order 5. By [2, Cor. 2 and Lemma 2], we have $\text{rank}(E) = \text{III}(5) = 0$, where $\text{III}(5)$ is the 5-torsion part of the Shafarevich-Tate group of E . Since the strict class number of k is one, we can show that $\text{End}(E) = \mathbb{Z}$ and E is isomorphic to E^σ over k by [5], where σ is a non-trivial automorphism of k .

REMARK. Recently, K. Shiota showed that this elliptic curve is isogenous to Shimura's elliptic curve which is attached to a cusp form of $S_2(\Gamma_0(37), \psi)$. (cf. [7].)

5.5. Let $k = \mathbb{Q}(\sqrt{29})$ and $u_0 = (5 + \sqrt{29})/2$.

$$E: y^2 + xy + u_0^2 y = x^3; \quad \Delta = -u_0^{10}.$$

This example is due to Tate (cf. [6]). The class number of $k(\sqrt{\Delta})$ is 3.

The following two examples are obtained as byproducts of Lemma 2.1.

5.6. Let $k = \mathbb{Q}(\sqrt{41})$ and $u_0 = 32 + 5\sqrt{41}$.

$$E: y^2 + xy = x^3 - u_0 x; \quad \Delta = u_0^4.$$

This example is due to Oort (cf. [9]). E has three rational points of order 2.

5.7. Let $k = \mathbb{Q}(\sqrt{65})$ and $u_0 = 8 + \sqrt{65}$.

$$E: y^2 + xy = x^3 + 8u_0 x^2 + u_0^3 x; \quad \Delta = u_0^6; \quad j = 257^3.$$

E has also three rational points of order 2.

APPENDIX

Let $k = \mathbb{Q}(\sqrt{d})$ be an imaginary quadratic field with class number 1 such that $k \neq \mathbb{Q}(\sqrt{-3})$. We give a table of elliptic curves of prime power conductor with rational point of order 3 defined over k . In case $k = \mathbb{Q}$, Miyawaki has given a table of such curves in [3]. So we are concerned only with elliptic curves which are not included in Miyawaki's table.

TABLE $E: y^2 + a_1xy + a_3y = x^3$.

k	a_3	a_1	Δ
$\mathbb{Q}(\sqrt{-1})$	$26 + 9\sqrt{-1}$	$\sqrt{-1} + 9$	$-\sqrt{-1}(26 + 9\sqrt{-1})^3$
$\mathbb{Q}(\sqrt{-1})$	$26 - 9\sqrt{-1}$	$-\sqrt{-1} + 9$	$\sqrt{-1}(26 - 9\sqrt{-1})^3$
$\mathbb{Q}(\sqrt{-1})$	1	$\sqrt{-1} + 3$	$-9 + 26\sqrt{-1}$
$\mathbb{Q}(\sqrt{-1})$	1	$-\sqrt{-1} + 3$	$-9 - 26\sqrt{-1}$
$\mathbb{Q}(\sqrt{11})$	1	$3 + \pi^3$	$\pi^6 \quad (\pi = (1 \pm \sqrt{-11})/2)$
$\mathbb{Q}(\sqrt{-2})$	1	$3 + \rho^3$	$-\rho^6 \quad (\rho = 1 \pm \sqrt{-2})$

We should note that $N_{k/\mathbb{Q}}(26 + 9\sqrt{-1}) = 757$ is a prime number.

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