On decaying entire positive solutions of semilinear elliptic equations

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§ 1. Introduction

This paper is concerned with the existence of positive entire solutions of semilinear equations of the form

$$(1.1) \quad Lu = \lambda f(x, u) \quad \text{in } \mathbb{R}^N, N \geq 2,$$

where L denotes a uniformly elliptic differential operator with smooth real valued coefficients defined by

$$(1.2) \quad Lu = - \left(\sum_{i,j=1}^N a_{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^N b_i(x) \frac{\partial u}{\partial x_i} \right) + c(x)u,$$

$f(x, u)$ is a locally Hölder continuous function in $\mathbb{R}^N \times \bar{\mathbb{R}}_+$, $\bar{\mathbb{R}}_+ = [0, \infty)$, and λ is a real constant.

By an *entire solution* of (1.1) we mean a function $u \in C^2(\mathbb{R}^N)$ which satisfies (1.1) throughout $\mathbb{R}^N$. Our purpose here is to give criteria for the existence of decaying entire positive solutions of (1.1), i.e., those entire solutions which are positive in $\mathbb{R}^N$ and have the uniform limit zero as $|x| \rightarrow \infty$, under suitable hypotheses on the nonlinearity of $f(x, u)$.

Our first theorem, Theorem 2.1 below, establishes the existence of a decaying entire positive solution $u(x)$ of (1.1) for sufficiently small $\lambda > 0$ under the hypotheses that $f(x, u) \geq 0$ and $\lim_{u \rightarrow 0^+} f(x, u)/u = \infty$ uniformly in some bounded domain in $\mathbb{R}^N$. On the other hand, in Theorem 2.2 we show that there exists a *maximal decaying entire positive solution* of (1.1) in the case where $f(x, u)$ has bounded nonlinearity at $u = \infty$ (see hypothesis (F_4) below); we also investigate the case where $\lim_{u \rightarrow 0^+} f(x, u)/u$ is finite. As a byproduct of Theorem 2.2 we can prove the uniqueness of decaying entire positive solution of the equation

$$-\Delta u + c(x)u = \lambda f(x, u) \quad \text{in } \mathbb{R}^N, N \geq 3,$$

when $f(x, u)/u$ is strictly decreasing in $u > 0$ (Theorem 5.1). Our Theorems 2.1

and 2.2 can be applied to the following mixed sublinear and superlinear equations

$$(1.3) \quad -\Delta u + c(x)u = \lambda(\Phi(x)u^r + \Psi(x)u^s) \quad \text{in } R^N, N \geq 3$$

and

$$(1.4) \quad -\Delta u + c(x)u = \lambda(\Phi(x)u^r - \Psi(x)u^s) \quad \text{in } R^N, N \geq 3,$$

where $0 \leq r < 1 \leq s$, $r \neq s$, and c , Φ and Ψ are nonnegative locally Hölder continuous functions in R^N . For (1.4) the possibility of $r=1$ is allowed. The statements of Theorems 2.1 and 2.2 are given in § 2. The proofs of these theorems are presented in § 4. Analytical tools which are needed in the proofs are provided in § 3. Section 5 concerns specialization of the main results to equations (1.3) and (1.4).

The existence of entire positive solutions of (1.1) has been proved under various structure hypotheses for L and $f(x, u)$ by many authors (see e.g. [3], [6], [7], [9], [11]–[19], [21]). Among them the results of Fukagai [6, 7], of Kusano and Trench [15] and of Noussair and Swanson [19] are closely related to this paper. Kusano and Trench [15, Theorem 2] gave sufficient conditions for (1.3) to have decaying entire positive solution in the case where $c(x) \equiv 0$, and Φ and Ψ are radially symmetric functions. As far as we are aware, there is no result on the existence of decaying entire solutions for the general equation (1.1) except for a very recent result in [19], where systems of equations generalizing (1.1) are considered.

§ 2. Statement of main theorems

Throughout this paper we assume that the coefficients of L of (1.2) satisfy $a_{ij} = a_{ji} \in C_{loc}^{1+\alpha}(R^N)$, $b_i, c \in C_{loc}^\alpha(R^N)$, $i, j = 1, \dots, N$, $0 < \alpha < 1$ and $c(x) \geq 0$ in R^N and that L is uniformly elliptic in R^N , i.e., there is a constant $a_0 > 0$ such that

$$\sum_{i,j=1}^N a_{ij}(x) \xi_i \xi_j \geq a_0 |\xi|^2 \quad \text{for } x, \xi \in R^N,$$

where $|\xi|$ denotes the Euclidean length of $\xi \in R^N$.

The following notation is employed:

$$R_+ = (0, \infty), \quad \bar{R}_+ = [0, \infty), \quad B_R = \{x \in R^N : |x| < R\} \quad \text{for } R > 0,$$

$$a(x) = \sum_{i,j=1}^N a_{ij}(x) x_i x_j / |x|^2,$$

$$b(x) = \left(\sum_{i=1}^N (b_i(x) x_i + a_{ii}(x)) - a(x) \right) / |x|,$$

$$b_*(r) = \min_{|x|=r} \frac{b(x)}{a(x)},$$

$$p_*(r) = \exp\left(\int_R^r b_*(s) ds\right), \quad \pi_*(r) = \int_r^\infty \frac{ds}{p_*(s)} \quad (\text{if the integral converges}),$$

$$h^*(r) = \max_{|x|=r} \frac{h(x)}{a(x)} \quad \text{for a continuous function } h.$$

Making use of the above a and b , we obtain an identity

$$Lu(r) = -a(x)u''(r) - b(x)u'(r) + c(x)u(r), \quad r = |x| > 0$$

for a smooth function $u(r)$ defined in R_+ , where $' = d/dr$.

In this paper, we make the following standing hypothesis (H) for the operator L :

$$(H) \quad \int_R^\infty \frac{ds}{p_*(s)} < \infty,$$

so that the integral $\pi_*(r)$ exists for any $r > 0$. It is easy to see that in the case where $L = -\Delta$, Δ being the N -dimensional Laplacian, hypothesis (H) holds if and only if $N \geq 3$. For other examples of L satisfying (H) we refer to [9] (see also Remark 4.1 below).

The conditions below will be imposed on the function $f(x, u)$ in (1.1).

(F₁) $f(x, u) \in C_{loc}^\theta(R^N \times \bar{R}_+)$, $0 < \theta < 1$. For any bounded domain $\Omega \subset R^N$ and any constant $U > 0$, there corresponds a positive constant M such that $f(x, u) + Mu$ is nondecreasing in $u \in [0, U]$ for any fixed $x \in \Omega$.

(F₂) There exists a positive function $m \in C_{loc}^\alpha(R^N)$ such that for any $U > 0$

$$0 \leq f(x, u) \leq C_U m(x), \quad x \in R^N, \quad 0 < u \leq U,$$

where C_U is a positive constant depending only on U .

(F₃) $\lim_{u \rightarrow 0+} (f(x, u)/u) = +\infty$ uniformly on some bounded domain Ω_0 in R^N .

Condition (F₃) is satisfied, for example, if $f(x, 0) > 0$ in R^N .

EXAMPLE 2.1. The function

$$f(x, u) = \Phi(x)u^r + \Psi(x)u^\delta, \quad 0 \leq r < 1, \quad \delta \geq r$$

satisfies (F₁)–(F₃) provided $\Phi, \Psi \in C_{loc}^\alpha(R^N)$, $\Phi(x) > 0$ and $\Psi(x) \geq 0$ in R^N . The possibility of $\delta \geq 1$ or $\Psi(x) \equiv 0$ in R^N is allowed.

The first main theorem in this paper is the following.

THEOREM 2.1. Assume that (H), (F₁)–(F₃) are satisfied. If

$$(2.1) \quad \int_R^\infty \pi_*(r) p_*(r) m^*(r) dr < \infty,$$

then there exists $\lambda^* > 0$ such that, for every $\lambda \in (0, \lambda^*)$, (1.1) possesses an entire positive solution u satisfying $\lim_{|x| \rightarrow \infty} u(x) = 0$.

Before stating the next result, we prepare some notation and terminology. Let λ^* and λ_* denote, respectively, the least upper bound and the greatest lower bound of $\lambda > 0$ for which (1.1) has an entire positive solution u satisfying $\lim_{|x| \rightarrow \infty} u(x) = 0$.

For a locally Hölder continuous positive function $\tilde{m}$ in $\mathbf{R}^N$ and $n = 1, 2, \dots$, let $\lambda_n(\tilde{m})$ be the principal eigenvalue of the Dirichlet problem

$$(2.2) \quad L\varphi = \lambda \tilde{m}(x)\varphi \quad \text{in } B_n, \quad \varphi = 0 \quad \text{on } \partial B_n.$$

It is well known that $\lambda_n(\tilde{m})$ is positive by Krein-Rutman's theory ([1, Theorem 4.3]) and decreases in n (see e.g. [9, p. 176]). Let us denote the limit of $\lambda_n(\tilde{m})$ as $n \rightarrow \infty$ by $\lambda_0(\tilde{m})$: $\lambda_0(\tilde{m}) = \lim_{n \rightarrow \infty} \lambda_n(\tilde{m})$.

An entire positive solution $\bar{u}$ of (1.1) is said to be a *maximal decaying entire positive solution* of (1.1) if $\lim_{|x| \rightarrow \infty} \bar{u}(x) = 0$ and $0 < u(x) \leq \bar{u}(x)$ in $\mathbf{R}^N$ for any entire positive solution u of (1.1) such that $\lim_{|x| \rightarrow \infty} u(x) = 0$.

The following conditions on $f(x, u)$ are needed in establishing the existence of a maximal decaying entire positive solution of (1.1).

(F₄) There exist a positive function $m \in C_{loc}^\alpha(\mathbf{R}^N)$ and a nonnegative function $g \in C_{loc}^\alpha(\mathbf{R}^N)$ such that, for any constant $\mu > 0$,

$$f(x, u) \leq \mu m(x)u + C(\mu)g(x), \quad x \in \mathbf{R}^N, \quad u \in \bar{\mathbf{R}}_+,$$

where $C(\mu)$ is a positive constant depending only on $\mu > 0$ but not on u .

(F₅) There exists a positive function $\tilde{m} \in C_{loc}^\alpha(\mathbf{R}^N)$ such that

$$\lim_{u \rightarrow 0+} \frac{f(x, u)}{\tilde{m}(x)u} = 1 \quad \text{uniformly in } \mathbf{R}^N.$$

EXAMPLE 2.2. Let Φ and Ψ be as in Example 2.1. Then the function

$$f(x, u) = \Phi(x)u^\gamma - \Psi(x)u^\delta, \quad 0 \leq \gamma \leq 1 \leq \delta, \quad \gamma \neq \delta$$

satisfies (F₄) and either (F₃) or (F₅) provided $\Psi(x)/\Phi(x)$ is bounded above and below by some positive constants in $\mathbf{R}^N$. In case $\gamma \neq 1$, $\Psi(x) \equiv 0$ is allowed.

THEOREM 2.2. Assume that (H) holds and f satisfies (F₁), (F₄) and either (F₃) or (F₅). Suppose for m and g in (F₄) that

$$(2.3) \quad \int_R^\infty \pi_*(r) p_*(r) \{m^*(r) + g^*(r)\} dr < \infty$$

for some $R > 0$.

(i) If f satisfies (F_3) , then $\lambda_* = 0$ and $\lambda^* = \infty$.

(ii) If f satisfies (F_5) , then $\lambda_* \leq \lambda_0(\tilde{m})$ and $\lambda^* = \infty$. In particular, λ_* is positive provided

$$(2.4) \quad \sup_{x \in \mathbb{R}^N, u > 0} \frac{f(x, u)}{m(x)u} < \infty.$$

Furthermore, for every $\lambda > \lambda_*$, (1.1) possesses a maximal decaying entire positive solution $\bar{u}$ in both cases (i) and (ii) above.

§ 3. Preliminaries

Let $f(x, u)$ be a locally Hölder continuous function defined in $\mathbb{R}^N \times \mathbb{R}$. A continuous function v in $\mathbb{R}^N$ is said to be a *supersolution* of

$$(3.1) \quad Lu = f(x, u) \quad \text{in } \mathbb{R}^N,$$

if there exists a smooth bounded domain G_1 in $\mathbb{R}^N$ such that $v_j = v|_{\bar{G}_j}$ (the restriction of v to $\bar{G}_j$, where $G_2 = \mathbb{R}^N \setminus \bar{G}_1$), $j = 1, 2$, satisfy the conditions

$$\begin{aligned} v_j &\in C^2(G_j) \cap C^1(\bar{G}_j), \\ Lv_j &\geq f(x, v_j), \quad x \in G_j, \\ \frac{\partial v_1}{\partial \nu} &\geq \frac{\partial v_2}{\partial \nu} \quad \text{on } \partial G_1, \end{aligned}$$

where ν is an outward (with respect to G_1) conormal vector to ∂G_1 associated with L .

A subsolution of (3.1) is defined by reversing the above inequality signs.

LEMMA 3.1. Assume that $f \in C_{loc}^\theta(\mathbb{R}^N \times \mathbb{R})$, $0 < \theta < 1$, and that, for any bounded domain $\Omega \subset \mathbb{R}^N$ and any finite interval $I \subset \mathbb{R}$, there is a constant $M > 0$ such that $f(x, u) + Mu$ is nondecreasing in $u \in I$ for any fixed $x \in \Omega$. Then the existence of a supersolution v and a subsolution w of (3.1) with the property $w(x) \leq v(x)$ in $\mathbb{R}^N$ implies the existence of a solution $\bar{u} \in C^2(\mathbb{R}^N)$ of (3.1) such that the inequalities

$$w(x) \leq u(x) \leq \bar{u}(x) \leq v(x), \quad x \in \mathbb{R}^N$$

hold for any solution u of (3.1) satisfying $w(x) \leq u(x) \leq v(x)$ in $\mathbb{R}^N$.

The proof of this lemma is the same as that of [9, Lemma 4.2], and will be omitted.

The next lemma is an analogue of Lemma 5.3 of [8] concerning the exterior boundary value problems for $Lu = \lambda f(x, u)$.

LEMMA 3.2. Let m be a continuous function in R^N and λ be a constant. Assume that there is a function $V \in C^2(R^N)$ satisfying

$$(3.2) \quad LV(x) \geq \lambda m(x)V(x), \quad V(x) > 0, \quad x \in R^N.$$

If $u \in C^2(R^N)$ satisfies

$$(3.3) \quad Lu(x) \geq \lambda m(x)u(x), \quad x \in R^N$$

and

$$(3.4) \quad \liminf_{|x| \rightarrow \infty} \frac{u(x)}{V(x)} \geq 0$$

then $u(x) > 0$ in R^N unless $u(x) \equiv 0$.

Proof. We first prove the nonnegativity of u . Fix an $\varepsilon > 0$ and choose an $R > 0$ so that

$$(3.5) \quad \frac{u(x)}{V(x)} \geq -\varepsilon, \quad |x| \geq R.$$

Then, by (3.2), (3.3) and (3.5) the function $\tilde{u}(x) = u(x) + \varepsilon V(x)$ satisfies

$$L\tilde{u}(x) - \lambda m(x)\tilde{u}(x) \geq 0 \quad \text{in } B_R, \quad \tilde{u}(x) \geq 0 \quad \text{on } \partial B_R.$$

Hence, $\tilde{u}(x)/V(x) \geq 0$ in R^N by the generalized maximum principle ([20, p. 73, Theorem 10]). Combining this with (3.5), we have

$$\tilde{u}(x) = u(x) + \varepsilon V(x) \geq 0 \quad \text{in } R^N.$$

Since $\varepsilon > 0$ is arbitrary, we conclude that $u(x) \geq 0$ in R^N .

Next, suppose that $u(x) \geq 0$ and $u(x) \not\equiv 0$. Then, by (3.3) and the generalized maximum principle mentioned above the function $u(x)/V(x)$ cannot attain a nonpositive minimum in R^N . This implies $u(x)/V(x) > 0$ and so $u(x) > 0$ in R^N .

The next lemma concerns the following initial value problem for an ordinary differential equation

$$(3.6) \quad \begin{aligned} (p(r)y')' + \lambda q(r)y + h(r) &= 0, & r > R, \\ y(R) &= \xi, & y'(R) = -\eta, \end{aligned}$$

where $' = d/dr$, $\lambda \in R$, and the functions p , q and h satisfy the conditions:

$$\begin{aligned} p \in C^1([R, \infty)) \text{ and } p(r) > 0 \text{ in } [R, \infty), \quad q, h \in C([R, \infty)) \text{ and} \\ q(r) \geq 0, \quad h(r) \geq 0 \text{ and } q(r) \not\equiv 0, \quad h(r) \not\equiv 0 \text{ in } [R, \infty). \end{aligned}$$

LEMMA 3.3. Assume that $\pi(r) \equiv \int_r^\infty ds/p(s) < \infty$ for $r \geq R$ and

$$(3.7) \quad \int_R^\infty \pi(r)\{q(r)+h(r)\}dr < \infty,$$

and let $0 \leq \lambda < \left(\int_R^\infty \pi(r)q(r)dr\right)^{-1}$. If $\xi_0 > 0$ is sufficiently large, then for every $\xi > \xi_0$, there corresponds a constant $\eta = \eta(\xi) > 0$ such that the solution y of (3.6) satisfies

$$y(r) > 0, \quad y'(r) < 0, \quad r \in [R, \infty) \quad \text{and} \quad \lim_{r \rightarrow \infty} y(r) = 0.$$

PROOF. Put $\kappa = \int_R^\infty \pi(r)q(r)dr$ and let $0 \leq \lambda < \kappa^{-1}$. We begin by proving that the solution z_0 of the initial value problem

$$(3.8) \quad (p(r)z')' + \lambda q(r)z = 0, \quad r > R,$$

$$(3.9) \quad z(R) = 0, \quad z'(R) = 1$$

is bounded and positive in (R, ∞) . In fact, by the change of variables

$$s = \frac{1}{\pi(r)}, \quad r \geq R, \quad \text{and} \quad Z(s) = \frac{z(r(s))}{\pi(r(s))}, \quad s \geq S = \frac{1}{\pi(R)},$$

(3.8) becomes

$$(3.10) \quad \frac{d^2 Z}{ds^2} + \lambda(p\pi^4 q)(r(s))Z = 0, \quad s > S.$$

Since

$$\int_S^\infty s(p\pi^4 q)(r(s))ds = \int_R^\infty \pi(r)q(r)dr < \infty,$$

(3.10) has a pair of solutions Z_1 and Z_2 such that

$$\lim_{s \rightarrow \infty} \frac{Z_1(s)}{s} = 1 \quad \text{and} \quad \lim_{s \rightarrow \infty} Z_2(s) = 1$$

([10, p. 428, Theorem 9.1.1]). Consequently, (3.8) has a pair of solutions z_1 and z_2 such that $\lim_{r \rightarrow \infty} z_1(r) = 1$ and $\lim_{r \rightarrow \infty} z_2(r) = 0$, which implies the boundedness of z_0 .

To prove the positivity of z_0 , suppose that z_0 has a zero in (R, ∞) . Then, there are $R_1 > R$ and $\tilde{R} \in (R, R_1)$ such that

$$z_0(r) > 0, \quad R < r < R_1, \quad z_0(R_1) = 0 \quad \text{and} \quad z'_0(\tilde{R}) = 0.$$

In view of the representation for z_0

$$z_0(r) = z_0(\tilde{R}) - \lambda \int_{\tilde{R}}^r (\pi(t) - \pi(r)) q(t) z_0(t) dt,$$

which follows from (3.8), we have

$$\begin{aligned} z_0(R_1) &= z_0(\tilde{R}) - \lambda \int_{\tilde{R}}^{R_1} (\pi(t) - \pi(R_1)) q(t) z_0(t) dt \\ &\geq z_0(\tilde{R}) \left(1 - \lambda \int_{\tilde{R}}^{R_1} \pi(t) q(t) dt \right) \geq z_0(\tilde{R}) (1 - \lambda \kappa) > 0, \end{aligned}$$

where we have used the fact that $z'_0(r) \leq 0$ for $r \in (\tilde{R}, R_1)$. This contradiction shows that $z_0(r) > 0$ throughout (R, ∞) .

Now, let us prove the assertion of the lemma. Put

$$\sigma = \int_R^\infty \pi(r) h(r) dr \quad \text{and} \quad \xi_0 = \sigma(1 - \lambda \kappa)^{-1} \quad \text{for} \quad 0 \leq \lambda < \kappa^{-1}.$$

For a fixed $\xi > \xi_0$ we denote the unique solution of (3.6) by $y(r; \eta)$. Integrating (3.6) twice from R to r , we have

$$\begin{aligned} (3.11) \quad y(r; \eta) &= \xi - p(R)(\pi(R) - \pi(r))\eta - \lambda \int_R^r (\pi(t) - \pi(r)) q(t) y(t; \eta) dt \\ &\quad - \int_R^r (\pi(t) - \pi(r)) h(t) dt, \quad r \geq R, \end{aligned}$$

from which it follows that $y(r; \eta) > 0$ for $r > R$ if $\eta < \{\xi(1 - \lambda \kappa) - \sigma\} / (p(R)\pi(R))$ and that $y(\tilde{r}; \eta) = 0$ for some $\tilde{r} > R$ if $\eta > \xi / (p(R)\pi(R))$. The first fact is shown analogously to the positivity of z_0 . To see the second fact it suffices to notice that if $y(r; \eta) > 0$ for all $r > R$, then we have $y(r; \eta) \leq \xi - p(R)\pi(R)\eta + p(R)\pi(r)\eta$, $r > R$, the right side of which tends to $\xi - p(R)\pi(R)\eta < 0$ as $r \rightarrow \infty$ provided $\eta > \xi / (p(R)\pi(R))$. Now, consider the set $A = \{\eta > 0 : y(\tilde{r}; \eta) = 0 \text{ for some } \tilde{r} > R\}$. Then, we see that A is open in R_+ by the continuous dependence of solutions of (3.6) on initial data and that

$$(3.12) \quad \eta' \in A \quad \text{if} \quad \eta' > \eta'' \quad \text{and} \quad \eta'' \in A$$

from the relation $y(r; \eta') - y(r; \eta'') = (\eta' - \eta'')z_0(r) > 0$ in (R, ∞) for $\eta' > \eta''$. For $\xi > \xi_0$ we define $\eta(\xi) = \inf\{\eta : \eta \in A\}$. Then, $\eta(\xi) \geq \{\xi(1 - \lambda \kappa) - \sigma\} / (p(R)\pi(R))$ and the following hold:

$$(3.13) \quad \eta \in A \quad \text{for} \quad \eta > \eta(\xi),$$

$$(3.14) \quad y(r; \eta(\xi)) > 0 \quad \text{and} \quad y'(r; \eta(\xi)) < 0, \quad r > R,$$

$$(3.15) \quad \lim_{r \rightarrow \infty} y(r; \eta(\xi)) = 0.$$

In fact, (3.13) follows from (3.12). The positivity of $y(r; \eta(\xi))$ follows from the definition of $\eta(\xi)$ and the fact that A is open. Differentiating (3.11) and using the positivity of $y(r; \eta(\xi))$, we get the second relation in (3.14). To prove (3.15), assume that $l_0 = \lim_{r \rightarrow \infty} y(r; \eta(\xi)) > 0$. Putting $\varepsilon = l_0 / (2 \sup_{r \geq R} z_0(r)) > 0$, we obtain

$$\begin{aligned} y(r; \eta) &= y(r; \eta(\xi)) - (y(r; \eta(\xi)) - y(r; \eta)) \geq l_0 - (\eta - \eta(\xi))z_0(r) \\ &\geq l_0 - \varepsilon z_0(r) \geq \frac{1}{2}l_0 > 0 \quad \text{in } (R, \infty) \end{aligned}$$

provided $\eta(\xi) < \eta < \eta(\xi) + \varepsilon$. This shows that $\eta \notin A$ for any $\eta \in (\eta(\xi), \eta(\xi) + \varepsilon)$ which contradicts (3.13). Therefore (3.15) holds. This finishes the proof.

From the above proof we have the following.

REMARK 3.1. For $\lambda \in [0, \kappa^{-1})$, ξ_0 in Lemma 3.3 can be taken to be $\xi_0 = \sigma / (1 - \lambda\kappa)$. Furthermore, if $0 < \lambda < \hat{\lambda} < \kappa^{-1}$, we have $\eta(\xi) \geq \{\xi(1 - \hat{\lambda}\kappa) - \sigma\} / (p(R)\pi(R))$ for $\xi > \xi_0$.

The following lemma, which is used in the proofs of Theorems 2.1 and 2.2, improves Theorem 1 of Friedman [5] and Theorem 1 of Noussair and Swanson [18].

LEMMA 3.4. Let m and g be locally Hölder continuous functions in R^N with exponent α , $0 < \alpha < 1$, and assume that $m(x) > 0$ and $g(x) \geq 0$, $\neq 0$ in R^N . If

$$(3.16) \quad \int_R^\infty \pi_*(r) p_*(r) \{m^*(r) + g^*(r)\} dr < \infty$$

for some $R > 0$, then there is a positive constant $\tilde{\lambda}$ such that, for every $\lambda \in [0, \tilde{\lambda})$, the linear equation

$$(3.17) \quad Lu = \lambda m(x)u + g(x) \quad \text{in } R^N$$

possesses a unique positive solution $u \in C_{loc}^{2+\alpha}(R^N)$ satisfying $\lim_{|x| \rightarrow \infty} u(x) = 0$.

PROOF. We first note that by [9, Theorem 4.3] one can choose a positive constant $\lambda_1 > 0$ such that, for every $\lambda \in [0, \lambda_1]$, there exists a function $V \in C_{loc}^{2+\alpha}(R^N)$ satisfying

$$(3.18) \quad LV \geq \lambda m(x)V, \quad 0 < C_1 \leq V(x) \leq C_2 < \infty \quad \text{in } R^N,$$

for some constants C_1 and C_2 .

Since $m(x) + g(x) > 0$ in $\bar{B}_R$, there is a constant $\lambda_2 > 0$ such that the Dirichlet problem

$$(3.19) \quad -\left(\sum_{i,j=1}^N a_{ij}(x) \frac{\partial^2 v}{\partial x_i \partial x_j} + \sum_{i=1}^N b_i(x) \frac{\partial v}{\partial x_i}\right) = \lambda(m(x) + g(x))v \quad \text{in } B_R,$$

$$v = 1 \quad \text{on } \partial B_R$$

has a unique solution $\hat{v}(x; \lambda) \in C^{2+\alpha}(\bar{B}_R)$ for every $\lambda < \lambda_2$ ([1, Theorem 4.4]). From the maximum principle it follows that for $0 < \lambda < \lambda_2$

$$(3.20) \quad 1 \equiv \hat{v}(x; 0) \leq \hat{v}(x; \lambda), \quad x \in B_R,$$

$$\frac{\partial \hat{v}(x; \lambda)}{\partial \nu} < 0, \quad x \in \partial B_R,$$

where $\nu = (\nu_1, \dots, \nu_N)$ is a conormal vector to ∂B_R whose components are

$$\nu_i = \left(\sum_{j=1}^N \left(\sum_{i=1}^N a_{ij}(x) x_j / R \right)^2 \right)^{-1/2} \sum_{j=1}^N a_{ij}(x) x_j / R, \quad i = 1, \dots, N.$$

Furthermore, by the L^p -estimates for the solution $\tilde{v}(x) = \hat{v}(x; \lambda) - 1$ of

$$-\left(\sum_{i,j=1}^N a_{ij}(x) \frac{\partial^2 \tilde{v}}{\partial x_i \partial x_j} + \sum_{i=1}^N b_i(x) \frac{\partial \tilde{v}}{\partial x_i}\right) = \lambda(m(x) + g(x))\tilde{v}(x; \lambda) \quad \text{in } B_R,$$

$$\tilde{v} = 0 \quad \text{on } \partial B_R$$

and Sobolev's imbedding theorem we obtain

$$\chi(\lambda) \equiv \max_{|x|=R} \left| \frac{\partial \hat{v}(x; \lambda)}{\partial \nu} \right| = O(\lambda) \quad \text{as } \lambda \rightarrow 0$$

and so $\chi(\lambda) \leq M\lambda$, $0 \leq \lambda < \lambda_2$, for some $M > 0$ (see e.g. the proof of Theorem 4.3 in [9]).

Put now

$$\tilde{\lambda} = \min \{ \lambda_1, \lambda_2, 1/(2\kappa), \hat{a}/(4Mp_*(R)\pi_*(R)) \},$$

where $\kappa = \int_R^\infty \pi_*(r)p_*(r)m^*(r)dr$ and $\hat{a} = \min_{|x|=R} \{ a(x) (\sum_{i=1}^N (\sum_{j=1}^N a_{ij}(x)x_j/R)^2)^{-1/2} \}$.

Then, we have

$$(3.21) \quad \chi(\lambda) \leq \frac{\hat{a}}{4p_*(R)\pi_*(R)} \quad \text{for } 0 \leq \lambda < \tilde{\lambda}.$$

Let $0 < \lambda < \tilde{\lambda}$ and choose a constant ξ so that

$$(3.22) \quad \xi > \max \left\{ 4\sigma, \frac{1}{\lambda} \right\},$$

where $\sigma = \int_R^\infty \pi_*(r)p_*(r)g^*(r)dr$. Since $\lambda < 1/2\kappa$ and $\xi > \sigma/(1 - \lambda\kappa)$, by Lemma 3.3

and Remark 3.1, we can take an $\eta(\xi) > 0$ such that the solution $y(r; \eta(\xi))$ of the problem

$$(3.23) \quad \begin{aligned} (p_*(r)y')' + \lambda p_*(r)m^*(r)y + p_*(r)g^*(r) &= 0, \quad r > R, \\ y(R) &= \xi, \quad y'(R) = -\eta(\xi) \end{aligned}$$

satisfies $y(r; \eta(\xi)) > 0$, $y'(r; \eta(\xi)) < 0$ for $r > R$ and $\lim_{r \rightarrow \infty} y(r; \eta(\xi)) = 0$. From Remark 3.1 and (3.22) it follows that

$$(3.24) \quad \eta(\xi) \geq \frac{\xi(1 - \tilde{\lambda}\kappa) - \sigma}{p_*(R)\pi_*(R)} \geq \frac{\xi}{4p_*(R)\pi_*(R)}.$$

For $0 < \lambda < \tilde{\lambda}$, define a function v by

$$v(x) = \xi \hat{v}(x; \lambda) \quad \text{for } x \in \bar{B}_R; \quad v(x) = y(|x|; \eta(\xi)) \quad \text{for } x \in \mathbb{R}^N \setminus \bar{B}_R.$$

Then v is a supersolution of (3.17). In fact, v is continuous in $\mathbb{R}^N$ and satisfies

$$\begin{aligned} Lv &\geq \lambda(m(x) + g(x))v = \lambda m(x)v + \lambda \xi g(x) \hat{v}(x; \lambda) \\ &\geq \lambda m(x)v + g(x) \quad \text{in } B_R \end{aligned}$$

by (3.19), (3.20) and (3.22). Furthermore we see that

$$\begin{aligned} Lv &= -a(x)y''(r) - b(x)y'(r) + c(x)y(r) \\ &= -a(x)(y''(r) + b_*(r)y'(r)) + (a(x)b_*(r) - b(x))y'(r) + c(x)y(r) \\ &\geq a(x)(\lambda m^*(r)y(r) + g^*(r)) \geq \lambda m(x)v + g(x) \quad \text{in } \mathbb{R}^N \setminus \bar{B}_R, \end{aligned}$$

where $y(r) = y(r; \eta(\xi))$, $r = |x|$, and we have used the relation $y'(r) < 0$ for $r > R$ and the reformulation of (3.23)

$$y'' + b_*(r)y' + \lambda m^*(r)y + g^*(r) = 0, \quad r > R.$$

Finally, by (3.21) and (3.24), $v_1 = v|_{\bar{B}_R}$ and $v_2 = v|_{\mathbb{R}^N \setminus \bar{B}_R}$ satisfy

$$\begin{aligned} \frac{\partial v_1}{\partial \nu} - \frac{\partial v_2}{\partial \nu} &= \xi \frac{\partial v(x; \lambda)}{\partial \nu} - \frac{\partial y(|x|; \eta(\xi))}{\partial \nu} \\ &\geq -\xi \chi(\lambda) - y'(R; \eta(\xi)) a(x) \left(\sum_{i=1}^N \left(\sum_{j=1}^N a_{ij}(x) x_j / |x| \right)^2 \right)^{-1/2} \\ &\geq -\xi \chi(\lambda) + \eta(\xi) \hat{a} \\ &\geq -\frac{\xi \hat{a}}{4p_*(R)\pi_*(R)} + \frac{\xi \hat{a}}{4p_*(R)\pi_*(R)} = 0 \quad \text{on } \partial B_R. \end{aligned}$$

Thus $v(x)$ is a positive supersolution of (3.17) tending to 0 as $|x| \rightarrow \infty$. Note that this v is also a supersolution of (3.17) with $\lambda = 0$.

Let us construct a subsolution of (3.17). By the hypothesis for g we may

assume that $g(x) > 0$ in $\bar{B}$ for some ball B in $\mathbb{R}^N$. Let $\mu_0 > 0$ be the principal eigenvalue of the operator L in B with the Dirichlet boundary condition and let $\varphi_0 \in C^{2+\alpha}(\bar{B})$ be an eigenfunction corresponding to μ_0 so chosen that

$$0 < \varphi_0(x) \leq \min \{g(x)/\mu_0, v(x)\}, \quad x \in B.$$

Then,

$$\begin{aligned} L\varphi_0(x) &= \mu_0\varphi_0(x) \leq g(x) \leq \lambda m(x)\varphi_0(x) + g(x) \quad \text{in } B, \\ \varphi_0(x) &= 0 \quad \text{and} \quad \frac{\partial \varphi_0(x)}{\partial \nu} < 0 \quad \text{on } \partial B, \end{aligned}$$

and so the function w defined by

$$w(x) = \varphi_0(x) \quad \text{for } x \in \bar{B}; \quad w(x) = 0 \quad \text{for } x \in \mathbb{R}^N \setminus B$$

is a subsolution of (3.17) satisfying $w(x) \leq v(x)$ in $\mathbb{R}^N$. Hence, by Lemma 3.1 (3.17) has a solution u such that $w(x) \leq u(x) \leq v(x)$ in $\mathbb{R}^N$ and $\lim_{|x| \rightarrow \infty} u(x) = 0$. Since $Lu \geq 0$, $u(x) \geq 0$ in $\mathbb{R}^N$ and $u(x) > 0$ in B , this u is positive throughout $\mathbb{R}^N$ by the maximum principle.

To prove the uniqueness, let u_1 and u_2 be two solutions of (3.17) for $0 \leq \lambda < \tilde{\lambda}$ which tend to 0 as $|x| \rightarrow \infty$. Then $\tilde{u} = u_1 - u_2$ satisfies

$$L\tilde{u} = \lambda m(x)\tilde{u} \quad \text{in } \mathbb{R}^N, \quad \lim_{|x| \rightarrow \infty} \frac{\tilde{u}(x)}{V(x)} = \lim_{|x| \rightarrow \infty} \tilde{u}(x) = 0,$$

where V is a function satisfying (3.18). Applying Lemma 3.2, we see that $\tilde{u}(x) = u_1(x) - u_2(x) \geq 0$ in $\mathbb{R}^N$. The same procedure yields $u_2(x) - u_1(x) \geq 0$ in $\mathbb{R}^N$, and consequently we have $u_1(x) = u_2(x)$ in $\mathbb{R}^N$.

COROLLARY 3.1. *Let $N \geq 3$ and let c , m and g be nonnegative locally Hölder continuous functions in $\mathbb{R}^N$ with exponent α , $0 < \alpha < 1$. Suppose that $m(x) > 0$, $g(x) \not\equiv 0$ in $\mathbb{R}^N$, and*

$$(3.25) \quad \int_R^\infty r \{m^*(r) + g^*(r)\} dr < \infty$$

for some $R > 0$, where $m^*(r) = \max_{|x|=r} m(x)$ and $g^*(r) = \max_{|x|=r} g(x)$.

Then, there is a $\tilde{\lambda} > 0$ such that, for every $\lambda \in [0, \tilde{\lambda})$, the equation

$$(3.26) \quad -\Delta u + c(x)u = \lambda m(x)u + g(x) \quad \text{in } \mathbb{R}^N$$

possesses a unique entire positive solution $u \in C_{loc}^{2+\alpha}(\mathbb{R}^N)$ satisfying $\lim_{|x| \rightarrow \infty} u(x) = 0$. Moreover, if

$$(3.27) \quad \int_R^\infty r^{1+\varepsilon} \{m^*(r) + g^*(r)\} dr < \infty$$

for some $\varepsilon \in (0, N-2]$, then the above solution u satisfies $u(x) = O(|x|^{-\varepsilon})$ as $|x| \rightarrow \infty$.

PROOF. Since the existence of $\tilde{\lambda} > 0$ with the desired property is guaranteed by Lemma 3.4, we need only to prove the last statement. From the proof of Lemma 3.4 it suffices to show that under (3.27) a positive solution y of the ordinary differential equation

$$(3.28) \quad (r^{N-1}y')' + r^{N-1}(\lambda m^*(r)y + g^*(r)) = 0, \quad r > R,$$

satisfies $y(r) = O(r^{-\varepsilon})$ as $r \rightarrow \infty$, provided $\lim_{r \rightarrow \infty} y(r) = 0$. Integrating (3.28) twice and using the boundedness of y , we have

$$(3.29) \quad \begin{aligned} y(r) &= -(N-2)^{-1} R^{N-1} y'(R) r^{-(N-2)} + \lambda \int_r^\infty s^{-(N-1)} \left(\int_R^s t^{N-1} m^*(t) y(t) dt \right) ds \\ &\quad + \int_r^\infty s^{-(N-1)} \left(\int_R^s t^{N-1} g^*(t) dt \right) ds \\ &\leq -(N-2)^{-1} R^{N-1} y'(R) r^{-(N-2)} + M \int_r^\infty s^{-(N-1)} \left(\int_R^s t^{N-1} (m^*(t) + g^*(t)) dt \right) ds, \\ &\quad r > R \end{aligned}$$

for some constant $M > 0$. It is not hard to show that

$$(3.30) \quad J(r) \equiv \int_r^\infty s^{-(N-1)} \left(\int_R^s t^{N-1} (m^*(t) + g^*(t)) dt \right) ds = O(r^{-\varepsilon}) \quad \text{as } r \rightarrow \infty$$

by using the identity

$$J(r) = \int_r^\infty s^{-(N-1)} \left(\int_R^s t^{N-2-\varepsilon} \left(\int_R^t \tau^{1+\varepsilon} (m^*(\tau) + g^*(\tau)) d\tau \right) dt \right) ds, \quad r > R,$$

L'Hospital's rule and (3.27). (3.29) and (3.30) imply $y(r) = O(r^{-\varepsilon})$ as $r \rightarrow \infty$.

§ 4. Proofs of Theorems 2.1 and 2.2

PROOF OF THEOREM 2.1. Let $v \in C_{loc}^{2+\alpha}(R^N)$ be a positive solution of the linear equation $Lv = m(x)$ in R^N such that $\lim_{|x| \rightarrow \infty} v(x) = 0$. Such a v exists by Lemma 3.4. Put $\lambda^* = 1/C_U$ and let $0 < \lambda < \lambda^*$, where $C_U > 0$ is the constant corresponding to $U = \sup_{x \in R^N} v(x)$ in (F_2) . Then, v is obviously a supersolution of (1.1) by (F_2) .

To construct a subsolution, choose a ball $B \subset \bar{B} \subset \Omega_0$, let $\mu_0 > 0$ be the principal eigenvalue of L on B with the Dirichlet boundary condition, and φ_0 be a positive eigenfunction corresponding to μ_0 satisfying

$$0 < \varphi_0(x) \leq \min \{u_0, v(x)\}, \quad x \in B,$$

where u_0 is a positive constant so chosen that

$$\frac{f(x, u)}{u} \geq \frac{\mu_0}{\lambda} \quad \text{for } x \in \bar{B}, 0 < u \leq u_0.$$

Then, it is easy to verify that the function w defined by

$$w(x) = \varphi_0(x) \quad \text{for } x \in \bar{B}; \quad w(x) = 0 \quad \text{for } x \in \mathbb{R}^N \setminus B$$

is a subsolution of (1.1) with the property $w(x) \leq v(x)$ in $\mathbb{R}^N$. Hence, by Lemma 3.1 (1.1) has a solution u satisfying $w(x) \leq u(x) \leq v(x)$ in $\mathbb{R}^N$. Because of (F_2) the positivity of u follows from the maximum principle as in the proof of Lemma 3.4. This completes the proof.

REMARK 4.1. It may happen that λ^* in Theorem 2.1 is finite. This is the case if, for example,

$$c_0 \equiv \inf_{x \in \Omega, u > 0} \frac{f(x, u)}{u} > 0$$

for some bounded domain Ω in $\mathbb{R}^N$. In this case (1.1) has no positive solution provided $\lambda(>0)$ is sufficiently large. In fact, assume that (1.1) has a positive solution u for $\lambda > 0$. Then u satisfies

$$Lu = \lambda f(x, u) \geq \lambda c_0 u \quad \text{in } B', \quad u > 0 \quad \text{on } \partial B'$$

for some ball $B' \subset \bar{B}' \subset \Omega$. Hence, λc_0 cannot be greater than the principal eigenvalue $\tilde{\mu}$ of L on B' with Dirichlet boundary condition (see e.g. [1, p. 636]), i.e., $\lambda \leq \tilde{\mu}/c_0$. This implies that (1.1) has no positive solution for $\lambda > \tilde{\mu}/c_0$.

PROOF OF THEOREM 2.2. The proof of (i) is essentially similar to those of Theorem 2.1 and of (ii) below, and will be omitted.

(ii) Let $\tilde{\lambda} > 0$ be a constant such that, for every $\lambda \in [0, \tilde{\lambda})$, there exists a unique entire positive solution of

$$(4.1) \quad Lu = \lambda m(x)u + g(x) \quad \text{in } \mathbb{R}^N, \quad \lim_{|x| \rightarrow \infty} u(x) = 0,$$

and let $\lambda_1 > 0$ be a constant for which there is a function $V \in C^2(\mathbb{R}^N)$ satisfying

$$(4.2) \quad LV(x) \geq \lambda_1 m(x)V(x), \quad 0 < M_1 \leq V(x) \leq M_2 < \infty \quad \text{in } \mathbb{R}^N,$$

for some constants M_1 and M_2 , where m and g are in (F_4) . The existence of $\tilde{\lambda}$ and λ_1 is guaranteed by (2.3), Lemma 3.4 and Theorem 4.3 of [9].

Denote $\lambda_0 = \lambda_0(\tilde{m})$ and let $\lambda > \lambda_0$. Choosing $\mu > 0$ so small that $\lambda\mu < \min\{\tilde{\lambda}, \lambda_1\}$, we can obtain a positive function $\hat{v} \in C_{loc}^{2+\alpha}(\mathbb{R}^N)$ such that

$$L\hat{v} = \lambda\mu m(x)\hat{v} + g(x) \quad \text{in } \mathbb{R}^N, \quad \lim_{|x| \rightarrow \infty} \hat{v}(x) = 0.$$

The function $v(x) = \lambda C(\mu) \hat{v}(x)$ satisfies

$$(4.3) \quad Lv = \lambda(\mu m(x)v + C(\mu)g(x)) \quad \text{in } \mathbb{R}^N, \quad \lim_{|x| \rightarrow \infty} v(x) = 0,$$

where $C(\mu)$ is a positive constant in (F_4) . By (4.3) and (F_4) this v is a supersolution of (1.1). Next, by the definition of λ_0 , we can choose $n \geq 1$ such that the principal eigenvalue $\lambda_n(\tilde{m})$ of (2.2) satisfies $\lambda_0 \leq \lambda_n(\tilde{m}) < \lambda$. From (F_5) it follows that

$$(4.4) \quad \lambda f(x, u) \geq \lambda_n(\tilde{m}) \tilde{m}(x) u \quad \text{for } x \in \mathbb{R}^N, \quad 0 < u \leq u_0$$

for sufficiently small $u_0 > 0$. Take now a positive eigenfunction φ_n of (2.2) corresponding to $\lambda_n(\tilde{m})$ which satisfies

$$0 < \varphi_n(x) \leq \min \{u_0, v(x)\}, \quad x \in B_n.$$

By this and (4.4) the function w defined by

$$w(x) = \varphi_n(x) \quad \text{for } x \in \bar{B}_n; \quad w(x) = 0 \quad \text{for } x \in \mathbb{R}^N \setminus B_n$$

is a subsolution of (1.1) satisfying $w(x) \leq v(x)$ in $\mathbb{R}^N$. So by Lemma 3.1 there is a solution u of (1.1) such that

$$0 < u(x) \leq v(x) \quad \text{in } \mathbb{R}^N \quad \text{and} \quad \lim_{|x| \rightarrow \infty} u(x) = 0.$$

Therefore, (1.1) has an entire positive solution tending to 0 as $|x| \rightarrow \infty$ for $\lambda > \lambda_0$. Since $\lambda > \lambda_0$ is arbitrary, we conclude that $\lambda^* = \infty$ and $\lambda_* \leq \lambda_0$.

To prove the positivity of λ_* under condition (2.4), put

$$\tilde{c} = \sup_{x \in \mathbb{R}^N, u > 0} f(x, u)/m(x)u$$

and assume that, for some $\lambda \in (0, \lambda_1/\tilde{c})$, (1.1) has a positive solution u satisfying $\lim_{|x| \rightarrow \infty} u(x) = 0$. Then, noting that this u satisfies

$$Lu = \lambda f(x, u) \leq \lambda \tilde{c} m(x) u \leq \lambda_1 m(x) u \quad \text{in } \mathbb{R}^N,$$

$$\lim_{|x| \rightarrow \infty} \frac{u(x)}{V(x)} = 0,$$

where V is as in (4.2), we get $u(x) \leq 0$ in $\mathbb{R}^N$ by Lemma 3.2, a contradiction. This shows that $\lambda_* \geq \lambda_1/\tilde{c} > 0$.

Finally, we claim the existence of a maximal decaying positive solution of (1.1) for every $\lambda > \lambda_*$. To this end, let $\lambda > \lambda_*$ and take a constant $\mu^* > 0$ so that $\lambda \mu^* < \min \{\tilde{\lambda}, \lambda_1\}$. Let $\bar{v}$ be the unique positive solution of

$$L\bar{v} = \lambda \mu^* m(x) \bar{v} + \lambda C(\mu^*) g(x) \quad \text{in } \mathbb{R}^N$$

satisfying $\lim_{|x| \rightarrow \infty} \bar{v}(x) = 0$, where $C(\mu^*)$ is as in (F_4) . Then, repeating the

same procedure as above, by Lemma 3.1 we obtain a positive solution $\bar{u}$ of (1.1) such that

$$(4.5) \quad u(x) \leq \bar{u}(x) \leq \bar{v}(x) \quad \text{in } R^N$$

for any solution u satisfying $0 \leq u(x) \leq \bar{v}(x)$ in R^N . This solution $\bar{u}$ is the desired maximal decaying entire positive solution of (1.1). To show this, let u be an arbitrary positive solution of (1.1) tending to 0 as $|x| \rightarrow \infty$. Then, by (1.1) and (F_4) ,

$$Lu = \lambda f(x, u(x)) \leq \lambda \mu^* m(x) u(x) + \lambda C(\mu^*) g(x) \quad \text{in } R^N,$$

and so $\bar{w}(x) = \bar{v}(x) - u(x)$ satisfies

$$L\bar{w} \geq \lambda \mu^* m(x) \bar{w} \quad \text{in } R^N \quad \text{and} \quad \lim_{|x| \rightarrow \infty} \frac{\bar{w}(x)}{V(x)} = 0$$

for the above mentioned function V . Lemma 3.2 then implies that $\bar{w}(x) \geq 0$, i.e., $u(x) \leq \bar{v}(x)$ in R^N . From this and (4.5) it follows that $u(x) \leq \bar{u}(x)$ in R^N . Thus the proof is complete.

COROLLARY 4.1. *Assume that (H) holds, and let Φ be a locally Hölder continuous positive function in R^N . If $0 \leq \gamma < 1$ and*

$$\int_R^\infty \pi_*(r) p_*(r) \Phi^*(r) dr < \infty$$

for some $R > 0$, then

$$Lu = \Phi(x) u^\gamma \quad \text{in } R^N$$

possesses a maximal decaying entire positive solution $\bar{u}$.

PROOF. Using the inequality $u^\gamma \leq \mu u + (1-\gamma)(\gamma/\mu)^{\gamma/(1-\gamma)}$ for $\mu > 0$, $u \geq 0$ and $0 < \gamma < 1$, we easily see that the function $f(x, u) = \Phi(x) u^\gamma$ satisfies all the conditions in Theorem 2.2. Therefore, the assertion follows from (i) of Theorem 2.2.

This corollary is a partial extension of [7, Theorem 1] dealing with the case $L = -\Delta$ and $c(x) \equiv 0$ in R^N , $N \geq 3$.

As a byproduct of Theorem 2.2 we obtain the following uniqueness result.

COROLLARY 4.2. *Assume that (H), (F_1) , (F_3) , (F_4) and (2.3) are satisfied and that $f(x, u)$ is nonincreasing in $u > 0$ for any fixed $x \in R^N$. Then, for every $\lambda > 0$, (1.1) has a unique entire positive solution u satisfying $\lim_{|x| \rightarrow \infty} u(x) = 0$.*

PROOF. By (i) of Theorem 2.2 (1.1) has a maximal decaying entire

positive solution $\bar{u}$ for every $\lambda > 0$. To prove the uniqueness let u be any entire positive solution of (1.1) such that $\lim_{|x| \rightarrow \infty} u(x) = 0$. Then, since $f(x, u)$ is nonincreasing in $u > 0$ and $\bar{u}(x) \geq u(x)$ in R^N , $\tilde{u}(x) = \bar{u}(x) - u(x)$ satisfies

$$L\tilde{u} = \lambda(f(x, \bar{u}) - f(x, u)) \leq 0 \quad \text{in } R^N \quad \text{and} \quad \lim_{|x| \rightarrow \infty} \tilde{u}(x) = 0.$$

This and Lemma 3.2 show that $\tilde{u}(x) \leq 0$ in R^N . Hence we see that $\tilde{u}(x) = 0$, i.e., $u(x) = \bar{u}(x)$ throughout R^N , proving the uniqueness.

REMARK 4.2. Let B_* , C^* , M^* and G^* be locally Hölder continuous functions in R_+ satisfying

$$B_*(r) \leq b_*(r), \quad c^*(r) \leq C^*(r), \quad m^*(r) \leq M^*(r) \quad \text{and} \quad g^*(r) \leq G^*(r), \quad r > 0,$$

respectively, and put $P_*(r) = \exp\left(\int_R^r B_*(s) ds\right)$, $r > 0$. Suppose that $\int_R^\infty ds/P_*(s) < \infty$ instead of condition (H) and put $\Pi_*(r) = \int_r^\infty ds/P_*(s)$, $r > 0$. Then, it is easy to see that all the conclusions of Theorems 2.1 and 2.2 remain true if in the hypotheses of these theorems p_* , π_* , c^* , m^* and g^* are replaced by P_* , Π_* , C^* , M^* and G^* , respectively.

EXAMPLE 4.1. Suppose that the coefficients a_{ij} and b_i of L are bounded in R^N and satisfy

$$(4.6) \quad \liminf_{|x| \rightarrow \infty} \sum_{i=1}^N b_i(x) x_i / |x| > 0.$$

If $f(x, u)$ satisfies (F_1) – (F_5) , and if

$$(4.7) \quad \int_R^\infty \{m^*(r) + g^*(r)\} dr < \infty$$

for some $R > 0$, where $m^*(r) = \max_{|x|=r} m(x)$ and $g^*(r) = \max_{|x|=r} g(x)$, then the conclusions in Theorems 2.1 and 2.2 still hold. In fact, by (4.6) and the boundedness of the coefficient of L we can choose a constant $\beta_* > 0$ so that $B_*(r) \equiv \beta_*$ satisfies $B_*(r) \leq b_*(r)$ for $r > R$ provided $R > 0$ is sufficiently large. Since

$$P_*(r) = \exp\left(\int_R^r B_*(s) ds\right) = e^{\beta_*(r-R)}, \quad r > R$$

satisfies

$$\int_R^\infty \frac{dr}{P_*(r)} < \infty, \quad \int_R^\infty \left(\int_r^\infty \frac{ds}{P_*(s)}\right) P_*(r) \{m^*(r) + g^*(r)\} dr < \infty$$

by the positivity of β_* and (4.7), the assertion follows from Remark 4.2.

§ 5. The case $L = -\Delta + c$

In this section, we treat (1.1) in the case where $L = -\Delta + c$, i.e.,

$$(5.1) \quad -\Delta u + c(x)u = \lambda f(x, u) \quad \text{in } \mathbf{R}^N, \quad N \geq 3.$$

We first consider the uniqueness of decaying entire positive solution of

(5.1). For that purpose the following conditions (F'_5) and (F_6) are imposed on f .

(F'_5) There are a positive function $\tilde{m} \in C_{loc}^\alpha(\mathbf{R}^N)$, $0 < \alpha < 1$, and a constant γ , $0 \leq \gamma \leq 1$, such that

$$(5.2) \quad \lim_{u \rightarrow 0+} \frac{f(x, u)}{\tilde{m}(x)u^\gamma} = 1 \quad \text{uniformly in } \mathbf{R}^N.$$

(F_6) $f(x, u)/u$ is strictly decreasing in $u > 0$ for any fixed $x \in \mathbf{R}^N$.

Throughout this section we employ the notation $h^*(r) = \max_{|x|=r} h(x)$, $r > 0$, for a continuous function h in $\mathbf{R}^N$ (cf. Section 2).

THEOREM 5.1. *Assume that $N \geq 3$ and c is nonnegative and locally Hölder continuous with exponent α , $0 < \alpha < 1$, in $\mathbf{R}^N$, and f satisfies (F_1) , (F_4) , (F'_5) and (F_6) . If*

$$(5.3) \quad \int_R^\infty r^{(N+\gamma-1)/(\gamma+1)} \{c^*(r) + m^*(r) + g^*(r)\} dr < \infty$$

for some $R > 0$, then there is a $\lambda_* \geq 0$ such that, for every $\lambda > \lambda_*$, (5.1) has a unique positive solution u satisfying $\lim_{|x| \rightarrow \infty} u(x) = 0$.

PROOF. In view of (F_1) , (F_4) and (F'_5) Theorem 2.2 guarantees the existence of a $\lambda_* > 0$ such that, for every $\lambda > \lambda_*$, (5.1) has a maximal decaying entire positive solution $\bar{u}$. We prove the uniqueness by using the idea of [2], [4] and [8]. Let u be any positive solution of (5.1) such that $\lim_{|x| \rightarrow \infty} u(x) = 0$. Then, from the proofs of Theorem 2.2 and Corollary 3.1 we see that u and $\bar{u}$ satisfy

$$(5.4) \quad u(x), \bar{u}(x) = O(|x|^{-\varepsilon}) \quad \text{as } |x| \rightarrow \infty,$$

where $\varepsilon = (N-2)/(\gamma+1)$. Take a function $\psi \in C^\infty(\bar{\mathbf{R}}_+)$ such that

$$\psi(r) = 1 \quad \text{for } 0 \leq r \leq 1; \quad 0 < \psi(r) < 1 \quad \text{for } 1 < r < 2; \quad \psi(r) = 0 \quad \text{for } r \geq 2$$

and put $\psi_\rho(r) = \psi(r/\rho)$ for $\rho > 0$. Then, by (5.1), (F_6) and the relation $0 < u(x) \leq \bar{u}(x)$, $x \in \mathbf{R}^N$, we get

$$(\bar{u}(x)\Delta u(x) - u(x)\Delta \bar{u}(x))\psi_\rho(|x|) = \lambda\{f(x, \bar{u})/\bar{u} - f(x, u)/u\}u(x)\bar{u}(x)\psi_\rho(|x|) \\ \leq 0 \quad \text{in } R^N.$$

Integrating the above over R^N and applying Green's theorem, we obtain

$$(5.5) \quad I_\rho \equiv \int_{R^N} (u(x)\nabla \bar{u}(x) - \bar{u}(x)\nabla u(x)) \cdot \nabla \psi_\rho(|x|) dx \\ = \lambda \int_{R^N} (f(x, \bar{u})/\bar{u} - f(x, u)/u) u(x)\bar{u}(x)\psi_\rho(|x|) dx \quad \rho > 0,$$

where $\nabla u(x)$ denotes the gradient of $u(x)$. The choice of ψ_ρ shows that

$$I_\rho = \int_{\rho \leq |x| \leq 2\rho} (u(x)\nabla \bar{u}(x) - \bar{u}(x)\nabla u(x)) \cdot \nabla \psi_\rho(|x|) dx.$$

It can be shown that u and $\bar{u}$ satisfy

- (a) $u, \bar{u} \in L^1_{loc}(R^N)$, (b) $\Delta u, \Delta \bar{u} \in L^1(R^N)$, and
(c) $\lim_{n \rightarrow \infty} \int_{1 \leq |x| \leq 2} u(nx) dx = \lim_{n \rightarrow \infty} \int_{1 \leq |x| \leq 2} \bar{u}(nx) dx = 0.$

We sketch only the proof of (b) for u . Since $\lim_{|x| \rightarrow \infty} u(x) = 0$, by (F'_5) there is $R_1 > 0$ such that $0 \leq f(x, u(x)) \leq 2\tilde{m}(x)u(x)^r$ for $|x| > R_1$. Using this and (5.4), we have

$$(5.6) \quad |\Delta u(x)| = c(x)u + \lambda f(x, u(x)) \leq M_1\{c(x)|x|^{-\varepsilon} + \tilde{m}(x)|x|^{-\varepsilon r}\}, \quad |x| > R_1$$

for some constant $M_1 > 0$. Since by (F_4) , (5.2) and (5.3) $\tilde{m}^*(r)$ satisfies $\int_{R_1}^\infty r^{1+\varepsilon} \tilde{m}^*(r) dr < \infty$, from (5.6) and (5.3) it follows that

$$\int_{R^N} |\Delta u(x)| dx = \left(\int_{|x| \leq R_1} + \int_{|x| > R_1} \right) |\Delta u(x)| dx \\ \leq \int_{|x| \leq R_1} |\Delta u(x)| dx + M_1 \int_{|x| > R_1} \{c(x)|x|^{-\varepsilon} + \tilde{m}(x)|x|^{-\varepsilon r}\} dx \\ \leq \int_{|x| \leq R_1} |\Delta u(x)| dx + M_2 \int_{R_1}^\infty \{r^{N-1-\varepsilon} c^*(r) + r^{N-1-\varepsilon r} \tilde{m}^*(r)\} dr < \infty,$$

where $M_2 > 0$ is a constant.

From (a)-(c) and [2, Lemma A.5] it follows that

$$\int_{\rho \leq |x| \leq 2\rho} (|\nabla u(x)| + |\nabla \bar{u}(x)|) dx = O(\rho) \quad \text{as } \rho \rightarrow \infty,$$

which, combined with (5.4) and the fact that $|\nabla \psi_\rho(|x|)| = O(\rho^{-1})$ as $\rho \rightarrow \infty$, implies that $\lim_{\rho \rightarrow \infty} I_\rho = 0$. Therefore, letting $\rho \rightarrow \infty$ in (5.5), we obtain

$$\int_{R^N} (f(x, \bar{u})/\bar{u} - f(x, u)/u) u(x)\bar{u}(x) dx = 0.$$

The nonpositivity of the integrand leads us to the equality $f(x, \bar{u}(x))/\bar{u}(x) = f(x, u(x))/u(x)$ in R^N . Hence, by (F_6) we conclude that $u(x) = \bar{u}(x)$ throughout R^N . This finishes the proof.

Now we apply our theorems to the following two special equations:

$$(5.7) \quad -\Delta u + c(x)u = \lambda(\Phi(x)u^\gamma + \Psi(x)u^\delta) \quad \text{in } R^N, N \geq 3$$

and

$$(5.8) \quad -\Delta u + c(x)u = \lambda(\Phi(x)u^\gamma - \Psi(x)u^\delta) \quad \text{in } R^N, N \geq 3,$$

where γ and δ are nonnegative constants, c , Φ and Ψ are nonnegative locally Hölder continuous functions in R^N with exponent α , $0 < \alpha < 1$, and especially Φ is positive throughout R^N .

THEOREM 5.2. *Let $0 \leq \gamma < 1$, $\gamma \leq \delta$ and suppose that*

$$(5.9) \quad \int_R^\infty r\{\Phi^*(r) + \Psi^*(r)\}dr < \infty$$

for some $R > 0$. Then, there is a $\lambda^ > 0$ such that, for every positive $\lambda < \lambda^*$, (5.7) has an entire positive solution u satisfying $\lim_{|x| \rightarrow \infty} u(x) = 0$. Furthermore, if $\Psi(x) > 0$ in R^N and $\delta > 1$, then the above λ^* is finite and (5.7) has no decaying entire positive solution provided $\lambda > \lambda^*$.*

PROOF. The function $f(x, u) = \Phi(x)u^\gamma + \Psi(x)u^\delta$ satisfies (F_1) – (F_3) with $m(x) = \Phi(x) + \Psi(x)$ (see Example 2.1). Moreover, condition (5.9) implies (2.1). Therefore the assertion is a consequence of Theorem 2.1 and Remark 4.1.

THEOREM 5.3. *Let $0 \leq \gamma \leq 1 \leq \delta$, $\gamma \neq \delta$, and suppose that $\Psi(x)/\Phi(x)$ is bounded in R^N and*

$$(5.10) \quad \int_R^\infty r\Phi^*(r)dr < \infty$$

for some $R > 0$.

(i) *Let $0 \leq \gamma < 1$. For every $\lambda > 0$, (5.8) has a maximal decaying entire positive solution.*

(ii) *Let $\gamma = 1$. If $\inf_{x \in R^N} \Psi(x)/\Phi(x) > 0$, then there is a $\lambda_* > 0$ such that (5.8) has a maximal decaying entire positive solution for every $\lambda > \lambda_*$ but has no decaying entire positive solution for any positive $\lambda < \lambda_*$.*

Furthermore, if c and Φ satisfy

$$(5.11) \quad \int_R^\infty r^{1+\varepsilon}\{c^*(r) + \Phi^*(r)\}dr < \infty, \quad \varepsilon = (N-2)/(\gamma+1),$$

then (5.8) has at most one decaying entire positive solution for any $\lambda > 0$.

PROOF. The function $f(x, u) = \Phi(x)u^r - \Psi(x)u^s$ satisfies conditions (F_1) , (F_4) , (F'_s) and (F'_s) with $m(x) = \tilde{m}(x) = g(x) = \Phi(x)$. The conditions (2.3) and (5.3) reduce to (5.10) and (5.11), respectively. Consequently, the assertion follows from Theorems 2.2 and 5.1.

REMARK 5.1. In the case where $0 < r < 1$ and $c(x) \equiv \Psi(x) \equiv 0$, $x \in \mathbb{R}^N$, that is, for the sublinear equation

$$-\Delta u = \Phi(x)u^r \quad \text{in } \mathbb{R}^N, \quad N \geq 3, \quad 0 < r < 1,$$

Fukagai [7, Theorems 1 and 2] proved the existence and uniqueness of decaying entire positive solutions under the condition (5.11) with ε replaced by $\varepsilon = (1-r)(N-2)$.

We conclude with some examples illustrating our theorems.

EXAMPLE 5.1. Let $N \geq 3$. Let c and m satisfy that $c, m \in C_{loc}^\alpha(\mathbb{R}^N)$, $0 < \alpha < 1$, $c(x) \geq 0$ and $m(x) > 0$ in $\mathbb{R}^N$. Moreover, suppose that $\int_R^\infty r m^*(r) dr < \infty$ for some $R > 0$.

(i) By (ii) of Theorem 2.2 the equation

$$-\Delta u + c(x)u = \lambda m(x) \sin u \quad \text{in } \mathbb{R}^N$$

has a maximal decaying entire positive solution for sufficiently large $\lambda > 0$ but has no decaying entire positive solution for sufficiently small $\lambda > 0$.

(ii) From (ii) of Theorem 2.2 and Theorem 5.1 it follows that if

$$\int_R^\infty r^{N/2} \{c^*(r) + m^*(r)\} dr < \infty,$$

the equation

$$-\Delta u + c(x)u = \lambda m(x) \log(1+u) \quad \text{in } \mathbb{R}^N$$

has exactly one decaying entire positive solution for sufficiently large $\lambda > 0$ but has no decaying entire positive solution provided $\lambda > 0$ is sufficiently small.

(iii) Consider the equation

$$(5.12) \quad -\Delta u + c(x)u = \lambda m(x)e^{Ku} \quad \text{in } \mathbb{R}^N,$$

where K is a nonzero constant.

(a) Let $K > 0$. Theorem 2.1 implies that, for sufficiently small $\lambda > 0$, (5.12) has a decaying entire positive solution.

(b) Let $K < 0$. For every $\lambda > 0$, (5.12) has exactly one decaying entire positive solution. This follows from (i) of Theorem 2.2 and Corollary 4.2.

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