Singular hyperbolic systems, VII. Asymptotic analysis for Fuchsian hyperbolic equations in Gevrey classes (2)

By Hidetoshi TAHARA

(Received February 29, 1988)

In the previous paper [5], the irregularity index $\sigma (\geq 1)$ was defined for some Fuchsian hyperbolic operators P , and the equation $Pu=f$ was investigated in the Gevrey class $\mathcal{E}^{(s)}$ under the condition $1 < s < \sigma/(\sigma-1)$. The following is a typical example:

$$L = (t\partial_t)^2 - \sum_{i=1}^n t^{2\kappa_i} \partial_{x_i}^2 + a(t, x)(t\partial_t) + \sum_{i=1}^n t^{l_i} b_i(t, x) \partial_{x_i} + c(t, x),$$

$$\sigma_L = \max \left\{ 1, \frac{2\kappa_1 - l_1}{\kappa_1}, \dots, \frac{2\kappa_n - l_n}{\kappa_n} \right\}$$

(where $(t, x) \in [0, T] \times \mathbf{R}^n$, $\kappa_i > 0$ ($1 \leq i \leq n$) and $l_i > 0$ ($1 \leq i \leq n$)).

The purpose of this paper is to study the following case:

$$s = \sigma/(\sigma-1).$$

Roughly speaking, the result established in this paper is as follows: if we treat the equation $Pu=f$ in the projective Gevrey class $\mathcal{E}^{(s)}$ (not in the classical one $\mathcal{E}^{\{s\}}$), we can obtain the same results as in [5] also in the case

$$s = \sigma/(\sigma-1) \quad \text{and} \quad \mathcal{A}_P \cap S_P = \emptyset$$

(see Theorem 2).

The reason why we must introduce the class $\mathcal{E}^{(s)}$ is illustrated as follows. Let $A = (t\partial_t + 1)(t\partial_t + 2) - t^{2k} \partial_x^2 + t^l a \partial_x$ (where $(t, x) \in [0, T] \times \mathbf{R}$, $k, l \in \{1, 2, \dots\}$, $l \geq 2$, $a \in C$ and $a \not\equiv 0$) and recall that the irregularity index σ of A is given by $\sigma = \max\{1, (2k-l)/k\}$. Then, we can see the following:

- (i) When $1 < s < \sigma/(\sigma-1)$, $Au=f$ is well-posed in $\mathcal{E}^{\{s\}}$ (by [5]).
- (ii) When $s = \sigma/(\sigma-1)$, $Au=f$ is not well-posed in $\mathcal{E}^{\{s\}}$ even in the local sense (by an application of [1]).
- (iii) When $s = \sigma/(\sigma-1)$, $Au=f$ is well-posed in $\mathcal{E}^{(s)}$ (by Theorem 2).

The condition $\mathcal{A}_P \cap S_P = \emptyset$ is introduced as a necessary and sufficient condition for some formal power series to converge under $s = \sigma/(\sigma-1)$ (see

Proposition 1). Most of our operators satisfy $\Delta_P \cap S_P = \emptyset$; in particular, if P is of the second order, $\Delta_P \cap S_P = \emptyset$ is trivially satisfied. But, there are examples of the case $\Delta_P \cap S_P \neq \emptyset$ and it seems impossible to apply our argument to the case

$$s = \sigma/(\sigma - 1) \quad \text{and} \quad \Delta_P \cap S_P \neq \emptyset$$

(by Proposition 1). Hence, it is still an open problem to decide whether the condition $\Delta_P \cap S_P = \emptyset$ (under $s = \sigma/(\sigma - 1)$) is essential or not.

§ 1. Preliminaries

First, let us explain the operator P treated here, the irregularity index $\sigma(\geq 1)$ of P , the function spaces and the problem.

The operator treated here is as follows:

$$(1.1) \quad P = (t\partial_t)^m + \sum_{\substack{j+|\alpha| \leq m \\ j < m}} t^{l(j, \alpha)} a_{j, \alpha}(t, x) (t\partial_t)^j \partial_x^\alpha,$$

where $(t, x) = (t, x_1, \dots, x_n) \in [0, T] \times \mathbf{R}^n$ ($T > 0$), $\partial_t = \partial/\partial t$, $\partial_x = (\partial/\partial x_1, \dots, \partial/\partial x_n)$, $m \in \mathbf{N} (= \{1, 2, 3, \dots\})$, $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbf{Z}_+^n (= \{0, 1, 2, \dots\}^n)$, $|\alpha| = \alpha_1 + \dots + \alpha_n$ and $\partial_x^\alpha = (\partial/\partial x_1)^{\alpha_1} \dots (\partial/\partial x_n)^{\alpha_n}$. On the coefficients, we assume that $a_{j, \alpha}(t, x) \in C^0([0, T] \times \mathbf{R}^n) \cap C^\infty((0, T) \times \mathbf{R}^n)$ ($j + |\alpha| \leq m$ and $j < m$) and that they satisfy $(t\partial_t)^i \partial_x^\beta a_{j, \alpha}(t, x) \in C^0([0, T] \times \mathbf{R}^n)$ for any $(i, \beta) \in \mathbf{Z}_+ \times \mathbf{Z}_+^n$. As to the hyperbolicity, we assume the following conditions (A _{κ}) and (B):

(A _{κ}) $l(j, \alpha) \in \mathbf{R}$ ($j + |\alpha| \leq m$ and $j < m$) satisfy

$$\begin{cases} l(j, \alpha) = \kappa_1 \alpha_1 + \dots + \kappa_n \alpha_n, & \text{when } j + |\alpha| = m \text{ and } j < m, \\ l(j, \alpha) > 0, & \text{when } j + |\alpha| < m \text{ and } |\alpha| > 0, \\ l(j, \alpha) \geq 0, & \text{when } j + |\alpha| < m \text{ and } |\alpha| = 0 \end{cases}$$

for some $\kappa = (\kappa_1, \dots, \kappa_n) \in \mathbf{R}^n$ such that $\kappa_i > 0$ ($1 \leq i \leq n$).

(B) All the roots $\lambda_i(t, x, \xi)$ ($1 \leq i \leq m$) of the equation (in λ)

$$\lambda^m + \sum_{\substack{j+|\alpha|=m \\ j < m}} a_{j, \alpha}(t, x) \lambda^j \xi^\alpha = 0$$

are *real, simple and bounded* on $\{(t, x, \xi) \in [0, T] \times \mathbf{R}^n \times \mathbf{R}^n; |\xi| = 1\}$.

Then, P is the Fuchsian hyperbolic operator discussed in [5]. Recall that the characteristic exponents $\rho_1(x), \dots, \rho_m(x)$ of P are defined by the roots of

$$\rho^m + \sum_{j < m} a_j(x) \rho^j = 0,$$

where $a_j(x) = [t^{l(j, (0, \dots, 0))} a_{j, (0, \dots, 0)}(t, x)]|_{t=0}$ ($j < m$). Also, recall that the irreg-

ularity index σ (≥ 1) of P is defined by

$$(1.2) \quad \sigma = \max[1, \max_{\substack{j+|\alpha| \leq m \\ |\alpha| > 0}} \{\min_{\tau \in \mathfrak{S}_n} (\max_{1 \leq r \leq n} M_{j, \alpha}(\tau, r))\}],$$

where $\mathfrak{S}_n$ is the permutation group of n -numbers and

$$(1.3) \quad M_{j, \alpha}(\tau, r) = \frac{\sum_{i=1}^r (\kappa_{\tau(i)} - \kappa_{\tau(r)}) \alpha_{\tau(i)} + (m-j) \kappa_{\tau(r)} - l(j, \alpha)}{(m-j-|\alpha|) \kappa_{\tau(r)}}.$$

In [5], we discussed the equation $Pu=f$ in $C^\infty([0, T], \mathcal{E}^{(s)}(\mathbf{R}^n))$ or $C^\infty((0, T), \mathcal{E}^{(s)}(\mathbf{R}^n))$ under the condition $1 < s < \sigma/(\sigma-1)$. But, in this paper, we will consider the equation $Pu=f$ in $C^\infty([0, T], \mathcal{E}^{(s)}(\mathbf{R}^n))$ or $C^\infty((0, T), \mathcal{E}^{(s)}(\mathbf{R}^n))$ under the condition $1 < s \leq \sigma/(\sigma-1)$.

Here, $\mathcal{E}^{(s)}(\mathbf{R}^n)$ [resp. $\mathcal{E}^{(s)}(\mathbf{R}^n)$] denotes the set of all the functions $f(x) \in C^\infty(\mathbf{R}^n)$ satisfying the following: for any compact subset K of $\mathbf{R}^n$ and for any $h > 0$, there is a $C > 0$ [resp. for any compact subset K of $\mathbf{R}^n$, there are $h > 0$ and $C > 0$] such that

$$\sup_{x \in K} |\partial_x^\alpha f(x)| \leq Ch^{|\alpha|} (|\alpha|!)^s \quad \text{for any } \alpha \in \mathbf{Z}_+^n.$$

When $s = \infty$, we put $\mathcal{E}^{(\infty)}(\mathbf{R}^n) = \mathcal{E}(\mathbf{R}^n)$. Note that $\mathcal{E}^{(s)}(\mathbf{R}^n) \subseteq \mathcal{E}^{(s)}(\mathbf{R}^n)$ holds for $1 < s < \infty$. For simplicity, we often write $\mathcal{E}^*$ instead of $\mathcal{E}^{(s)}$ or $\mathcal{E}^{(s)}$; that is, $\mathcal{E}^*$ with $* = \{s\}$ means $\mathcal{E}^{(s)}$, and $\mathcal{E}^*$ with $* = (s)$ means $\mathcal{E}^{(s)}$. By $C^\infty([0, T], \mathcal{E}^*(\mathbf{R}^n))$ [resp. $C^\infty((0, T), \mathcal{E}^*(\mathbf{R}^n))$] we denote the set of all infinitely differentiable functions on $[0, T]$ [resp. $(0, T)$] with values in $\mathcal{E}^*$ equipped with the usual topology.

Then, by following the discussion in [5] we can obtain the following result.

THEOREM 1. *Let P be the operator in (1.1) satisfying (A _{ϵ}) and (B). Assume the following conditions:*

- (1) $1 < s < \sigma/(\sigma-1)$.
- (2) $* = \{s\}$ or (s) .
- (3) $t^{l(j, \alpha)} a_{j, \alpha}(t, x) \in C^\infty([0, T], \mathcal{E}^*(\mathbf{R}^n))$ ($j + |\alpha| \leq m$ and $j < m$).

Then we have the following results (I) and (II).

(I) (Unique solvability). *If $\rho_i(x) \notin \mathbf{Z}_+$ holds for any $x \in \mathbf{R}^n$ and $1 \leq i \leq m$, the equation*

$$(1.4) \quad Pu = f \quad \text{in } C^\infty([0, T], \mathcal{E}^*(\mathbf{R}^n))$$

is uniquely solvable.

(II) (Asymptotic expansions). *If $\rho_i(x) - \rho_j(x) \notin \mathbf{Z}$ holds for any $x \in \mathbf{R}^n$*

and $1 \leq i \leq j \leq m$, the general solution of the equation

$$(1.5) \quad Pu=0 \quad \text{in } C^\infty((0, T), \mathcal{E}^*(\mathbf{R}^n))$$

is characterized as follows. (II-1) Any solution $u(t, x) (=u) \in C^\infty((0, T), \mathcal{E}^*(\mathbf{R}^n))$ of (1.5) is expanded asymptotically into the form

$$u(t, x) \sim \sum_{i=1}^m \left\{ \varphi_i(x) t^{\rho_i(x)} + \sum_{k=1}^{\infty} \sum_{h=0}^{mk} \varphi_{i, k, h}(x) t^{\rho_i(x) + k} (\log t)^h \right\}$$

(as $t \rightarrow +0$) for some unique $\varphi_i(x)$, $\varphi_{i, k, h}(x) \in \mathcal{E}^*(\mathbf{R}^n)$ ($1 \leq i \leq m$, $1 \leq k < \infty$ and $0 \leq h \leq mk$). (II-2) Conversely, for any $\varphi_1(x), \dots, \varphi_m(x) \in \mathcal{E}^*(\mathbf{R}^n)$ there exist a unique solution $u(t, x) (=u) \in C^\infty((0, T), \mathcal{E}^*(\mathbf{R}^n))$ of (1.5) and unique coefficients $\varphi_{i, k, h}(x) \in \mathcal{E}^*(\mathbf{R}^n)$ ($1 \leq i \leq m$, $1 \leq k < \infty$ and $0 \leq h \leq mk$) such that the asymptotic relation in (II-1) holds.

In fact, the case $*=\{s\}$ was already proved in [5] and the case $*=(s)$ may be proved by an argument quite parallel to the case $*=\{s\}$ (see also §§ 4 and 5).

Hence, our target is only the case $s=\sigma/(\sigma-1)$. Since it seems impossible to have good results in the case $*=\{s\}$ with $s=\sigma/(\sigma-1)$ (as is seen in the introduction), we will discuss only the case $*=(s)$ with $s=\sigma/(\sigma-1)$ from now on.

§ 2. Main theorem

Now, let us consider the equation $Pu=f$ in $C^\infty([0, T], \mathcal{E}^{(s)}(\mathbf{R}^n))$ or $C^\infty((0, T), \mathcal{E}^{(s)}(\mathbf{R}^n))$ under the condition $s=\sigma/(\sigma-1)$.

Let P be the operator in (1.1) satisfying (A _{κ}) and (B). Put

$$(2.1) \quad \mathcal{J} = \{(j, \alpha) \in \mathbf{Z}_+ \times \mathbf{Z}_+^n; j + |\alpha| < m \text{ and } |\alpha| > 0\},$$

let $\sigma (\geq 1)$ be as in (1.2), and let $M_{j, \alpha}(\tau, r)$ be as in (1.3). Define $\sigma_{j, \alpha} (\geq 1)$ by

$$(2.2) \quad \sigma_{j, \alpha} = \max[1, \min(\max_{\tau \in \mathfrak{S}_n} \max_{1 \leq r \leq n} M_{j, \alpha}(\tau, r))].$$

Then, we have $\sigma = \max\{\sigma_{j, \alpha}; (j, \alpha) \in \mathcal{J}\}$. Put $\mathcal{A}_P$ as follows:

$$(2.3) \quad \mathcal{A}_P = \{(j, \alpha) \in \mathcal{J}; \sigma_{j, \alpha} = \sigma\}.$$

Let $\kappa = (\kappa_1, \dots, \kappa_n) \in \mathbf{R}^n$ be the one in (A _{κ}). For $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbf{Z}_+^n$, denote by $S_\kappa(\alpha)$ the set of all $l \in \mathbf{R}$ satisfying the following (i) and (ii): (i) $0 < l < \langle \kappa, \alpha \rangle$, and (ii) there are $\tau \in \mathfrak{S}_n$ and $p \in \{1, \dots, n-1\}$ such that

$$(2.4) \quad \begin{cases} l = \kappa_{\tau(1)} \alpha_{\tau(1)} + \dots + \kappa_{\tau(p)} \alpha_{\tau(p)}, \\ \{\kappa_{\tau(1)}, \dots, \kappa_{\tau(p)}\} < \{\kappa_{\tau(p+1)}, \dots, \kappa_{\tau(n)}\}. \end{cases}$$

Here, $\langle \kappa, \alpha \rangle = \kappa_1 \alpha_1 + \dots + \kappa_n \alpha_n$, and $\{a_1, \dots, a_p\} < \{b_1, \dots, b_q\}$ means that $a_i < b_j$

holds for any i and j . Put S_P as follows:

$$(2.5) \quad S_P = \{(j, \alpha) \in \mathcal{J}; l(j, \alpha) \in S_s(\alpha)\}.$$

Then, the main theorem of this paper is stated as follows.

THEOREM 2. *Let P be the operator in (1.1) satisfying (A_s) and (B). Assume the following conditions:*

- (1) $s = \sigma/(\sigma - 1)$ and $\Delta_P \cap S_P = \emptyset$.
- (2) $* = (s)$.
- (3) $t^{l(j, \alpha)} a_{j, \alpha}(t, x) \in C^\infty([0, T], \mathcal{E}^{(s)}(\mathbf{R}^n))$ ($j + |\alpha| \leq m$ and $j < m$).

Then, the same results as in Theorem 1: —(I) (Unique solvability) and (II) (Asymptotic expansions)—are valid.

In the case $\sigma = 1$ (i.e. $s = \infty$), Theorem 2 was already proved in [3, 4]. In the case $\sigma > 1$ (i.e. $1 < s < \infty$), Theorem 2 will be proved in §§ 4 and 5.

As to the condition $\Delta_P \cap S_P = \emptyset$, we remark the following.

REMARK. (1) When $\sigma = 1$, $l(j, \alpha) \geq \langle \kappa, \alpha \rangle$ holds for any $(j, \alpha) \in \mathcal{J}$; therefore in this case the condition $\Delta_P \cap S_P = \emptyset$ is trivially satisfied.

(2) When $\kappa_1 = \dots = \kappa_n$, $S_s(\alpha) = \emptyset$ holds for any $\alpha \in \mathbf{Z}_+^n$; therefore in this case the condition $\Delta_P \cap S_P = \emptyset$ is trivially satisfied.

(3) When $|\alpha| = 1$, $S_s(\alpha) = \emptyset$ holds for any $\kappa = (\kappa_1, \dots, \kappa_n)$; therefore in the case $m = 2$ the condition $\Delta_P \cap S_P = \emptyset$ is trivially satisfied.

(4) When $m = 3$, $n = 2$ and $0 < \kappa_1 < \kappa_2$, $S_s((1, 0)) = S_s((0, 1)) = S_s((2, 0)) = S_s((0, 2)) = \emptyset$ and $S_s((1, 1)) = \{\kappa_1\}$ hold; therefore in this case the condition $\Delta_P \cap S_P = \emptyset$ is equivalent to the following: $\sigma \neq 2$ or $l(0, (1, 1)) \neq \kappa_1$.

(5) Put $\Sigma_m = \{k/h; k, h \in \mathbf{Z} \text{ and } 1 \leq h < k \leq m-1\}$. Then we can see the following: when $\sigma \notin \Sigma_m$, we have $\Delta_P \cap S_P = \emptyset$. Note that for any $k/h \in \Sigma_m$ we can find an example such that $\sigma = k/h$ and $\Delta_P \cap S_P \neq \emptyset$.

By Theorems 1 and 2, we have obtained good results in the following two cases:

- (C-1) $1 < s < \sigma/(\sigma - 1)$.
- (C-2) $s = \sigma/(\sigma - 1)$ and $\Delta_P \cap S_P = \emptyset$.

But, our method employed here cannot be applied to the case

$$(C-3) \quad s = \sigma/(\sigma - 1) \quad \text{and} \quad \Delta_P \cap S_P \neq \emptyset;$$

therefore, it is still an open problem to study the case (C-3).

In §§ 4 and 5 [resp. in [5]], we will prove Theorem 2 [resp. we proved

Theorem 1] by reducing the problem to the one of the convergence of a majorant series $M_a(\{\beta(J_k)\}, C, R, t)$ which appears naturally in the estimation of our formal solution (based on the method of successive approximations). In our proof, the following fact is essential: *the majorant series $M_a(\{\beta(J_k)\}, C, R, t)$ converges, if and only if (C-1) or (C-2) holds.*

Precisely, it is stated as follows. Let $\mathcal{J}$ be as in (2.1), and put $\mathcal{J}^1 = \mathcal{J}$, $\mathcal{J}^2 = \mathcal{J} \times \mathcal{J}$, $\mathcal{J}^3 = \mathcal{J} \times \mathcal{J} \times \mathcal{J}$, $\dots$. For $J_k = ((j_1, \alpha_{(1)}), \dots, (j_k, \alpha_{(k)})) \in \mathcal{J}^k$, we denote by $\mathcal{M}(J_k)$ the set of all $(\beta_{(1)}, \dots, \beta_{(k)}) \in \mathbf{Z}_+^n \times \dots \times \mathbf{Z}_+^n$ satisfying the following:

$$(2.6) \quad \begin{cases} (0, \dots, 0) \leq \beta_{(i)} \leq \alpha_{(i)} & (i=1, \dots, k), \\ \langle \kappa, \beta_{(1)} + \dots + \beta_{(k)} \rangle \leq l(j_1, \alpha_{(1)}) + \dots + l(j_k, \alpha_{(k)}) & (i=1, \dots, k). \end{cases}$$

Let $a > 0$. For $J_k = ((j_1, \alpha_{(1)}), \dots, (j_k, \alpha_{(k)})) \in \mathcal{J}^k$, $\beta(J_k) = (\beta_{(1)}, \dots, \beta_{(k)}) \in \mathcal{M}(J_k)$, $C > 0$ and $R > 0$, we put

$$(2.7) \quad \begin{aligned} & F_a(J_k, \beta(J_k), C, R, t) \\ &= C^{k+1} t^{l_1 + \dots + l_k - \langle \kappa, \beta_{(1)} + \dots + \beta_{(k)} \rangle} \times R^{(|\alpha_{(1)}| - |\beta_{(1)}|) + \dots + (|\alpha_{(k)}| - |\beta_{(k)}|)} \\ & \quad \times \frac{1}{a_{k+1}} \times \frac{[(|\alpha_{(1)}| - |\beta_{(1)}|) + \dots + (|\alpha_{(k)}| - |\beta_{(k)}|)]!^s}{(a_1)^{m-j_1-|\beta_{(1)}|} \times \dots \times (a_k)^{m-j_k-|\beta_{(k)}|}}, \end{aligned}$$

where $l_i = l(j_i, \alpha_{(i)})$ ($i=1, \dots, k$), $a_1 = a$ and

$$a_{i+1} = a + l_1 + \dots + l_i - \langle \kappa, \beta_{(1)} + \dots + \beta_{(i)} \rangle \quad (i=1, \dots, k).$$

Then, the majorant series $M_a(\{\beta(J_k)\}, C, R, t)$ which appears in § 4 is given by

$$(2.8) \quad M_a(\{\beta(J_k)\}, C, R, t) = \sum_{k=1}^{\infty} \sum_{J_k \in \mathcal{J}^k} F_a(J_k, \beta(J_k), C, R, t)$$

(see (4.7)). As to the convergence of $M_a(\{\beta(J_k)\}, C, R, t)$, we have the following result (the proof will be given in § 6).

PROPOSITION 1. *Let $1 < s < \infty$, $a > 0$, $C > 0$ and $T > 0$. Then, we have the following results (I) and (II).*

(I) *The following three conditions (I-1), (I-2) and (I-3) are equivalent to each other.*

(I-1) *(C-1) or (C-2) holds.*

(I-2) *There are $R_0 > 0$ and $\beta(J_k) \in \mathcal{M}(J_k)$ ($J_k \in \mathcal{J}^k$, $k=1, 2, \dots$) such that*

$$M_a(\{\beta(J_k)\}, C, R_0, T) < +\infty.$$

(I-3) *There are $R_0 > 0$, $T_0 > 0$ and $\beta(J_k) \in \mathcal{M}(J_k)$ ($J_k \in \mathcal{J}^k$, $k=1, 2, \dots$) such that*

$$M_a(\{\beta(J_k)\}, C, R_0, T_0) < +\infty.$$

(II) The following three conditions (II-1), (II-2) and (II-3) are equivalent to each other.

(II-1) (C-1) holds.

(II-2) For any $R > 0$, there are $\beta(J_k) \in \mathcal{M}(J_k)$ ($J_k \in \mathcal{J}^k$, $k=1, 2, \dots$) such that

$$M_a(\{\beta(J_k)\}, C, R, T) < +\infty.$$

(II-3) For any $R > 0$, there are $T_0 > 0$ and $\beta(J_k) \in \mathcal{M}(J_k)$ ($J_k \in \mathcal{J}^k$, $k=1, 2, \dots$) such that

$$M_a(\{\beta(J_k)\}, C, R, T_0) < +\infty.$$

In the proof of Theorem 1 (in [5]), we used the condition (II-2) in Proposition 1. In the proof of Theorem 2 (in §§ 4 and 5), we will use the condition (I-2) in Proposition 1. This is the reason why our method cannot be applied to the case (C-3).

§ 3. Some lemmas

Before the proof of Theorem 2, let us recall here formal norms introduced in [5, § 5] (see also [2]), and let us present some preparatory lemmas.

Let $p, l \in \mathbb{Z}_+$, $r \in \mathbf{R}$ and $\kappa = (\kappa_1, \dots, \kappa_n) \in \mathbf{R}^n$ such that $\kappa_i > 0$ ($1 \leq i \leq n$). For $f(t, x) \in C^\infty((0, T), H^\infty(\mathbf{R}^n))$ and $a(t, x) \in C^\infty((0, T) \times \mathbf{R}^n)$, we write

$$\begin{aligned} \|\mathcal{V}_r^{p, \infty} f(t)\| &= \sum_{q=0}^{\infty} \sum_{j+|\alpha| \leq p} \frac{1}{j! \alpha!} \sum_{|\beta|=q} \|(t\partial_t + r)^j \partial_x^{\alpha+\beta} f(t)\|_{L_2(\mathbf{R}^n)} \frac{\rho^q}{q!}, \\ \|\mathcal{V}_r^{p, \infty} \mathcal{V}_{\kappa, r}^l f(t)\| &= \sum_{j+|\alpha| \leq l} \|\mathcal{V}_r^{p, \infty} t^{\langle \kappa, \alpha \rangle} (t\partial_t + r)^j \partial_x^\alpha f(t)\|, \\ \|\mathcal{V}^{p, \infty} a\|_\infty &= \sum_{q=0}^{\infty} \sum_{j+|\alpha| \leq p} \frac{1}{j! \alpha!} \sum_{|\beta|=q} \|(t\partial_t)^j \partial_x^{\alpha+\beta} a\|_{L^\infty(\Omega)} \frac{\rho^q}{q!} \end{aligned}$$

(where $\langle \kappa, \alpha \rangle = \kappa_1 \alpha_1 + \dots + \kappa_n \alpha_n$ and $\Omega = (0, T) \times \mathbf{R}^n$). Also, for a differential operator R of the form

$$R = \sum_{j+|\alpha| \leq m} t^{\langle \kappa, \alpha \rangle} a_{j, \alpha}(t, x) (t\partial_t)^j \partial_x^\alpha$$

we write

$$\|\mathcal{V}^{p, \infty} R\|_\infty = \sum_{j+|\alpha| \leq m} \|\mathcal{V}^{p, \infty} a_{j, \alpha}\|_\infty.$$

For basic properties of these formal norms, see [5, § 5].

Let $1 \leq s < \infty$ and put

$$\theta_s(\rho) = \sum_{q=0}^{\infty} (q!)^s \frac{\rho^q}{q!}.$$

For a formal power series $\varphi(t, \rho)$ in ρ , we write

$$(3.1) \quad \varphi(t, \rho) \in \mathcal{E}^{(s)} \text{ uniformly on } [0, T],$$

if $\varphi(t, \rho)$ satisfies the following: for any $\varepsilon > 0$ there is a $C_\varepsilon > 0$ such that $\varphi(t, \rho) \ll C_\varepsilon \theta_s(\varepsilon \rho)$ holds on $[0, T]$. Here, $\sum_{q=0}^{\infty} a_q \rho^q \ll \sum_{q=0}^{\infty} b_q \rho^q$ means that $|a_q| \leq b_q$ holds for any q . Then, by Sobolev's lemma we can see

LEMMA 1. *Let $f(t, x) \in C^0([0, T], H^\infty(\mathbf{R}^n))$. Assume that $\text{supp}(f) \subset [0, T] \times K$ holds for some compact subset K of $\mathbf{R}^n$. Then, the following conditions (i) and (ii) are equivalent:*

- (i) $f(t, x) \in C^0([0, T], \mathcal{E}^{(s)}(\mathbf{R}^n))$.
- (ii) $\|F_r^{0, \infty} f(t)\| \in \mathcal{E}^{(s)}$ uniformly on $[0, T]$.

For a positive-valued function $C(\varepsilon)$ in $\varepsilon > 0$, we write

$$\theta_{(s)}(\rho; C(\varepsilon)) = \inf_{\varepsilon > 0} (C(\varepsilon) \theta_s(\varepsilon \rho)).$$

Then, (3.1) is equivalent to the condition that $\varphi(t, \rho) \ll \theta_{(s)}(\rho; C(\varepsilon))$ holds on $[0, T]$ for some $C(\varepsilon)$. Note that by putting $A = C(1)$ and $C_0(\varepsilon) = C(\varepsilon)/A$ we have $\theta_{(s)}(\rho; C(\varepsilon)) = A \theta_{(s)}(\rho; C_0(\varepsilon))$ and $C_0(1) = 1$.

We say that $\varphi(t, \rho) = \sum_{q=0}^{\infty} \varphi_q(t) \rho^q$ satisfies $(M_{(s)})$ with respect to $C(\varepsilon)$, if $\varphi(t, \rho)$ satisfies the following condition:

$$(M_{(s)}) \quad \left(\inf_{\varepsilon > 0} (C(\varepsilon) \varepsilon^p) \right) \frac{(p!)^s}{p!} \varphi_q(t) \leq \varphi_{p+q}(t) \quad \text{on } (0, T) \text{ for any } p, q \in \mathbf{Z}_+.$$

Then, by the same argument as in [5, Lemma 5] we have

LEMMA 2. *Let $1 \leq s < \infty$. Assume that $\varphi(t, \rho) (\gg 0)$ satisfies $(M_{(s)})$ with respect to $C(\varepsilon)$. Then, for any $0 < 2k \leq h$ we have*

$$\theta_{(s)}(k\rho; C(\varepsilon)) \varphi(t, h\rho) \ll 2\varphi(t, h\rho).$$

The following lemma guarantees that the arguments in [5] with $\theta_s(\rho)$ replaced by $\theta_{(s)}(\rho; C(\varepsilon))$ are also valid.

LEMMA 3. *Let $1 < s < \infty$, let $\varphi(t, \rho) (\gg 0)$ be a formal power series in ρ such that $\varphi(t, \rho) \in \mathcal{E}^{(s)}$ uniformly on $[0, T]$, and let $C_0(\varepsilon)$ be a positive-valued function in $\varepsilon > 0$ such that $C_0(1) = 1$. Then, we can find a positive-valued function $C(\varepsilon)$ in $\varepsilon > 0$ which satisfies the following conditions (i) and (ii):*

- (i) $\varphi(t, \rho) \ll \theta_{(s)}(\rho; C(\varepsilon))$ holds on $[0, T]$.
- (ii) $\theta_{(s)}(\rho; C(\varepsilon))$ satisfies $(M_{(s)})$ with respect to $C_0(\varepsilon)$.

PROOF. First, let us make clear the meaning of $\theta_{(s)}(\rho; C(\varepsilon))$. Denote by

$\mathcal{F}$ the set of all functions $f(z) \in C^2([0, \infty))$ satisfying the following: $f(z)=0$ on $[0, 1]$, $(d/dz)f(z) \leq 0$ on $[0, \infty)$, $(d/dz)f(z) \rightarrow -\infty$ (as $z \rightarrow +\infty$) and $(d/dz)^2 f(z) \leq 0$ on $[0, \infty)$. For $f(z) \in \mathcal{F}$, we write

$$\theta_{s,f}(\rho) = \sum_{q=0}^{\infty} e^{f(q)} (q!)^s \frac{\rho^q}{q!}.$$

Then, by comparing the convex sets of the following two types

$$\bigcap_{\varepsilon > 0} \{(z, y) \in \mathbf{R}^2; z \geq 0 \text{ and } y \leq (\log \varepsilon)z + \log C(\varepsilon)\},$$

$$\{(z, y) \in \mathbf{R}^2; z \geq 0 \text{ and } y \leq f(z)\},$$

we can see the following facts (F-1) and (F-2).

(F-1) For any positive-valued function $C(\varepsilon)$ in $\varepsilon > 0$ satisfying $C(1)=1$, we can find an $f(z) \in \mathcal{F}$ such that $\theta_{(s)}(\rho; C(\varepsilon)) \ll \theta_{s,f}(\rho)$.

(F-2) For any $f(z) \in \mathcal{F}$, we can find a positive-valued function $C(\varepsilon)$ in $\varepsilon > 0$ such that $C(1)=1$ and $\theta_{s,f}(\rho) = \theta_{(s)}(\rho; C(\varepsilon))$.

Now, let us show Lemma 3. Let $\varphi(t, \rho) (\gg 0)$ and $C_0(\varepsilon)$ be as in Lemma 3. Choose a positive-valued function $C_i(\varepsilon)$ in $\varepsilon > 0$ such that $C_i(1)=1$ and

$$(3.2) \quad \varphi(t, \rho) \ll A\theta_{(s)}(\rho; C_i(\varepsilon)) \quad \text{on } [0, T]$$

for some $A > 0$. Since $C_i(\varepsilon)$ ($i=0, 1$) satisfy $C_i(1)=1$, by (F-1) we can find an $f(z) \in \mathcal{F}$ such that

$$(3.3) \quad \theta_{(s)}(\rho; C_i(\varepsilon)) \ll \theta_{s,f}(\rho) \quad (i=0, 1).$$

Therefore, to have Lemma 3 it is sufficient to construct a function $g(z) \in \mathcal{F}$ satisfying the following (g-1) and (g-2):

$$(g-1) \quad f(z) \leq g(z) \quad \text{on } [0, \infty).$$

$$(g-2) \quad (d/dz)^2(g(z) + (s-1)\Gamma(1+z)) \geq 0 \quad \text{on } [0, \infty)$$

(where $\Gamma(z)$ denotes the gamma-function of Euler). In fact, if such a function $g(z) \in \mathcal{F}$ is obtained, by (3.2), (3.3) and (g-1) we have

$$\varphi(t, \rho) \ll A\theta_{s,f}(\rho) \ll A\theta_{s,g}(\rho) \quad \text{on } [0, T],$$

and by (3.3), (g-1) and (g-2) we have

$$\begin{aligned} (\inf_{\varepsilon > 0} (C_0(\varepsilon)\varepsilon^p)) \left(\frac{(p!)^s}{p!} \right) \left(e^{g(q)} \frac{(q!)^s}{q!} \right) &\leq \left(e^{f(p)} \frac{(p!)^s}{p!} \right) \left(e^{g(q)} \frac{(q!)^s}{q!} \right) \\ &\leq \left(e^{g(p)} \frac{(p!)^s}{p!} \right) \left(e^{g(q)} \frac{(q!)^s}{q!} \right) \\ &\leq e^{g(p+q)} \frac{[(p+q)!]^s}{(p+q)!} \end{aligned}$$

(for any $p, q \in \mathbf{Z}_+$) which means that $\theta_{s, g}(\rho)$ satisfies $(M_{(s)})$ with respect to $C_0(\varepsilon)$; hence, by choosing $C(\varepsilon)$ so that $A\theta_{s, g}(\rho) = \theta_{(s)}(\rho; C(\varepsilon))$ (by (F-2)) we can obtain Lemma 3.

Here, we note the following lemma (the proof will be given later).

LEMMA 4. *Let $a(z) \in C^0([0, \infty))$ be such that $a(z) \geq 0$ on $[0, \infty)$ and $\int_0^z a(y) dy \rightarrow \infty$ (as $z \rightarrow \infty$), and let $h(z) \in C^0([0, \infty))$ be such that $h(z) \geq 0$ on $[0, \infty)$, $h(z)$ is increasing in z and $h(z) \rightarrow \infty$ (as $z \rightarrow \infty$). Then, we can find a $b(z) \in C^0([0, \infty))$ which satisfies the following conditions: (b-1) $b(z) = 0$ on $[0, 1]$, (b-2) $0 \leq b(z) \leq a(z)$ on $[0, \infty)$, (b-3) $\int_0^z b(y) dy \rightarrow \infty$ (as $z \rightarrow \infty$), and (b-4) $\int_0^z b(y) dy \leq h(z)$ on $[0, \infty)$.*

By using this lemma, let us construct a function $g(z) \in \mathcal{F}$ satisfying (g-1) and (g-2). Put

$$a(z) = (s-1) \left(\frac{d}{dz} \right)^2 \Gamma(1+z),$$

$$h(z) = (-1) \left(\frac{d}{dz} \right) f(z).$$

Then, $a(z)$ and $h(z)$ satisfy the conditions in Lemma 4. Therefore, by Lemma 4 we have a $b(z) \in C^0([0, \infty))$ which satisfies (b-1) ~ (b-4). Hence, by putting

$$g(z) = (-1) \int_0^z \int_0^y b(\tau) d\tau dy$$

we can obtain a $g(z) \in \mathcal{F}$ satisfying (g-1) and (g-2). Note that the fact $g(z) \in \mathcal{F}$ follows from (b-1) ~ (b-3), that (g-1) follows from (b-4), and that (g-2) follows from (b-2). Q.E.D.

PROOF OF LEMMA 4. When there is a $p \geq 1$ such that $\int_p^z a(y) dy \leq h(z)$ for any $z > p$, by modifying the function

$$b_0(z) = \begin{cases} 0, & \text{for } 0 \leq z \leq p, \\ a(z), & \text{for } z > p \end{cases}$$

we can obtain a function $b(z)$ in Lemma 4.

When for any $p \geq 1$ there is a $q > p$ such that $\int_p^q a(y) dy > h(q)$, we can choose a sequence $p_1, q_1, p_2, q_2, \dots$ successively so that the following conditions are satisfied: $p_1 \geq 1$, $h(p_1) \geq 1$, $q_k > p_k$,

$$\sum_{i=1}^{k-1} \int_{p_i}^{q_i} a(y) dy + \int_{p_k}^z a(y) dy \begin{cases} < h(z), & \text{for } p_k \leq z < q_k, \\ = h(z), & \text{for } z = q_k, \end{cases}$$

$$p_{k+1} > q_k + 1,$$

$$\sum_{i=1}^k \int_{p_i}^{q_i} a(y) dy + 1 \leq h(p_{k+1})$$

($k=1, 2, \dots$). Therefore, in this case, by modifying the function

$$b_0(z) = \begin{cases} 0, & \text{for } q_{k-1} \leq z \leq p_k \quad (k=1, 2, \dots), \\ a(z), & \text{for } p_k < z < q_k \quad (k=1, 2, \dots) \end{cases}$$

(where $q_0=0$) we can obtain a function $b(z)$ in Lemma 4.

Q.E.D.

§ 4. Proof of (I): Unique solvability

To have the part (I) in Theorems 1 and 2, it is sufficient to establish the $\mathcal{E}^{(s)}$ -version of [5, Proposition 4] (see also [5, §§ 3 and 7]).

Let P be the operator in (1.1), and let us consider

$$(4.1)_r \quad P(t^r v) = t^r g,$$

where $v=v(t, x)$, $g=g(t, x)$, and $r \in \mathbf{R}$ is a parameter. As in [5, Proposition 4], we impose here the following conditions:

(B') (B) is satisfied. In addition, there is a $c>0$ such that

$$|\lambda_i(t, x, \xi) - \lambda_j(t, x, \xi)| \geq c$$

holds on $\{(t, x, \xi) \in [0, T] \times \mathbf{R}^n \times \mathbf{R}^n; |\xi|=1\}$ for any $1 \leq i \neq j \leq m$.

$$(D_p^{(s)}) \quad \|\nabla^{p, \infty} a_{j, \alpha}\|_{\infty} \in \mathcal{E}^{(s)} \quad (j+|\alpha| \leq m \text{ and } j < m).$$

Then, our $\mathcal{E}^{(s)}$ -version of [5, Proposition 4] is given as follows.

PROPOSITION 2. *Let P be the operator in (1.1), let $1 < s < \infty$, and let $p \in \mathbf{N}$. Assume that P satisfies (A_{*}), (B') and (D_p^(s)), and that s satisfies (C-1) or (C-2). Then, there is an $a_s > 0$ which satisfies the following condition. If $r > a_s$, and if $g(t, x) \in C^\infty((0, T), H^\infty(\mathbf{R}^n))$ satisfies $(t\partial_t)^l g(t, x) \in C^0([0, T], H^\infty(\mathbf{R}^n))$ for any $l \in \mathbf{Z}_+$ and $\|\nabla_r^{p, \infty} g(t)\| \in \mathcal{E}^{(s)}$ uniformly on $[0, T]$, then the equation (4.1)_r has a unique solution $v(t, x) \in C^\infty((0, T), H^\infty(\mathbf{R}^n))$ such that $(t\partial_t)^l v(t, x) \in C^0([0, T], H^\infty(\mathbf{R}^n))$ for any $l \in \mathbf{Z}_+$ and $\|\nabla_r^{p, \infty} \nabla_{x, r}^{m-1} v(t)\| \in \mathcal{E}^{(s)}$ uniformly on $[0, T]$. In addition, if $g(t, x)$ satisfies $\text{supp}(g) \subset C_\mu(0, K)$ for some compact subset K of $\mathbf{R}^n$, $v(t, x)$ also satisfies $\text{supp}(v) \subset C_\mu(0, K)$.*

Here, $C_\mu(0, K)$ is defined by the case $t_0=0$ of

$$(4.2) \quad C_\mu(t_0, K) = \left\{ (t, x) \in [0, T] \times \mathbf{R}^n; \min_{y \in K} |x - y| \leq \frac{\lambda_{\max} T^{k-\mu}}{\mu} |t^\mu - t_0^\mu| \right\},$$

where $\mu = \min\{\kappa_1, \dots, \kappa_n\}$, $k = \max\{\kappa_1, \dots, \kappa_n\}$ and $\lambda_{\max}$ is the least upper bound of $|\lambda_i(t, x, \xi)|$ ($1 \leq i \leq m$) on $\{(t, x, \xi) \in [0, T] \times \mathbf{R}^n \times \mathbf{R}^n; |\xi| = 1\}$.

PROOF OF PROPOSITION 2. Let $\mathcal{J}$ be as in (2.1). Put $Q_r(j, \alpha)$ ($(j, \alpha) \in \mathcal{J}$) and L as follows:

$$(4.3) \quad \begin{aligned} Q_r(j, \alpha) &= (-1)^{t^{(j, \alpha)}} a_{j, \alpha}(t, x) (t \partial_t + r)^j \partial_x^\alpha, \\ L &= P + \sum_{(j, \alpha) \in \mathcal{J}} Q_0(j, \alpha). \end{aligned}$$

Then, $(4.1)_r$ is equivalent to

$$L(t^r v) = t^r (g + \sum_{(j, \alpha) \in \mathcal{J}} Q_r(j, \alpha) v).$$

Therefore, to solve $(4.1)_r$ we can use the method of successive approximations: first we solve (by [5, Lemma 11])

$$\begin{aligned} L(t^r v_0) &= t^r g, \\ L(t^r v((j_1, \alpha_{(1)}))) &= t^r Q_r(j_1, \alpha_{(1)}) v_0, \\ L(t^r v((j_1, \alpha_{(1)}), (j_2, \alpha_{(2)}))) &= t^r Q_r(j_2, \alpha_{(2)}) v((j_1, \alpha_{(1)})), \\ &\vdots \\ L(t^r v((j_1, \alpha_{(1)}), \dots, (j_k, \alpha_{(k)}))) &= t^r Q_r(j_k, \alpha_{(k)}) v((j_1, \alpha_{(1)}), \dots, (j_{k-1}, \alpha_{(k-1)})), \end{aligned}$$

and then we show the convergence of the formal solution

$$(4.4) \quad v = v_0 + \sum_{k=1}^{\infty} \sum_{\substack{(j_i, \alpha_{(i)}) \in \mathcal{J} \\ i=1, \dots, k}} v((j_1, \alpha_{(1)}), \dots, (j_k, \alpha_{(k)})).$$

The proof of the convergence is as follows. By $(D_p^{(s)})$, we have

$$\|F^{p, \infty} L\|_{\infty}, \quad \|F^{p, \infty} a_{j, \alpha}\|_{\infty} \ll B_0 \theta_{(s)}(\rho; C_0(\varepsilon))$$

for some $B_0 > 0$ and some positive-valued function $C_0(\varepsilon)$ in $\varepsilon > 0$ satisfying $C_0(1) = 1$. Since $\|F_r^{p, \infty} g(t)\| \in \mathcal{C}^{(s)}$ uniformly on $[0, T]$, by Lemma 3 we can find a positive-valued function $C(\varepsilon)$ in $\varepsilon > 0$ such that

$$\|F_r^{p, \infty} g(t)\| \ll \theta_{(s)}(\rho; C(\varepsilon)) \ll \theta_{(s)}(2\rho; C(\varepsilon)) \quad \text{on } [0, T]$$

and that $\theta_{(s)}(\rho; C(\varepsilon))$ satisfies $(M_{(s)})$ with respect to $C_0(\varepsilon)$. Therefore, by the same argument as in [5, formula (7.8) or (7.10)] we have the following: there are $C > 0$, $H > 0$ and $a > 0$ such that for any $J_k = ((j_1, \alpha_{(1)}), \dots, (j_k, \alpha_{(k)})) \in \mathcal{J}^k$ we have

$$(4.5) \quad \begin{aligned} &\|F_r^{p, \infty} F_{\varepsilon, r}^{m-1} v(J_k)(t)\| \ll C^{k+1} t^{l_1 + \dots + l_k - \langle \varepsilon, \beta_{(1)} + \dots + \beta_{(k)} \rangle} \\ &\times \frac{1}{a_{k+1}} \times \frac{1}{(a_1)^{m-j_1-|\beta_{(1)}|} \times \dots \times (a_k)^{m-j_k-|\beta_{(k)}|}} \\ &\times \partial_\rho^{(|\alpha_{(1)}| - |\beta_{(1)}|) + \dots + (|\alpha_{(k)}| - |\beta_{(k)}|)} \theta_{(s)}(H\rho; C(\varepsilon)) \end{aligned}$$

on $[0, T]$ for any $\beta(J_k) = (\beta_{(1)}, \dots, \beta_{(k)}) \in \mathcal{M}(J_k)$, where we put $l_i = l(j_i, \alpha_{(i)})$ ($i=1, \dots, k$), $a_1 = a$ and

$$a_{i+1} = a + l_1 + \dots + l_i - \langle \kappa, \beta_{(1)} + \dots + \beta_{(i)} \rangle \quad (i=1, \dots, k).$$

Here, we note the following:

$$(4.6) \quad \begin{aligned} & \partial_\rho^{(|\alpha_{(1)}| - |\beta_{(1)}|) + \dots + (|\alpha_{(k)}| - |\beta_{(k)}|)} \theta_{(s)}(H\rho; C(\varepsilon)) \\ & \ll C(\varepsilon)(\varepsilon 2^s H)^{(|\alpha_{(1)}| - |\beta_{(1)}|) + \dots + (|\alpha_{(k)}| - |\beta_{(k)}|)} \\ & \quad \times [(|\alpha_{(1)}| - |\beta_{(1)}|) + \dots + (|\alpha_{(k)}| - |\beta_{(k)}|)]^s \theta_s(\varepsilon 2^s H\rho) \end{aligned}$$

holds for any $\varepsilon > 0$. Therefore, by (4.5), (4.6) and (2.7) we have

$$\|\nabla_{\kappa, r}^{p, \infty} \nabla_{\kappa, r}^{m-1} v(J_k)(t)\| \ll C(\varepsilon) F_a(J_k, \beta(J_k), C, \varepsilon R, T) \theta_s(\varepsilon R\rho)$$

(where $R = 2^s H$) for any $\varepsilon > 0$ and $\beta(J_k) \in \mathcal{M}(J_k)$, that is,

$$(4.7) \quad \sum_{k=1}^{\infty} \sum_{J_k \in \mathcal{J}^k} \|\nabla_{\kappa, r}^{p, \infty} \nabla_{\kappa, r}^{m-1} v(J_k)(t)\| \ll C(\varepsilon) M_a(\{\beta(J_k)\}, C, \varepsilon R, T) \theta_s(\varepsilon R\rho)$$

for any $\varepsilon > 0$ (by (2.8)). Hence, by the condition (I-2) in Proposition 1 we obtain

$$\sum_{k=1}^{\infty} \sum_{J_k \in \mathcal{J}^k} \|\nabla_{\kappa, r}^{p, \infty} \nabla_{\kappa, r}^{m-1} v(J_k)(t)\| \in \mathcal{O}^{(s)} \quad \text{uniformly on } [0, T].$$

Thus, the existence part of Proposition 2 is proved. The other part may be proved in the same way. Q.E.D.

§ 5. Proof of (II): Asymptotic expansions

To have the part (II) in Theorems 1 and 2, it is sufficient to establish the $\mathcal{O}^{(s)}$ -version of [5, Proposition 5] (see also [5, §§ 3 and 8]).

Let P be the operator in (1.1), and let us consider

$$(5.1) \quad \begin{cases} Pu = f, \\ \partial_t^i u|_{t=T} = 0 \quad \text{for } i=0, 1, \dots, m-1, \end{cases}$$

where $u = u(t, x)$ and $f = f(t, x)$. Then, our $\mathcal{O}^{(s)}$ -version of [5, Proposition 5] is given as follows.

PROPOSITION 3. *Let P be the operator in (1.1), let $1 < s < \infty$, and let $p \in \mathbb{N}$. Assume that P satisfies (A_s), (B') and (D_p^(s)), and that s satisfies (C-1) or (C-2). Then, if $f(t, x) \in C^\infty((0, T], H^\infty(\mathbb{R}^n))$ satisfies $\partial_t^i f|_{t=T} = 0$ for $i=0, 1, \dots, p-1$ and $t^A \|\nabla_0^{p, \infty} f(t)\| \in \mathcal{O}^{(s)}$ uniformly on $(0, T]$ for some $A > 0$, then the equation (5.1) has a unique solution $u(t, x) \in C^\infty((0, T], H^\infty(\mathbb{R}^n))$ such that $\partial_t^i u|_{t=T} = 0$ for*

$i=0, 1, \dots, m+p-1$ and that $t^B \|\mathcal{V}_0^{p,\infty} \mathcal{V}_{\varepsilon,0}^{m-1} u(t)\| \in \mathcal{E}^{(s)}$ uniformly on $(0, T]$ for some $B>0$. In addition, if $f(t, x)$ satisfies $\text{supp}(f) \subset C_\mu(T, K)$ for some compact subset K of $\mathbf{R}^n$, $u(t, x)$ also satisfies $\text{supp}(u) \subset C_\mu(T, K)$. (Here, $C_\mu(T, K)$ is defined by the case $t_0=T$ of (4.2).)

PROOF. Let $Q_\tau(j, \alpha)$ and L be as in (4.3). Then, to solve (5.1) we can use the method of successive approximations as follows: first we solve

$$\begin{cases} Lu_0 = f, \\ \partial_i^j u_0|_{t=T} = 0 & \text{for } i=0, 1, \dots, m+p-1, \\ Lu((j_1, \alpha_{(1)})) = Q_0(j_1, \alpha_{(1)})u_0, \\ \partial_i^j u((j_1, \alpha_{(1)}))|_{t=T} = 0 & \text{for } i=0, 1, \dots, m+p-1, \\ \vdots \\ Lu((j_1, \alpha_{(1)}), \dots, (j_k, \alpha_{(k)})) = Q_0(j_k, \alpha_{(k)})u((j_1, \alpha_{(1)}), \dots, (j_{k-1}, \alpha_{(k-1)})), \\ \partial_i^j u((j_1, \alpha_{(1)}), \dots, (j_k, \alpha_{(k)}))|_{t=T} = 0 & \text{for } i=0, 1, \dots, m+p-1, \end{cases}$$

and then we show the convergence of the formal solution

$$u = u_0 + \sum_{k=1}^{\infty} \sum_{\substack{(j_i, \alpha_{(i)}) \in \mathcal{J} \\ i=1, \dots, k}} u((j_1, \alpha_{(1)}), \dots, (j_k, \alpha_{(k)})).$$

The proof of the convergence is as follows. Since $(D_p^{(s)})$ and $t^A \|\mathcal{V}_0^{p,\infty} f(t)\| \in \mathcal{E}^{(s)}$ (uniformly on $(0, T]$) are assumed, we have

$$\begin{aligned} \|\mathcal{V}^{p,\infty} L\|_{\infty}, \|\mathcal{V}^{p,\infty} a_{j,\alpha}\|_{\infty} &\ll B_0 \theta_{(s)}(\rho; C_0(\varepsilon)), \\ \|\mathcal{V}_0^{p,\infty} f(t)\| &\ll t^{-A} \theta_{(s)}(2\rho; C(\varepsilon)) \quad \text{on } (0, T] \end{aligned}$$

for some $B_0>0$, $C_0(\varepsilon)$ and $C(\varepsilon)$ such that $\theta_{(s)}(\rho; C(\varepsilon))$ satisfies $(M_{(s)})$ with respect to $C_0(\varepsilon)$. Therefore, by the same argument as in (4.5) (see also [5, §§ 5 and 8]) we can obtain the following: there are $C>0$, $H>0$ and $a>0$ such that for any $J_k = ((j_1, \alpha_{(1)}), \dots, (j_k, \alpha_{(k)})) \in \mathcal{J}^k$ we have

$$\begin{aligned} \|\mathcal{V}_0^{p,\infty} \mathcal{V}_{\varepsilon,0}^{m-1} u(J_k)(t)\| &\ll C^{k+1} t^{-A} \left(\frac{T}{t}\right)^{a+1} T^{l_1 + \dots + l_k - \langle \kappa, \beta_{(1)} + \dots + \beta_{(k)} \rangle} \\ &\times \frac{1}{a+b_1} \times \frac{1}{(b_1)^{m-j_1-|\beta_{(1)}|} \times \dots \times (b_k)^{m-j_k-|\beta_{(k)}|}} \\ &\times \partial_p^{(|\alpha_{(1)}|-|\beta_{(1)}|) + \dots + (|\alpha_{(k)}|-|\beta_{(k)}|)} \theta_{(s)}(H\rho; C(\varepsilon)) \end{aligned}$$

on $(0, T]$ for any $(\beta_{(1)}, \dots, \beta_{(k)}) \in \mathbf{Z}_+^n \times \dots \times \mathbf{Z}_+^n$ satisfying

$$\begin{cases} (0, \dots, 0) \leq \beta_{(i)} \leq \alpha_{(i)} & (i=1, \dots, k), \\ \langle \kappa, \beta_{(i)} + \dots + \beta_{(k)} \rangle \leq l_i + \dots + l_k & (i=1, \dots, k), \end{cases}$$

where we put $l_i = l(j_i, \alpha_{(i)})$ ($i=1, \dots, k$) and

$$b_i = 1 + l_i + \cdots + l_k - \langle \kappa, \beta_{(i)} + \cdots + \beta_{(k)} \rangle \quad (i=1, \dots, k).$$

Hence, by an argument quite parallel to the proof of Proposition 2 we obtain

$$t^{a+1+A} \sum_{k=1}^{\infty} \sum_{J_k \in \mathcal{J}^k} \|V_0^{p, \infty} V_{\kappa, 0}^{m-1} u(J_k)(t)\| \in \mathcal{O}^{(s)} \quad \text{uniformly on } (0, T].$$

Thus, the existence part of Proposition 3 is proved. The other part may be proved in the same way. Q.E.D.

§ 6. Proof of Proposition 1

Note that “(I-2) $\Rightarrow$ (I-3)” is trivial, that “(II-1) $\Rightarrow$ (II-2)” was already proved in [5], and that “(II-2) $\Rightarrow$ (II-3)” is trivial. Therefore, to have Proposition 1 it is sufficient to prove the following parts: “(I-1) $\Rightarrow$ (I-2)”, “(I-3) $\Rightarrow$ (I-1)” and “(II-3) $\Rightarrow$ (II-1)”.

For $z \geq 0$, we write

$$z! = \Gamma(1+z),$$

where $\Gamma(z)$ is the gamma-function of Euler. Then, we have

LEMMA 5. (1) For any $p, q > 0$ we have

$$\frac{1}{1+p+q} \leq \frac{(p+q)!}{p!q!} \leq 2^{p+q}.$$

(2) For any $p_1, p_2, \dots, p_k > 0$ we have

$$1 \leq \frac{(1+p_1+p_2+\cdots+p_k)^{1+p_1+p_2+\cdots+p_k}}{(1+p_1)^{p_1}(1+p_1+p_2)^{p_2}\cdots(1+p_1+p_2+\cdots+p_k)^{p_k}} \leq e^{p_1+p_2+\cdots+p_k}.$$

(3) For any $0 < a < b$, there are $A_1 > 0$, $A_2 > 0$, $C_1 > 0$ and $C_2 > 0$ which satisfy the following: for any $p_1, p_2, \dots, p_k > 0$ satisfying $a \leq p_i \leq b$ ($i=1, 2, \dots, k$) we have

$$A_1 C_1^k \leq \frac{(p_1+p_2+\cdots+p_k)!}{(1)^{p_1}(2)^{p_2}\cdots(k)^{p_k}} \leq A_2 C_2^k.$$

In [5, § 4], we gave an interpretation of our irregularity condition. The following lemma is easily obtained by the proof of [5, Proposition 1]. Let $\mathcal{J}$, $l(j, \alpha)$ and $\kappa = (\kappa_1, \dots, \kappa_n)$ be as in § 2.

LEMMA 6. Let $(j, \alpha) \in \mathcal{J}$, let $1 \leq s < \infty$ and let $\sigma_{j, \alpha} (\geq 1)$ be as in (2.2). Then, we have the following results.

(1) $\sigma_{j, \alpha} = 1$ is equivalent to the condition $l(j, \alpha) \geq \langle \kappa, \alpha \rangle$.

(2) When $1 \leq s < \sigma_{j, \alpha} / (\sigma_{j, \alpha} - 1)$, there is a $z \in \mathbb{R}^n$ such that $(0, \dots, 0) \leq$

$z \leq \alpha$, $0 < |z| < |\alpha|$, $\langle \kappa, z \rangle \leq l(j, \alpha)$ and

$$s < \frac{m-j-|z|}{|\alpha|-|z|}.$$

(3) When $s = \sigma_{j, \alpha} / (\sigma_{j, \alpha} - 1)$, there is a $z \in \mathbf{R}^n$ such that $(0, \dots, 0) \leq z \leq \alpha$, $0 < |z| < |\alpha|$, $\langle \kappa, z \rangle = l(j, \alpha)$ and

$$s = \frac{m-j-|z|}{|\alpha|-|z|}.$$

Moreover, we can find $\tau \in \mathfrak{S}_n$ and $p \in \{1, \dots, n\}$ such that $\kappa_{\tau(1)} \leq \kappa_{\tau(2)} \leq \dots \leq \kappa_{\tau(n)}$, $z_{\tau(k)} = \alpha_{\tau(k)}$ for $1 \leq k < p$, $0 < z_{\tau(p)} \leq \alpha_{\tau(p)}$, and $z_{\tau(k)} = 0$ for $p < k \leq n$ (where $z = (z_1, \dots, z_n)$ and $\alpha = (\alpha_1, \dots, \alpha_n)$).

Here, we used the following notations for $a = (a_1, \dots, a_n)$ and $b = (b_1, \dots, b_n)$: $a \leq b$ means that $a_i \leq b_i$ holds for any i , $|a| = |a_1| + \dots + |a_n|$ and $\langle a, b \rangle = a_1 b_1 + \dots + a_n b_n$.

To prove Proposition 1, we need further information on z . Let us give a refinement of Lemma 6. For $(j, \alpha) \in \mathcal{J}$, denote by $\mathcal{Z}(j, \alpha)$ the set of all $z \in \mathbf{R}^n$ satisfying the following:

$$\begin{cases} (0, \dots, 0) \leq z \leq \alpha, \\ |z| < |\alpha|, \\ \langle \kappa, z \rangle \leq l(j, \alpha). \end{cases}$$

Let $\kappa = (\kappa_1, \dots, \kappa_n)$ be the one in (A_κ) . Define $p \sim q$ by $\kappa_p = \kappa_q$. Then, the $\sim$ defines an equivalence relation in $\{1, 2, \dots, n\}$; therefore, we obtain $K_1, \dots, K_\nu \subset \{1, 2, \dots, n\}$ satisfying the following: $K_1 \cup \dots \cup K_\nu = \{1, 2, \dots, n\}$, $K_i \cap K_j = \emptyset$ for $i \neq j$, and

$$\begin{cases} \kappa_p = \kappa_q & \text{for } p, q \in K_i, \\ \kappa_p \neq \kappa_q & \text{for } p \in K_i, q \in K_j \text{ and } i \neq j. \end{cases}$$

Under this notation, for $x = (x_1, \dots, x_n)$ we write

$$(6.1) \quad |x|_i = \sum_{p \in K_i} |x_p| \quad (i = 1, \dots, \nu).$$

Then, our refinement of Lemma 6 is given as follows. Let $S_i(\alpha)$ be as in (2.5).

LEMMA 7. Let $(j, \alpha) \in \mathcal{J}$, let $1 \leq s < \infty$ and let $\kappa = (\kappa_1, \dots, \kappa_n)$ be the one in (A_κ) . Then, we have the following results.

(1) When $1 \leq s < \sigma_{j, \alpha} / (\sigma_{j, \alpha} - 1)$, there is a $z \in \mathcal{Z}(j, \alpha)$ such that

$$(6.2) \quad s < \frac{m-j-|z|}{|\alpha|-|z|}$$

and that $0 < |z|_i < |\alpha|_i$ holds for some $i \in \{1, \dots, \nu\}$.

(2) When $s = \sigma_{j, \alpha} / (\sigma_{j, \alpha} - 1)$ and $l(j, \alpha) \in S_k(\alpha)$, there is a $z \in \mathcal{Z}(j, \alpha)$ such that

$$s = \frac{m-j-|z|}{|\alpha|-|z|}$$

and that $0 < |z|_i < |\alpha|_i$ holds for some $i \in \{1, \dots, \nu\}$.

(3) When $s = \sigma_{j, \alpha} / (\sigma_{j, \alpha} - 1)$ and $l(j, \alpha) \in S_k(\alpha)$, there is a $z \in \mathcal{Z}(j, \alpha)$ such that

$$s = \frac{m-j-|z|}{|\alpha|-|z|}$$

and that $|z|_i = 0$ or $|z|_i = |\alpha|_i$ holds for any $i \in \{1, \dots, \nu\}$.

(4) When $s > \sigma_{j, \alpha} / (\sigma_{j, \alpha} - 1)$, there are no $z \in \mathcal{Z}(j, \alpha)$ such that

$$s \leq \frac{m-j-|z|}{|\alpha|-|z|}.$$

PROOF. Note that $\kappa_i > 0$ ($1 \leq i \leq n$) and $l(j, \alpha) > 0$ are assumed (as in § 2). Put $\alpha = (\alpha_1, \dots, \alpha_n)$.

When $1 \leq s < \sigma_{j, \alpha} / (\sigma_{j, \alpha} - 1)$, by (2) in Lemma 6 we can find a $w = (w_1, \dots, w_n) \in \mathbf{R}^n$ such that $(0, \dots, 0) \leq w \leq \alpha$, $0 < |w| < |\alpha|$, $\langle \kappa, w \rangle \leq l(j, \alpha)$ and

$$s < \frac{m-j-|w|}{|\alpha|-|w|}.$$

Then, we can obtain (1) in Lemma 7 by defining $z = (z_1, \dots, z_n) \in \mathbf{R}^n$ as follows: when $w_k = 0$, we put $z_k = 0$; when $w_k > 0$, we choose $z_k \in \mathbf{R}$ so that $0 < z_k < w_k$ and that z_k is sufficiently close to w_k . In fact, if $z \in \mathbf{R}^n$ is as above, $z \in \mathcal{Z}(j, \alpha)$ is clear, (6.2) is verified by the fact that

$$\frac{m-j-|z|}{|\alpha|-|z|} \longrightarrow \frac{m-j-|w|}{|\alpha|-|w|}$$

as $|z| \rightarrow |w|$, and the last condition is verified as follows. Since $|w| > 0$, we have $w_k > 0$ for some k and therefore $0 < z_k < w_k \leq \alpha_k$; hence by taking $i \in \{1, \dots, \nu\}$ such that $k \in K_i$ we obtain $0 < |z|_i < |\alpha|_i$. Thus, (1) is proved.

When $s = \sigma_{j, \alpha} / (\sigma_{j, \alpha} - 1)$, by (3) in Lemma 6 we can find $z = (z_1, \dots, z_n) \in \mathbf{R}^n$, $\tau \in \mathfrak{S}_n$ and $p \in \{1, \dots, n\}$ such that $(0, \dots, 0) \leq z \leq \alpha$, $0 < |z| < |\alpha|$, $\langle \kappa, z \rangle = l(j, \alpha)$,

$$s = \frac{m-j-|z|}{|\alpha|-|z|},$$

$\kappa_{\tau(1)} \leq \kappa_{\tau(2)} \leq \cdots \leq \kappa_{\tau(n)}$, $z_{\tau(k)} = \alpha_{\tau(k)}$ for $1 \leq k < p$, $0 < z_{\tau(p)} \leq \alpha_{\tau(p)}$, and $z_{\tau(k)} = 0$ for $p < k \leq n$. Take $i_0 \in \{1, \dots, \nu\}$ such that $\tau(p) \in K_{i_0}$. Then we have

$$\begin{cases} 0 < |z|_{i_0} \leq |\alpha|_{i_0}, \\ |z|_h = 0 \quad \text{or} \quad |z|_h = |\alpha|_h \quad \text{for } h \neq i_0. \end{cases}$$

Therefore, to have (2) and (3) in Lemma 7 it is sufficient to prove the following fact:

$$(6.3) \quad |z|_{i_0} < |\alpha|_{i_0}, \quad \text{when } l(j, \alpha) \notin S_s(\alpha),$$

$$(6.4) \quad |z|_{i_0} = |\alpha|_{i_0}, \quad \text{when } l(j, \alpha) \in S_s(\alpha).$$

Note that we have the condition $l(j, \alpha) < \langle \kappa, \alpha \rangle$, since $s = \sigma_{j, \alpha} / (\sigma_{j, \alpha} - 1) < \infty$ and therefore $\sigma_{j, \alpha} > 1$ (see (1) in Lemma 6).

Let us show (6.3). Assume that $l(j, \alpha) \notin S_s(\alpha)$. When $0 < z_{\tau(p)} < \alpha_{\tau(p)}$, (6.3) is trivial. Therefore, in the discussion below we may assume that $z_{\tau(p)} = \alpha_{\tau(p)}$. In this case, by the condition $l(j, \alpha) = \langle \kappa, z \rangle$ we have

$$(6.5) \quad l(j, \alpha) = \kappa_{\tau(1)}\alpha_{\tau(1)} + \cdots + \kappa_{\tau(p)}\alpha_{\tau(p)}.$$

Moreover, we can see the following: $p \leq n-1$, $\kappa_{\tau(p)} = \kappa_{\tau(p+1)}$ and $\alpha_{\tau(k)} > 0$ for some $k \in \{p+1, \dots, n\}$. In fact, these are verified as follows: if $p = n$, (6.5) contradicts the condition $l(j, \alpha) < \langle \kappa, \alpha \rangle$; if $\kappa_{\tau(p)} < \kappa_{\tau(p+1)}$, (6.5) contradicts the condition $l(j, \alpha) \notin S_s(\alpha)$; if $\alpha_{\tau(p+1)} = \cdots = \alpha_{\tau(n)} = 0$, (6.5) contradicts the condition $l(j, \alpha) < \langle \kappa, \alpha \rangle$. Hence, by putting

$$q = \min\{k \in \{p+1, \dots, n\}; \alpha_{\tau(k)} > 0\}$$

we have $\alpha_{\tau(q)} > 0$, $\alpha_{\tau(k)} = 0$ for $p+1 \leq k < q$ and

$$(6.6) \quad l(j, \alpha) = \kappa_{\tau(1)}\alpha_{\tau(1)} + \cdots + \kappa_{\tau(h)}\alpha_{\tau(h)}$$

for any $h \in \{p, \dots, q-1\}$. If $\kappa_{\tau(h)} < \kappa_{\tau(h+1)}$ holds for some $h \in \{p, \dots, q-1\}$, (6.6) contradicts the condition $l(j, \alpha) \notin S_s(\alpha)$; therefore we have $\kappa_{\tau(p)} = \kappa_{\tau(p+1)} = \cdots = \kappa_{\tau(q)}$. This implies that $\tau(k) \in K_{i_0}$ for $k = p, p+1, \dots, q$. Since $z_{\tau(q)} = 0$ and $\alpha_{\tau(q)} > 0$, we obtain

$$|\alpha|_{i_0} - |z|_{i_0} \geq \alpha_{\tau(q)} - z_{\tau(q)} = \alpha_{\tau(q)} > 0$$

which proves (6.3). As a consequence, we obtain (2) in Lemma 7.

Let us next show (6.4). Assume that $l(j, \alpha) \in S_s(\alpha)$. Then, there are $\nu \in \mathfrak{S}_n$ and $q \in \{1, \dots, n-1\}$ such that

$$(6.7) \quad l(j, \alpha) = \kappa_{\nu(1)}\alpha_{\nu(1)} + \cdots + \kappa_{\nu(q)}\alpha_{\nu(q)},$$

$$(6.8) \quad \{\kappa_{\nu(1)}, \dots, \kappa_{\nu(q)}\} < \{\kappa_{\nu(q+1)}, \dots, \kappa_{\nu(n)}\}.$$

Since $\kappa_{\tau(1)} \leq \kappa_{\tau(2)} \leq \cdots \leq \kappa_{\tau(n)}$ is assumed, by (6.8) we have $\{\nu(1), \dots, \nu(q)\} = \{\tau(1), \dots, \tau(q)\}$ and therefore

$$(6.9) \quad l(j, \alpha) = \kappa_{\tau(1)}\alpha_{\tau(1)} + \cdots + \kappa_{\tau(q)}\alpha_{\tau(q)},$$

$$(6.10) \quad \{\kappa_{\tau(1)}, \dots, \kappa_{\tau(q)}\} < \{\kappa_{\tau(q+1)}, \dots, \kappa_{\tau(n)}\}.$$

On the other hand, by the condition $l(j, \alpha) = \langle \kappa, z \rangle$ and by the choice of z we have

$$(6.11) \quad l(j, \alpha) = \kappa_{\tau(1)}\alpha_{\tau(1)} + \cdots + \kappa_{\tau(p-1)}\alpha_{\tau(p-1)} + \kappa_{\tau(p)}z_{\tau(p)}.$$

If $q < p$, by (6.9) and (6.11) we have

$$\kappa_{\tau(q+1)}\alpha_{\tau(q+1)} + \cdots + \kappa_{\tau(p-1)}\alpha_{\tau(p-1)} + \kappa_{\tau(p)}z_{\tau(p)} = 0$$

and therefore we have $z_{\tau(p)} = 0$; this contradicts the condition $0 < z_{\tau(p)} \leq \alpha_{\tau(p)}$. Hence, we may assume that $q \geq p$. In this case, by (6.9) and (6.11) we have

$$\kappa_{\tau(p)}(\alpha_{\tau(p)} - z_{\tau(p)}) + \kappa_{\tau(p+1)}\alpha_{\tau(p+1)} + \cdots + \kappa_{\tau(q)}\alpha_{\tau(q)} = 0;$$

therefore, we have $\alpha_{\tau(p)} = z_{\tau(p)}$ and $\alpha_{\tau(k)} = 0$ ($= z_{\tau(k)}$) for $p < k \leq q$. Thus, by combining this with the known condition $\kappa_{\tau(q)} < \kappa_{\tau(q+1)}$ (by (6.10)) we obtain

$$z_{\tau(k)} = \alpha_{\tau(k)} \quad \text{for any } \tau(k) \in K_{i_0},$$

which proves (6.4). As a consequence, we obtain (3) in Lemma 7.

Lastly, let us prove (4) in Lemma 7 by showing the following: under the conditions $s > \sigma_{j, \alpha}/(\sigma_{j, \alpha} - 1)$, $z \in \mathcal{Z}(j, \alpha)$ and

$$(6.12) \quad s \leq \frac{m-j-|z|}{|\alpha|-|z|},$$

we can obtain a contradiction. Assume that $s > \sigma_{j, \alpha}/(\sigma_{j, \alpha} - 1)$ and that $z = (z_1, \dots, z_n) \in \mathcal{Z}(j, \alpha)$ satisfies (6.12). Then, we have

$$(6.13) \quad \sigma_{j, \alpha}/(\sigma_{j, \alpha} - 1) < \frac{m-j-|z|}{|\alpha|-|z|}.$$

Since $\sigma_{j, \alpha} > 1$, by (6.13) we have

$$\frac{m-j-|z|}{m-j-|\alpha|} < \min_{\tau \in \mathfrak{S}_n} (\max_{1 \leq r \leq n} M_{j, \alpha}(\tau, r)).$$

Therefore, for any $\tau \in \mathfrak{S}_n$ we can find a $p \in \{1, \dots, n\}$ such that

$$\frac{m-j-|z|}{m-j-|\alpha|} < M_{j,a}(\tau, p),$$

and this is equivalent to

$$(6.14) \quad l(j, \alpha) < \sum_{i=1}^p (\kappa_{\tau(i)} - \kappa_{\tau(p)}) \alpha_{\tau(i)} + \kappa_{\tau(p)} \sum_{i=1}^n z_{\tau(i)}.$$

On the other hand, by the condition $\langle \kappa, z \rangle \leq l(j, \alpha)$ we have

$$(6.15) \quad \sum_{i=1}^n \kappa_{\tau(i)} z_{\tau(i)} \leq l(j, \alpha).$$

Hence, by (6.14) and (6.15) we obtain

$$(6.16) \quad \sum_{i=1}^p (\kappa_{\tau(p)} - \kappa_{\tau(i)}) (\alpha_{\tau(i)} - z_{\tau(i)}) + \sum_{i=p+1}^n (\kappa_{\tau(i)} - \kappa_{\tau(p)}) z_{\tau(i)} < 0.$$

Thus, by choosing $\tau \in \mathfrak{S}_n$ so that $\kappa_{\tau(1)} \leq \kappa_{\tau(2)} \leq \cdots \leq \kappa_{\tau(n)}$ we can obtain a contradiction from (6.16). Q.E.D.

COROLLARY TO (3) IN LEMMA 7. Assume that $s = \sigma_{j,a}/(\sigma_{j,a} - 1)$ and $l(j, \alpha) \in S_s(\alpha)$ hold. Let $\nu \in \mathfrak{S}_n$ and $q \in \{1, \dots, n-1\}$ be such that

$$\begin{cases} l(j, \alpha) = \kappa_{\nu(1)} \alpha_{\nu(1)} + \cdots + \kappa_{\nu(q)} \alpha_{\nu(q)}, \\ \{\kappa_{\nu(1)}, \dots, \kappa_{\nu(q)}\} < \{\kappa_{\nu(q+1)}, \dots, \kappa_{\nu(n)}\}. \end{cases}$$

Then, $z^* = (z_1^*, \dots, z_n^*) \in Z_+^n$ defined by

$$(6.17) \quad z_{\nu(k)}^* = \begin{cases} \alpha_{\nu(k)}, & \text{for } 1 \leq k \leq q, \\ 0, & \text{for } q+1 \leq k \leq n \end{cases}$$

satisfies $z^* \in \mathcal{Z}(j, \alpha)$, $\langle \kappa, z^* \rangle = l(j, \alpha)$ and

$$(6.18) \quad s = \frac{m-j-|z^*|}{|\alpha|-|z^*|}.$$

PROOF. In the proof of (3) in Lemma 7 we already have the following: $\{\nu(1), \dots, \nu(q)\} = \{\tau(1), \dots, \tau(q)\}$, $\{\nu(q+1), \dots, \nu(n)\} = \{\tau(q+1), \dots, \tau(n)\}$, $z_{\tau(k)} = \alpha_{\tau(k)}$ for $1 \leq k \leq q$, and $z_{\tau(k)} = 0$ for $q+1 \leq k \leq n$. Hence, we obtain $z = z^*$. Q.E.D.

Note that the condition “(C-1) or (C-2)” is equivalent to the following: for any $(j, \alpha) \in \mathcal{J}$ we have the condition (i) or (ii) given below:

- (i) $1 < s < \sigma_{j,a}/(\sigma_{j,a} - 1)$,
- (ii) $s = \sigma_{j,a}/(\sigma_{j,a} - 1)$ and $l(j, \alpha) \notin S_s(\alpha)$.

Hence, by (1) and (2) in Lemma 7 we can obtain the following result which yields a proof of “(I-1) $\Rightarrow$ (I-2)” in Proposition 1.

LEMMA 8. *Let $1 < s < \infty$, let $l(j, \alpha) > 0$ ($(j, \alpha) \in \mathcal{J}$) and let $\kappa_i > 0$ ($1 \leq i \leq n$). Assume that (C-1) or (C-2) holds. Then, there are $c_1 > 0$, $c_2 > 0$ and $c_3 > 0$ such that for any $J_k = ((j_1, \alpha_{(1)}), \dots, (j_k, \alpha_{(k)})) \in \mathcal{J}^k$ we can find a $\beta(J_k) = (\beta_{(1)}, \dots, \beta_{(k)}) \in \mathcal{M}(J_k)$ which satisfies the following conditions (i)~(iii):*

$$(i) \quad (|\alpha_{(1)}| - |\beta_{(1)}|) + \dots + (|\alpha_{(k)}| - |\beta_{(k)}|) \geq c_1 k.$$

$$(ii) \quad \frac{c_2 + (m - j_1 - |\beta_{(1)}|) + \dots + (m - j_k - |\beta_{(k)}|)}{(|\alpha_{(1)}| - |\beta_{(1)}|) + \dots + (|\alpha_{(k)}| - |\beta_{(k)}|)} \geq s.$$

(iii) *There are $I_1, \dots, I_p \subset \{1, \dots, k\}$ and $N_1, \dots, N_k \in \mathbf{Z}_+$ such that $p \leq 2\nu$, $I_r \cap I_q = \emptyset$ for $r \neq q$, $I_1 \cup \dots \cup I_p = \{1, \dots, k\}$, $\{N_i; i \in I_r\} = \{0, 1, \dots, |I_r| - 1\}$ for $r = 1, \dots, p$, and*

$$(6.19) \quad l(j_1, \alpha_{(1)}) + \dots + l(j_i, \alpha_{(i)}) - \langle \kappa, \beta_{(1)} + \dots + \beta_{(i)} \rangle \geq c_3 N_i$$

for $i = 1, \dots, k$, where ν is the one in (6.1) and $|I_r|$ means the number of elements in I_r .

PROOF. Since (C-1) or (C-2) holds, by (1) and (2) in Lemma 7 we have the following: for any $(j, \alpha) \in \mathcal{J}$ we can find a $z(j, \alpha) \in \mathcal{Z}(j, \alpha)$ such that

$$s \leq \frac{m - j - |z(j, \alpha)|}{|\alpha| - |z(j, \alpha)|}$$

and that $0 < |z(j, \alpha)|_i < |\alpha|_i$ holds for some $i \in \{1, \dots, \nu\}$. We take $z(j, \alpha) \in \mathcal{Z}(j, \alpha)$ (for any $(j, \alpha) \in \mathcal{J}$) as above and fix them hereafter.

Take any $J_k = ((j_1, \alpha_{(1)}), \dots, (j_k, \alpha_{(k)})) \in \mathcal{J}^k$ and fix it. Put $\alpha_{(i)} = (\alpha_{(i), 1}, \dots, \alpha_{(i), n})$, $l(j_i, \alpha_{(i)}) = l_i$ and $z(j_i, \alpha_{(i)}) = z_{(i)} = (z_{(i), 1}, \dots, z_{(i), n})$ ($i = 1, \dots, k$). Then, we have $(0, \dots, 0) \leq z_{(i)} \leq \alpha_{(i)}$, $|z_{(i)}| < |\alpha_{(i)}|$, $\langle \kappa, z_{(i)} \rangle \leq l_i$,

$$(6.20) \quad s \leq \frac{m - j_i - |z_{(i)}|}{|\alpha_{(i)}| - |z_{(i)}|}$$

and the following condition

$$(6.21) \quad 0 < |z_{(i)}|_{p_i} < |\alpha_{(i)}|_{p_i} \quad \text{for some } p_i \in \{1, \dots, \nu\}$$

($i = 1, \dots, k$). Put $\alpha_{(0)} = (1, \dots, 1)$ and $\alpha_{(k+1)} = (0, \dots, 0)$. Let $K_1, \dots, K_\nu$ be the ones used in (6.1). Choose $r_1, \dots, r_\nu \in \{0, 1, \dots, k\}$ such that

$$(6.22) \quad |\alpha_{(r_j+1)}|_j + \dots + |\alpha_{(k+1)}|_j \leq |z_{(1)}|_j + \dots + |z_{(k)}|_j \\ < |\alpha_{(r_j)}|_j + |\alpha_{(r_j+1)}|_j + \dots + |\alpha_{(k+1)}|_j$$

($j = 1, \dots, \nu$), define $\beta_{(i)} = (\beta_{(i), 1}, \dots, \beta_{(i), n}) \in \mathbf{Z}_+^n$ ($i = 1, \dots, k$) by

$$(6.23) \quad \beta_{(i), h} = \begin{cases} \alpha_{(i), h}, & \text{if } h \in K_j \text{ and } r_j + 1 \leq i \leq k, \\ 0, & \text{if otherwise} \end{cases}$$

($i=1, \dots, k$ and $h=1, \dots, n$), and put $\beta(J_k) = (\beta_{(1)}, \dots, \beta_{(k)})$. Then, we can see that this $\beta(J_k)$ satisfies the conditions $\beta(J_k) \in \mathcal{M}(J_k)$ and (i)~(iii) (as proved below). Note that by (6.23) we have

$$(6.24) \quad |\beta_{(i)}|_j = \begin{cases} 0, & \text{when } 1 \leq i \leq r_j, \\ |\alpha_{(i)}|_j, & \text{when } r_j + 1 \leq i \leq k \end{cases}$$

($i=1, \dots, k$ and $j=1, \dots, \nu$).

Let us first show the condition $\beta(J_k) \in \mathcal{M}(J_k)$. Since $(0, \dots, 0) \leq \beta_{(i)} \leq \alpha_{(i)}$ ($i=1, \dots, k$) is clear from (6.23), what we must prove is the following:

$$(6.25) \quad \langle \kappa, \beta_{(1)} + \dots + \beta_{(i)} \rangle \leq l_1 + \dots + l_i \quad (i=1, \dots, k).$$

Recall that by the definition of $\{K_1, \dots, K_\nu\}$ we can choose $\kappa(K_j) > 0$ ($j=1, \dots, \nu$) such that $\kappa(K_j) = \kappa_p$ for any $p \in K_j$. Define $\psi_j^{(i)}$ ($i=1, \dots, k$ and $j=1, \dots, \nu$) by

$$(6.26) \quad \psi_j^{(i)} = \begin{cases} \kappa(K_j)(|z_{(1)}|_j + \dots + |z_{(i)}|_j), & \text{when } 1 \leq i \leq r_j, \\ \kappa(K_j)(|z_{(1)}|_j + \dots + |z_{(i)}|_j) \\ \quad - \kappa(K_j)(|\alpha_{(r_j+1)}|_j + \dots + |\alpha_{(i)}|_j), & \text{when } r_j + 1 \leq i \leq k, \end{cases}$$

Then, by (6.24), (6.26) and the condition $\langle \kappa, z_{(i)} \rangle \leq l_i$ ($i=1, \dots, k$) we have

$$(6.27) \quad \begin{aligned} & l_1 + \dots + l_i - \langle \kappa, \beta_{(1)} + \dots + \beta_{(i)} \rangle \\ &= l_1 + \dots + l_i - \langle \kappa, z_{(1)} + \dots + z_{(i)} \rangle + \psi_1^{(i)} + \dots + \psi_\nu^{(i)} \\ &\geq \psi_1^{(i)} + \dots + \psi_\nu^{(i)} \end{aligned}$$

($i=1, \dots, k$). Moreover, we can see the following:

$$(6.28) \quad \psi_j^{(i)} \geq 0 \quad (i=1, \dots, k \text{ and } j=1, \dots, \nu).$$

In fact, this is verified as follows: when $1 \leq i \leq r_j$, $\psi_j^{(i)} \geq 0$ is clear from (6.26); when $r_j + 1 \leq i \leq k$, $\psi_j^{(i)} \geq 0$ is verified by (6.22) and

$$\begin{aligned} \psi_j^{(i)} &= \kappa(K_j)[(|z_{(1)}|_j + \dots + |z_{(k)}|_j) - (|\alpha_{(r_j+1)}|_j + \dots + |\alpha_{(k)}|_j)] \\ &\quad + \kappa(K_j)[(|\alpha_{(i+1)}|_j - |z_{(i+1)}|_j) + \dots + (|\alpha_{(k)}|_j - |z_{(k)}|_j)]. \end{aligned}$$

Hence, by (6.27) and (6.28) we obtain (6.25). Thus, the condition $\beta(J_k) \in \mathcal{M}(J_k)$ is proved.

Let us next show the conditions (i) and (ii). Note that by (6.22), (6.23) and (6.24) we have

$$\begin{aligned}
& (|\alpha_{(1)}| - |\beta_{(1)}|) + \cdots + (|\alpha_{(k)}| - |\beta_{(k)}|) \\
& = (|\alpha_{(1)}| - |z_{(1)}|) + \cdots + (|\alpha_{(k)}| - |z_{(k)}|) \\
& \quad + \sum_{j=1}^{\nu} [(|z_{(1)}|_j + \cdots + |z_{(k)}|_j) - (|\alpha_{(r_j+1)}|_j + \cdots + |\alpha_{(k)}|_j)] \\
& \geq (|\alpha_{(1)}| - |z_{(1)}|) + \cdots + (|\alpha_{(k)}| - |z_{(k)}|).
\end{aligned}$$

Therefore, by putting

$$c_1 = \min\{|\alpha| - |z(j, \alpha)|; (j, \alpha) \in \mathcal{J}\} \quad (>0)$$

we can obtain (i). Note that by (6.20) we have

$$(6.29) \quad s(|\alpha_{(i)}| - |\beta_{(i)}|) \leq (m - j_i - |\beta_{(i)}|) + (s-1)(|z_{(i)}| - |\beta_{(i)}|)$$

($i=1, \dots, k$). Therefore, by (6.22), (6.23), (6.24) and (6.29) we have

$$\begin{aligned}
& s[(|\alpha_{(1)}| - |\beta_{(1)}|) + \cdots + (|\alpha_{(k)}| - |\beta_{(k)}|)] \\
& \leq (m - j_1 - |\beta_{(1)}|) + \cdots + (m - j_k - |\beta_{(k)}|) \\
& \quad + (s-1) \sum_{j=1}^{\nu} [(|z_{(1)}|_j + \cdots + |z_{(k)}|_j) - (|\alpha_{(r_j+1)}|_j + \cdots + |\alpha_{(k)}|_j)] \\
& \leq (m - j_1 - |\beta_{(1)}|) + \cdots + (m - j_k - |\beta_{(k)}|) + (s-1) \sum_{j=1}^{\nu} |\alpha_{(r_j)}|_j.
\end{aligned}$$

Hence, by putting $c_2 = (s-1)m\nu$ (>0) we can obtain (ii).

Let us lastly show the condition (iii). Put $\mathcal{J}_{(\mu)}$, $\mathcal{J}^{(\mu)}$, $\varepsilon_{(\mu)}$, $\varepsilon^{(\mu)}$ ($\mu=1, \dots, \nu$) as follows:

$$\begin{aligned}
\mathcal{J}_{(\mu)} &= \{(j, \alpha) \in \mathcal{J}; |z(j, \alpha)|_{\mu} > 0\}, \\
\mathcal{J}^{(\mu)} &= \{(j, \alpha) \in \mathcal{J}; |\alpha|_{\mu} > |z(j, \alpha)|_{\mu}\}, \\
\varepsilon_{(\mu)} &= \min\{|z(j, \alpha)|_{\mu}; (j, \alpha) \in \mathcal{J}_{(\mu)}\} \quad (>0), \\
\varepsilon^{(\mu)} &= \min\{|\alpha|_{\mu} - |z(j, \alpha)|_{\mu}; (j, \alpha) \in \mathcal{J}^{(\mu)}\} \quad (>0).
\end{aligned}$$

Put $I = \{1, \dots, k\}$, and put $J_{(j)}$, $J^{(j)}$ ($j=1, \dots, \nu$) as follows:

$$\begin{aligned}
J_{(j)} &= \{i \in I; |z_{(i)}|_j > 0 \text{ and } 1 \leq i \leq r_j\}, \\
J^{(j)} &= \{i \in I; |\alpha_{(i)}|_j - |z_{(i)}|_j > 0 \text{ and } r_j + 1 \leq i \leq k\}.
\end{aligned}$$

Then, we can see the following conditions (iii-1)~(iii-3):

(iii-1) $(J_{(1)} \cup J^{(1)}) \cup \cdots \cup (J_{(\nu)} \cup J^{(\nu)}) = I (= \{1, \dots, k\})$.

(iii-2) If $J_{(j)} = \{h_1, h_2, \dots, h_p\}$ and $h_1 < h_2 < \cdots < h_p$, we have

$$\psi_j^{(h_d)} \geq \kappa(K_j) \varepsilon_{(j)} d \quad \text{for } d=1, 2, \dots, p.$$

(iii-3) If $J^{(j)} = \{h_1, h_2, \dots, h_p\}$ and $h_1 < h_2 < \cdots < h_p$, we have

$$\psi_j^{(h_d)} \geq \kappa(K_j) \varepsilon^{(j)}(p-d) \quad \text{for } d=1, 2, \dots, p.$$

In fact, these are verified as follows. (iii-1) is clear from (6.21). (iii-2) is verified by the following: by (6.26) we have

$$\begin{aligned} \psi_j^{(h_d)} &= \kappa(K_j)(|z_{(1)}|_j + \dots + |z_{(h_d)}|_j) \\ &\geq \kappa(K_j)(|z_{(h_1)}|_j + |z_{(h_2)}|_j + \dots + |z_{(h_d)}|_j) \\ &\geq \kappa(K_j) \varepsilon_{(j)} d. \end{aligned}$$

(iii-3) is verified by the following: by (6.22) and (6.26) we have

$$\begin{aligned} \psi_j^{(h_d)} &= \kappa(K_j)[(|z_{(1)}|_j + \dots + |z_{(k)}|_j) - (|\alpha_{(r_j+1)}|_j + \dots + |\alpha_{(k)}|_j)] \\ &\quad + \kappa(K_j)[(|\alpha_{(h_d+1)}|_j - |z_{(h_d+1)}|_j) + \dots + (|\alpha_{(k)}|_j - |z_{(k)}|_j)] \\ &\geq \kappa(K_j)[(|\alpha_{(h_d+1)}|_j - |z_{(h_d+1)}|_j) + (|\alpha_{(h_d+2)}|_j - |z_{(h_d+2)}|_j) \\ &\quad + \dots + (|\alpha_{(h_p)}|_j - |z_{(h_p)}|_j)] \\ &\geq \kappa(K_j) \varepsilon^{(j)}(p-d). \end{aligned}$$

By using (iii-1) ~ (iii-3), we can prove (iii) as follows. Put

$$c_3 = \min\{\kappa(K_j) \varepsilon_{(j)}, \kappa(K_j) \varepsilon^{(j)}; j=1, \dots, \nu\} \quad (>0),$$

choose a disjoint family $\{I_{(j)}, I^{(j)}; j=1, \dots, \nu\}$ such that $I_{(j)} \subset J_{(j)}$, $I^{(j)} \subset J^{(j)}$ and $(I_{(1)} \cup I^{(1)}) \cup \dots \cup (I_{(\nu)} \cup I^{(\nu)}) = I$, and define $N_1, \dots, N_k \in \mathbf{Z}_+$ as follows: if $I_{(j)} = \{i_1, i_2, \dots, i_q\}$ and $i_1 < i_2 < \dots < i_q$, we put $N_{i_d} = (d-1)$ ($d=1, 2, \dots, q$); if $I^{(t)} = \{i_1, i_2, \dots, i_q\}$ and $i_1 < i_2 < \dots < i_q$, we put $N_{i_d} = (q-d)$ ($d=1, 2, \dots, q$). Then, by (6.27), (6.28), (iii-2) and (iii-3) we obtain

$$\begin{aligned} l_1 + \dots + l_i - \langle \kappa, \beta_{(1)} + \dots + \beta_{(i)} \rangle &\geq \psi_1^{(i)} + \dots + \psi_\nu^{(i)} \\ &\geq c_3 N_i \end{aligned}$$

($i=1, \dots, k$). Thus, (iii) is proved.

Q.E.D.

As a consequence of Lemma 8, we obtain

PROOF OF “(I-1) $\Rightarrow$ (I-2)” IN PROPOSITION 1. Assume that (C-1) or (C-2) holds. Let $c_1 > 0$, $c_2 > 0$ and $c_3 > 0$ be as in Lemma 8. Take $J_k = ((j_1, \alpha_{(1)}), \dots, (j_k, \alpha_{(k)})) \in \mathcal{J}^k$, let $\beta(J_k) = (\beta_{(1)}, \dots, \beta_{(k)}) \in \mathcal{M}(J_k)$ and $N_1, \dots, N_k \in \mathbf{Z}_+$ be the ones chosen for J_k in Lemma 8, and let a_i ($i=1, \dots, k$) be as in (2.7). Then, by (6.19) we have

$$(6.30) \quad \frac{1}{a_i} \leq \frac{1}{a + c_3 N_{i-1}} \leq \frac{1}{1 + N_{i-1}} \frac{1}{\min\{a, c_3\}} \quad (i=1, \dots, k)$$

(where $N_0 = 0$), and by (6.30), (iii) in Lemma 8 and (3) in Lemma 5 we have

$$\begin{aligned}
(6.31) \quad & \frac{1}{(a_1)^{m-j_1-|\beta_{(1)}|} \times \dots \times (a_k)^{m-j_k-|\beta_{(k)}|}} \\
& \leq A_1^k \frac{1}{(1+N_0)^{m-j_1-|\beta_{(1)}|} \times \dots \times (1+N_{k-1})^{m-j_k-|\beta_{(k)}|}} \\
& \leq A_2^{k+1} \frac{1}{[(m-j_1-|\beta_{(1)}|) + \dots + (m-j_k-|\beta_{(k)}|)]!}
\end{aligned}$$

for some $A_1 > 0$ and $A_2 > 0$ (depending only on a , c_3 , m and ν). Therefore, by (6.31), (ii) in Lemma 8 and (1) in Lemma 5 we have

$$(6.32) \quad \frac{[(|\alpha_{(1)}| - |\beta_{(1)}|) + \dots + (|\alpha_{(k)}| - |\beta_{(k)}|)]!^s}{(a_1)^{m-j_1-|\beta_{(1)}|} \times \dots \times (a_k)^{m-j_k-|\beta_{(k)}|}} \leq B^{k+1}$$

for some $B > 0$ (which is independent of J_k , $\beta(J_k)$ and k). Hence, if we assume that $0 < R \leq 1$, by (2.7), (6.32) and (i) in Lemma 8 we obtain

$$F_a(J_k, \beta(J_k), C, R, T) \leq H^{k+1} R^{c_1 k}$$

for some $H > 0$ (which is independent of J_k , $\beta(J_k)$ and k). This leads us to the condition (I-2) in Proposition 1. Q.E.D.

In the above proof, the following fact is essential: the conditions (i), (ii) in Lemma 8 and (6.32) imply (I-2) (and hence (I-3)). The following lemmas 9 and 10 assert that the converse relation holds.

LEMMA 9. *There is a $d > 0$ which satisfies the following condition: if $J_k = ((j_1, \alpha_{(1)}), \dots, (j_k, \alpha_{(k)})) \in \mathcal{J}^k$ satisfies $0 < l(j_i, \alpha_{(i)}) < \langle \kappa, \alpha_{(i)} \rangle$ ($i = 1, \dots, k$), we have*

$$(6.33) \quad (|\alpha_{(1)}| - |\beta_{(1)}|) + \dots + (|\alpha_{(k)}| - |\beta_{(k)}|) \geq dk$$

for any $\beta(J_k) = (\beta_{(1)}, \dots, \beta_{(k)}) \in \mathcal{M}(J_k)$.

PROOF. Put

$$\begin{aligned}
(6.34) \quad & \mathcal{J}_0 = \{(j, \alpha) \in \mathcal{J}; 0 < l(j, \alpha) < \langle \kappa, \alpha \rangle\}, \\
& c = \min\{\langle \kappa, \alpha \rangle - l(j, \alpha); (j, \alpha) \in \mathcal{J}_0\} \quad (> 0), \\
& \kappa^* = \max\{\kappa_1, \dots, \kappa_n\}.
\end{aligned}$$

Take any $\beta(J_k) = (\beta_{(1)}, \dots, \beta_{(k)}) \in \mathcal{M}(J_k)$. Then, by (2.6), (6.34) and the condition $(j_i, \alpha_{(i)}) \in \mathcal{J}_0$ ($i = 1, \dots, k$) we have

$$\begin{aligned}
& \kappa^* [(|\alpha_{(1)}| - |\beta_{(1)}|) + \dots + (|\alpha_{(k)}| - |\beta_{(k)}|)] \geq \langle \kappa, \alpha_{(1)} - \beta_{(1)} \rangle + \dots + \langle \kappa, \alpha_{(k)} - \beta_{(k)} \rangle \\
& = (\langle \kappa, \alpha_{(1)} \rangle - l(j_1, \alpha_{(1)})) + \dots + (\langle \kappa, \alpha_{(k)} \rangle - l(j_k, \alpha_{(k)})) \\
& \quad + (l(j_1, \alpha_{(1)}) + \dots + l(j_k, \alpha_{(k)}) - \langle \kappa, \beta_{(1)} + \dots + \beta_{(k)} \rangle) \geq ck.
\end{aligned}$$

Hence, by putting $d=c/\kappa^*$ we obtain (6.33).

Q.E.D.

LEMMA 10. Assume the condition (I-3) in Proposition 1. Then, there are $A>0$ and $B>0$ such that for any $J_k=((j_1, \alpha_{(1)}), \dots, (j_k, \alpha_{(k)})) \in \mathcal{J}^k$ we can find $\beta(J_k)=(\beta_{(1)}, \dots, \beta_{(k)}) \in \mathcal{M}(J_k)$ and $c(J_k) \geq 0$ which satisfy the following conditions (i)~(iii).

$$(i) \quad \frac{[(|\alpha_{(1)}|-|\beta_{(1)}|)+\dots+(|\alpha_{(k)}|-|\beta_{(k)}|)]!^s}{(a_1)^{m-j_1-|\beta_{(1)}|} \times \dots \times (a_k)^{m-j_k-|\beta_{(k)}|}} \leq AB^k,$$

where $a_1=a (>0)$ and

$$a_{i+1}=l(j_1, \alpha_{(1)})+\dots+l(j_i, \alpha_{(i)})-\langle \kappa, \beta_{(1)}+\dots+\beta_{(i)} \rangle \quad (i=1, \dots, k).$$

$$(ii) \quad \frac{c(J_k)+(m-j_1-|\beta_{(1)}|)+\dots+(m-j_k-|\beta_{(k)}|)}{(|\alpha_{(1)}|-|\beta_{(1)}|)+\dots+(|\alpha_{(k)}|-|\beta_{(k)}|)} \geq s.$$

$$(iii) \quad \max_{J_k \in \mathcal{J}^k} \{c(J_k)\} = o(k) \quad (\text{as } k \rightarrow \infty).$$

PROOF. Assume the condition (I-3) in Proposition 1. Then,

$$M_a(\{\beta(J_k)\}, C, R_0, T_0) < +\infty$$

for some $\beta(J_k)=(\beta_{(1)}, \dots, \beta_{(k)}) \in \mathcal{M}(J_k)$ (where $J_k=((j_1, \alpha_{(1)}), \dots, (j_k, \alpha_{(k)})) \in \mathcal{J}^k$, $k=1, 2, \dots$), $C>0$, $R_0>0$ and $T_0>0$. Therefore, by (2.7) and (2.8) we have

$$(6.35) \quad \frac{[(|\alpha_{(1)}|-|\beta_{(1)}|)+\dots+(|\alpha_{(k)}|-|\beta_{(k)}|)]!^s}{(a_1)^{m-j_1-|\beta_{(1)}|} \times \dots \times (a_k)^{m-j_k-|\beta_{(k)}|}} \leq AB^k$$

for some $A>0$ and $B>0$ (which are independent of J_k , $\beta(J_k)$ and k); this is (i). Put $l=\max\{l(j, \alpha); (j, \alpha) \in \mathcal{J}\}$. Since

$$(6.36) \quad \frac{1}{a_i} \geq \frac{1}{a+l(i-1)} \geq \frac{1}{i} \frac{1}{\max\{a, l\}} \quad (i=1, \dots, k),$$

by (6.35) and (6.36) we have

$$(6.37) \quad \frac{[(|\alpha_{(1)}|-|\beta_{(1)}|)+\dots+(|\alpha_{(k)}|-|\beta_{(k)}|)]!^s}{(1)^{m-j_1-|\beta_{(1)}|} \times \dots \times (k)^{m-j_k-|\beta_{(k)}|}} \leq AB_1^k$$

for $B_1=B(\max\{1, a, l\})^m$. Hence, by applying (3) in Lemma 5 and Stirling's formula to (6.37) we obtain

$$(6.38) \quad \frac{[s(|\alpha_{(1)}|-|\beta_{(1)}|)+\dots+s(|\alpha_{(k)}|-|\beta_{(k)}|)]!}{[(m-j_1-|\beta_{(1)}|)+\dots+(m-j_k-|\beta_{(k)}|)]!} \leq CH^k$$

for some $C>0$ and $H>0$ (which are independent of J_k , $\beta(J_k)$ and k).

Here, we put $b(J_k)$ and $c(J_k)$ as follows:

$$\begin{aligned}
(6.39) \quad b(J_k) &= [s(|\alpha_{(1)}| - |\beta_{(1)}|) + \cdots + s(|\alpha_{(k)}| - |\beta_{(k)}|)] \\
&\quad - [(m - j_1 - |\beta_{(1)}|) + \cdots + (m - j_k - |\beta_{(k)}|)], \\
c(J_k) &= \max\{0, b(J_k)\} \quad (\geq 0).
\end{aligned}$$

Then, we have $c(J_k) \geq b(J_k)$ and therefore by (6.39) we obtain (ii). Moreover, we can see the following: there are $C_1 > 0$ and $C_2 > 0$ such that

$$(6.40) \quad \log \Gamma(1 + c(J_k)) \leq C_1 k + C_2$$

holds for any J_k and k . In fact, this is verified as follows: by (6.38) and (6.39) we have

$$\frac{[b(J_k) + (m - j_1 - |\beta_{(1)}|) + \cdots + (m - j_k - |\beta_{(k)}|)]!}{[(m - j_1 - |\beta_{(1)}|) + \cdots + (m - j_k - |\beta_{(k)}|)]!} \leq CH^k$$

and therefore by (1) in Lemma 5 we obtain

$$[c(J_k)]! \leq C_0 H_0^k$$

(which implies (6.40)) for $C_0 = \max\{1, 2C\}$ and $H_0 = \max\{1, 2^{m_s} H\}$.

Hence, by using (6.40) we can verify the condition (iii) as follows. Assume that $\{c(J_k)\}$ does not satisfy (iii). Then, there are $\varepsilon > 0$, $k_1 < k_2 < \cdots$ and $J^{(i)} \in \mathcal{J}^{k_i}$ ($i = 1, 2, \dots$) such that $c(J^{(i)}) \geq \varepsilon k_i$ ($i = 1, 2, \dots$) and $k_i \rightarrow \infty$ (as $i \rightarrow \infty$); therefore, by (6.40) we have

$$\log \Gamma(1 + \varepsilon k_i) \leq C_1 k_i + C_2 \quad (i = 1, 2, \dots).$$

This contradicts the condition $k_i \rightarrow \infty$ (as $i \rightarrow \infty$). Thus, (iii) is also verified.

Q.E.D.

COROLLARY TO LEMMAS 9 AND 10. Assume the condition (I-3) in Proposition 1. Let $(j, \alpha) \in \mathcal{J}$. Then we have the following results.

(1) If $0 < l(j, \alpha) < \langle \kappa, \alpha \rangle$ holds, there are $d > 0$, $A > 0$, $B > 0$, $z_{(k)} \in \mathcal{Z}(j, \alpha)$ ($k = 1, 2, \dots$) and $c_k \geq 0$ ($k = 1, 2, \dots$) which satisfy the following conditions (i) ~ (iv).

(i) $|\alpha| - |z_{(k)}| \geq d$ ($k = 1, 2, \dots$).

(ii) For any k , we can find a $(\beta_{(1)}, \dots, \beta_{(k)}) \in \mathbf{Z}_+^n \times \cdots \times \mathbf{Z}_+^n$ such that $(0, \dots, 0) \leq \beta_{(i)} \leq \alpha$ ($i = 1, \dots, k$), $\beta_{(1)} + \cdots + \beta_{(k)} = k z_{(k)}$, $\langle \kappa, \beta_{(1)} + \cdots + \beta_{(i)} \rangle \leq il(j, \alpha)$ ($i = 1, \dots, k$) and

$$(6.41) \quad \frac{[k(|\alpha| - |z_{(k)}|)]!^s}{(a_1)^{m - j_1 - |\beta_{(1)}|} \times \cdots \times (a_k)^{m - j_k - |\beta_{(k)}|}} \leq AB^k,$$

where $a_1 = a$ and $a_{i+1} = a + il(j, \alpha) - \langle \kappa, \beta_{(1)} + \cdots + \beta_{(i)} \rangle$ ($i = 1, \dots, k$).

(iii) $s \leq c_k + \frac{m - j - |z_{(k)}|}{|\alpha| - |z_{(k)}|}$ ($k = 1, 2, \dots$).

(iv) $c_k = o(1)$ (as $k \rightarrow \infty$).

(2) Moreover, if $s = \sigma_{j, a}/(\sigma_{j, a} - 1)$ and $l(j, \alpha) \in S_\epsilon(\alpha)$ hold, the $z_{(k)}$ ($k = 1, 2, \dots$) in (1) satisfy also the following condition:

$$z_{(k)} \rightarrow z^* \quad (\text{as } k \rightarrow \infty),$$

where z^* is the one in Corollary to (3) in Lemma 7.

PROOF. Assume the condition (I-3) in Proposition 1. Let $(j, \alpha) \in \mathcal{J}$ and assume that $0 < l(j, \alpha) < \langle \kappa, \alpha \rangle$ holds. Let $d > 0$ be as in Lemma 9, let $A > 0$, $B > 0$ be as in Lemma 10, put $J_k^0 = ((j, \alpha), \dots, (j, \alpha)) \in \mathcal{J}^k$, let $\beta(J_k^0) = (\beta_{(1)}, \dots, \beta_{(k)}) \in \mathcal{M}(J_k^0)$ be the one chosen for J_k^0 in Lemma 10, and define $z_{(k)}$, c_k by

$$z_{(k)} = \frac{1}{k} (\beta_{(1)} + \dots + \beta_{(k)}),$$

$$c_k = \frac{c(J_k^0)}{dk}$$

($k = 1, 2, \dots$). Then (i) is verified by Lemma 9 and (ii)~(iv) are verified by Lemma 10. Hence, we obtain (1).

Next, let us prove (2). Assume that $s = \sigma_{j, a}/(\sigma_{j, a} - 1)$ and $l(j, \alpha) \in S_\epsilon(\alpha)$ hold. Let $\nu \in \mathfrak{S}_n$, $q \in \{1, \dots, n-1\}$ and $z^* = (z_1^*, \dots, z_n^*) \in Z_+^n$ be as in Corollary to (3) in Lemma 7. Put $z_{(k)} = (z_{(k), 1}, \dots, z_{(k), n})$, $a = \max\{\kappa_{\nu(1)}, \dots, \kappa_{\nu(q)}\}$ and $b = \min\{\kappa_{\nu(q+1)}, \dots, \kappa_{\nu(n)}\}$. Then, we have $z_{\nu(i)}^* = \alpha_{\nu(i)}$ for $1 \leq i \leq q$, $z_{\nu(i)}^* = 0$ for $q+1 \leq i \leq n$, $\langle \kappa, z^* \rangle = l(j, \alpha) \geq \langle \kappa, z_{(k)} \rangle$, $(0, \dots, 0) \leq z_{(k)} \leq \alpha$, $(1/a)\kappa_{\nu(i)} \leq 1$ for $1 \leq i \leq q$, $(1/b)\kappa_{\nu(i)} \geq 1$ for $q+1 \leq i \leq n$, $0 < a < b$ and

$$\begin{aligned} |z^*| - |z_{(k)}| &= (\alpha_{\nu(1)} - z_{(k), \nu(1)}) + \dots + (\alpha_{\nu(q)} - z_{(k), \nu(q)}) \\ &\quad - (z_{(k), \nu(q+1)} + \dots + z_{(k), \nu(n)}) \\ &\geq \frac{1}{a} [\kappa_{\nu(1)}(\alpha_{\nu(1)} - z_{(k), \nu(1)}) + \dots + \kappa_{\nu(q)}(\alpha_{\nu(q)} - z_{(k), \nu(q)})] \\ &\quad - \frac{1}{b} (\kappa_{\nu(q+1)} z_{(k), \nu(q+1)} + \dots + \kappa_{\nu(n)} z_{(k), \nu(n)}) \\ (6.42) \quad &= \frac{1}{a} (\langle \kappa, z^* \rangle - \langle \kappa, z_{(k)} \rangle) \\ &\quad + \left(\frac{1}{a} - \frac{1}{b} \right) (\kappa_{\nu(q+1)} z_{(k), \nu(q+1)} + \dots + \kappa_{\nu(n)} z_{(k), \nu(n)}) \\ &\geq \left(\frac{b}{a} - 1 \right) (z_{(k), \nu(q+1)} + \dots + z_{(k), \nu(n)}). \end{aligned}$$

On the other hand, we know from (6.18), (iii), (iv) and the condition $|z^*| \geq |z_{(k)}|$ (by (6.42)) that

$$\begin{aligned}
(6.43) \quad |z^*| - |z_{(k)}| &= \frac{(|\alpha| - |z^*|)(|\alpha| - |z_{(k)}|)}{(m-j-|\alpha|)} \left(\frac{m-j-|z^*|}{|\alpha| - |z^*|} - \frac{m-j-|z_{(k)}|}{|\alpha| - |z_{(k)}|} \right) \\
&= \frac{(|\alpha| - |z^*|)(|\alpha| - |z_{(k)}|)}{(m-j-|\alpha|)} \left(s - \frac{m-j-|z_{(k)}|}{|\alpha| - |z_{(k)}|} \right) \\
&= o(1) \quad (\text{as } k \rightarrow \infty).
\end{aligned}$$

Therefore, by (6.42) and (6.43) we have

$$(6.44) \quad z_{(k), \nu(q+1)} + \cdots + z_{(k), \nu(n)} = o(1) \quad (\text{as } k \rightarrow \infty),$$

and by (6.43) and (6.44) we have

$$\begin{aligned}
(6.45) \quad &(\alpha_{\nu(1)} - z_{(k), \nu(1)}) + \cdots + (\alpha_{\nu(q)} - z_{(k), \nu(q)}) \\
&= |z^*| - |z_{(k)}| + (z_{(k), \nu(q+1)} + \cdots + z_{(k), \nu(n)}) \\
&= o(1) \quad (\text{as } k \rightarrow \infty).
\end{aligned}$$

Hence, we obtain $z_{(k)} \rightarrow z^*$ (as $k \rightarrow \infty$), since (6.44) and (6.45) imply

$$z_{(k), \nu(i)} \longrightarrow \begin{cases} \alpha_{\nu(i)}, & \text{when } 1 \leq i \leq q, \\ 0, & \text{when } q+1 \leq i \leq n \end{cases}$$

(as $k \rightarrow \infty$).

Q.E.D.

As a consequence of the above corollary and the condition (4) in Lemma 7, we obtain

PROOF OF “(I-3) $\Rightarrow$ (I-1)” IN PROPOSITION 1. Assume the condition (I-3) in Proposition 1. Put (C-3) and (C-4) as follows:

$$(C-3) \quad s = \sigma/(\sigma-1) \quad \text{and} \quad \Delta_P \cap S_P \neq \emptyset,$$

$$(C-4) \quad s > \sigma/(\sigma-1).$$

Then, to have (I-1) in Proposition 1 it is sufficient to prove the following: if we assume that (C-3) or (C-4) holds, we can obtain a contradiction. Let us show this.

First, let us discuss the case (C-4). Assume that (C-4) holds; that is, $s > \sigma_{j, \alpha}/(\sigma_{j, \alpha} - 1)$ holds for some $(j, \alpha) \in \mathcal{J}$. Take such a $(j, \alpha) \in \mathcal{J}$ and fix it. Then, we have $\sigma_{j, \alpha} > 1$ and hence $0 < l(j, \alpha) < \langle \kappa, \alpha \rangle$ (by (1) in Lemma 6). Therefore, by (1) in Corollary to Lemmas 9 and 10 we can find $z_{(k)} \in \mathcal{Z}(j, \alpha)$ ($k=1, 2, \dots$) such that

$$s \leq o(1) + \frac{m-j-|z_{(k)}|}{|\alpha| - |z_{(k)}|}$$

(as $k \rightarrow \infty$). Thus, by choosing $s_0 > 0$ so that $s > s_0 > \sigma_{j, \alpha}/(\sigma_{j, \alpha} - 1)$ and by

taking k sufficiently large we obtain

$$\sigma_{j, \alpha}/(\sigma_{j, \alpha}-1) < s_0 < \frac{m-j-|z_{(k)}|}{|\alpha|-|z_{(k)}|},$$

which contradicts the condition (4) in Lemma 7.

Next, let us discuss the case (C-3). Assume that (C-3) holds; that is, $s = \sigma_{j, \alpha}/(\sigma_{j, \alpha}-1)$ and $l(j, \alpha) \in S_s(\alpha)$ hold for some $(j, \alpha) \in \mathcal{J}$. Take such a $(j, \alpha) \in \mathcal{J}$ and fix it. Let $A > 0$, $B > 0$ and $z_{(k)} \in \mathcal{Z}(j, \alpha)$ ($k=1, 2, \dots$) be as in Corollary to Lemmas 9 and 10, and let $z^* \in \mathbb{Z}_+^n$ be the one in Corollary to (3) in Lemma 7. Then, we can see that

$$(6.46) \quad \frac{[k(|\alpha|-|z_{(k)}|)]!^s}{(a+k|\langle \kappa, z^*-z_{(k)} \rangle|)^{k(m-j-|z_{(k)}|)}} \leq AB^k$$

holds for any k . In fact, this is verified as follows. Note that $\langle \kappa, z^* \rangle = l(j, \alpha)$, and let $(\beta_{(1)}, \dots, \beta_{(k)}) \in \mathbb{Z}_+^n \times \dots \times \mathbb{Z}_+^n$ be the one in (ii) in Corollary to Lemmas 9 and 10. Then, we have $(0, \dots, 0) \leq \beta_{(i)} \leq \alpha$ ($i=1, \dots, k$), $kz_{(k)} = (\beta_{(1)} + \dots + \beta_{(k)})$,

$$(6.47) \quad (m-j-|\beta_{(1)}|) + \dots + (m-j-|\beta_{(k)}|) = k(m-j-|z_{(k)}|),$$

and by (6.17) we have

$$(6.48) \quad \begin{aligned} 0 &\leq il(j, \alpha) - \langle \kappa, \beta_{(1)} + \dots + \beta_{(i)} \rangle = \sum_{h=1}^i \langle \kappa, z^* - \beta_{(h)} \rangle \\ &\leq \sum_{h=1}^i |\langle \kappa, z^* - \beta_{(h)} \rangle| \\ &\leq \sum_{h=1}^k |\langle \kappa, z^* - \beta_{(h)} \rangle| = k |\langle \kappa, z^* - z_{(k)} \rangle| \end{aligned}$$

($i=1, \dots, k$). Hence, by (6.41), (6.47) and (6.48) we obtain (6.46).

Further, we can also see that

$$(6.49) \quad \frac{m-j-|z_{(k)}|}{|\alpha|-|z_{(k)}|} \leq s \quad \text{for any } k \in \mathbb{N}.$$

In fact, if otherwise, we can find a $k \in \mathbb{N}$ such that

$$\frac{m-j-|z_{(k)}|}{|\alpha|-|z_{(k)}|} > s = \sigma_{j, \alpha}/(\sigma_{j, \alpha}-1),$$

which contradicts the condition (4) in Lemma 7.

Hence, by (6.46), (6.49) and (2) in Corollary to Lemmas 9 and 10 we have the following: for any $\varepsilon > 0$ there is a $C_\varepsilon > 0$ such that

$$(6.50) \quad \begin{aligned} [k(|\alpha| - |z_{(k)}|)]!^s &\leq AB^k(\varepsilon k + C_\varepsilon)^{k(m-j-|z_{(k)}|)} \\ &\leq AB^k(\varepsilon k + C_\varepsilon)^{sk(|\alpha| - |z_{(k)}|)} \end{aligned}$$

holds for any k . In addition, by applying Stirling's formula to the left hand side of (6.50) and by using the condition $d \leq (|\alpha| - |z_{(k)}|) \leq m$ ($k=1, 2, \dots$) we have

$$(6.51) \quad k \leq H(\varepsilon k + C_\varepsilon) \quad (k=1, 2, \dots)$$

for some $H > 0$ (which is independent of ε and k). Thus, by choosing $\varepsilon > 0$ so that $\varepsilon < 1/H$ and by letting $k \rightarrow \infty$ in (6.51) we can obtain a contradiction.

Q.E.D.

Lastly, let us show “(II-3) $\Rightarrow$ (II-1)” in Proposition 1 by the argument similar to that above. Note that the difference between (II-3) and (I-3) lies in whether we can take $R > 0$ sufficiently large or not, and that T_0 depends on R in (II-3).

PROOF OF “(II-3) $\Rightarrow$ (II-1)” IN PROPOSITION 1. Our purpose is to prove the following: if we assume that (II-3) in Proposition 1 and $s \geq \sigma/(\sigma-1)$ hold, we can obtain a contradiction.

Assume the condition (II-3) in Proposition 1. Let $R > 1$ be sufficiently large. Then, we have

$$M_a(\{\beta(J_k)\}, C, R, T_0) < +\infty$$

for some $T_0 > 0$ and $\beta(J_k) = (\beta_{(1)}, \dots, \beta_{(k)}) \in \mathcal{M}(J_k)$ (where $J_k = ((j_1, \alpha_{(1)}), \dots, (j_k, \alpha_{(k)})) \in \mathcal{J}^k$, $k=1, 2, \dots$). Therefore, by (2.7), (2.8) and (6.36) we have

$$(6.52) \quad \begin{aligned} &\frac{[(|\alpha_{(1)}| - |\beta_{(1)}|) + \dots + (|\alpha_{(k)}| - |\beta_{(k)}|)]!^s}{(1)^{m-j_1-|\beta_{(1)}|} \times \dots \times (k)^{m-j_k-|\beta_{(k)}|}} \\ &\leq AB^k \left(\frac{1}{R} \right)^{(|\alpha_{(1)}| - |\beta_{(1)}|) + \dots + (|\alpha_{(k)}| - |\beta_{(k)}|)} \\ &\quad \times \left(\frac{1}{T_0} \right)^{l(j_1, \alpha_{(1)}) + \dots + l(j_k, \alpha_{(k)}) - \langle \kappa, \beta_{(1)} + \dots + \beta_{(k)} \rangle} \end{aligned}$$

for some $A > 0$ and $B > 0$ (which are independent of J_k , $\beta(J_k)$, k , R and T_0).

Assume that $s \geq \sigma/(\sigma-1)$ holds, that is, $s \geq \sigma_{j, \alpha}/(\sigma_{j, \alpha} - 1)$ holds for some $(j, \alpha) \in \mathcal{J}$. Take such a $(j, \alpha) \in \mathcal{J}$ and fix it. Then, by the same argument as in Corollary to Lemmas 9 and 10 we can find $d > 0$, $A_1 > 0$, $B_1 > 0$ (independent of k , R and T_0), $z_{(k)} \in \mathcal{Z}(j, \alpha)$ ($k=1, 2, \dots$) and $c_k \geq 0$ ($k=1, 2, \dots$) which satisfy the following conditions (i) $\sim$ (iv):

$$(i) \quad |\alpha| - |z_{(k)}| \geq d \quad (k=1, 2, \dots).$$

- (ii) $\frac{[k(|\alpha| - |z_{(k)}|)]!^s}{[k(m-j-|z_{(k)}|)]!} \leq A_1 B_1^k \left(\frac{1}{R}\right)^{k(|\alpha| - |z_{(k)}|)} \left(\frac{1}{T_0}\right)^{k(l(j, \alpha) - \langle \kappa, z_{(k)} \rangle)}$
 $(k=1, 2, \dots).$
- (iii) $s \leq c_k + \frac{m-j-|z_{(k)}|}{|\alpha| - |z_{(k)}|} \quad (k=1, 2, \dots).$
- (iv) $c_k = o(1) \quad (\text{as } k \rightarrow \infty).$

By using these conditions, we can obtain a contradiction as follows.

When $s > \sigma_{j, \alpha} / (\sigma_{j, \alpha} - 1)$, by choosing $s_0 > 0$ so that $s > s_0 > \sigma_{j, \alpha} / (\sigma_{j, \alpha} - 1)$ and by taking k sufficiently large we have

$$\sigma_{j, \alpha} / (\sigma_{j, \alpha} - 1) < s_0 < \frac{m-j-|z_{(k)}|}{|\alpha| - |z_{(k)}|}$$

(by (iii) and (iv)), which contradicts the condition (4) in Lemma 7.

When $s = \sigma_{j, \alpha} / (\sigma_{j, \alpha} - 1)$ and when

$$s < \frac{m-j-|z_{(k)}|}{|\alpha| - |z_{(k)}|} \quad \text{for some } k \in N,$$

we can find $s_0 > 0$ such that

$$\sigma_{j, \alpha} / (\sigma_{j, \alpha} - 1) = s < s_0 < \frac{m-j-|z_{(k)}|}{|\alpha| - |z_{(k)}|},$$

which contradicts the condition (4) in Lemma 7.

When $s = \sigma_{j, \alpha} / (\sigma_{j, \alpha} - 1)$ and when

$$(6.53) \quad \frac{m-j-|z_{(k)}|}{|\alpha| - |z_{(k)}|} \leq s \quad \text{for any } k \in N,$$

by (i), (ii), (6.53) and Stirling's formula we have

$$(6.54) \quad 1 \leq H \left(\frac{1}{R}\right)^d \left(\frac{1}{T_0}\right)^{l(j, \alpha) - \langle \kappa, z_{(k)} \rangle} \quad (k=1, 2, \dots)$$

for some $H > 0$ (which is independent of k , R and T_0). Since R is sufficiently large, we may assume that $H/R^d < 1$. Therefore, to have a contradiction from (6.54) it is sufficient to prove the following: there is a subsequence $\{z_{(k_i)}\}$ of $\{z_{(k)}\}$ such that

$$\langle \kappa, z_{(k_i)} \rangle \longrightarrow l(j, \alpha) \quad (\text{as } k_i \rightarrow \infty).$$

Let us show this now. Since $\{z_{(k)}\}$ is a bounded sequence in $\mathbf{R}^n$, we can take a convergent subsequence $\{z_{(k_i)}\}$ of $\{z_{(k)}\}$. Then, the limit $z_* \in \mathbf{R}^n$ of $\{z_{(k_i)}\}$ satisfies $z_* \in \mathcal{Z}(j, \alpha)$ (by (i)),

$$s = \frac{m-j-|z_*|}{|\alpha|-|z_*|}$$

(by (6.53), (iii) and (iv)), and $\langle \kappa, z_* \rangle \leq l(j, \alpha)$. Furthermore, we can see the following: if $\langle \kappa, z_* \rangle < l(j, \alpha)$ holds, we can find a $w \in \mathcal{Z}(j, \alpha)$ satisfying $|z_*| < |w|$ and this yields

$$\sigma_{j, \alpha} / (\sigma_{j, \alpha} - 1) = s = \frac{m-j-|z_*|}{|\alpha|-|z_*|} < \frac{m-j-|w|}{|\alpha|-|w|},$$

which contradicts the condition (4) in Lemma 7. Hence, we obtain $\langle \kappa, z_* \rangle = l(j, \alpha)$ and therefore

$$\langle \kappa, z_{(k_i)} \rangle \longrightarrow \langle \kappa, z_* \rangle = l(j, \alpha) \quad (\text{as } k_i \rightarrow \infty).$$

Q.E.D.

REFERENCES

- [1] V. Ja. Ivrii, Cauchy problem conditions for hyperbolic operators with characteristic of variable multiplicity for Gevrey classes, *Sibirsk. Mat. Ž.*, **17** (1976), 1256–1270. (English translation, *Siberian Math. J.*, **17** (1976), 921–931.)
- [2] J. Leray and Y. Ohya, Systèmes linéaires, hyperboliques non stricts, *Colloque CBRM*, 1964, 105–144.
- [3] H. Tahara, Singular hyperbolic systems, III. On the Cauchy problem for Fuchsian hyperbolic partial differential equations, *J. Fac. Sci. Univ. Tokyo, Sect. IA Math.*, **27** (1980), 465–507.
- [4] H. Tahara, Singular hyperbolic systems, V. Asymptotic expansions for Fuchsian hyperbolic partial differential equations, *J. Math. Soc. Japan*, **36** (1984), 449–473.
- [5] H. Tahara, Singular hyperbolic systems, VI. Asymptotic analysis for Fuchsian hyperbolic equations in Gevrey classes, *J. Math. Soc. Japan*, **39** (1987), 551–580.

DEPARTMENT OF MATHEMATICS
SOPHIA UNIVERSITY
KIOICHO, CHIYODA-KU
TOKYO 102
JAPAN