Singular hyperbolic systems, VII.
Asymptotic analysis for Fuchsian hyperbolic equations in Gevrey classes (2)

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In the previous paper [5], the irregularity index $\sigma (\geq 1)$ was defined for some Fuchsian hyperbolic operators $P$, and the equation $Pu=f$ was investigated in the Gevrey class $G^{(s)}$ under the condition $1<s<\sigma/(\sigma-1)$. The following is a typical example:

$$L = (t\partial_t)^2 - \sum_{i=1}^{n} t^{x_i} \partial_{x_i} + a(t, x)(t\partial_t) + \sum_{i=1}^{n} t^{l_i} b_i(t, x) \partial_{x_i} + c(t, x),$$

$$\sigma = \max\left\{1, \frac{2\kappa_i - l_i}{\kappa_i}, \ldots, \frac{2\kappa_n - l_n}{\kappa_n}\right\}$$

(see (1) and (2)).

The purpose of this paper is to study the following case:

$$s = \sigma/(\sigma-1).$$

Roughly speaking, the result established in this paper is as follows: if we treat the equation $Pu=f$ in the projective Gevrey class $G^{(s)}$ (not in the classical one $G^{(s')}$), we can obtain the same results as in [5] also in the case

$$s = \sigma/(\sigma-1)$$

and $\mathcal{D}_p \cap S_p = \emptyset$ (see Theorem 2).

The reason why we must introduce the class $G^{(s)}$ is illustrated as follows. Let $A = (t\partial_t + 1)(t\partial_t + 2) - t^{\kappa} \partial_x^2 + t^a \partial_x$ (where $(t, x) \in [0, T] \times \mathbb{R}$, $\kappa > 0 (1 \leq i \leq n)$ and $l_i > 0 (1 \leq i \leq n)$).

The condition $\mathcal{J} = 0$ is introduced as a necessary and sufficient condition for some formal power series to converge under $s = \sigma/(\sigma-1)$ (see...
Proposition 1). Most of our operators satisfy $\mathcal{D}_p \cap S_p = \emptyset$; in particular, if $P$ is of the second order, $\mathcal{D}_p \cap S_p = \emptyset$ is trivially satisfied. But, there are examples of the case $\mathcal{D}_p \cap S_p \neq \emptyset$ and it seems impossible to apply our argument to the case $s = \sigma/(\sigma - 1)$ and $\mathcal{D}_p \cap S_p \neq \emptyset$

(by Proposition 1). Hence, it is still an open problem to decide whether the condition $\mathcal{D}_p \cap S_p = \emptyset$ (under $s = \sigma/(\sigma - 1)$) is essential or not.

§ 1. Preliminaries

First, let us explain the operator $P$ treated here, the irregularity index $\sigma(\geq 1)$ of $P$, the function spaces and the problem.

The operator treated here is as follows:

\[ P = (t \partial_t)^m + \sum_{j, |\alpha| \leq m} t^{(l, \alpha)} a_{j, \alpha}(t, x)(t \partial_t)^{\alpha} \partial_x^{\beta}, \]

where $(t, x) = (t, x_1, \ldots, x_n) \in [0, T] \times \mathbb{R}^n (T > 0)$, $\partial_t = \partial/\partial t$, $\partial_x = (\partial/\partial x_1, \ldots, \partial/\partial x_n)$, $m \in \mathbb{N}(=\{1, 2, 3, \cdots\})$, $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}_+^n (=\{0, 1, 2, \cdots\})^n$, $|\alpha| = \alpha_1 + \cdots + \alpha_n$ and $\partial_x^{\beta} = (\partial/\partial x_1)^{\beta_1} \cdots (\partial/\partial x_n)^{\beta_n}$. On the coefficients, we assume that $a_{j, \alpha}(t, x) \in C^\infty([0, T] \times \mathbb{R}^n)$ and $C^\infty((0, T) \times \mathbb{R}^n)$ ($j + |\alpha| \leq m$ and $j < m$) and that they satisfy\( (t \partial_t)^{l} \partial_x^{\beta} a_{j, \alpha}(t, x) \in C^\infty([0, T] \times \mathbb{R}^n) \) for any $(l, \beta) \in \mathbb{Z}_+ \times \mathbb{Z}_+^n$. As to the hyperbolicity, we assume the following conditions (A) and (B):

(A) $l(j, \alpha) \in \mathbb{R}$ ($j + |\alpha| \leq m$ and $j < m$) satisfy

\[
\begin{align*}
& l(j, \alpha) = \kappa_1 \alpha_1 + \cdots + \kappa_n \alpha_n, \quad \text{when } j + |\alpha| = m \text{ and } j < m, \\
& l(j, \alpha) > 0, \quad \text{when } j + |\alpha| < m \text{ and } |\alpha| > 0, \\
& l(j, \alpha) \geq 0, \quad \text{when } j + |\alpha| < m \text{ and } |\alpha| = 0
\end{align*}
\]

for some $\kappa = (\kappa_1, \ldots, \kappa_n) \in \mathbb{R}^n$ such that $\kappa_i > 0$ ($1 \leq i \leq n$).

(B) All the roots $\lambda_i(t, x, \xi) (1 \leq i \leq m)$ of the equation (in $\lambda$)

\[
\lambda^m + \sum_{j, |\alpha| = m} a_{j, \alpha}(t, x) \lambda^\alpha \xi^\beta = 0
\]

are real, simple and bounded on $\{(t, x, \xi) \in [0, T] \times \mathbb{R}^n \times \mathbb{R}^n; |\xi| = 1\}$.

Then, $P$ is the Fuchsian hyperbolic operator discussed in [5]. Recall that the characteristic exponents $\rho_1(x), \ldots, \rho_m(x)$ of $P$ are defined by the roots of

\[
\rho^m + \sum_{j, |\alpha| = m} a_{j, \alpha}(x) \rho^\alpha = 0,
\]

where $a_{j, \alpha}(x) = [t^{(l, \alpha)} a_{j, \alpha}(t, x)]_{t=0} (j < m)$. Also, recall that the irreg-
ularity index $\sigma (\geq 1)$ of $P$ is defined by

$$\sigma = \max \left[ 1, \max_{j+|\alpha|<m} \{ \min(\max M_{j,\alpha}(\tau, r)) \} \right],$$

where $\mathfrak{S}_n$ is the permutation group of $n$-numbers and

$$M_{j,\alpha}(\tau, r) = \sum_{t=1}^r (\rho_{t,\alpha} - \rho_{t,\alpha}^r) \alpha_{t,\alpha} + (m-j)\rho_{t,\alpha} - l(j, \alpha) \frac{(m-j-|\alpha|)\rho_{t,\alpha}}{(m-j-|\alpha|)\rho_{t,\alpha}}.$$

In [5], we discussed the equation $P\nu = f$ in $C^\infty([0, T], \delta^*(R^n))$ or $C^\infty((0, T), \delta^*(R^n))$ under the condition $1 < s < \sigma/(\sigma - 1)$. But, in this paper, we will consider the equation $P\nu = f$ in $C^\infty([0, T], \delta^{(s)}(R^n))$ or $C^\infty((0, T), \delta^{(s)}(R^n))$ under the condition $1 < s \leq \sigma/(\sigma - 1)$.

Here, $\delta^{(s)}(R^n)$ [resp. $\delta^{(s)}(R^n)$] denotes the set of all the functions $f(x) \in C^\infty(R^n)$ satisfying the following: for any compact subset $K$ of $R^n$ and for any $h > 0$, there is a $C > 0$ [resp. for any compact subset $K$ of $R^n$, there are $h > 0$ and $C > 0$] such that

$$\sup_{x \in K} |\partial_\alpha^r f(x)| \leq Ch^{|\alpha|/(|\alpha|)!} \quad \text{for any } \alpha \in Z^n.$$

When $s = \infty$, we put $\delta^{(\infty)}(R^n) = \delta(R^n)$. Note that $\delta^{(s)}(R^n) \subseteq \delta^{(s)}(R^n)$ holds for $1 < s < \infty$. For simplicity, we often write $\delta^*$ instead of $\delta^{(s)}$ or $\delta^{(s)}$; that is, $\delta^*$ with $* = \{s\}$ means $\delta^{(s)}$, and $\delta^*$ with $* = (s)$ means $\delta^{(s)}$. By $C^\infty([0, T], \delta^*(R^n))$ [resp. $C^\infty((0, T), \delta^*(R^n))$] we denote the set of all infinitely differentiable functions on $[0, T]$ [resp. $(0, T)$] with values in $\delta^*$ equipped with the usual topology.

Then, by following the discussion in [5] we can obtain the following result.

**Theorem 1.** Let $P$ be the operator in (1.1) satisfying (A), (B) and the following conditions:

1. $1 < s < \sigma/(\sigma - 1)$.
2. $* = \{s\}$ or $(s)$.
3. $\tau^{(s, x)}_{j,\alpha}(t, x) \in C^\infty([0, T], \delta^*(R^n)) (j + |\alpha| \leq m$ and $j < m)$.

Then we have the following results (I) and (II).

(I) (Unique solvability). If $\rho_i(x) \in Z_+$ holds for any $x \in R^n$ and $1 \leq i \leq m$, the equation

$$P \nu = f \quad \text{in } C^\infty([0, T], \delta^*(R^n))$$

is uniquely solvable.

(II) (Asymptotic expansions). If $\rho_i(x) - \rho_j(x) \in Z$ holds for any $x \in R^n$.
and $1 \leq i \leq j \leq m$, the general solution of the equation

$$(1.5) \quad Pu = 0 \quad \text{in } C^\infty((0, T), \mathcal{E}^s(R^n))$$

is characterized as follows. (II-1) Any solution $u(t, x)(=u) \in C^\infty((0, T), \mathcal{E}^s(R^n))$ of (1.5) is expanded asymptotically into the form

$$u(t, x) \sim \sum_{i=1}^{m} \left\{ \varphi_i(x) t^{s_i(x)} + \sum_{k=1}^{m} \sum_{h=0}^{m_k} (\log t)^h \right\}$$

(as $t \to +0$) for some unique $\varphi_i(x), \varphi_{i, k, h}(x) \in \mathcal{E}^s(R^n)$, $1 \leq i \leq m, 1 \leq k < \infty$ and $0 \leq h \leq m_k$. (II-2) Conversely, for any $\varphi_i(x), \cdots, \varphi_m(x) \in \mathcal{E}^s(R^n)$ there exist a unique solution $u(t, x)(=u) \in C^\infty((0, T), \mathcal{E}^s(R^n))$ of (1.5) and unique coefficients $\varphi_{i, k, h}(x) \in \mathcal{E}^s(R^n)$, $1 \leq i \leq m, 1 \leq k < \infty$ and $0 \leq h \leq m_k$ such that the asymptotic relation in (II-1) holds.

In fact, the case $* = \{s\}$ was already proved in [5] and the case $* = (s)$ may be proved by an argument quite parallel to the case $* = \{s\}$ (see also §§ 4 and 5). Hence, our target is only the case $s = a/(b - 1)$. Since it seems impossible to have good results in the case $* = \{s\}$ with $s = a/(b - 1)$ (as is seen in the introduction), we will discuss only the case $* = (s)$ with $s = a/(b - 1)$ from now on.

§ 2. Main theorem

Now, let us consider the equation $Pu = f$ in $C^\infty([0, T], \mathcal{E}^{(s)}(R^n))$ or $C^\infty((0, T), \mathcal{E}^{(s)}(R^n))$ under the condition $s = a/(b - 1)$.

Let $P$ be the operator in (1.1) satisfying (A) and (B). Put

$$(2.1) \quad \mathcal{J} = \{(j, \alpha) \in Z_n \times Z_n^n; j + |\alpha| < m \text{ and } |\alpha| > 0\},$$

let $\beta$ be as in (1.2), and let $M_{j, \alpha}(\tau, r)$ be as in (1.3). Define $\sigma_{j, \alpha} (\geq 1)$ by

$$(2.2) \quad \sigma_{j, \alpha} = \max \{1, \min \{\max M_{j, \alpha}(\tau, r)\}\}.$$ 

Then, we have $\alpha = \max \{\sigma_{j, \alpha}; (j, \alpha) \in \mathcal{J}\}$. Put $\mathcal{J}_{p}$ as follows:

$$(2.3) \quad \mathcal{J}_{p} = \{(j, \alpha) \in \mathcal{J}; \sigma_{j, \alpha} = p\}.$$ 

Let $\kappa = (\kappa_1, \cdots, \kappa_n) \in R^n$ be the one in (A). For $\alpha = (\alpha_1, \cdots, \alpha_n) \in Z_n^n$, denote by $S_\alpha$ the set of all $l \in R$ satisfying the following (i) and (ii): (i) $0 < l < \langle \kappa, \alpha \rangle$, and (ii) there are $\tau \in S_\alpha$ and $p \in \{1, \cdots, n-1\}$ such that

$$(2.4) \quad \{l = \kappa_{r(1)} \alpha_{r(1)} + \cdots + \kappa_{r(p)} \alpha_{r(p)},$$

$$\{\kappa_{r(1)}, \cdots, \kappa_{r(p)}\} \subset \{\kappa_{r(p+1)}, \cdots, \kappa_{r(n)}\}.$$ 

Here, $\langle \kappa, \alpha \rangle = \kappa_1 \alpha_1 + \cdots + \kappa_n \alpha_n$, and $\{a_1, \cdots, a_p\} \subset \{b_1, \cdots, b_q\}$ means that $a_i < b_j$.
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holds for any i and j. Put $S_p$ as follows:

\[(2.5) \quad S_p = \{(j, \alpha) \in \mathcal{J} : l(j, \alpha) \in S_{\lambda}(\alpha)\}.
\]

Then, the main theorem of this paper is stated as follows.

**Theorem 2.** Let $P$ be the operator in (1.1) satisfying (A) and (B). Assume the following conditions:

1. $s = a/(a - 1)$ and $A_p \cap S_p = \emptyset$.
2. $*= (s)$.
3. $t^{(\beta)}a_{j,*,}(t, x) \in C^\infty([0, T], \mathcal{C}^\infty(R^n)) \ (j + |\alpha| \leq m$ and $j < m)$.

Then, the same results as in Theorem 1: (I) (Unique solvability) and (II) (Asymptotic expansions)—are valid.

In the case $a = 1$ (i.e. $s = \infty$), Theorem 2 was already proved in [3, 4]. In the case $a > 1$ (i.e. $1 < s < \infty$), Theorem 2 will be proved in §4 and 5.

As to the condition $A_p \cap S_p = \emptyset$, we remark the following.

**Remark.** (1) When $a = 1$, $l(j, \alpha) \leq \langle \kappa, \alpha \rangle$ holds for any $(j, \alpha) \in \mathcal{J}$; therefore in this case the condition $A_p \cap S_p = \emptyset$ is trivially satisfied.
(2) When $\kappa_1 = \cdots = \kappa_n$, $S_{\lambda}(\alpha) = \emptyset$ holds for any $\alpha \in \mathbb{Z}_+^n$; therefore in this case the condition $A_p \cap S_p = \emptyset$ is trivially satisfied.
(3) When $|\alpha| = 1$, $S_{\lambda}(\alpha) = \emptyset$ holds for any $\kappa = (\kappa_1, \cdots, \kappa_n)$; therefore in this case the condition $A_p \cap S_p = \emptyset$ is trivially satisfied.
(4) When $m = 3$, $n = 2$ and $0 < \kappa_1 < \kappa_2$, $S_{\lambda}((1, 0)) = S_{\lambda}((0, 1)) = S_{\lambda}((2, 0)) = S_{\lambda}((0, 2)) = \emptyset$ and $S_{\lambda}((1, 1)) = \{\kappa\}$ hold; therefore in this case the condition $A_p \cap S_p = \emptyset$ is equivalent to the following: $\sigma \neq 2$ or $l(0, (1, 1)) \neq \kappa$.
(5) Put $\Sigma_m = \{k/h : k, h \in \mathbb{Z}$ and $1 \leq h < k \leq m - 1\}$. Then we can see the following: when $\sigma \in \Sigma_m$, we have $A_p \cap S_p = \emptyset$. Note that for any $k/h \in \Sigma_m$ we can find an example such that $\sigma = k/h$ and $A_p \cap S_p = \emptyset$.

By Theorems 1 and 2, we have obtained good results in the following two cases:

(C-1) $1 < s < a/(a - 1)$.
(C-2) $s = a/(a - 1)$ and $A_p \cap S_p = \emptyset$.

But, our method employed here cannot be applied to the case

(C-3) $s = a/(a - 1)$ and $A_p \cap S_p = \emptyset$.

therefore, it is still an open problem to study the case (C-3).

In §§4 and 5 [resp. in [5]], we will prove Theorem 2 [resp. we proved
Theorem 1] by reducing the problem to the one of the convergence of a majorant series $M_a(\{\beta(J_k)\}, C, R, t)$ which appears naturally in the estimation of our formal solution (based on the method of successive approximations). In our proof, the following fact is essential: the majorant series $M_a(\{\beta(J_k)\}, C, R, t)$ converges, if and only if (C-1) or (C-2) holds.

Precisely, it is stated as follows. Let $\mathcal{J}$ be as in (2.1), and put $\mathcal{J}^1=\mathcal{J}$, $\mathcal{J}^2=\mathcal{J} \times \mathcal{J}$, $\mathcal{J}^3=\mathcal{J} \times \mathcal{J} \times \mathcal{J}$, $\cdots$. For $J_k=((j_1, \alpha(1)), \cdots, (j_k, \alpha(k))) \in \mathcal{J}^k$, we denote by $\mathcal{M}(J_k)$ the set of all $(\beta(1), \cdots, \beta(k)) \in \mathbb{Z}^+ \times \cdots \times \mathbb{Z}^+$ satisfying the following:

\[(2.6)\]

\[
\left\{(0, \cdots, 0) \leq \beta(i) \leq \alpha(i) \quad (i=1, \cdots, k)\right. \]
\[
\left.\langle \kappa, \beta(1) + \cdots + \beta(k) \rangle \leq l(j_1, \alpha(1)) + \cdots + l(j_k, \alpha(k)) \quad (i=1, \cdots, k).\right. 
\]

Let $a>0$. For $J_k=((j_1, \alpha(1)), \cdots, (j_k, \alpha(k))) \in \mathcal{J}^k$, $\beta(J_k)=(\beta(1), \cdots, \beta(k)) \in \mathcal{M}(J_k)$, $C>0$ and $R>0$, we put

\[
(2.7)
\]

\[
F_a(J_k, \beta(J_k), C, R, t) = C^{k+1} l_1 \cdots l_k \left(\frac{1}{a^k} \right) \times \prod_{i=1}^{k} \frac{1}{a_{i+1}} \times \prod_{i=1}^{k} \frac{1}{a_i}, 
\]

where $l_i=l(j_i, \alpha(i))$ ($i=1, \cdots, k$), $a_i=a$ and $a_{i+1}=a_1+\cdots+l_i-\langle \kappa, \beta(1) + \cdots + \beta(i) \rangle$ ($i=1, \cdots, k$).

Then, the majorant series $M_a(\{\beta(J_k)\}, C, R, t)$ which appears in §4 is given by

\[
(2.8) \quad M_a(\{\beta(J_k)\}, C, R, t) = \sum_{J_k \in \mathcal{J}^k} \sum_{\beta(J_k) \in \mathcal{M}(J_k)} F_a(J_k, \beta(J_k), C, R, t) 
\]

(see (4.7)). As to the convergence of $M_a(\{\beta(J_k)\}, C, R, t)$, we have the following result (the proof will be given in §6).

**Proposition 1.** Let $1<s<\infty$, $a>0$, $C>0$ and $T>0$. Then, we have the following results (I) and (II).

(I) The following three conditions (I-1), (I-2) and (I-3) are equivalent to each other.

(I-1) (C-1) or (C-2) holds.

(I-2) There are $R_0>0$ and $\beta(J_k) \in \mathcal{M}(J_k)$ ($J_k \in \mathcal{J}^k$, $k=1, 2, \cdots$) such that

$\quad M_a(\{\beta(J_k)\}, C, R_0, T) < +\infty$.

(I-3) There are $R_0>0$, $T_0>0$ and $\beta(J_k) \in \mathcal{M}(J_k)$ ($J_k \in \mathcal{J}^k$, $k=1, 2, \cdots$) such that

$\quad M_a(\{\beta(J_k)\}, C, R_0, T_0) < +\infty$.
The following three conditions (II-1), (II-2) and (II-3) are equivalent to each other.

(II-1) \((C-1)\) holds.

(II-2) For any \(R > 0\), there are \(\beta(J_k) \in \mathcal{M}(J_k) (J_k \in \mathcal{J}^k, k=1, 2, \ldots)\) such that

\[
M_\varepsilon(\{\beta(J_k)\}, C, R, T) < +\infty.
\]

(II-3) For any \(R > 0\), there are \(T_0 > 0\) and \(\beta(J_k) \in \mathcal{M}(J_k) (J_k \in \mathcal{J}^k, k=1, 2, \ldots)\) such that

\[
M_\varepsilon(\{\beta(J_k)\}, C, R, T_0) < +\infty.
\]

In the proof of Theorem 1 (in [5]), we used the condition (II-2) in Proposition 1. In the proof of Theorem 2 (in §4 and 5), we will use the condition (I-2) in Proposition 1. This is the reason why our method cannot be applied to the case (C-3).

§ 3. Some lemmas

Before the proof of Theorem 2, let us recall here formal norms introduced in [5, §5] (see also [2]), and let us present some preparatory lemmas.

Let \(p, l \in \mathbb{Z}^+, r \in \mathbb{R}\) and \(\epsilon=(\epsilon_1, \ldots, \epsilon_n) \in \mathbb{R}^n\) such that \(\epsilon_i > 0 (1 \leq i \leq n)\). For \(f(t, x) \in C^\omega((0, \alpha), H^\omega(\mathbb{R}^n))\) and \(a(t, x) \in C^\omega((0, T) \times \mathbb{R}^n)\), we write

\[
\|P_{p, \infty}^\omega f(t)\| = \sum_{\|\alpha\| \leq p} \sum_{\|\beta\| \leq q} \sum_{j=0}^m \sum_{|j| = q} \frac{|t_\beta| + r|}{j! \alpha!} \|f(t)\|_{L^\infty(\mathbb{R}^n)} \frac{\rho^q}{q!},
\]

\[
\|P_{p, l}^{\infty} f(t)\| = \sum_{\|\beta\| \leq q} \sum_{|j| = q} \frac{|t_\beta| + r|}{j! \alpha!} \|f(t)\|_{L^\infty(\mathbb{R}^n)},
\]

\[
\|P_{p, \infty} a\| = \sum_{\|\alpha\| \leq p} \sum_{|j| = q} \sum_{|j| = q} \frac{|t_\beta|}{j! \alpha!} \|f(t)\|_{L^\infty(\mathbb{R}^n)} \frac{\rho^q}{q!},
\]

(where \(\langle \epsilon, \alpha \rangle = \epsilon_1 \alpha_1 + \cdots + \epsilon_n \alpha_n\) and \(\Omega = (0, T) \times \mathbb{R}^n\)). Also, for a differential operator \(R\) of the form

\[
R = \sum_{j + |\alpha| \leq m} \epsilon^{\langle \alpha, \gamma \rangle} a_{j, \alpha}(t, x) (t_\alpha) \partial_x^\alpha
\]

we write

\[
\|P_{p, \infty}^\omega R\| = \sum_{j + |\alpha| \leq m} \|P_{p, \infty}^\omega a_{j, \alpha}\|.
\]

For basic properties of these formal norms, see [5, §5].

Let \(1 \leq s < \infty\) and put

\[
\theta_s(\rho) = \sum_{q=0}^{\infty} \frac{(q!)^s \rho^q}{q!}.
\]
For a formal power series \( \varphi(t, \rho) \) in \( \rho \), we write

\[
\varphi(t, \rho) \in \mathcal{E}^{(s)} \text{ uniformly on } [0, T],
\]

if \( \varphi(t, \rho) \) satisfies the following: for any \( \varepsilon > 0 \) there is a \( C_\varepsilon > 0 \) such that \( \varphi(t, \rho) \ll C_\varepsilon \theta_\varepsilon(t, \rho) \) holds on \( [0, T] \). Here, \( \sum_{q=0}^\infty a_q \rho^q \ll \sum_{q=0}^\infty b_q \rho^q \) means that \( |a_q| \leq b_q \) holds for any \( q \). Then, by Sobolev's lemma we can see

**Lemma 1.** Let \( f(t, x) \in C^\infty([0, T], H^{-s}(\mathbb{R}^n)) \). Assume that \( \text{supp}(f) \subset [0, T] \times K \) holds for some compact subset \( K \) of \( \mathbb{R}^n \). Then, the following conditions (i) and (ii) are equivalent:

(i) \[ \| f^{(s)}(t) \| \in \mathcal{E}^{(s)} \text{ uniformly on } [0, T]. \]

(ii) \[ \| f^{(s)}(t) \| \in \mathcal{E}^{(s)} \text{ uniformly on } [0, T]. \]

For a positive-valued function \( C(\varepsilon) \) in \( \varepsilon > 0 \), we write

\[
\theta_\varepsilon(\rho; C(\varepsilon)) = \inf_{\varepsilon > 0} (C(\varepsilon) \theta_\varepsilon(\varepsilon \rho)).
\]

Then, (3.1) is equivalent to the condition that \( \varphi(t, \rho) \ll \theta_\varepsilon(\rho; C(\varepsilon)) \) holds on \( [0, T] \) for some \( C(\varepsilon) \). Note that by putting \( A = C(1) \) and \( C(\varepsilon) = C(\varepsilon)/A \) we have \( \theta_\varepsilon(\rho; C(\varepsilon)) = A \theta_\varepsilon(\rho; C(\varepsilon)) \) and \( C(1) = 1 \).

We say that \( \varphi(t, \rho) = \sum_{q=0}^\infty \varphi_q(t) \rho^q \) satisfies (M(\varepsilon)) with respect to \( C(\varepsilon) \), if \( \varphi(t, \rho) \) satisfies the following condition:

\[
(M(\varepsilon)) \quad (\inf_{\varepsilon > 0} C(\varepsilon)) \varphi_p(t) \leq \varphi_{p+q}(t) \quad \text{on } (0, T) \text{ for any } p, q \in \mathbb{Z}_+.\]

Then, by the same argument as in [5, Lemma 5] we have

**Lemma 2.** Let \( 1 \leq s < \infty \). Assume that \( \varphi(t, \rho) \gg \theta_\varepsilon(\rho; C(\varepsilon)) \) satisfies (M(\varepsilon)) with respect to \( C(\varepsilon) \). Then, for any \( 0 < 2k \leq h \) we have

\[
\theta_{\epsilon^k}(k \rho; C(\epsilon)) \varphi(t, h \rho) \ll 2 \varphi(t, h \rho).
\]

The following lemma guarantees that the arguments in [5] with \( \theta_\varepsilon(\rho; C(\varepsilon)) \) replaced by \( \theta_{\epsilon^k}(\rho; C(\varepsilon)) \) are also valid.

**Lemma 3.** Let \( 1 < s < \infty \), let \( \varphi(t, \rho) \gg \theta_\varepsilon(\rho; C(\varepsilon)) \) be a formal power series in \( \rho \) such that \( \varphi(t, \rho) \in \mathcal{E}^{(s)} \) uniformly on \( [0, T] \), and let \( C(\varepsilon) \) be a positive-valued function in \( \varepsilon > 0 \) which satisfies the following conditions (i) and (ii):

(i) \( \varphi(t, \rho) \ll \theta_{\epsilon^k}(\rho; C(\varepsilon)) \) holds on \( [0, T] \).

(ii) \( \theta_{\epsilon^k}(\rho; C(\varepsilon)) \) satisfies (M(\varepsilon)) with respect to \( C(\varepsilon) \).

**Proof.** First, let us make clear the meaning of \( \theta_{\epsilon^k}(\rho; C(\varepsilon)) \). Denote by
The set of all functions \( f(z) \in \mathcal{C}([0, \infty)) \) satisfying the following: \( f(z) = 0 \) on \([0, 1]\), \((d/dz)f(z) \leq 0 \) on \([0, \infty)\), \((d/dz)f(z) \to -\infty \) as \( z \to +\infty \) and \((d/dz)^2f(z) \leq 0 \) on \([0, \infty)\). For \( f(z) \in \mathcal{F} \), we write

\[
\theta_{\epsilon, f}(\rho) = \sum_{q=0}^{\infty} e^{\epsilon(q)} \left( \frac{q!}{q!} \right)^{\rho^q}.
\]

Then, by comparing the convex sets of the following two types

\[
\{ (z, y) \in \mathbb{R}^2 ; z \geq 0 \text{ and } y \leq (\log \varepsilon)z + \log C(\varepsilon) \},
\]

\[
\{ (z, y) \in \mathbb{R}^2 ; z \geq 0 \text{ and } y \leq f(z) \},
\]

we can see the following facts (F-1) and (F-2).

(F-1) For any positive-valued function \( C(\varepsilon) \) in \( \varepsilon > 0 \) satisfying \( C(1) = 1 \), we can find an \( f(z) \in \mathcal{F} \) such that \( \theta_{\epsilon, f}(\rho; C(\varepsilon)) \ll \theta_{\epsilon, f}(\rho) \).

(F-2) For any \( f(z) \in \mathcal{F} \), we can find a positive-valued function \( C(\varepsilon) \) in \( \varepsilon > 0 \) such that \( C(1) = 1 \) and \( \theta_{\epsilon, f}(\rho; C(\varepsilon)) = \theta_{\epsilon, f}(\rho) \).

Now, let us show Lemma 3. Let \( \varphi(t, \rho) > 0 \) and \( C_{\epsilon}(\varepsilon) \) be as in Lemma 3. Choose a positive-valued function \( C_{\epsilon}(\varepsilon) \) in \( \varepsilon > 0 \) such that \( C_{\epsilon}(1) = 1 \) and

\[
(3.2) \quad \varphi(t, \rho) \ll A \theta_{t, f}(\rho; C_{\epsilon}(\varepsilon)) \quad \text{on } [0, T]
\]

for some \( A > 0 \). Since \( C_{\epsilon}(\varepsilon) (i = 0, 1) \) satisfy \( C_{\epsilon}(1) = 1 \), by (F-1) we can find an \( f(z) \in \mathcal{F} \) such that

\[
(3.3) \quad \theta_{\epsilon, f}(\rho; C_{\epsilon}(\varepsilon)) \ll \theta_{\epsilon, f}(\rho) \quad (i = 0, 1).
\]

Therefore, to have Lemma 3 it is sufficient to construct a function \( g(z) \in \mathcal{F} \) satisfying the following (g-1) and (g-2):

(g-1) \( f(z) \leq g(z) \) on \([0, \infty)\).

(g-2) \( (d/dz)^2(g(z) + (s-1)\Gamma(1+z)) \geq 0 \) on \([0, \infty)\)

(\text{where } \Gamma(z) \text{ denotes the gamma-function of Euler}). In fact, if such a function \( g(z) \in \mathcal{F} \) is obtained, by (3.2), (3.3) and (g-1) we have

\[
\varphi(t, \rho) \ll A \theta_{t, f}(\rho) \ll A \theta_{t, g}(\rho) \quad \text{on } [0, T],
\]

and by (3.3), (g-1) and (g-2) we have

\[
\left( \inf_{\varepsilon > 0} (C_{\epsilon}(\varepsilon))^z \right) \left( \frac{(p!)^r}{p!} \right) \left( \frac{(q!)^s}{q!} \right) \leq \left( e^{\epsilon(p)} \left( \frac{p!}{p!} \right)^r \right) \left( e^{\epsilon(q)} \left( \frac{q!}{q!} \right)^s \right)
\]

\[
\leq \left( e^{\epsilon(p)} \left( \frac{p!}{p!} \right)^r \right) \left( e^{\epsilon(q)} \left( \frac{q!}{q!} \right)^s \right)
\]

\[
\leq e^{\epsilon(p+q)} \frac{(p+q)!}{(p+q)!}.
\]
(for any \( p, q \in \mathbb{Z}^+ \)) which means that \( \theta_{s, \rho} \) satisfies \((M_{s, \rho})\) with respect to \( C_C(z) \); hence, by choosing \( C(s) \) so that \( A\theta_{s, \rho} = \theta_{s, \rho} ; C(s) \) (by \((F-2)\)) we can obtain Lemma 3.

Here, we note the following lemma (the proof will be given later).

**Lemma 4.** Let \( a(z) \in C^0([0, \infty)) \) be such that \( a(z) \geq 0 \) on \([0, \infty)\) and \( \int_0^\infty a(y)dy \rightarrow \infty \) (as \( z \rightarrow \infty \)), and let \( h(z) \in C^0([0, \infty)) \) be such that \( h(z) \geq 0 \) on \([0, \infty)\), \( h(z) \) is increasing in \( z \) and \( h(z) \rightarrow \infty \) (as \( z \rightarrow \infty \)). Then, we can find a \( b(z) \in C^0([0, \infty)) \) which satisfies the following conditions: \((b-1)\) \( b(z) = 0 \) on \([0, 1]\), \((b-2)\) \( 0 \leq b(z) \leq a(z) \) on \([0, \infty)\), \((b-3)\) \( \int_0^\infty b(y)dy \rightarrow \infty \) (as \( z \rightarrow \infty \)), and \((b-4)\) \( \int_0^z b(y)dy \leq h(z) \) on \([0, \infty)\).

By using this lemma, let us construct a function \( g(z) \in \mathcal{F} \) satisfying \((g-1)\) and \((g-2)\). Put

\[
\begin{align*}
a(z) &= (s-1)\left( \frac{d}{dz} \right)^s \Gamma(1+z), \\
h(z) &= (-1)\left( \frac{d}{dz} \right) f(z).
\end{align*}
\]

Then, \( a(z) \) and \( h(z) \) satisfy the conditions in Lemma 4. Therefore, by Lemma 4 we have a \( b(z) \in C^0([0, \infty)) \) which satisfies \((b-1) \sim (b-4)\). Hence, by putting

\[
g(z) = (-1) \int_0^z \int_0^y b(z) \, dx \, dy
\]

we can obtain a \( g(z) \in \mathcal{F} \) satisfying \((g-1)\) and \((g-2)\). Note that the fact \( g(z) \in \mathcal{F} \) follows from \((b-1) \sim (b-3)\), that \((g-1)\) follows from \((b-4)\), and that \((g-2)\) follows from \((b-2)\).

**Proof of Lemma 4.** When there is a \( p \geq 1 \) such that \( \int_p^\infty a(y)dy \leq h(z) \) for any \( z > p \), by modifying the function

\[
b_b(z) = \begin{cases} 
0, & \text{for } 0 \leq z \leq p, \\
a(z), & \text{for } z > p
\end{cases}
\]

we can obtain a function \( b(z) \) in Lemma 4.

When for any \( p \geq 1 \) there is a \( q > p \) such that \( \int_p^q a(y)dy > h(q) \), we can choose a sequence \( p_1, q_1, p_2, q_2, \ldots \) successively so that the following conditions are satisfied: \( p_k \geq 1 \), \( h(p_k) \geq 1 \), \( q_k > p_k \),

\[
\sum_{i=1}^{k-1} \int_{p_i}^{q_i} a(y)dy + \int_{p_k}^{q_k} a(y)dy \begin{cases} < h(z), & \text{for } p_k \leq z < q_k, \\
= h(z), & \text{for } z = q_k
\end{cases}
\]

Q.E.D.
\[ p_{k+1} > q_k + 1, \]
\[
\sum_{i=1}^{k} \int_{p_i}^{q_i} a(y) dy + 1 \leq h(p_{k+1})
\]

\( k = 1, 2, \ldots \). Therefore, in this case, by modifying the function

\[
b_i(z) = \begin{cases} 0, & \text{for } q_{k-1} \leq z \leq p_k \quad (k = 1, 2, \ldots), \\ a(z), & \text{for } p_k < z < q_k \quad (k = 1, 2, \ldots) \end{cases}
\]

(where \( q_0 = 0 \)) we can obtain a function \( b(z) \) in Lemma 4. Q.E.D.

\section{4. Proof of (I): Unique solvability}

To have the part (I) in Theorems 1 and 2, it is sufficient to establish the \( \mathcal{E}^{(s)} \)-version of [5, Proposition 4] (see also [5, §§ 3 and 7]).

Let \( P \) be the operator in (1.1), and let us consider

\[
P(t'v) = t'g,
\]

where \( v = v(t, x), g = g(t, x) \), and \( r \in \mathbb{R} \) is a parameter. As in [5, Proposition 4], we impose here the following conditions:

(B') (B) is satisfied. In addition, there is a \( c > 0 \) such that

\[
|\lambda_i(t, x, \xi) - \lambda_i(t, x, \xi')| \geq c
\]

holds on \( \{(t, x, \xi) \in [0, T] \times \mathbb{R}^n \times \mathbb{R}^n \mid |\xi'| = 1\} \) for any \( 1 \leq i \neq j \leq m \).

\[
(D^{(s)}_{p}) \quad \|F_{p}^{-1}a_{j, s}\|_{e} \in \mathcal{E}^{(s)} \quad (j + |\alpha| \leq m \text{ and } j \leq m).
\]

Then, our \( \mathcal{E}^{(s)} \)-version of [5, Proposition 4] is given as follows.

\begin{proposition}
Let \( P \) be the operator in (1.1), let \( 1 < s < \infty \), and let \( p \in \mathbb{N} \). Assume that \( P \) satisfies (A\(_{s}\)), (B') and (D\(_{p}^{(s)}\)), and that \( s \) satisfies (C-1) or (C-2). Then, there is an \( a_{s} > 0 \) which satisfies the following condition. If \( r > a_{s} \), and if \( g(t, x) \in C^{-\infty}((0, T), H^{s}(\mathbb{R}^n)) \) satisfies \( (t_0)_{1}g(t, x) \in C^{0}([0, T], H^{s}(\mathbb{R}^n)) \) for any \( l \in \mathbb{Z}_{+} \), and \( \|F_{p}^{-1}g(t)\| \in \mathcal{E}^{(s)} \) uniformly on \([0, T]\), then the equation (4.1) has a unique solution \( v(t, x) \in C^{-\infty}((0, T), H^{s}(\mathbb{R}^n)) \) such that \( (t_0)_{1}v(t, x) \in C^{0}([0, T], H^{s}(\mathbb{R}^n)) \) for any \( l \in \mathbb{Z}_{+} \), and \( \|F_{p}^{-1}F_{p}^{-1}v(t)\| \in \mathcal{E}^{(s)} \) uniformly on \([0, T]\). In addition, if \( g(t, x) \) satisfies \( \text{supp}(g) \subset C_{\rho}(0, K) \) for some compact subset \( K \) of \( \mathbb{R}^n \), \( v(t, x) \) also satisfies \( \text{supp}(v) \subset C_{\rho}(0, K) \).
\end{proposition}

Here, \( C_{\rho}(0, K) \) is defined by the case \( t_0 = 0 \) of

\[
C_{\rho}(t_0, K) = \{ (t, x) \in [0, T] \times \mathbb{R}^n ; \min_{y \in K} |x - y| \leq \frac{\lambda_{\max} T^{\omega - \rho}}{\mu} |t' - t_0'| \},
\]
where \( \mu = \min\{\varepsilon_1, \ldots, \varepsilon_n\} \), \( k = \max\{\varepsilon_1, \ldots, \varepsilon_n\} \) and \( \lambda_{\max} \) is the least upper bound of \( |\lambda(t, x, \xi)| \) \((1 \leq i \leq m)\) on \( \{(t, x, \xi) \in [0, T] \times \mathbb{R}^n \times \mathbb{R}^n \mid |\xi| = 1\} \).

**Proof of Proposition 2.** Let \( \mathcal{J} \) be as in (2.1). Put \( Q_j(j, \alpha) \) \((j, \alpha) \in \mathcal{J}\) and \( L \) as follows:

\[
Q_j(j, \alpha) = (-1)^{t(j)} a_{j, \alpha}(t, x)(t \partial_t + r)^j \partial_x^j,
\]

\[
L = P + \sum_{(j, \alpha) \in \mathcal{J}} Q_j(j, \alpha).
\]

Then, (4.1), is equivalent to

\[
L(t'v) = t'(g + \sum_{(j, \alpha) \in \mathcal{J}} Q_j(j, \alpha)v).
\]

Therefore, to solve (4.1), we can use the method of successive approximations: first we solve (by [5, Lemma 11])

\[
L(t'v_0) = t'g,
\]

\[
L(t'v((j_1, \alpha_{(1)}))) = t'Q_j(j_1, \alpha_{(1)})v_0,
\]

\[
L(t'v((j_1, \alpha_{(1)}), (j_2, \alpha_{(2)}))) = t'Q_{(j_2, \alpha_{(2)})}v((j_1, \alpha_{(1)})),
\]

\[
\vdots
\]

\[
L(t'v((j_1, \alpha_{(1)}), \ldots, (j_k, \alpha_{(k)}))) = t'Q_{(j_k, \alpha_{(k)})}v((j_1, \alpha_{(1)}), \ldots, (j_{k-1}, \alpha_{(k-1)})),
\]

and then we show the convergence of the formal solution

\[
v = v_0 + \sum_{k=1}^{\infty} \sum_{(j_1, \alpha_{(1)}), \ldots, (j_k, \alpha_{(k)}) \in \mathcal{J}} v((j_1, \alpha_{(1)}), \ldots, (j_k, \alpha_{(k)})).
\]

The proof of the convergence is as follows. By \((D_{\varepsilon}^\alpha)\), we have

\[
\|F_{\varepsilon}^{p, n}L\|_{\infty}, \quad \|F_{\varepsilon}^{p, n}a_{j, \alpha}\|_{\infty} \lesssim B_{\varepsilon} \theta_{\varepsilon}^1(p; C_{\varepsilon}(\varepsilon))
\]

for some \( B_{\varepsilon} > 0 \) and some positive-valued function \( C_{\varepsilon}(\varepsilon) \) in \( \varepsilon > 0 \) satisfying \( C_{\varepsilon}(1) = 1 \). Since \( \|F_{\varepsilon}^{p, n}g(t)\|_{\mathcal{E}(\varepsilon)} \) uniformly on \([0, T]\), by Lemma 3 we can find a positive-valued function \( C_{\varepsilon}(\varepsilon) \) in \( \varepsilon > 0 \) such that

\[
\|F_{\varepsilon}^{p, n}g(t)\|_{\mathcal{E}(\varepsilon)} \lesssim \theta_{\varepsilon}(2p; C_{\varepsilon}(\varepsilon)) \quad \text{on } [0, T]
\]

and that \( \theta_{\varepsilon}^1(p; C_{\varepsilon}(\varepsilon)) \) satisfies \((M_{\varepsilon})\) with respect to \( C_{\varepsilon}(\varepsilon) \). Therefore, by the same argument as in [5, formula (7.8) or (7.10)] we have the following: there are \( C > 0, H > 0 \) and \( a > 0 \) such that for any \( J_{k} = ((j_1, \alpha_{(1)}), \ldots, (j_k, \alpha_{(k)})) \in \mathcal{J}^k \) we have

\[
\left|F_{\varepsilon}^{p, n} = \frac{1}{a_{k+1}} \times \frac{1}{(a_j)^{n-\lambda_1-|\lambda_{(1)}|}} \times \cdots \times (a_k)^{n-\lambda_k-|\lambda_{(k)}|}
\times \partial_{\varepsilon}^{(1, \alpha_{(1)}-\lambda_{(1)})} \times \cdots \times (1, \alpha_{(k)}-\lambda_{(k)}) \theta_{\varepsilon}(H\rho; C_{\varepsilon}(\varepsilon))
\right|
\]

(4.5)
on \([0, T]\) for any \(\beta(J_s) = (\beta_1, \ldots, \beta_k) \in \mathcal{M}(J_s)\), where we put \(l_i = l(j_i, \alpha(j_i))\) \((i=1, \ldots, k)\), \(a_i = a\) and

\[
a_{i+1} = a + l_i + \cdots + l_i - \langle \kappa, \beta_1 + \cdots + \beta_i \rangle \quad (i=1, \ldots, k).
\]

Here, we note the following:

\[
\varepsilon \sum_{k=1}^{m} \left| \beta_1 - \beta_2 \right|^{\sigma} \leq C(\varepsilon) \left( |\beta_1 - \beta_2| + \cdots + |\beta_1 - \beta_k| \right) \left( \varepsilon c^2 + \varepsilon c^{2s} R \right)
\]

(4.6)

holds for any \(\varepsilon > 0\). Therefore, by (4.5), (4.6) and (2.7) we have

\[
\| P_{\tau} - P_{\tau - \varepsilon} \|_{C^s([0, T])} \leq C(\varepsilon) \varepsilon \sum_{k=1}^{m} \left( |\beta_1 - \beta_2| + \cdots + |\beta_1 - \beta_k| \right) \left( \varepsilon c^2 + \varepsilon c^{2s} R \right)
\]

(4.7)

for any \(\varepsilon > 0\) (by (2.8)). Hence, by the condition (I-2) in Proposition 1 we obtain

\[
\sum_{k=1}^{m} \sum_{\beta(J_s) \in \mathcal{M}(J_s)} \| P_{\tau - \varepsilon} P_{\tau - \varepsilon} \|_{C^s([0, T])} \leq C(\varepsilon) \sum_{\beta(J_s) \in \mathcal{M}(J_s)} \left( |\beta_1 - \beta_2| + \cdots + |\beta_1 - \beta_k| \right) \left( \varepsilon c^2 + \varepsilon c^{2s} R \right)
\]

(4.8)

Thus, the existence part of Proposition 2 is proved. The other part may be proved in the same way. Q.E.D.

\section{5. Proof of (II): Asymptotic expansions}

To have the part (II) in Theorems 1 and 2, it is sufficient to establish the \(\varepsilon^{(s)}\)-version of [5, Proposition 5] (see also [5, §§ 3 and 8]).

Let \(P\) be the operator in (1.1), and let us consider

\[
\begin{align*}
Pu &= f, \\
\varepsilon \sum_{k=1}^{m} \left| \beta_1 - \beta_2 \right|^{\sigma} &= 0 \quad \text{for } i=0, 1, \ldots, m-1,
\end{align*}
\]

(5.1)

where \(u = u(t, x)\) and \(f = f(t, x)\). Then, our \(\varepsilon^{(s)}\)-version of [5, Proposition 5] is given as follows.

**Proposition 3.** Let \(P\) be the operator in (1.1), let \(1 \leq s < \infty\), and let \(p \in N\). Assume that \(P\) satisfies (A\(_1\)), (B\(_r\)) and (D\(_p^{(1)}\)), and that \(s\) satisfies (C-1) or (C-2). Then, if \(f(t, x) \in C_{-c}(0, T], H^{s}(\mathbb{R}^n)\) satisfies \(\varepsilon \sum_{k=1}^{m} \left| \beta_1 - \beta_2 \right|^{\sigma} = 0 \) for \(i=0, 1, \ldots, p-1\) and \(t^p \| P_{\tau - \varepsilon} f(t) \|_{C^s([0, T])} \leq A\(\varepsilon\)\) uniformly on \((0, T]\) for some \(A>0\), then the equation (5.1) has a unique solution \(u(t, x) \in C_{-c}(0, T], H^{s}(\mathbb{R}^n)\) such that \(\varepsilon \sum_{k=1}^{m} \left| \beta_1 - \beta_2 \right|^{\sigma} = 0 \) for
\[ i=0, 1, \ldots, m+p-1 \text{ and that } t^B \| \mathcal{F}_{\theta}^a u(t) \| \in \mathcal{E}(i) \text{ uniformly on } (0, T) \text{ for some } B>0. \] In addition, if \( f(t, x) \) satisfies \( \text{supp}(f) \subset C_\alpha(T, K) \) for some compact subset \( K \) of \( \mathbb{R}^n \), \( u(t, x) \) also satisfies \( \text{supp}(u) \subset C_\alpha(T, K) \). (Here, \( C_\alpha(T, K) \) is defined by the case \( t_0 = T \) of \( (4.2) \).)

**Proof.** Let \( Q_\alpha(j, a) \) and \( L \) be as in \( (4.3) \). Then, to solve \( (5.1) \) we can use the method of successive approximations as follows: first we solve

\[
\begin{align*}
Lu_0 &= f, \\
\partial_t u_0 |_{t=0} &= 0 \\
Lu((j_1, \alpha_1)) &= Q_\alpha(j_1, \alpha_1) u_0, \\
\partial_t u((j_1, \alpha_1)) |_{t=0} &= 0 \\
&\vdots \\
Lu((j_1, \alpha_1), \ldots, (j_k, \alpha_k)) &= Q_\alpha(j_1, \alpha_1) \cdots Q_\alpha(j_k, \alpha_k) u((j_1, \alpha_1), \ldots, (j_k, \alpha_k)) \\
\partial_t u((j_1, \alpha_1), \ldots, (j_k, \alpha_k)) |_{t=0} &= 0
\end{align*}
\]

and then we show the convergence of the formal solution

\[ u = u_0 + \sum_{k=1}^{\infty} \sum_{(j_1, \alpha_1), \ldots, (j_k, \alpha_k) \in I} u((j_1, \alpha_1), \ldots, (j_k, \alpha_k)). \]

The proof of the convergence is as follows. Since \((D^\gamma)^- \) and \( t^B \| \mathcal{F}_{\theta}^a f(t) \| \in \mathcal{E}(i) \) (uniformly on \( (0, T) \)) are assumed, we have

\[
\begin{align*}
\| \mathcal{F}_{\theta}^a L \| , \| \mathcal{F}_{\theta}^a a_j, a \| &< B_0 \theta(\rho; C_\gamma) \\
\| \mathcal{F}_{\theta}^\infty f(t) \| &< t^{-\alpha} \theta(2\rho; C_\gamma)
\end{align*}
\]

for some \( B_0 > 0, C_\gamma \) and \( C_\gamma \) such that \( \theta(\rho; C_\gamma) \) satisfies \( (M_{\alpha}) \) with respect to \( C_\gamma \). Therefore, by the same argument as in \( (4.5) \) (see also \([5, 8] \)) we can obtain the following: there are \( C > 0, H > 0 \) and \( a > 0 \) such that for any \( J_k = ((j_1, \alpha_1), \ldots, (j_k, \alpha_k)) \in \mathcal{E}^k \) we have

\[
\| \mathcal{F}_{\theta}^a \mathcal{F}_{\theta}^{\infty} u(J_k)(t) \| < C^{k+1} t^{-\alpha} \left( \frac{T}{t} \right)^{k+1} T^{l_1 + \cdots + l_k - \sum_{i=1}^k \beta_1} \theta(\rho; C_\gamma)
\]

on \( (0, T) \) for any \( (\beta_1, \ldots, \beta_k) \in \mathbb{Z}_+^n \times \cdots \times \mathbb{Z}_+^n \) satisfying

\[
\begin{align*}
(0, \ldots, 0) &\leq (\beta_1, \ldots, \beta_k) \\
(\xi, \beta_1 + \cdots + \beta_k) &\leq (l_1 + \cdots + l_k)
\end{align*}
\]

where we put \( l_i = l(j_i, \alpha_i) \) for \( i = 1, \ldots, k \).
Hence, by an argument quite parallel to the proof of Proposition 2 we obtain

$$t^{n+1} + \sum_{k=1}^{\infty} \sum_{J \in \mathcal{J}^k} || P_{\phi} \circ P_{\gamma_{n-1}} u(J_k)(t) || \in \mathcal{E}(t)$$

uniformly on $(0, T]$. Thus, the existence part of Proposition 3 is proved. The other part may be proved in the same way. Q.E.D.

§ 6. Proof of Proposition 1

Note that "(I-2)⇒(I-3)" is trivial, that "(II-1)⇒(II-2)" was already proved in [5], and that "(II-2)⇒(II-3)" is trivial. Therefore, to have Proposition 1 it is sufficient to prove the following parts: "(I-1)⇒(I-2)", "(I-3)⇒(I-1)" and "(II-3)⇒(II-1)".

For $z \geq 0$, we write

$$z! = \Gamma(1+z),$$

where $\Gamma(z)$ is the gamma-function of Euler. Then, we have

**Lemma 5.** (1) For any $p, q > 0$ we have

$$\frac{1}{1+p+q} \leq \frac{(p+q)!}{p! q!} \leq 2^{p+q}.$$

(2) For any $p_1, p_2, \ldots, p_k > 0$ we have

$$1 \leq \frac{(1+p_1+p_2+\cdots+p_k)^{p_1+p_2+\cdots+p_k}}{(1+p_1+p_2+p_3)\cdots(1+p_1+p_2+\cdots+p_k)}\leq 2^{p_1+p_2+\cdots+p_k}.$$

(3) For any $0 < a < b$, there are $A_1 > 0$, $A_2 > 0$, $C_1 > 0$ and $C_2 > 0$ which satisfy the following: for any $p_1, p_2, \ldots, p_k > 0$ satisfying $a \leq p_i \leq b(i=1, 2, \ldots, k)$ we have

$$A_1 C_i \preceq \frac{(p_1+p_2+\cdots+p_k)!}{(1^p)(2^p)\cdots(k^p)} \preceq A_2 C_i.$$

In [5, § 4], we gave an interpretation of our irregularity condition. The following lemma is easily obtained by the proof of [5, Proposition 1]. Let $J, \ell(j, a)$ and $k=(k_1, \ldots, k_n)$ be as in § 2.

**Lemma 6.** Let $(j, a) \in J$, let $1 \leq s < \infty$ and let $\sigma_{j, a} (\geq 1)$ be as in (2.2). Then, we have the following results.

(1) $\sigma_{j, a} = 1$ is equivalent to the condition $l(j, a) \geq \langle k, a \rangle$.

(2) When $1 \leq s < \sigma_{j, a}((\sigma_{j, a} - 1)$, there is a $z \in \mathbb{R}^n$ such that $(0, \ldots, 0) \leq$
When $s = \sigma_{j, \alpha}(\sigma_{j, \alpha} - 1)$, there is a $z \in \mathbb{R}^n$ such that $(0, \ldots, 0) \leq z \leq \alpha$, $0 < |z| < \alpha$, $\langle \kappa, z \rangle = l(j, \alpha)$ and
\[
 s = \frac{m - j - |z|}{|\alpha| - |z|}.
\]

Moreover, we can find $\tau \in \mathcal{Z}_\alpha$ and $p \in \{1, \ldots, n\}$ such that $\kappa_{x(1)} \leq \kappa_{x(2)} \leq \cdots \leq \kappa_{x(n)}$, $z_{x(k)} = \alpha_{x(k)}$ for $1 \leq k < p$, $0 < z_{x(p)} \leq \alpha_{x(p)}$, and $z_{x(q)} = 0$ for $p < k \leq n$ (where $z = (z_1, \ldots, z_n)$ and $\alpha = (\alpha_1, \ldots, \alpha_n)$).

Here, we used the following notations for $a = (a_1, \ldots, a_n)$ and $b = (b_1, \ldots, b_n)$: $a \leq b$ means that $a_i \leq b_i$ holds for any $i$, $|a| = |a_1| + \cdots + |a_n|$ and $\langle a, b \rangle = a_1b_1 + \cdots + a_nb_n$.

To prove Proposition 1, we need further information on $z$. Let us give a refinement of Lemma 6. For $(j, \alpha) \in \mathcal{F}$, denote by $\mathcal{L}(j, \alpha)$ the set of all $z \in \mathbb{R}^n$ satisfying the following:

\[
\begin{aligned}
(0, \ldots, 0) & \leq z \leq \alpha, \\
|z| & < |\alpha|, \\
\langle \kappa, z \rangle & \leq l(j, \alpha).
\end{aligned}
\]

Let $\kappa = (\kappa_1, \ldots, \kappa_n)$ be the one in (A). Define $p \sim q$ by $\kappa_p \equiv \kappa_q$. Then, the $\sim$ defines an equivalence relation in $\{1, 2, \ldots, n\}$; therefore, we obtain $K_1, \ldots, K_n \in \{1, 2, \ldots, n\}$ satisfying the following: $K_1 \cup \cdots \cup K_n = \{1, 2, \ldots, n\}$, $K_i \cap K_j = \emptyset$ for $i \neq j$, and
\[
\begin{aligned}
\kappa_p &= \kappa_q & & \text{for } p, q \in K_i, \\
\kappa_p &\equiv \kappa_q & & \text{for } p \in K_i, q \in K_j, \text{ and } i \neq j.
\end{aligned}
\]

Under this notation, for $x = (x_1, \ldots, x_n)$ we write
\[
|x_i| = \sum_{p \in K_i} |x_p| \quad (i = 1, \ldots, n).
\]

Then, our refinement of Lemma 6 is given as follows. Let $S_\alpha(\alpha)$ be as in (2.5).

**Lemma 7.** Let $(j, \alpha) \in \mathcal{F}$, let $1 \leq s < \infty$ and let $\kappa = (\kappa_1, \ldots, \kappa_n)$ be the one in (A). Then, we have the following results.

(1) When $1 \leq s < \sigma_{j, \alpha}(\sigma_{j, \alpha} - 1)$, there is a $z \in \mathcal{L}(j, \alpha)$ such that
(6.2) \[ s < \frac{m-j-|z|}{|\alpha|-|z|} \]
and that \(0 < |z| < |\alpha|\) holds for some \(i \in \{1, \ldots, \nu\}\).

(2) When \(s = \sigma_{j, \alpha} / (\sigma_{j, \alpha} - 1)\) and \(l(j, \alpha) \in \mathcal{S}_\nu(\alpha)\), there is a \(z \in \mathcal{Z}(j, \alpha)\) such that
\[ s = \frac{m-j-|z|}{|\alpha|-|z|} \]
and that \(0 < |z| < |\alpha|\) holds for some \(i \in \{1, \ldots, \nu\}\).

(3) When \(s = \sigma_{j, \alpha} / (\sigma_{j, \alpha} - 1)\) and \(l(j, \alpha) \in \mathcal{S}_\nu(\alpha)\), there is a \(z \in \mathcal{Z}(j, \alpha)\) such that
\[ s = \frac{m-j-|z|}{|\alpha|-|z|} \]
and that \(|z| = 0\) or \(|z| = |\alpha|\) holds for any \(i \in \{1, \ldots, \nu\}\).

(4) When \(s > \sigma_{j, \alpha} / (\sigma_{j, \alpha} - 1)\), there are no \(z \in \mathcal{Z}(j, \alpha)\) such that
\[ s \leq \frac{m-j-|z|}{|\alpha|-|z|} \]

**Proof.** Note that \(\epsilon_k > 0\) \((1 \leq i \leq n)\) and \(l(j, \alpha) > 0\) are assumed (as in § 2). Put \(\alpha = (\alpha_1, \ldots, \alpha_n)\).

When \(1 \leq s < \sigma_{j, \alpha} / (\sigma_{j, \alpha} - 1)\), by (2) in Lemma 6 we can find a \(w = (w_1, \ldots, w_n) \in \mathbb{R}^n\) such that \((0, \ldots, 0) \leq w \leq \alpha\), \(0 < |w| < |\alpha|\), \(<\kappa, w> \leq l(j, \alpha)\) and
\[ s < \frac{m-j-|w|}{|\alpha|-|w|} \]
Then, we can obtain (1) in Lemma 7 by defining \(z = (z_1, \ldots, z_n) \in \mathbb{R}^n\) as follows: when \(w_k = 0\), we put \(z_k = 0\); when \(w_k > 0\), we choose \(z_k \in \mathbb{R}\) so that \(0 < z_k < w_k\) and that \(z_k\) is sufficiently close to \(w_k\). In fact, if \(z \in \mathbb{R}^n\) is as above, \(z \in \mathcal{Z}(j, \alpha)\) is clear, (6.2) is verified by the fact that
\[ \frac{m-j-|z|}{|\alpha|-|z|} \rightarrow \frac{m-j-|w|}{|\alpha|-|w|} \]
as \(|z| \rightarrow |w|\), and the last condition is verified as follows. Since \(|w| > 0\), we have \(w_k > 0\) for some \(k\) and therefore \(0 < z_k < w_k \leq \alpha_k\); hence by taking \(i \in \{1, \ldots, \nu\}\) such that \(k \in K_i\), we obtain \(0 < |z| < |\alpha|\). Thus, (1) is proved.

When \(s = \sigma_{j, \alpha} / (\sigma_{j, \alpha} - 1),\) by (3) in Lemma 6 we can find \(z = (z_1, \ldots, z_n) \in \mathbb{R}^n, \tau \in \mathbb{S}_n\) and \(p \in \{1, \ldots, n\}\) such that \((0, \ldots, 0) \leq z \leq \alpha\), \(0 < |z| < |\alpha|, \langle \kappa, z \rangle = l(j, \alpha)\),
\[ s = \frac{m - j - |z|}{|\alpha| - |z|}, \]

\[ \kappa_{r(1)} \leq \kappa_{r(2)} \leq \cdots \leq \kappa_{r(n)} \]

\[ z_{r(k)} = \alpha_{r(k)} \]

for \( 1 \leq k < p \), \( 0 < z_{r(p)} \leq \alpha_{r(p)} \), and \( z_{r(k)} = 0 \) for \( p < k \leq n \). Take \( i_0 \in \{1, \ldots, v\} \) such that \( \tau(p) \in K_{i_0} \). Then we have

\[ \begin{cases} 
0 < |z|_{i_0} < |\alpha|_{i_0}, \\
|z|_h = 0 \quad \text{or} \quad |z|_h = |\alpha|_h 
\end{cases} \quad \text{for } h \neq i_0. \]

Therefore, to have (2) and (3) in Lemma 7 it is sufficient to prove the following fact:

(6.3) \[ |z|_{i_0} < |\alpha|_{i_0}, \quad \text{when } \ell(j, \alpha) \in S_1(\alpha), \]

(6.4) \[ |z|_{i_0} = |\alpha|_{i_0}, \quad \text{when } \ell(j, \alpha) \in S_2(\alpha). \]

Note that we have the condition \( \ell(j, \alpha) < \langle \kappa, \alpha \rangle \), since \( s = s_{j, a} (\sigma_{j, a} - 1) < \infty \) and therefore \( \sigma_{j, a} > 1 \) (see (1) in Lemma 6).

Let us show (6.3). Assume that \( \ell(j, \alpha) \in S_2(\alpha) \). When \( 0 < z_{r(p)} < \alpha_{r(p)} \), (6.3) is trivial. Therefore, in the discussion below we may assume that \( z_{r(p)} = \alpha_{r(p)} \).

In this case, by the condition \( \ell(j, \alpha) < \langle \kappa, \alpha \rangle \) we have

(6.5) \[ \ell(j, \alpha) = \kappa_{r(1)} \alpha_{r(1)} + \cdots + \kappa_{r(p)} \alpha_{r(p)}, \]

Moreover, we can see the following: \( p \leq n - 1 \), \( \kappa_{r(p)} = \kappa_{r(p+1)} \) and \( \alpha_{r(k)} > 0 \) for some \( k \in \{p+1, \ldots, n\} \). In fact, these are verified as follows: if \( p = n \), (6.5) contradicts the condition \( \ell(j, \alpha) < \langle \kappa, \alpha \rangle \); if \( \kappa_{r(p)} < \kappa_{r(p+1)} \), (6.5) contradicts the condition \( \ell(j, \alpha) \in S_1(\alpha) \); if \( \alpha_{r(p+1)} = \cdots = \alpha_{r(n)} = 0, (6.5) \) contradicts the condition \( \ell(j, \alpha) < \langle \kappa, \alpha \rangle \). Hence, by putting

\[ q = \min \{k \in \{p+1, \ldots, n\}; \alpha_{r(k)} > 0\} \]

we have \( \alpha_{r(q)} > 0, \alpha_{r(k)} = 0 \) for \( p + 1 \leq k < q \) and

(6.6) \[ \ell(j, \alpha) = \kappa_{r(1)} \alpha_{r(1)} + \cdots + \kappa_{r(h)} \alpha_{r(h)} \]

for any \( h \in \{p, \ldots, q-1\} \). If \( \kappa_{r(h)} < \kappa_{r(h+1)} \) holds for some \( h \in \{p, \ldots, q-1\}, (6.6) \) contradicts the condition \( \ell(j, \alpha) \in S_1(\alpha) \); therefore we have \( \kappa_{r(p)} = \kappa_{r(p+1)} = \cdots = \kappa_{r(q)} \). This implies that \( \tau(k) \in K_{i_0} \) for \( k = p, p+1, \ldots, q \). Since \( z_{r(q)} = 0 \) and \( \alpha_{r(q)} > 0 \), we obtain

\[ |\alpha|_{i_0} - |z|_{i_0} \leq |\alpha|_{q} - |z|_{q} = \alpha_{r(q)} > 0 \]

which proves (6.3). As a consequence, we obtain (2) in Lemma 7.

Let us next show (6.4). Assume that \( \ell(j, \alpha) \in S_2(\alpha) \). Then, there are \( \nu \in S_n \) and \( q \in \{1, \ldots, n-1\} \) such that
Since \( K_1(1) \leq K_1(2) \leq \ldots \leq K_1(n) \) is assumed, by (6.8) we have \( \{\nu(1), \ldots, \nu(q)\} = \{\tau(1), \ldots, \tau(q)\} \) and therefore

(6.9) \[ l(j, \alpha) = k_{\tau(1)}(x_{1:1}) + \cdots + k_{\tau(q)}(x_{1:q}), \]

(6.10) \[ \{k_{\tau(1)}, \ldots, k_{\tau(q)}\} \subset \{k_{\tau(q+1)}, \ldots, k_{\tau(n)}\}. \]

On the other hand, by the condition \( l(j, \alpha) = \langle k, z \rangle \) and by the choice of \( z \) we have

(6.11) \[ l(j, \alpha) = k_{\tau(1)}(x_{1:1}) + \cdots + k_{\tau(p-1)}(x_{1:p-1}) + k_{\tau(p)}(z_{1:p}) \]

If \( q < p \), by (6.9) and (6.11) we have

\[ k_{\tau(q+1)}(x_{1:q+1}) + \cdots + k_{\tau(p-1)}(x_{1:p-1}) + k_{\tau(p)}(z_{1:p}) = 0, \]

and therefore we have \( z_{1:p} = 0 \); this contradicts the condition \( 0 < z_{1:p} \leq x_{1:p} \). Hence, we may assume that \( q \geq p \). In this case, by (6.9) and (6.11) we have

\[ k_{\tau(q)}(x_{1:p}) - k_{\tau(p)}(z_{1:p}) + k_{\tau(p+1)}(x_{1:p+1}) + \cdots + k_{\tau(q)}(x_{1:q}) = 0; \]

therefore, we have \( x_{1:p} = z_{1:p} \) and \( x_{1:k} = 0 \) (\( = z_{1:k} \)) for \( p < k \leq q \). Thus, by combining this with the known condition \( x_{1:q} < x_{1:q+1} \) (by (6.10)) we obtain

\[ z_{1:k} = x_{1:k} \quad \text{for any } \tau(k) \in K_1, \]

which proves (6.4). As a consequence, we obtain (3) in Lemma 7.

Lastly, let us prove (4) in Lemma 7 by showing the following: under the conditions \( s > \sigma(j, \alpha)/(\sigma(j, \alpha) - 1) \), \( z \in \mathcal{X}(j, \alpha) \) and

(6.12) \[ s \leq \frac{m - j - |z|}{|\alpha| - |z|}, \]

we can obtain a contradiction. Assume that \( s > \sigma(j, \alpha)/(\sigma(j, \alpha) - 1) \) and that \( z = (z_1, \ldots, z_n) \in \mathcal{X}(j, \alpha) \) satisfies (6.12). Then, we have

(6.13) \[ \sigma(j, \alpha)/(\sigma(j, \alpha) - 1) < \frac{m - j - |z|}{|\alpha| - |z|}. \]

Since \( \sigma(j, \alpha) > 1 \), by (6.13) we have

\[ \frac{m - j - |z|}{m - j - |\alpha|} < \min \left( \max_{r \in \mathbb{S}_n} M_{j, \alpha}(\tau, r) \right). \]

Therefore, for any \( \tau \in \mathbb{S}_n \) we can find a \( p \in \{1, \ldots, n\} \) such that
\[
\frac{m-j-|z|}{m-j-|\alpha|} < M_{j, \sigma}(\tau, p),
\]

and this is equivalent to

\[
(l(j, \alpha))^{\sum_{i=1}^{p} (\kappa_{(i)} - \kappa_{(p)}) \alpha_{(i)} + \kappa_{(p)} \sum_{i=1}^{n} z_{(i)}},
\]

On the other hand, by the condition \( \langle \kappa, z \rangle \leq l(j, \alpha) \) we have

\[
\sum_{i=1}^{n} \kappa_{(i)} z_{(i)} \leq l(j, \alpha).
\]

Hence, by (6.14) and (6.15) we obtain

\[
\sum_{i=1}^{p} (\kappa_{(p)} - \kappa_{(i)}) (\alpha_{(i)} - z_{(i)}) + \sum_{i=p+1}^{n} (\kappa_{(i)} - \kappa_{(p)}) z_{(i)} < 0.
\]

Thus, by choosing \( \tau \in \mathcal{S}_x \) so that \( \kappa_{(1)} \leq \kappa_{(2)} \leq \cdots \leq \kappa_{(n)} \) we can obtain a contradiction from (6.16). Q.E.D.

**Corollary to (3) in Lemma 7.** Assume that \( s = \sigma_{j, \sigma}/(\sigma_{j, \sigma} - 1) \) and \( l(j, \alpha) \in S_{e}(\alpha) \) hold. Let \( \nu \in \mathcal{S}_x \) and \( q \in \{1, \cdots, n-1\} \) be such that

\[
\{\nu_{(q)} \leq \nu_{(1)} \leq \cdots \leq \nu_{(p)} \} = \{\nu_{(1)} \leq \cdots \leq \nu_{(q)} \}, \quad \{\nu_{(p+1)} \leq \cdots \leq \nu_{(n)} \} = \{\nu_{(p+1)} \leq \cdots \leq \nu_{(n)} \}.
\]

Then, \( z^* = (z_{1}^*, \cdots, z_{n}^*) \in Z^\alpha_+ \) defined by

\[
z_{k}^* = \begin{cases} 
\alpha_{(k)}, & \text{for } 1 \leq k \leq q, \\
0, & \text{for } q+1 \leq k \leq n
\end{cases}
\]

satisfies \( z^* \in \mathcal{Z}(j, \alpha) \), \( \langle \kappa, z^* \rangle = l(j, \alpha) \) and

\[
s = \frac{m-j-|z^*|}{|\alpha|-|z^*|}.
\]

**Proof.** In the proof of (3) in Lemma 7 we already have the following:

\[
\{\nu(1), \cdots, \nu(q)\} = \{\tau(1), \cdots, \tau(q)\}, \quad \{\nu(q+1), \cdots, \nu(n)\} = \{\tau(q+1), \cdots, \tau(n)\}.
\]

\( z_{(k)} = \alpha_{(k)} \) for \( 1 \leq k \leq q \), and \( z_{(k)} = 0 \) for \( q+1 \leq k \leq n \). Hence, we obtain \( z = z^* \).

Q.E.D.

Note that the condition "(C-1) or (C-2)" is equivalent to the following:

for any \( (j, \alpha) \in \mathcal{J} \) we have the condition (i) or (ii) given below:

( i ) \( 1 < s < \sigma_{j, \sigma}/(\sigma_{j, \sigma} - 1) \),

(ii) \( s = \sigma_{j, \sigma}/(\sigma_{j, \sigma} - 1) \) and \( l(j, \alpha) \in S_{e}(\alpha) \).
Hence, by (1) and (2) in Lemma 7 we can obtain the following result which yields a proof of "(I-1)⇒(I-2)" in Proposition 1.

**Lemma 8.** Let $1 < s < \infty$, let $l(j, \alpha) > 0 ((j, \alpha) \in \mathcal{J})$ and let $\kappa_i > 0 (1 \leq i \leq n)$. Assume that (C-1) or (C-2) holds. Then, there are $c_i > 0, c_i > 0$ and $c_i > 0$ such that for any $J_k=((j_1, \alpha_{(1)}), \ldots, (j_k, \alpha_{(k)})) \in \mathcal{J}$ we can find a $\beta(J_k)=(\beta_{(1)}, \ldots, \beta_{(k)}) \in \mathcal{M}(J_k)$ which satisfies the following conditions (i)~(iii):

(i) \[ \frac{c_1 + (m - j_1 - |\beta_{(1)}|) + \cdots + (m - j_k - |\beta_{(k)}|)}{|\alpha_{(1)}| - |\beta_{(1)}|} + \cdots + \frac{|\alpha_{(k)}| - |\beta_{(k)}|}{|\alpha_{(k)}| - |\beta_{(k)}|} \geq s. \]

(ii) \[ \frac{c_2 + (m - j_1 - |\beta_{(1)}|) + \cdots + (m - j_k - |\beta_{(k)}|)}{|\alpha_{(1)}| - |\beta_{(1)}|} + \cdots + \frac{|\alpha_{(k)}| - |\beta_{(k)}|}{|\alpha_{(k)}| - |\beta_{(k)}|} \geq s. \]

(iii) There are $I_1, \ldots, I_p \subseteq \{1, \ldots, k\}$ and $N_1, \ldots, N_p \in \mathbb{Z}$, such that $p \leq 2\nu$, $I_r \cap I_q = \emptyset$ for $r \neq q$, $I_1 \cup \cdots \cup I_p = \{1, \ldots, k\}$, $\{N_i; i \in I_r\} = \{0, 1, \ldots, |I_r| - 1\}$ for $r = 1, \ldots, p$, and

\[ l(j_1, \alpha_{(1)}) + \cdots + l(j_k, \alpha_{(k)}) - \langle \kappa, \beta_{(1)} + \cdots + \beta_{(k)} \rangle \geq c_3 N_i \]

for $i = 1, \ldots, k$, where $\nu$ is the one in (6.1) and $|I_r|$ means the number of elements in $I_r$.

**Proof.** Since (C-1) or (C-2) holds, by (1) and (2) in Lemma 7 we have the following: for any $(j, \alpha) \in \mathcal{J}$ we can find a $z(j, \alpha) \in \mathcal{Z}(j, \alpha)$ such that

\[ s \leq \frac{m - j - |z(j, \alpha)|}{|\alpha| - |z(j, \alpha)|} \]

and that $0 < |z(j, \alpha)| < |\alpha|_t$ holds for some $i \in \{1, \ldots, \nu\}$. We take $z(j, \alpha) \in \mathcal{Z}(j, \alpha)$ (for any $(j, \alpha) \in \mathcal{J}$) as above and fix them hereafter.

Take any $J_k=((j_1, \alpha_{(1)}), \ldots, (j_k, \alpha_{(k)})) \in \mathcal{J}$ and fix it. Put $\alpha_{(s)}=(\alpha_{(1), s}, \ldots, \alpha_{(k), s})$, $l(j_1, \alpha_{(1)})=l_1$ and $z(j_1, \alpha_{(1)})=z(1, \alpha_{(1)}, \ldots, z(1, \alpha_{(1)})$ $(i=1, \ldots, k)$. Then, we have

\[ 0 \leq z(1) \leq \alpha_{(1)}, \quad |z(1)| < |\alpha_{(1)}|, \quad \langle \kappa, z(1) \rangle \leq l_1, \]

and the following condition

\[ 0 < |z(s)|_p < |\alpha_{(s)}|_p \quad \text{for some } p_i \in \{1, \ldots, \nu\} \quad (i=1, \ldots, k). \]

Put $\alpha_{(0)}=(1, \ldots, 1)$ and $\alpha_{(0)}=(0, \ldots, 0)$. Let $K_1, \ldots, K_k$ be the ones used in (6.1). Choose $r_1, \ldots, r_\nu \in \{0, 1, \ldots, k\}$ such that

\[ |\alpha_{(r_1)}|_j + \cdots + |\alpha_{(r_\nu)}|_j \leq |z(\nu)|_j + \cdots + |z(\nu)|_j \]

\[ < |\alpha_{(r_1)}|_j + |\alpha_{(r_\nu)}|_j + \cdots + |\alpha_{(r_\nu)}|_j \]

and the following condition

\[ (6.22) \]

(j=1, \ldots, \nu), define $\beta_{(i)}=(\beta_{(i), 1}, \ldots, \beta_{(i), n}) \in \mathbb{Z}_n^+ (i=1, \ldots, k)$ by
if $h \in K_j$ and $r_j + 1 \leq i \leq k$, 
if otherwise.

Then, we can see that this $\beta(J_k)$ satisfies the conditions $\beta(J_k) \in \mathcal{M}(J_k)$ and (i) $\sim$ (iii) (as proved below). Note that by (6.23) we have

$$\beta_{(i)} = \begin{cases} \alpha_{(i)}, & \text{if } h \in K_j \text{ and } r_j + 1 \leq i \leq k, \\ 0, & \text{if otherwise} \end{cases}$$

$(i = 1, \ldots, k$ and $h = 1, \ldots, n)$, and put $\beta(J_k) = (\beta_{(1)}, \ldots, \beta_{(k)})$. Then, we see that $\beta(J_k) \in \mathcal{M}(J_k)$ and (i) $\sim$ (iii) (as proved below). Note that by (6.23) we have

$$\beta_{(i)} = \begin{cases} 0, & \text{when } 1 \leq i \leq r_j, \\ |\alpha_{(i)}|, & \text{when } r_j + 1 \leq i \leq k \end{cases}$$

$(i = 1, \ldots, k$ and $j = 1, \ldots, v)$.

Let us first show the condition $\beta(J_k) \in \mathcal{M}(J_k)$. Since $(0, \ldots, 0) \leq \beta_{(i)} \leq \alpha_{(i)}$ $(i = 1, \ldots, k)$ is clear from (6.23), what we must prove is the following:

$$\langle \varepsilon, \beta_{(i)} + \cdots + \beta_{(k)} \rangle \leq l_i + \cdots + l_i \quad (i = 1, \ldots, k).$$

Recall that by the definition of $\{K_1, \ldots, K_v\}$ we can choose $\kappa(K_j) > 0$ $(j = 1, \ldots, v)$ such that $\kappa(K_j) = \kappa_p$ for any $p \in K_j$. Define $\psi_{(i)}$ $(i = 1, \ldots, k$ and $j = 1, \ldots, v)$ by

$$\psi_{(i)} = \begin{cases} \kappa(K_j)(|z_{(i)}| + \cdots + |z_{(i)}|), & \text{when } 1 \leq i \leq r_j, \\ \kappa(K_j)(|z_{(i)}| + \cdots + |z_{(i)}|) \\ - \kappa(K_j)(|\alpha_{(i)}| + \cdots + |\alpha_{(i)}|), & \text{when } r_j + 1 \leq i \leq k, \end{cases}$$

Then, by (6.24), (6.26) and the condition $\langle \varepsilon, z_{(i)} \rangle \leq l_i \quad (i = 1, \ldots, k)$ we have

$$l_i + \cdots + l_i - \langle \varepsilon, \beta_{(i)} + \cdots + \beta_{(i)} \rangle$$

$$= l_i + \cdots + l_i - \langle \varepsilon, z_{(i)} + \cdots + z_{(i)} \rangle + \psi_{(i)} + \cdots + \psi_{(i)}$$

$$\geq \psi_{(i)} + \cdots + \psi_{(i)}$$

$(i = 1, \ldots, k)$. Moreover, we can see the following:

$$\psi_{(i)} \geq 0 \quad (i = 1, \ldots, k$ and $j = 1, \ldots, v)$$

In fact, this is verified as follows: when $1 \leq i \leq r_j$, $\psi_{(i)} \geq 0$ is clear from (6.26); when $r_j + 1 \leq i \leq k$, $\psi_{(i)} \geq 0$ is verified by (6.22) and

$$\psi_{(i)} = \kappa(K_j)(|z_{(i)}| + \cdots + |z_{(i)}|) - (|\alpha_{(i)}| + \cdots + |\alpha_{(i)}|)$$

$$+ \kappa(K_j)(|\alpha_{(i)}| - |z_{(i)}| + \cdots + |\alpha_{(i)}| - |z_{(i)}|).$$

Hence, by (6.27) and (6.28) we obtain (6.25). Thus, the condition $\beta(J_k) \in \mathcal{M}(J_k)$ is proved.

Let us next show the conditions (i) and (ii). Note that by (6.22), (6.23) and (6.24) we have
Therefore, by putting
\[ c_i = \min \{ |\alpha| - |z(j, \alpha)|; (j, \alpha) \in \mathcal{F} \} \quad (>0) \]
we can obtain (i). Note that by (6.20) we have
\[ (6.29) \quad s(\alpha(j) - |\beta(j)|) \leq (m - j_i - |\beta(j)|) + (s - 1)(|z(j) - |\beta(j)|) \]
\[ (i=1, \ldots, k). \] Therefore, by (6.22), (6.23), (6.24) and (6.29) we have
\[ s [(\alpha(j) - |\beta(j)|) + \cdots + (|\alpha_i - |\beta_i|)] \]
\[ \leq (m - j_i - |\beta_i|) + \cdots + (m - j_k - |\beta_k|) \]
\[ + (s - 1) \sum_{j=1}^{k} [(|z(j) + \cdots + z(k)| - (|\alpha_{(j+1)}| + \cdots + |\alpha(k)|)] \]
\[ \leq (m - j_i - |\beta_i|) + \cdots + (m - j_k - |\beta_k|) + (s - 1) \sum_{j=1}^{k} |\alpha_{(j)}|. \]
Hence, by putting \( c_i = (s - 1)m\nu > 0 \) we can obtain (ii).

Let us lastly show the condition (iii). Put \( \mathcal{F}_{(\mu)}, \mathcal{F}^{(\mu)}, \varepsilon_{(\mu)}, \varepsilon^{(\mu)} \) \((\mu = 1, \ldots, v)\) as follows:
\[ \mathcal{F}_{(\mu)} = \{ (j, \alpha) \in \mathcal{F} ; |z(j, \alpha)| > 0 \}, \]
\[ \mathcal{F}^{(\mu)} = \{ (j, \alpha) \in \mathcal{F} ; |\alpha| > |z(j, \alpha)| \}, \]
\[ \varepsilon_{(\mu)} = \min \{ |z(j, \alpha)| ; (j, \alpha) \in \mathcal{F}_{(\mu)} \} \quad (>0), \]
\[ \varepsilon^{(\mu)} = \min \{ |\alpha| - |z(j, \alpha)| ; (j, \alpha) \in \mathcal{F}^{(\mu)} \} \quad (>0). \]

Put \( I = \{1, \ldots, k\} \), and put \( J_{(\mu)}, J^{(\mu)} \) \((j=1, \ldots, \nu)\) as follows:
\[ J_{(\mu)} = \{ i \in I ; |z(\alpha)| > 0 \quad \text{and} \quad 1 \leq i \leq r_j \}, \]
\[ J^{(\mu)} = \{ i \in I ; |\alpha_{(i)}| - |z(\alpha)| > 0 \quad \text{and} \quad r_j + 1 \leq i \leq k \}. \]
Then we can see the following conditions (iii-1)\hypertarget{iii-2}{\ conspiratory} (iii-3):
\[ (iii-1) \quad (J_{(\mu)} \cup J^{(\mu)}) \cup \cdots \cup (J_{(v)} \cup J^{(v)}) = I (= \{1, \ldots, k\}). \]
\[ (iii-2) \quad \text{If } J_{(\mu)} = \{ h_1, h_2, \ldots, h_p \} \text{ and } h_1 < h_2 < \cdots < h_p, \text{ we have} \]
\[ \psi_{(\mu)}^{(\delta_1)} \geq \varepsilon^{(\mu)} e^{(\delta_1)} \]
\[ \text{for } d = 1, 2, \ldots, p. \]
\[ (iii-3) \quad \text{If } J^{(\mu)} = \{ h_1, h_2, \ldots, h_p \} \text{ and } h_1 < h_2 < \cdots < h_p, \text{ we have} \]

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\[ \psi_j^{(d)} \geq \kappa(K_j) \varepsilon_j^{(p-d)} \quad \text{for } d = 1, 2, \ldots, p. \]

In fact, these are verified as follows. (iii-1) is clear from (6.21). (iii-2) is verified by the following: by (6.26) we have

\[ \psi_j^{(d)} = \kappa(K_j)(|z_{(1)}|_j + \cdots + |z_{(d)}|_j) \]
\[ \geq \kappa(K_j)(|z_{(1)}|_j + |z_{(2)}|_j + \cdots + |z_{(d)}|_j) \]
\[ \geq \kappa(K_j) \varepsilon_j^{(d)} d. \]

(iii-3) is verified by the following: by (6.22) and (6.26) we have

\[ \psi_j^{(d)} = \kappa(K_j)(|z_{(1)}|_j + \cdots + |z_{(d)}|_j - |\alpha_{(1)}|_j + \cdots + |\alpha_{(d)}|_j) \]
\[ + \kappa(K_j)(|z_{(1)}|_j - |z_{(2)}|_j) + \cdots + (|\alpha_{(d)}|_j - |z_{(d)}|_j) \]
\[ \geq \kappa(K_j)(|\alpha_{(1)}|_j - |z_{(1)}|_j) + (|\alpha_{(2)}|_j - |z_{(2)}|_j) \]
\[ + \cdots + (|\alpha_{(d)}|_j - |z_{(d)}|_j) \]
\[ \geq \kappa(K_j) \varepsilon_j^{(d)} (p-d). \]

By using (iii-1) - (iii-3), we can prove (iii) as follows. Put

\[ c_3 = \min \{ \kappa(K_j) \varepsilon_j^{(d)} ; j = 1, \ldots, v \} > 0, \]

choose a disjoint family \( \{ I_{(j)} \}, I_{(j)} ; j = 1, \ldots, v \} \) such that \( I_{(j)} \subset I_{(j)} \), \( I_{(j)} \subset J_{(j)} \)

and \( I_{(1)} \cup I_{(2)} \cup \cdots \cup I_{(v)} \) is a subset of \( J_{(1)} \), and define \( N_1, \ldots, N_v \in \mathbb{Z}_+ \) as follows: if \( I_{(j)} = \{ i_1, i_2, \ldots, i_q \} \) and \( i_1 < i_2 < \cdots < i_q \), we put \( N_{i_1} = (d-1) (d=1, 2, \ldots, q) \); if \( I_{(j)} = \{ i_1, i_2, \ldots, i_q \} \) and \( i_1 < i_q < \cdots < i_q \), we put \( N_{i_q} = (q-d) (d=1, 2, \ldots, q) \). Then, by (6.27), (6.28), (iii-2) and (iii-3) we obtain

\[ l_1 + \cdots + l_q - \langle \kappa, \beta_{(1)} + \cdots + \beta_{(v)} \rangle \geq \psi_{(1)}^{(i)} + \cdots + \psi_{(v)}^{(i)} \]
\[ \geq c_3 N_i \]

\( i = 1, \ldots, k \). Thus, (iii) is proved.

Q.E.D.

As a consequence of Lemma 8, we obtain

**Proof of (I-1) \Rightarrow (I-2)** in Proposition 1. Assume that (C-1) or (C-2) holds. Let \( c_1 > 0, c_2 > 0 \) and \( c_3 > 0 \) be as in Lemma 8. Take \( J_x = ((j_1, \alpha_{(1)}), \ldots, (j_x, \alpha_{(x)})) \in \mathcal{F}^x \), let \( \beta(J_x) = (\beta_{(1)}, \ldots, \beta_{(x)}) \in \mathcal{M}(J_x) \) and \( N_1, \ldots, N_x \in \mathbb{Z}_+ \) be the ones chosen for \( J_x \) in Lemma 8, and let \( \alpha_i (i = 1, \ldots, k) \) be as in (2.7). Then, by (6.19) we have

\[ \frac{1}{a_i} \leq \frac{1}{a + c_2 N_{i+1}} \leq \frac{1}{1 + N_{i+1}} \min \{ a, c_2 \} \]

\( (i = 1, \ldots, k) \)

(where \( N_0 = 0 \)), and by (6.30), (iii) in Lemma 8 and (3) in Lemma 5 we have
for some $A_1 > 0$ and $A_2 > 0$ (depending only on $a, c_2, m$ and $\nu$). Therefore, by (6.31), (ii) in Lemma 8 and (1) in Lemma 5 we have

$$
\frac{1}{(a_1)^{m-j_1-|\beta_{(1)}|} \times \cdots \times (a_k)^{m-j_k-|\beta_{(k)}|}}
\leq A^{k+1}_2
\left[ (m-j_1-|\beta_{(1)}|) + \cdots + (m-j_k-|\beta_{(k)}|) \right] !
$$

for some $B > 0$ (which is independent of $J_k, \beta(J_k)$ and $k$). Hence, if we assume that $0 < R \leq 1$, by (2.7), (6.32) and (i) in Lemma 8 we obtain

$$
\frac{1}{(a_1)^{m-j_1-|\beta_{(1)}|} \times \cdots \times (a_k)^{m-j_k-|\beta_{(k)}|}} \leq B^{k+1}
$$

for some $B > 0$ (which is independent of $J_k, \beta(J_k)$ and $k$). This leads us to the condition (I-2) in Proposition 1. Q. E. D.

In the above proof, the following fact is essential: the conditions (i), (ii) in Lemma 8 and (6.32) imply (I-2) (and hence (I-3)). The following lemmas 9 and 10 assert that the converse relation holds.

**Lemma 9.** There is a $d > 0$ which satisfies the following conditions if $J_k=((j_1, \alpha_{(1)}), \ldots, (j_k, \alpha_{(k)})) \in \mathcal{J}^k$ satisfies $0 < l(j_i, \alpha_{(i)}) < \langle \kappa, \alpha_{(i)} \rangle$ (i = 1, ..., k),

then we have

$$
(|\alpha_{(1)}| - |\beta_{(1)}|) + \cdots + (|\alpha_{(k)}| - |\beta_{(k)}|) \geq dk
$$

for any $\beta(J_k)=(\beta_{(1)}, \ldots, \beta_{(k)}) \in \mathcal{H}(J_k)$.

**Proof.** Put

$$
\mathcal{J}_0 = \{ (j, \alpha) \in \mathcal{J} ; 0 < l(j, \alpha) < \langle \kappa, \alpha \rangle \},
$$

$$
c = \min\{ \langle \kappa, \alpha \rangle - l(j, \alpha) ; (j, \alpha) \in \mathcal{J}_0 \} > 0, \quad \kappa^* = \max\{ \kappa_1, \ldots, \kappa_n \}.
$$

Take any $\beta(J_k)=(\beta_{(1)}, \ldots, \beta_{(k)}) \in \mathcal{H}(J_k)$. Then, by (2.6), (6.34) and the condition $(j_i, \alpha_{(i)}) \in \mathcal{J}_0$ (i = 1, ..., k) we have

$$
\kappa^*[(|\alpha_{(1)}| - |\beta_{(1)}|) + \cdots + (|\alpha_{(k)}| - |\beta_{(k)}|)] \geq \langle \kappa, \alpha_{(1)} - \beta_{(1)} \rangle + \cdots + \langle \kappa, \alpha_{(k)} - \beta_{(k)} \rangle
$$

$$
= \langle \kappa, \alpha_{(1)} \rangle - l(j_1, \alpha_{(1)}) + \cdots + \langle \kappa, \alpha_{(k)} \rangle - l(j_k, \alpha_{(k)})
$$

$$
+ l(j_1, \alpha_{(1)}) + \cdots + l(j_k, \alpha_{(k)}) - \langle \kappa, \beta_{(1)} + \cdots + \beta_{(k)} \rangle \geq ck.
$$
Hence, by putting \( d = c/\kappa^* \) we obtain (6.33).

**Q.E.D.**

**LEMA 10.** Assume the condition (I-3) in Proposition 1. Then, there are \( A > 0 \) and \( B > 0 \) such that for any \( J_k = ((j_1, \alpha_{(1)}), \ldots, (j_k, \alpha_{(k)})) \in \mathcal{J} \) we can find \( \beta(J_k) = (\beta_1, \ldots, \beta_k) \in \mathbb{R}^{(j_k)} \) and \( c(J_k) \geq 0 \) which satisfy the following conditions (i) \( \sim \) (iii).

\[
(i) \quad \frac{\sum \left( \frac{1}{|\beta_{(i)}|} \right) \cdots \left( \frac{1}{|\beta_{(k)}|} \right) \prod \left( \frac{1}{|\alpha_{(i)}|} \right) \cdots \left( \frac{1}{|\alpha_{(k)}|} \right)}{(a_i)^{m-j_i-|\beta_{(i)}|}} \leq AB^x,
\]

where \( a_i = a (i > 0) \) and

\[
(ii) \quad \frac{1}{a_i} \leq \frac{1}{a + l(i-1)} \leq \frac{1}{i \cdot \max[a, l]} \quad (i = 1, \ldots, k),
\]

\[
(iii) \quad \max_{J_k} c(J_k) = o(k) \quad (as \quad k \to \infty).
\]

**PROOF.** Assume the condition (I-3) in Proposition 1. Then,

\[
M_x((\beta(J_k)), C, R_5, T_0) < +\infty
\]

for some \( \beta(J_k) = (\beta_1, \ldots, \beta_k) \in \mathbb{R}^{(j_k)} \) (where \( J_k = ((j_1, \alpha_{(1)}), \ldots, (j_k, \alpha_{(k)})) \in \mathcal{J} \), \( k = 1, 2, \ldots, C > 0, R_5 > 0 \) and \( T_0 > 0 \). Therefore, by (2.7) and (2.8) we have

\[
(6.35) \quad \frac{\sum \left( \frac{1}{|\beta_{(i)}|} \right) \cdots \left( \frac{1}{|\beta_{(k)}|} \right) \prod \left( \frac{1}{|\alpha_{(i)}|} \right) \cdots \left( \frac{1}{|\alpha_{(k)}|} \right)}{(a_i)^{m-j_i-|\beta_{(i)}|}} \leq AB^x
\]

for some \( A > 0 \) and \( B > 0 \) (which are independent of \( J_k, \beta(J_k) \) and \( k \)); this is (i). Put \( l = \max \{l(j, \alpha); (j, \alpha) \in \mathcal{J} \} \). Since

\[
(6.36) \quad \frac{1}{a_i} \leq \frac{1}{a + l(i-1)} \leq \frac{1}{i \cdot \max[a, l]} \quad (i = 1, \ldots, k),
\]

by (6.35) and (6.36) we have

\[
(6.37) \quad \frac{\sum \left( \frac{1}{|\beta_{(i)}|} \right) \cdots \left( \frac{1}{|\beta_{(k)}|} \right) \prod \left( \frac{1}{|\alpha_{(i)}|} \right) \cdots \left( \frac{1}{|\alpha_{(k)}|} \right)}{(1)^{m-j_1-|\beta_{(1)}|}} \cdots \times \frac{1}{(k)^{m-j_k-|\beta_{(k)}|}} \leq AB^x
\]

for \( B_1 = B(\max \{1, a, l \})^x \). Hence, by applying (3) in Lemma 5 and Stirling’s formula to (6.37) we obtain

\[
(6.38) \quad \frac{\sum \left( \frac{1}{|\beta_{(i)}|} \right) \cdots \left( \frac{1}{|\beta_{(k)}|} \right) \prod \left( \frac{1}{|\alpha_{(i)}|} \right) \cdots \left( \frac{1}{|\alpha_{(k)}|} \right)}{(m-j_1-|\beta_{(1)}|)} \cdots \times \frac{1}{(k)^{m-j_k-|\beta_{(k)}|}} \leq CH^x
\]

for some \( C > 0 \) and \( H > 0 \) (which are independent of \( J_k, \beta(J_k) \) and \( k \)).

Here, we put \( b(J_k) \) and \( c(J_k) \) as follows:
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Then, we have $c(J_k)\geq b(J_k)$ and therefore by (6.39) we obtain (ii). Moreover, we can see the following: there are $C_1>0$ and $C_2>0$ such that

$$c(J_k)\leq C_0 H_k (\text{which implies (6.40)})$$

for $C_0=\max\{1, 2C\}$ and $H_0=\max\{1, 2^{n_1} H\}$.

Hence, by using (6.40) we can verify the condition (iii) as follows. Assume that $\{c(J_k)\}$ does not satisfy (iii). Then, there are $\varepsilon>0, k_1<k_2<\cdots$ and $J(i) \in \mathcal{F}^1$ $(i=1, 2, \cdots)$ such that $c(J(i))>\varepsilon k_i (i=1, 2, \cdots)$ and $k_i \to \infty$ (as $i \to \infty$); therefore, by (6.40) we have

$$\log \Gamma(1+c(J_k)) \leq C_1 k + C_2 \quad (i=1, 2, \cdots).$$

This contradicts the condition $\varepsilon \to \infty$ (as $i \to \infty$). Thus, (iii) is also verified.

Q.E.D.

**Corollary to Lemmas 9 and 10.** Assume the condition (I-3) in Proposition 1. Let $(j, \alpha) \in \mathcal{F}$. Then we have the following results.

1. If $0<\ell(j, \alpha)<\langle \kappa, \alpha \rangle$ holds, there are $d>0$, $A>0$, $B>0$, $z_{(k)} \in \mathcal{Z}(j, \alpha)$ $(k=1, 2, \cdots)$ and $c_k \geq 0$ $(k=1, 2, \cdots)$ which satisfy the following conditions
   
   (i) $|\alpha|-|z_{(k)}| \geq d \quad (k=1, 2, \cdots)$.
   
   (ii) For any $k$, we can find a $(\beta_0, \cdots, \beta_{(k)}) \in \mathbb{Z}^+ \times \cdots \times \mathbb{Z}^+$ such that
   
   $$0, \cdots, 0 \leq \beta_{(i)} \leq \alpha \quad (i=1, \cdots, k), \quad \beta_0 + \cdots + \beta_{(k)} = k z_{(k)}, \quad \langle \kappa, \beta_0 + \cdots + \beta_{(k)} \rangle \leq il(j, \alpha) \quad (i=1, \cdots, k)$$

   and

   $$\frac{[k(|\alpha|-|z_{(k)}|)]^i}{(a_1)^{m-j_1-|z_{(k)}|} \cdots (a_k)^{m-j_k-|z_{(k)}|}} \leq A B^k,$$

   where $a_i = a$ and $a_{i+1} = a + il(j, \alpha) - \langle \kappa, \beta_0 + \cdots + \beta_{(k)} \rangle (i=1, \cdots, k)$.

2. $s \leq c_k + \frac{m-j-|z_{(k)}|}{|\alpha|-|z_{(k)}|} \quad (k=1, 2, \cdots)$. 


(ii) $c_k = o(1)$ (as $k \to \infty$).

Moreover, if $s = \sigma_j, a/(\sigma_j, n-1)$ and $l(j, \alpha) \in S_\alpha(\alpha)$ hold, the $z(k)$ ($k = 1, 2, \ldots$) in (1) satisfy also the following condition:

$$z(k) \to z^* \quad \text{(as } k \to \infty\text{)},$$

where $z^*$ is the one in Corollary to (3) in Lemma 7.

**Proof.** Assume the condition (I-3) in Proposition 1. Let $(j, \alpha) \in \mathcal{S}$ and assume that $0 < l(j, \alpha) < \langle \kappa, \alpha \rangle$ holds. Let $d > 0$ be as in Lemma 9, let $A > 0$, $B > 0$ be as in Lemma 10, put $J_k^0 = ((j, \alpha), \ldots, (j, \alpha)) \in \mathcal{S}$, let $\beta(J_k^0) = (\beta_1, \ldots, \beta_q)$ and $(\beta_{(k)}(q)) \in \mathcal{M}(J_k^0)$ be the one chosen for $J_k^0$ in Lemma 10, and define $z(k), c_k$ by

$$z(k) = \frac{1}{k} (\beta_1 + \cdots + \beta_q),$$

$$c_k = \frac{\alpha(J_k^0)}{dk},$$

($k = 1, 2, \ldots$). Then (i) is verified by Lemma 9 and (ii)−(iv) are verified by Lemma 10. Hence, we obtain (1).

Next, let us prove (2). Assume that $s = \sigma_j, a/(\sigma_j, n-1)$ and $l(j, \alpha) \in S_\alpha(\alpha)$ hold. Let $\nu \in \mathcal{S}_\alpha$, $q \in \{1, \ldots, n-1\}$ and $z^* = (z^*_1, \ldots, z^*_q) \in Z^\nu$ be as in Corollary to (3) in Lemma 7. Then, we have $z^*_i = \alpha_i$ for $1 \leq i \leq q$, $z^*_i = 0$ for $q+1 \leq i \leq n$, $\langle \kappa, z^* \rangle = \langle \kappa, z^*_i \rangle$, $(0, \ldots, 0) \leq z^*_i \leq \alpha$, $(1/a) \kappa_i \leq 1$ for $1 \leq i \leq q$, $(1/b) \kappa_i \geq 1$ for $q+1 \leq i \leq n$, $0 < a < b$ and

$$\begin{align*}
|z^*| - |z(k)| &= (\alpha_1 - z(k, s(1))) + \cdots + (\alpha_q - z(k, s(q))) \\
&\leq \frac{1}{a} [\kappa_1 (\alpha_1 - z(k, s(1))) + \cdots + \kappa_q (\alpha_q - z(k, s(q)))] \\
&= \frac{1}{a} \langle \alpha, z^* \rangle \\
&= \frac{1}{a} \langle \kappa, z^* \rangle - \langle \kappa, z(k) \rangle \\
&\leq \frac{1}{a} \langle \kappa, z^* \rangle - \langle \kappa, z(k) \rangle \\
&\leq \left( \frac{b}{a} - 1 \right) (z(k, s(q+1))) + \cdots + z(k, s(n)).
\end{align*}$$

(6.42)

On the other hand, we know from (6.18), (iii), (iv) and the condition $|z^*| \geq |z(k)|$ (by (6.42)) that
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Therefore, by (6.42) and (6.43) we have

\[ (6.44) \]

\[ |z^* - z_{(k)}| = \frac{(|\alpha| - |z^*|)(|\alpha| - |z_{(k)}|)}{(m - j - |\alpha|)} \left( \frac{m - j - |z^*|}{|\alpha| - |z^*|} - \frac{m - j - |z_{(k)}|}{|\alpha| - |z_{(k)}|} \right) \]

and by (6.43) and (6.44) we have

\[ (6.45) \]

\[ = o(1) \quad (as \ k \to \infty). \]

Hence, we obtain \( z_{(k)} \to z^* \) (as \( k \to \infty \)), since (6.44) and (6.45) imply

\[ z_{(k), (q+1)} + \cdots + z_{(k), (n)} = o(1) \quad (as \ k \to \infty), \]

and by (6.43) and (6.44) we have

\[ (6.45) \]

\[ = |z^* - z_{(k)}| + (z_{(k), (q+1)} + \cdots + z_{(k), (n)}) \]

\[ = o(1) \quad (as \ k \to \infty). \]

As a consequence of the above corollary and the condition (4) in Lemma 7, we obtain

PROOF OF "(I-3) \Rightarrow (I-1)" IN PROPOSITION 1. Assume the condition (I-3) in Proposition 1. Put (C-3) and (C-4) as follows:

\[ (c-3) \]

\[ (c-4) \]

Then, to have (14) in Proposition 1 it is sufficient to prove the following: if we assume that (C-3) or (C-4) holds, we can obtain a contradiction. Let us show this.

First, let us discuss the case (C-4). Assume that (C-4) holds; that is, \( s > \sigma_j, a/(\sigma_j, a - 1) \) holds for some \((j, a) \in J\). Take such a \((j, a) \in J\) and fix it. Then, we have \( \sigma_j, a > 1 \) and hence \( 0 < l(j, a) < \langle k, a \rangle \) (by (1) in Lemma 6). Therefore, by (1) in Corollary to Lemmas 9 and 10 we can find \( z_{(k)} \in \mathcal{Z}(j, a) \) \((k = 1, 2, \cdots)\) such that

\[ s \leq o(1) + \frac{m - j - |z_{(k)}|}{|\alpha| - |z_{(k)}|} \]

(as \( k \to \infty \)). Thus, by choosing \( s_0 > 0 \) so that \( s > s_0 > \sigma_j, n/(\sigma_j, a - 1) \) and by
taking $k$ sufficiently large we obtain
\[
\sigma_{j, a}/(a_j, a - 1) s < \frac{|m - j - |z_{(k)}|}{|\alpha - |z_{(k)}|},
\]
which contradicts the condition (4) in Lemma 7.

Next, let us discuss the case (C-3). Assume that (C-3) holds; that is, $s = \sigma_{j, a}/(a_j, a - 1)$ and $l(j, \alpha) \in S_2(\alpha)$ hold for some $(j, \alpha) \in \mathcal{F}$. Take such a $(j, \alpha) \in \mathcal{F}$ and fix it. Let $A > 0$, $B > 0$ and $z_{(k)} \in \mathcal{Z}(j, \alpha)$ ($k = 1, 2, \ldots$) be as in Corollary to Lemmas 9 and 10, and let $z^* \in Z^*_+$ be the one in Corollary to (3) in Lemma 7. Then, we can see that
\[
\frac{k(|\alpha - |z_{(k)}|)!^r}{(\alpha + k)|\langle \kappa, z^* - z_{(k)} \rangle|^{k(m - j - |z_{(k)}|)}} \leq A B^k
\]
holds for any $k$. In fact, this is verified as follows. Note that $\langle \kappa, z^* \rangle = l(j, \alpha)$, and let $(\beta_{(1)}, \ldots, \beta_{(k)}) \in Z^*_+ \times \cdots \times Z^*_+$ be the one in (ii) in Corollary to Lemmas 9 and 10. Then, we have $0 \leq \beta_{(i)} \leq \alpha$ ($i = 1, \ldots, k$), $k z_{(k)} := (\beta_{(1)} + \cdots + \beta_{(k)})$,
\[
(m - j - |\beta_{(1)}|) + \cdots + (m - j - |\beta_{(k)}|) = k(m - j - |z_{(k)}|),
\]
and by (6.17) we have
\[
0 \leq i l(j, \alpha) - \langle \kappa, \beta_{(1)} + \cdots + \beta_{(i)} \rangle = \sum_{h=1}^{i} \langle \kappa, z^* - \beta_{(h)} \rangle
\]
\[
\leq \sum_{h=1}^{i} |\langle \kappa, z^* - \beta_{(h)} \rangle|
\]
\[
\leq \sum_{h=1}^{k} |\langle \kappa, z^* - \beta_{(h)} \rangle| = k|\langle \kappa, z^* - z_{(k)} \rangle|
\]
($i = 1, \ldots, k$). Hence, by (6.41), (6.47) and (6.48) we obtain (6.46).

Further, we can also see that
\[
(m - j - |z_{(k)}|) s \leq \frac{|m - j - |z_{(k)}|}{|\alpha - |z_{(k)}|} \leq \frac{s}{\tau} \quad \text{for any } k \in N.
\]
In fact, if otherwise, we can find a $k \in N$ such that
\[
(m - j - |z_{(k)}|) > s = \sigma_{j, a}/(a_j, a - 1),
\]
which contradicts the condition (4) in Lemma 7.

Hence, by (6.46), (6.49) and (2) in Corollary to Lemmas 9 and 10 we have the following: for any $\varepsilon > 0$ there is a $C_\varepsilon > 0$ such that
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(6.50) \[ [k(|\alpha| - |z_{(k)}|)]! \leq AB^k(\varepsilon k + C_v)^{k(\alpha - |z_{(k)}|)} \leq AB^k(\varepsilon k + C_v)^{k(|\alpha| - |z_{(k)}|)} \]

holds for any \( k \). In addition, by applying Stirling’s formula to the left hand side of (6.50) and by using the condition \( d \leq (|\alpha| - |z_{(k)}|) \leq m (k=1, 2, \ldots) \) we have

(6.51) \[ k \leq H(\varepsilon k + C_v) \quad (k=1, 2, \ldots) \]

for some \( H > 0 \) (which is independent of \( \varepsilon \) and \( k \)). Thus, by choosing \( \varepsilon > 0 \) so that \( \varepsilon < 1/H \) and by letting \( k \to \infty \) in (6.51) we can obtain a contradiction. Q.E.D.

Lastly, let us show “(II-3)⇒(II-1)” in Proposition 1 by the argument similar to that above. Note that the difference between (II-3) and (I-3) lies in whether we can take \( R > 0 \) sufficiently large or not, and that \( T_0 \) depends on \( R \) in (II-3).

PROOF OF “(II-3)⇒(II-1)” IN PROPOSITION 1. Our purpose is to prove the following: if we assume that (II-3) in Proposition 1 and \( s \geq \sigma(\sigma - 1) \) hold, we can obtain a contradiction.

Assume the condition (II-3) in Proposition 1. Let \( R > 1 \) be sufficiently large. Then, we have

\[ M_s(\hat{\beta}(J_k), C, R, T_0) < +\infty \]

for some \( T_0 > 0 \) and \( \hat{\beta}(J_k) = (\hat{\beta}_{(1)}, \ldots, \hat{\beta}_{(k)}) \in \mathcal{J}(J_k) \) (where \( J_k = ((j_1, \alpha_{(1)}), \ldots, (j_k, \alpha_{(k)})) \in \mathcal{J}^k, k = 1, 2, \ldots \)). Therefore, by (2.7), (2.8) and (6.36) we have

(6.52) \[ \frac{[(|\alpha_{(1)}| - |\hat{\beta}_{(1)}|) + \cdots + (|\alpha_{(k)}| - |\beta_{(k)}|)]!}{(1)^{m-1-|\beta_{(1)}|} \times \cdots \times (k)^{m-2-|\beta_{(k)}|}} \leq AB^k \left( 1 \right) \frac{1}{R} \times \left( 1 \right) \frac{1}{T_0} \times \prod (\sigma_{(1)} - \hat{\beta}_{(1)}) \cdots \times \prod (\sigma_{(k)} - \hat{\beta}_{(k)}) \]

for some \( A > 0 \) and \( B > 0 \) (which are independent of \( J_k, \hat{\beta}(J_k), k, R \) and \( T_0 \)).

Assume that \( s \geq \sigma(\sigma - 1) \) holds, that is, \( s \geq \sigma_{j,s}(\sigma_{j,s} - 1) \) holds for some \( (j, \alpha) \in \mathcal{J} \). Take such a \( (j, \alpha) \in \mathcal{J} \) and fix it. Then, by the same argument as in Corollary to Lemmas 9 and 10 we can find \( d > 0, A_i > 0, B_i > 0 \) (independent of \( k, R \) and \( T_0 \), \( z_{(k)} \in \mathcal{R}(j, \alpha) \) \( k=1, 2, \ldots \)) and \( c_k \geq 0 (k=1, 2, \ldots) \) which satisfy the following conditions (i)−(iv):

( i ) \(|\alpha| - |z_{(k)}| \geq d \quad (k=1, 2, \ldots)\).
By using these conditions, we can obtain a contradiction as follows.

When \( s > \sigma_{j, a}(\sigma_{j, a} - 1) \), by choosing \( s_0 > 0 \) so that \( s > s_0 > \sigma_{j, a}(\sigma_{j, a} - 1) \) and by taking \( k \) sufficiently large we have

\[
\sigma_{j, a}(\sigma_{j, a} - 1) < s < \frac{m - j - |z_{(k)}|}{|\alpha| - |z_{(k)}|}
\]

(by (iii) and (iv)), which contradicts the condition (4) in Lemma 7.

When \( s = \sigma_{j, a}(\sigma_{j, a} - 1) \) and when

\[
s < \frac{m - j - |z_{(k)}|}{|\alpha| - |z_{(k)}|}
\]

for some \( k \in \mathbb{N} \), we can find \( s_0 > 0 \) such that

\[
\sigma_{j, a}(\sigma_{j, a} - 1) = s < s_0 < \frac{m - j - |z_{(k)}|}{|\alpha| - |z_{(k)}|}
\]

which contradicts the condition (4) in Lemma 7.

When \( s = \sigma_{j, a}(\sigma_{j, a} - 1) \) and when

\[
(6.53) \quad \frac{m - j - |z_{(k)}|}{|\alpha| - |z_{(k)}|} \leq s
\]

for any \( k \in \mathbb{N} \), by (i), (ii), (6.53) and Stirling’s formula we have

\[
(6.54) \quad 1 \leq H \left( \frac{1}{R} \right)^4 \left( \frac{1}{T_0} \right)^{(j, a) - \langle \alpha, z_{(k)} \rangle} (k = 1, 2, \ldots)
\]

for some \( H > 0 \) (which is independent of \( k, R \) and \( T_0 \)). Since \( R \) is sufficiently large, we may assume that \( H/R^d < 1 \). Therefore, to have a contradiction from (6.54) it is sufficient to prove the following: there is a subsequence \( \{z_{(k_i)}\} \) of \( \{z_{(k)}\} \) such that

\[
\langle k, z_{(k_i)} \rangle \overset{\longrightarrow}{\longrightarrow} i(j, a) \quad (\text{as } k_i \to \infty).
\]

Let us show this now. Since \( \{z_{(k)}\} \) is a bounded sequence in \( \mathbb{R}^n \), we can take a convergent subsequence \( \{z_{(k_i)}\} \) of \( \{z_{(k)}\} \). Then, the limit \( z_{\ast} \in \mathbb{R}^n \) of \( \{z_{(k_i)}\} \) satisfies \( z_{\ast} \in \mathcal{P}(j, \alpha) \) (by (i)),

\[
(\text{ii}) \quad \frac{[k(|\alpha| - |z_{(k)}|)]!}{[k(m - j - |z_{(k)}|)]!} \leq A_i B_i \left( \frac{1}{R} \right)^k \left( \frac{1}{T_0} \right)^{(j, a) - \langle \alpha, z_{(k)} \rangle}
\]

\[
(\text{iii}) \quad s \leq c_k \left( \frac{m - j - |z_{(k)}|}{|\alpha| - |z_{(k)}|} \right) (k = 1, 2, \ldots).
\]

\[
(\text{iv}) \quad c_k = o(1) \quad (\text{as } k \to \infty).
\]
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\[ s = \frac{m - j - |z_*|}{|\alpha| - |z|} \]

(by (6.53), (iii) and (iv)), and \( \langle \kappa, z \rangle \leq l(j, \alpha) \). Furthermore, we can see the following: if \( \langle \kappa, z \rangle < l(j, \alpha) \) holds, we can find a \( w \in \mathcal{X}(j, \alpha) \) satisfying \( |z_*| < |w| \) and this yields

\[ a_j \left( a_{j, a} - s \right) = s = \frac{m - j - |z_*|}{|\alpha| - |z|} < \frac{m - j - |w|}{|\alpha| - |w|} \]

which contradicts the condition (4) in Lemma 7. Hence, we obtain \( \langle \kappa, z_* \rangle = l(j, \alpha) \) and therefore

\[ \langle \kappa, z_{(k_t)} \rangle \to \langle \kappa, z_* \rangle = l(j, \alpha) \quad (\text{as } k_t \to \infty). \]

Q.E.D.

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