Nonintegrability of harmonic functions in a domain

By Noriaki SUZUKI

(Received February 9, 1989)

§1. Let D be a domain in the Euclidean space R^n with $n \geq 2$. We define a function δ_D in D by

$$\delta_D(x) = \text{dist}(x, \partial D),$$

and for $p > 0$ we put

$$\alpha(p) = \begin{cases} 1 & \text{if } 0 < p \leq (n-2)/(n-1) \\ (n-1)(1-p) & \text{if } (n-2)/(n-1) < p \leq 1 \\ 0 & \text{if } p > 1. \end{cases}$$

In contrast to [5], our concern in this paper is the nonintegrability of harmonic functions. We state our result in the following form:

THEOREM A. *Let D be a bounded domain in R^n with the exterior sphere property. Assume in addition that there is a countable closed set F in ∂D such that D has the locally uniform exterior sphere property on $\partial D \setminus F$ and that D satisfies the Lipschitz condition at every point in $\partial D \setminus F$. If a non-negative harmonic function h in D satisfies*

$$(1) \quad \int_D \frac{h(x)^p}{\delta_D(x)^{1+\alpha(p)}} dx < \infty$$

with some $p > 0$, then $h \equiv 0$ in D .

Here, we recall some definitions; we say that D has the exterior sphere property if for each boundary points $\xi \in \partial D$, there exists a ball B_ξ such that $B_\xi \subset \bar{D}^c$ and $\bar{B}_\xi \cap \partial D = \{\xi\}$. Further, for an open set E in ∂D , if the radius of B_ξ can be taken uniformly on any compact set in E , we say that D has the locally uniform exterior sphere property on E ; in case $E = \partial D$ we say that D has the uniform exterior sphere property. Again, let ξ be a boundary point of D . We say that D satisfies the Lipschitz condition at ξ if there is a neighbourhood N of ξ such that $D \cap N$ is a Lipschitz domain.

The proof of Theorem A is carried out in Section 5 by combining three theorems below. Indeed, each result is interesting in its own right. The

first result, which is discussed in Section 2, concerns the nonintegrability of positive superharmonic functions in arbitrary domains. On the other hand, in Section 3, we deal with the nonintegrability of subharmonic functions in $C^{1,1}$ -domains. Finally, in Section 4, we study the non-tangential positiveness of positive harmonic functions in certain Lipschitz domains. In particular, as the simplest case, we obtain the following result:

Let h be a non-negative harmonic function in the unit ball B in $\mathbf{R}^n$. If $\liminf_{r \rightarrow 1} h(r\xi) = 0$ for every boundary point $\xi \in \partial B$, then h vanishes identically.

§2. Nonintegrability of superharmonic functions

Throughout this paper we denote by $B(x, r)$ the open ball of radius $r > 0$ centred at $x \in \mathbf{R}^n$ and by G_D the Green function of a domain D . We use $c(p, q, \dots)$ to denote positive constants depending only on $p, q, \dots$ not necessarily the same on any two occurrences.

We have the following theorem.

THEOREM 1. *Let D be a bounded domain in $\mathbf{R}^n$. If a non-negative superharmonic function u in D satisfies*

$$(2) \quad \int_D \frac{u(x)}{\delta_D(x)^2} dx < \infty$$

then $u \equiv 0$ in D .

PROOF. We consider

$$v(y) = \int_D G_D(y, x) \delta_D(x)^{-2} dx.$$

For $y \in D$, put $B = B(y, \delta_D(y))$ and $B' = B(y, \delta_D(y)/2)$. Since $G_D(x, y) \geq G_{B'}(x, y) \geq c(n)|x-y|^{-n+2}$ for $x \in B'$, we have

$$v(y) \geq c(n) \delta_D(y)^{-2} \int_{B'} |x-y|^{-n+2} dx \geq c(n) > 0.$$

This implies $v \equiv \infty$; for, otherwise v is a potential which contradicts the fact that v has a positive harmonic minorant. Hence, for any potential $p =$

$\int G_D(\cdot, y) d\nu(y)$ with $\nu \neq 0$, we also see

$$\int_D p(x) \delta_D(x)^{-2} dx = \int_D \left(\int_D G_D(x, y) \delta_D(y)^{-2} dy \right) d\nu(y) = \infty.$$

Since every positive superharmonic function has a non-zero potential as its

minorant, the assertion of Theorem 1 follows as an easy consequence.

3. Nonintegrability of subharmonic functions

For $p > 0$ we put $\beta(p) = \max \{(n-1)(1-p), 0\}$. We have the following theorem.

THEOREM 2. *Let D be a bounded $C^{1,1}$ -domain in $\mathbb{R}^n$ and $p > 0$. If a non-negative subharmonic function s in D satisfies*

$$(3) \quad \int_D \frac{s(x)^p}{\delta_D(x)^{1+\beta(p)}} dx < \infty$$

then $s \equiv 0$ in D .

Furthermore, if the absolute value $|s|$ of a subharmonic function s in D satisfies (3) instead of s for some $0 < p < (n-1)/n$, then $s \equiv 0$.

PROOF. We first quote some properties of $C^{1,1}$ -domains which are shown in [8, Theorems 2.3, 2.4, 2.5] (see also [6, Lemmas 5, 6]). Let x_0 be a fixed point in D . For $r > 0$, we set

$$\Omega_r = \{x \in D; G_D(x, x_0) > r\}.$$

Then there are positive numbers r_0 and $c_0 = c(D, x_0)$ such that for any $0 < r < r_0$,

$$(a) \quad c_0^{-1}r \leq \delta_D(x) \leq c_0r \quad \text{for any } x \in \partial\Omega_r;$$

(b) Ω_r is a smooth domain and satisfies

$$c_0^{-1}dx \leq d\omega_r^0 dr \leq c_0 dx \quad \text{on } D \setminus \Omega_{r_0},$$

where ω_r^0 is the harmonic measure on $\partial\Omega_r$ at x_0 .

Now let $s \geq 0$ be a subharmonic function in D satisfying (3) and suppose $0 < p \leq 1$. For $x \in D$, let $B = B(x, \delta_D(x))$. Then

$$s(x)^p \leq c(n, p) \delta_D(x)^{-n} \int_B s(y)^p dy.$$

This inequality is proved in the same manner as in [1, p. 172]. Remarking $\delta_D(y) \leq 2\delta_D(x)$ for $y \in B$, we have

$$s(x)^p \leq c(n, p) \delta_D(x)^{(1-n)p} \int_D s(y)^p \delta_D(y)^{-1-\beta(p)} dy$$

and hence $s^{p-1} \geq c(n, p, s) \delta_D^{\beta(p)}$ holds. Therefore by (a) and (b),

$$\infty > \int_{D \setminus \Omega_{r_0}} s(x)^p \delta_D(x)^{-1-\beta(p)} dx$$

$$\begin{aligned}
&\geq c(D, x_0, p, s) \int_0^{r_0} \int_{\partial \Omega_r} s(r\xi)^p r^{-1-\beta(p)} d\omega_r^0(\xi) dr \quad (x=r\xi) \\
&\geq c(D, x_0, p, s) \int_0^{r_0} \left(\int_{\partial \Omega_r} s(r\xi)^p d\omega_r^0(\xi) \right) r^{-1-\beta(p)} (\min_{\xi \in \partial \Omega_r} s(r\xi)^{p-1}) dr \\
&\geq c(D, x_0, p, s) s(x_0) \int_0^{r_0} r^{-1-\beta(p)} r^{\beta(p)} dr,
\end{aligned}$$

which shows $s(x_0)=0$. Since this holds for all $x_0 \in D$ we have $s \equiv 0$.

The case $p > 1$ follows from the case $p=1$, since s^p is also subharmonic.

Next let s be a subharmonic function in D satisfying

$$\int_D |s(x)|^p \delta_D(x)^{-1-\beta(p)} dx < \infty.$$

Applying the previous argument to the positive part s^+ of s , we see $s^+ \equiv 0$, because s^+ is also subharmonic. Thus $-s$ is a non-negative superharmonic function in D . Suppose $s \neq 0$. Since a $C^{1,1}$ -domain is associated with a ball in the sense of [7], by [7, Théorème B] we have $-s \geq c\delta_D$ with some $c > 0$. However, if $0 < p < n/(n-1)$, then $\beta(p) \geq p$, so that we have

$$\int_D |s(x)|^p \delta_D(x)^{-1-\beta(p)} dx \geq c^p \int_D \delta_D(x)^{p-1-\beta(p)} dx = \infty,$$

which contradicts our assumption. Hence $s \equiv 0$ and the proof is completed.

Before concluding this section, we make two remarks. First, the constant $\beta(p)$ in Theorem 2 can not be replaced by any smaller number.

REMARK 1. Let $B=B(0, 1)$ be the unit ball in $\mathbb{R}^n$ and $p > 0$. For any $\varepsilon > 0$, there is a non-negative non-zero subharmonic function s on B satisfying

$$\int_B s(x)^p \delta_B(x)^{\varepsilon-1-\beta(p)} dx < \infty.$$

In fact, we may consider a constant function in the case $p \geq 1$. In the case $0 < p < 1$, let $\xi_0 = (1, 0, \dots, 0) \in \mathbb{R}^n$. For $0 < \theta < \pi/2$, we choose $r_\theta > 0$ and $c(\theta) > 0$ such that for the truncated circular cone

$$\Gamma(\theta) = \{x \in B; (\xi_0 - x, \xi_0) > |x - \xi_0| \cos \theta\} \cap B(\xi_0, r_\theta)$$

we have $\delta_B(x) \geq c(\theta)|x - \xi_0|$ whenever $x \in \Gamma(\theta)$. Let h_θ be a kernel function at ξ_0 in $\Gamma(\theta)$ (i.e., h_θ is a positive harmonic function in $\Gamma(\theta)$ vanishing continuously on $\partial\Gamma(\theta) \setminus \{\xi_0\}$). Then

$$h_\theta(x) \leq c(\theta) |x - \xi_0|^{1-n-\gamma(\theta)}$$

with $\gamma(\theta) > 0$, which satisfies $\gamma(\theta) \rightarrow 0$ as $\theta \rightarrow \pi/2$ (cf. [5, Lemma 3]). Choose θ such that $p\gamma(\theta) < \varepsilon$. Then the function s in B defined by

$$s(x) = \begin{cases} h_\theta(x) & x \in \Gamma(\theta) \\ 0 & x \in B \setminus \Gamma(\theta) \end{cases}$$

is subharmonic and satisfies

$$\int_B s(x)^p \delta_B(x)^{\varepsilon-1-\beta(p)} dx \leq c(\theta, p) \int_{\Gamma(\theta)} |x - \xi_0|^{(1-n-\gamma(\theta))p} |x - \xi_0|^{\varepsilon-1-\beta(p)} dx < \infty.$$

In the following remark, let $\beta'(p) = \min \{\beta(p), p\}$.

REMARK 2. If a non-negative harmonic function h on a $C^{1,1}$ -domain D satisfies

$$\int_D h(x)^p \delta_D(x)^{-1-\beta'(p)} dx < \infty$$

for some $p > 0$, then $h \equiv 0$ holds.

In fact, this follows from Theorem 2 (in the case $p \geq (n-1)/n$) and from the argument as in the last part of the proof of Theorem 2 (in the case $0 < p < (n-1)/n$).

Furthermore, by the direct computation, we see that the Poisson kernel of $B(O, 1)$, that is,

$$h(x) = \frac{1 - |x|^2}{|\xi_0 - x|^n} \quad \xi_0 = (1, 0, \dots, 0)$$

is an example proving the sharpness of the constant $\beta'(p)$ for $p \leq 1$ (for $p > 1$, we may consider the constant function as before).

§ 4. Non-tangential positiveness of harmonic functions

In this section D is assumed to be a Lipschitz domain in $\mathbb{R}^n$. Let $\xi \in \partial D$. By a cone at ξ we mean an open truncated circular cone with vertex at ξ . We say that a line segment T in D is a non-tangential (abbreviated to n.t.) segment at ξ if there is a cone Γ at ξ in D such that its axis of symmetry is T . A cone Γ at ξ is also called an n.t. cone in D if there is a cone Γ' such that $\bar{\Gamma} \setminus \{\xi\} \subset \Gamma' \subset D$.

Let h be a non-negative harmonic function in D . We say that h is non-tangentially positive at $\xi \in \partial D$, if there is an n.t. segment T at ξ such that

$$\liminf_{\substack{x \rightarrow \xi \\ x \in T}} h(x) > 0.$$

Then by the Harnack inequality, h has the positive infimum on any n.t. cone at ξ . Hence this positivity is independent of the choice of n.t. segment at ξ , since D is a Lipschitz domain.

We denote by $P_D(x, \xi)$ the Martin (Poisson) kernel on $D \times \partial D$ (normalized at some point $x_0 \in D$) (cf. [3]). We have the following result.

THEOREM 3. *Let D be a bounded Lipschitz domain in $\mathbb{R}^n$ with the uniform exterior sphere property. Let h be a non-negative harmonic function in D and let μ_h be the non-negative Borel measure on ∂D which represents h , that is,*

$$h = \int_{\partial D} P_D(\cdot, \xi) d\mu_h(\xi).$$

Then h is non-tangentially positive μ_h -a.e., namely if we put $A_h = \{\xi \in \partial D; h \text{ is non-tangentially positive at } \xi\}$, then A_h is Borel measurable and $\mu_h(\partial D \setminus A_h) = 0$.

For the proof, we prepare the following lemma.

LEMMA 1. *Let D and Ω be two Lipschitz domains with $x_0 \in D \subset \Omega$. Suppose that $\xi \in \partial D \cap \partial \Omega$ and let T be an n.t. segment at ξ in D . If $u = P_D(\cdot, \xi)$ and $v = P_\Omega(\cdot, \xi)$ satisfy $u(x_0) = v(x_0) = 1$, then*

$$(4) \quad \liminf_{\substack{x \rightarrow \xi \\ x \in T}} \frac{u(x)}{v(x)} \geq 1.$$

Furthermore, for any n.t. cone Γ at ξ in D , we have $u(x) \geq c(\Gamma)v(x)$ for $x \in \Gamma$ with some constant $c(\Gamma) > 0$.

PROOF. We follow the argument in the proof of [3, Lemma (3.7)]. Suppose that (4) does not hold. Then there exist a constant $\varepsilon > 0$ and a sequence $\{x_n\}$ in T such that $\lim_{n \rightarrow \infty} x_n = \xi$ and $u(x_n) < (1 - 2\varepsilon)v(x_n)$ for $n \geq 1$. By the Harnack inequality we can take $r_0 > 0$ satisfying $B := \bigcup_{n=1}^{\infty} B(x_n, r_0 |x_n - \xi|) \subset D$ and

$$u(x) < (1 - \varepsilon)v(x) \quad \text{for } x \in B.$$

From [3, Lemma (3.6)] it follows that

$$\begin{aligned} 1 &= u(x_0) = R_u^{B, D}(x_0) \\ &:= \inf \{s(x_0); \{s \geq 0 \text{ superharmonic in } D \text{ with } s \geq u \text{ on } B\} \\ &\leq \inf \{\tilde{s}(x_0); \tilde{s} \geq 0 \text{ superharmonic in } \Omega \text{ with } \tilde{s} \geq (1 - \varepsilon)v \text{ on } B\} \\ &:= R_{(1-\varepsilon)v}^{B, \Omega}(x_0) \leq (1 - \varepsilon)v(x_0) = 1 - \varepsilon. \end{aligned}$$

This contradiction proves (4). The latter assertion of the lemma follows immediately by the Harnack inequality.

PROOF OF THEOREM 3. First we shall show that $A_h = \emptyset$ implies $h \equiv 0$. By the uniform exterior sphere property of D , there exists $r_0 > 0$ such that for each $\xi \in \partial D$ there corresponds a point ξ' such that $|\xi - \xi'| = r_0$ and $B(\xi', r_0) \subset \bar{D}^c$. Take $r_1 > 0$ sufficiently large so that

$$\Omega_\xi := B(\xi', r_1) \setminus \overline{B(\xi', r_0)}$$

contains D for any $\xi \in \partial D$.

Let η be an arbitrary fixed point of ∂D . Since D is a Lipschitz domain, there exist an n.t. segment T_0 at η in D and a constant $r_2 > 0$ such that for any $\xi \in \partial D \cap B(\eta, r_2)$,

$$T_\xi = \{x - \eta + \xi; x \in T_0\}$$

is an n.t. segment at ξ . Denote $\xi(r) = re_0 + \xi$ ($\subset D$), where e_0 is the unit vector parallel to T_0 . Since $\{\Omega_\xi\}$ are mutually congruent C^∞ -domains, we see ([8, Theorem 2.3])

$$P_{\Omega_\xi}(\xi(r), \xi) \geq c(D, \eta)r^{1-n}$$

for any $0 < r < r_2$ and for any $\xi \in \partial D \cap B(\eta, r)$ (P_{Ω_ξ} is also normalized at $x_0 \in D \subset \Omega_\xi$). Hence by Lemma 1 and by the Harnack inequality, we have

$$P_D(\eta(r), \xi) \geq c(D, \eta)P_{\Omega_\xi}(\xi(r), \xi) \geq c(D, \eta)r^{1-n}$$

for any $\xi \in \partial D \cap B(\eta, r/M)$, where M is a positive constant depending only on D . Therefore

$$\begin{aligned} h(\eta(r)) &\geq \int_{\partial D \cap B(\eta, r/M)} P_D(\eta(r), \xi) d\mu_h(\xi) \\ &\geq c(D, \eta)r^{1-n} \mu_h(B(\eta, r/M)). \end{aligned}$$

Denoting by σ the surface measure on ∂D , we have $\sigma(B(\eta, r)) \geq c(D)r^{n-1}$ and hence

$$h(\eta(r))\sigma(B(\eta, r/5M)) \geq c(D, \eta)\mu_h(B(\eta, r/M)).$$

Let $\varepsilon > 0$. By our assumption on h , $\liminf_{r \rightarrow 0} h(\eta(r)) = 0$ and hence there is $r_h > 0$ satisfying $\varepsilon\sigma(B(\eta, r_h/5)) \geq \mu_h(B(\eta, r_h))$. Since $\{B(\eta, r_h/5); \eta \in \partial D\}$ is an open covering of ∂D , we can choose finite points $\eta_1, \eta_2, \dots, \eta_m$ in ∂D such that

$$\partial D = \bigcup_{i=1}^m B(\eta_i, r_{\eta_i})$$

and $\{B(\eta_i, r_{\eta_i}/5)\}_{i=1}^m$ are mutually disjoint. Thus we have

$$\mu_h(\partial D) \leq \left(\sum_{i=1}^m \mu_h(B(\eta_i, r_{\eta_i}))\right) \leq \varepsilon \sum_{i=1}^m \sigma(B(\eta_i, r_{\eta_i}/5)) \leq \varepsilon \sigma(\partial D).$$

Since $\varepsilon > 0$ is arbitrary, $\mu_h = 0$ and hence $h \equiv 0$.

Next we consider the case $A_h \neq \emptyset$. We see that A_h is a Borel subset of ∂D , since h is a continuous function in D and since we can choose n.t. segments T_ξ which vary continuously in $\xi \in \partial D$ (note the argument in the second paragraph of this proof). Suppose that $\mu_h(\partial D \setminus A_h) > 0$. Since μ_h is regular, there exists an open set U in ∂D such that $U \supset A_h$ and $\mu_h(\partial D \setminus U) > 0$. Letting $\tilde{\mu}$ be the restriction of μ_h to $\partial D \setminus U$ and letting

$$\tilde{h} = \int_{\partial D} P_D(\cdot, \xi) d\tilde{\mu}(\xi),$$

we see $A_{\tilde{h}} \subset A_h \subset U$ and $A_{\tilde{h}} \neq \emptyset$, because $\tilde{h} \neq 0$. On the other hand, since $\text{supp}(\tilde{\mu}) \subset \partial D \setminus U$, $\tilde{h}$ vanishes continuously on U , which implies $A_{\tilde{h}} \subset \partial D \setminus U$. This contradiction proves $\mu_h(\partial D \setminus A_h) = 0$ and the proof is completed.

§ 5. Proof of Theorem A

The case $0 < p \leq (n-2)/(n-1)$: The result follows from Theorem 1, since h^p is superharmonic in D .

The case $(n-2)/(n-1) < p \leq 1$: We denote by $\mathcal{A}$ the Martin boundary of D and by $K(x, \lambda)$ the Martin kernel on $D \times \mathcal{A}$. The set of all minimal boundary points of $\mathcal{A}$ is denoted by $\mathcal{A}_1$. Then there exists a non-negative Borel measure μ on $\mathcal{A}_1$ such that

$$h = \int_{\mathcal{A}_1} K(\cdot, \lambda) d\mu(\lambda)$$

(see [2, Ch. 12]). We may assume $\mu(\mathcal{A}_1) \leq 1$.

Let $\lambda \in \mathcal{A}_1$. Since $K(\cdot, \lambda)$ is minimal we see by the assumption on ∂D that there is a point ξ_λ in ∂D such that $K(\cdot, \lambda)$ vanishes continuously on $\partial D \setminus \{\xi_\lambda\}$ (see [4]). Take two balls B_λ, B'_λ such that

$$B_\lambda \subset \bar{D}^c, \quad \bar{B}_\lambda \cap \partial D = \{\xi_\lambda\} \quad \text{and} \quad \bar{D} \cup \bar{B}_\lambda \subset B'_\lambda.$$

Let $D_\lambda = B'_\lambda \setminus \bar{B}_\lambda$ and define a function s_λ on D_λ by

$$s_\lambda(x) = \begin{cases} K(x, \lambda) & x \in D \\ 0 & x \in D_\lambda \setminus D, \end{cases}$$

then s_λ is a positive subharmonic function on a $C^{1,1}$ -domain D_λ . Noting that $\delta_{D_\lambda}(x) \geq \delta_D(x)$ for $x \in D$ and $\alpha(p) = \beta(p)$, we have

$$\int_D K(x, \lambda)^p \delta_D(x)^{-1-\alpha(p)} dx \geq \int_{D_\lambda} s_\lambda(x)^p \delta_{D_\lambda}(x)^{-1-\beta(p)} dx = \infty$$

by Theorem 2. By the Hölder inequality and the Fubini theorem,

$$\begin{aligned}\int_D h(x)^p \delta_D(x)^{-1-\alpha(p)} dx &\geq \int_D \delta_D(x)^{-1-\alpha(p)} \left(\int_{A_1} K(x, \lambda)^p d\mu(\lambda) \right) dx \\ &= \int_{A_1} \int_D K(x, \lambda)^p \delta_D(x)^{-1-\alpha(p)} dx d\mu(\lambda),\end{aligned}$$

so that (1) implies $\mu=0$ and consequently $h \equiv 0$.

The case $p > 1$: As before, we represent h as

$$h = \int_{A_1} K(\cdot, \lambda) d\mu(\lambda).$$

For $m \geq 1$, put $F_m = \{x \in \mathbb{R}^n; \text{dist}(x, F) \geq 1/m\}$, where F is the countable closed set in Theorem A. We take a Lipschitz domain D_m with the uniform exterior property such that $D \cap F_{2m} \subset D_m \subset D$. Then in D_m we have $h \int_{\partial D_m} P_m(\cdot, \xi) d\mu_m(\xi)$ with some positive measure μ_m on ∂D_m , where $P_m = P_{D_m}$. Now put

$$h_m(x) = \int_{\partial D_m \cap F_m} P_m(x, \xi) d\mu_m(\xi).$$

We shall show that $h_m \equiv 0$ in D_m for every $m \geq 1$. By Theorem 3, we have $\mu_m(\partial D_m \cap F_m \setminus A_m) = 0$, where $A_m = \{\xi \in \partial D_m; h_m \text{ is non-tangentially positive at } \xi\}$. For $\xi \in A_m$, let Γ_ξ be an n.t. cone at ξ in D_m . Then $h_m \geq c_\xi$ on Γ_ξ with some constant $c_\xi > 0$. On the other hand, by the exterior sphere property of D_m , $P_m(x, \xi) \geq c(D_m, \Gamma_\xi) |x - \xi|^{1-n}$ for $x \in \Gamma_\xi$ (see Lemma 1) and hence

$$\int_{\Gamma_\xi} P_m(x, \xi) \delta_D(x)^{-1} dx \geq c(D_m, \Gamma_\xi) \int_{\Gamma_\xi} |x - \xi|^{-n} dx = \infty.$$

Therefore

$$\begin{aligned}\int_D h(x)^p \delta_D(x)^{-1} dx &\geq \int_{D_m} h_m(x)^p \delta_D(x)^{-1} dx \\ &= \int_{\partial D_m} \int_{D_m} P_m(x, \xi) \delta_D(x)^{-1} (h_m(x)^{p-1}) dx d\mu_m(\xi) \\ &\geq \int_{A_m} \left(\int_{\Gamma_m} P_m(x, \xi) \delta_D(x)^{-1} c_\xi^{p-1} dx \right) d\mu_m(\xi),\end{aligned}$$

so that (1) implies $\mu_m(\partial D_m \cap F_m) = \mu_m(A_m) = 0$, and hence $h_m \equiv 0$. Hence h vanishes continuously on $\partial D \cap F_m^c$ for all $m \geq 1$, that is, h has boundary value zero on $\partial D \setminus F$.

For $\xi \in F$, we put

$$M_\xi = \{\lambda \in A_1; K(\cdot, \lambda) \text{ vanishes continuously on } \partial D \setminus \{\xi\}\}.$$

Then M_ξ is closed in A_1 and

$$\mu(A_1) = \sum_{\xi \in F} \mu(M_\xi).$$

We take two balls B_1 and B_2 such that $B_1 \subset D^c$, $\bar{B}_1 \cap \partial D = \{\xi\}$ and $B_2 \supset \overline{D \cup B_1}$. Define a function s in $\tilde{D} = B_2 \setminus \bar{B}_1$ by

$$s(x) = \begin{cases} \left(\int_{M_\xi} K(\cdot, \lambda) d\mu(\lambda) \right)^p & x \in D \\ 0 & x \in \tilde{D} \setminus D, \end{cases}$$

then s is a non-negative subharmonic function in a $C^{1,1}$ -domain $\tilde{D}$. Since

$$\int_D s(x) \delta_D(x)^{-1} dx \leq \int_D h(x)^p \delta_D(x)^{-1} dx < \infty,$$

Theorem 2 shows $s \equiv 0$ or $\mu(M_\xi) = 0$. Since this holds for every $\xi \in F$ we have $\mu = 0$ and consequently $h \equiv 0$ in D . The proof is completed.

ADDED IN PROOF. After submitted this paper, Prof. H. Aikawa pointed out that the assertion of Theorem 3 was given in a paper of J. M. Wu [Ann. Inst. Fourier **28**, 4 (1978) 147–167]. She proved this result by comparing representing measure and harmonic measure instead of surface measure.

References

- [1] C. Fefferman and E. M. Stein, H^p spaces of several variables, Acta Math., **129** (1972), 137–193.
- [2] L. L. Helms, Introduction to Potential Theory, Wiley-Interscience, New York, 1969.
- [3] R. A. Hunt and R. L. Wheeden, Positive harmonic functions on Lipschitz domains, Trans. Amer. Math. Soc., **147** (1970), 507–527.
- [4] P. J. Kostense, On the behaviour of minimal positive harmonic functions at the boundary of their domain, Nederl. Akad. Wetensch. Proc. Ser. A, **80**, Indag. Math. (1977), 23–28.
- [5] F.-Y. Maeda and N. Suzuki, The integrability of superharmonic functions on Lipschitz domains, Bull. London Math. Soc., **21** (1989), 270–278.
- [6] N. Suzuki, Martin boundary for $\mathcal{A}-P$, Hiroshima Math. J., **14** (1984), 67–74.
- [7] N. Suzuki, L'allure à la frontière des fonctions L-harmoniques positives dans un domaine, C. R. Acad. Sci. Paris Sér. I, **303** (1986), 621–624.
- [8] K.-O. Widman, Inequalities for the Green function and boundary continuity of the gradient of solutions of elliptic differential equations, Math. Scand., **21** (1967), 17–37.

DEPARTMENT OF MATHEMATICS
FACULTY OF SCIENCE
HIROSHIMA UNIVERSITY

CURRENT ADDRESS:
DEPARTMENT OF MATHEMATICS
COLLEGE OF GENERAL EDUCATION
NAGOYA UNIVERSITY