An application of the second microlocalization at the boundary to the extension of solutions of differential systems

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§ 0. We use the theory of microlocalization of sheaves of [K-S2], and especially its formulation in [S] for boundary value problems, to treat the extension of regular solutions of systems of P.D.E. across 1-codimensional singular sets. Let M be a real analytic manifold, X a complexification of M , N an analytic hypersurface of M , Y a complexification of N , Ω an open component of $M \setminus N$ at $x \in N$. For a suitable involutive manifold $V \subset \dot{T}_M^*x$, invariant under the Hamiltonian flow of $N \times_M \dot{T}_M^*x$, we introduce a new complex $\mathcal{B}_{\partial|X}^a$ of hyperfunctions in Ω with real analytic parameters (of also [Ô2]) and study its applications to non-characteristic boundary value problems. In particular we show that the trace morphism preserves the analytic parameters. The analysis of $\mathcal{B}_{\partial|X}^a$ could be performed from the viewpoint of the 2nd microlocalization at the boundary along V developed in [U-Z]; but we do not need to refer to such a general theory for the purpose of the present paper. We then consider a differential system $\mathcal{M}$ at x , $x \in N$, for which Y is non-characteristic. We denote by $\tilde{V}$ the union of the leaves of V° issued from V , and by ρ the projection $Y \times_x T^*X \rightarrow T^*Y$. We take a closed set $S \subset N$ with $x \in \partial S$ and make the following hypotheses: $\mathcal{M}$ is microhyperbolic relative to $\tilde{V}$ to any $\theta \in \dot{T}_N^*M_x$; $\text{char } \mathcal{M} \cap \rho^{-1}\rho(\{x\} \times_M V) \subset V$; $iN_x^*(S) \subset \rho(\{x\} \times_M V)$. We then prove that $H^0(\mathcal{B}_{M|X}^a)$ -solutions of $\mathcal{M}$ on $M \setminus S$ extend to M at x . (Here $H^0(\mathcal{B}_{M|X}^a)$ is the sheaf of hyperfunctions depending real analytically on the variables transversal to the leaves of V .) Under some additional hypotheses on "propagation in the interior", we also obtain the extension of $\mathcal{A}_M$ -solutions. The basic method of the paper, namely the estimation of microsupport for traces of solutions to single operators (resp. to systems), is due to Kaneko [Kan 1] (resp. Oaku [Ô 1]).

§ 1. The complexes $\mathcal{B}_{\partial|X}^2$ and $\mathcal{B}_{\partial|X}^a$

Let $M = M' \times L$ be real analytic manifolds with complexifications $X = X' \times Z$ and dimensions $n = n_1 + n_2$. For a locally closed set $A = A' \times L$ of M , put $\tilde{A} = A' \times Z$ and define (cf. [K-S-2], [S])

$$(1.1) \quad \mathcal{C}_{A|X}^h = \mu_A(\mathcal{O}_X) \otimes \omega_{M'/X}[n_1],$$

$$(1.2) \quad \mathcal{B}_{A|X}^2 = R\Gamma_{T^*X' \times L}(\mathcal{C}_{A|X}^h) \otimes \omega_{L/Z}[n_2].$$

We often consider the case $A=M$ or $A=N$ for an analytic submanifold $N=N' \times L$ of M of codimension 1, or else $A=\Omega$ where $\Omega=\Omega^*$ are the components of $M \setminus N$. The following triangle will play an essential role:

$$(1.3) \quad \mathcal{B}_{N|X}^2 \longrightarrow \mathcal{B}_{M|X}^2 \longrightarrow \mathcal{B}_{\Omega^+|X}^2 \oplus \mathcal{B}_{\Omega^-|X}^2 \longrightarrow {}^{+1}.$$

REMARK 1.1. By the results of [U-Z] we could give a canonical definition of the complexes $\mathcal{C}_{*|X}^2$ and $\mathcal{B}_{*|X}^2$, $*$ = M, N, Ω , associated to a smooth conic regular involutive manifold $V \subset \dot{T}_M^*X$ such that

$$(1.4) \quad \begin{aligned} &V \text{ and } N \times_M \dot{T}_M^*X \text{ intersect transversally,} \\ &\text{and } N \times_M V \text{ is regular involutive.} \end{aligned}$$

We recall from [K-L] that for $*$ = M, N , $\mathcal{B}_{*|X}^2$ is concentrated in degree 0 and the natural morphism $\mathcal{C}_{*|X}^2|_{T^*X' \times L} \hookrightarrow \mathcal{B}_{*|X}^2$ is injective, $\mathcal{C}_{*|X}^2$ being the sheaf of usual microfunctions. (As for the case $*$ = Ω it is proven in [U-Z] that $(\mathcal{B}_{\Omega|X}^2)_{T_{M',X' \times L}^*}$ is concentrated in degree 0 but that the corresponding result on injectivity does not hold any more. However this is not needed here.)

We set now:

$$(1.5) \quad \mathcal{B}_{\Omega|X}^a = R\Gamma_M((\mathcal{O}_X)_{\Omega' \times Z}) \otimes \omega_{L/Z}[n_2].$$

We have a distinguished triangle

$$(1.6) \quad \mathcal{B}_{\Omega|X}^a \longrightarrow R\Gamma_{\Omega}(\mathcal{B}_M) \longrightarrow R\dot{\pi}_*(\mathcal{B}_{\Omega|X}^2) \longrightarrow {}^{+1},$$

$\mathcal{B}_M$ being the sheaf of hyperfunctions. (We remark that (1.6) is obtained by applying $R\Gamma_M(\cdot) \otimes \omega_{M/X}[n]$ to the triangle

$$(\mathcal{O}_X)_{\Omega' \times Z} \otimes \omega_{M'/X}[-n_1] \longrightarrow R\Gamma_{\Omega' \times Z}(\mathcal{O}_X) \longrightarrow R\dot{\pi}_* \mu_{\Omega' \times Z}(\mathcal{O}_X) \longrightarrow {}^{+1}.$$

We similarly define

$$\mathcal{B}_{M|X}^a = R\Gamma_M((\mathcal{O}_X)_{M' \times Z}) \otimes \omega_{L/Z}[n_2],$$

$$(\text{resp. } \mathcal{B}_{N|X}^a = R\Gamma_M((\mathcal{O}_X)_{N' \times Z}) \otimes \omega_{L/Z}[n_2]),$$

and get (1.6) with Ω replaced by M (resp. N and with the first term shifted by -1). One also proves that $\mathcal{B}_{*|X}^a$, $*$ = M, N or Ω , are concentrated in degrees 0, 1 with $H^1(\mathcal{B}_{*|X}^a) \neq 0$. For $*$ = M this follows from the analogous of (1.6), from the flabbiness of $\mathcal{B}_{M|X}^2$ and the non-surjectivity of $\mathcal{C}_M|_{T_{M',X' \times L}^*} \hookrightarrow \mathcal{B}_{M|X}^2$. As for $*$ = N the degree is ≤ 1 (by the flabby dimension 1 of $\mathcal{B}_{N|X}^2$) and

this is in fact 0, 1 (by the edge of the wedge theorem for $(\mathcal{O}_X)_{N' \times Z}$). Moreover $H^1(\mathcal{B}_{N|X}^2) \neq 0$ by an argument of the type of the dummy variable (i.e. by microlocalization of the exact sequence: $0 \rightarrow \mathcal{O}_X \rightarrow \mathcal{O}_X \rightarrow j_* \mathcal{O}_Y \rightarrow 0$, Y being a complexification of N and $j: Y \hookrightarrow X$ the embedding) and by the corresponding result for $\mathcal{B}_{M|X}^2$. Finally the case $* = \Omega$ obviously follows from those $* = M$ and $* = N$. The detailed study of the complex (1.5) is left to [U-Z]; we only treat here its applications to boundary value problems. Thus let $\mathcal{M}$ be a coherent $\mathcal{D}_X$ -module on an open set of M . We assume all through this section that Y , the complexification of N , is non-characteristic for $\mathcal{M}$.

PROPOSITION 1.2. *The natural morphisms*

$$(1.7) \quad H^0(R \operatorname{Hom}_{\mathcal{D}_X}(\mathcal{M}, \mathcal{C}_{\mathcal{Q}|X}))|_{T^*X' \times L} \longrightarrow H^0(R \operatorname{Hom}_{\mathcal{D}_X}(\mathcal{M}, \mathcal{B}_{\mathcal{Q}|X}^2)),$$

and

$$(1.8) \quad H^0(R \operatorname{Hom}_{\mathcal{D}_X}(\mathcal{M}, \mathcal{B}_{\mathcal{Q}|X}^a)) \longrightarrow \operatorname{Hom}_{\mathcal{D}_X}(\mathcal{M}, \Gamma_{\mathcal{Q}}(\mathcal{B}_M)),$$

are injective. (cf. [S-Z 1] as for (1.7))

PROOF. By the results of [K-L] it is enough to prove (1.7) and (1.8) in $T_{N'}^*X' \times L$ and N respectively. As for (1.7) let us consider the commutative diagram

$$(1.9) \quad \begin{array}{ccc} \mathcal{C}_{\mathcal{Q}|X} & \longrightarrow & \mathcal{B}_{\mathcal{Q}|X}^2 \\ \downarrow & & \downarrow \\ \mathcal{C}_{N|X}[1] & \longrightarrow & \mathcal{B}_{N|X}^2[1]. \end{array}$$

We observe that

$$H^0(R \operatorname{Hom}_{\mathcal{D}_X}(\mathcal{M}, \mathcal{C}_{\mathcal{Q}|X})) \longrightarrow H^1(R \operatorname{Hom}_{\mathcal{D}_X}(\mathcal{M}, \mathcal{C}_{N|X}))$$

is injective, due to the triangle: $\mathcal{C}_{N|X} \rightarrow \mathcal{C}_{\mathcal{Q}|X} \rightarrow \mathcal{C}_{\mathcal{Q}|X} \rightarrow^{+1}$ and to:

$$\operatorname{Hom}_{\mathcal{D}_X}(\mathcal{M}, \mathcal{C}_{\mathcal{Q}|X})_{\pi^{-1}(N)} = 0.$$

(The latter vanishing follows in turn from the unique continuation property for $\mathcal{C}_{\mathcal{Q}|X}|_{\pi^{-1}(N)}$ due to Kataoka [Kat] together with the non-characteristicity of Y .) We also have

$$(1.10) \quad \operatorname{Hom}_{\mathcal{D}_X}(\mathcal{M}, \mathcal{B}_{N|X}^2 / \mathcal{C}_{N|X})|_{T_{N'}^*X' \times L} = 0,$$

by division formulas for $\mathcal{C}_{N|X}$ and $\mathcal{B}_{N|X}^2$ (cf. [S-Z 2]). The injectivity of (1.7) follows at once.

As for the injectivity of (1.8), this is equivalent, in view of (1.6) to:

$$H^{-1}(R \operatorname{Hom}_{\mathcal{D}_X}(\mathcal{M}, R\hat{\pi}_*(\mathcal{B}_{\mathcal{Q}|X}^2))) = 0.$$

On the other hand this follows from: $\text{Hom}_{\mathcal{D}_X}(\mathcal{M}, \mathcal{B}_{N|X}^2) = 0$ which is still a consequence of division formula for $\mathcal{B}_{N|X}^2$.

Let $\mathcal{M}_Y$ denote the induced system of $\mathcal{M}$ on Y , and let $\gamma: \text{Hom}_{\mathcal{D}_X}(\mathcal{M}, \Gamma_{\mathcal{Q}}(\mathcal{B}_M))|_N \rightarrow \text{Hom}_{\mathcal{D}_Y}(\mathcal{M}_Y, \mathcal{B}_N)$ be the trace morphism (cf. [S]). By collecting all above results we get

PROPOSITION 1.3. *We have*

$$(1.11) \quad H^0(R \text{Hom}_{\mathcal{D}_X}(\mathcal{M}, \mathcal{B}_{\mathcal{Q}|X}^a))_x \\ = \{u \in \text{Hom}_{\mathcal{D}_X}(\mathcal{M}, \Gamma_{\mathcal{Q}}(\mathcal{B}_M))_x : SS(\gamma(u)) \cap (\dot{T}_N^* Y' \times L) = \emptyset\}, \quad x \in N.$$

PROOF. Let $\mathcal{F} = R \text{Hom}_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X)$, put $\tilde{\mathcal{Q}} = \mathcal{Q}' \times Z$, and note that the natural diagram

$$(1.12) \quad \begin{array}{ccc} \pi^{-1} R \Gamma_{\mathcal{Q}}(\mathcal{F}) & \longrightarrow & R \Gamma_{T^* X' \times L} \mu_{\tilde{\mathcal{Q}}}(\mathcal{F}) \\ & \searrow & \nearrow \\ & \mu_{\mathcal{Q}}(\mathcal{F}) & T^* X' \times L \end{array}$$

is commuting. Thus recalling (1.6) and applying Proposition 1.2, we get

$$(1.13) \quad H^0(R \text{Hom}_{\mathcal{D}_X}(\mathcal{M}, \mathcal{B}_{\mathcal{Q}|X}^a))_x \\ = \{u \in \text{Hom}_{\mathcal{D}_X}(\mathcal{M}, \Gamma_{\mathcal{Q}}(\mathcal{B}_M))_x : SS_{\mathcal{Q}}^{\mathcal{F}, 0}(u) \cap (\dot{T}^* X' \times L) = \emptyset\},$$

where $SS_{\mathcal{Q}}^{\mathcal{F}, 0}(u)$ is the support of u identified to a section of $H^0(R \text{Hom}_{\mathcal{D}_X}(\mathcal{M}, \mathcal{C}_{\mathcal{Q}|X}))$ (cf. [S]). According to [S] this is in turn equivalent to (1.11).

REMARK 1.4. When considering $\mathcal{B}_{M|X}^a$ one can use the injectivity of $\mathcal{C}_{M|X}|_{T^* M, X' \times L} \hookrightarrow \mathcal{B}_{M|X}^2$ and $H^0(\mathcal{B}_{M|X}^a) \hookrightarrow \mathcal{B}_M$ as a substitute of Proposition 1.2. (Note that the latter injectivity follows from the analogous of (1.6) for $\mathcal{B}_{M|X}^2$ and the concentration in degree 0 for $\mathcal{B}_{M|X}^2$ (cf. [K-L]). This gives

$$(1.14) \quad H^0(\mathcal{B}_{M|X}^a)_x = \{u \in (\mathcal{B}_M)_x : SS(u) \cap (\dot{T}_{M'}^* X' \times L) = \emptyset\}, \quad x \in M.$$

REMARK 1.5. For a regular involutive manifold V defined on the whole $\dot{T}_M^* X$ and satisfying (1.4), we can intrinsically define $\mathcal{B}_{*|X}^a$, $*$ = $M, N, \mathcal{Q}$, by replacing in (1.5) $\bar{*} \times Z$ by $\pi(\tilde{V}_{\bar{*}})$ (and $\omega_{L/Z}$ by $\omega_{V/\mathcal{P}_M}$), where $\tilde{V}_{\bar{*}}$ is the union of the leaves of V^c issued from $\bar{*} \times_M \dot{T}_M^* X$; (we also write $\tilde{V} = \tilde{V}_M$). One can also intrinsically define the right hand sides of (1.11), (1.14) just by replacing $\dot{T}_N^* Y' \times L$ and $\dot{T}_{M'}^* X' \times L$ by $\rho \bar{\omega}^{-1}(V)$ and V respectively (ρ and $\bar{\omega}$ being the natural mappings from $Y \times_X \dot{T}^* X$ to $\dot{T}^* Y$ and $\dot{T}^* X$ resp.). It is then clear that if for some coordinates on M we can write

$$V = \dot{T}_{M'}^* X' \times L, \quad N = N' \times L,$$

then (1.11) and (1.14) still hold. More generally owing to the invariance of

$\mathcal{B}_{*|M}^2$ under contact transformation preserving $V, N \times_M V$, and $\omega_{N/M}$ (cf. [U-Z]), one could prove that (1.6) (and the corresponding triangles for $*=M, N$) hold. But this refined argument is not needed here.

§ 2. Extension of solutions with real analytic parameters

Let M be a real analytic manifold with complexification X , N an analytic hypersurface of M with complexification Y , $\Omega = \Omega^\pm$ the two components of $M \setminus N$ at x , $x \in N$, ρ and $\bar{\omega}$ the canonical mappings from $Y \times_X T^*X$ to T^*Y and T^*X respectively.

Let $U \subset M$ be a neighborhood of x , and let V be a manifold in $U \times_M \dot{T}_M^*X$. We assume that, in suitable coordinates on U :

$$(2.1) \quad \begin{aligned} M &= M' \times L, & X &= X' \times Z, & N &= N' \times L, \\ V &= \dot{T}_{M'}^*X' \times L, & \tilde{V} &= \dot{T}_{M'}^*X' \times Z, & \tilde{V}^\pm &= (\bar{\Omega}^\pm \times_{M'} \dot{T}_{M'}^*X') \times Z. \end{aligned}$$

Recall the complexes $\mathcal{B}_{M|X}^a, \mathcal{B}_{\partial|X}^a$ (intrinsically associated to V) and remember (1.11), (1.14). For any $p \in \dot{\pi}^{-1}(x)$ recall the identification $T_x^*M \xrightarrow{\sim} T_p T^*X$ obtained through the embedding $T^*X \times_X T^*X \hookrightarrow T^*T^*X$ and the Hamiltonian isomorphism, and observe that $(\dot{T}_N^*M)_x/R^+$ is just a pair of vectors $\pm\theta$.

THEOREM 2.1. *Let N and V be defined, in suitable coordinates by (2.1), and let $\mathcal{M}$ be a coherent $\mathcal{D}_X$ -module at x which verifies*

$$(2.2) \quad \pm\theta \notin C_p(\text{char } \mathcal{M}, \tilde{V}^\mp) \text{ for } \pm\theta \in (\dot{T}_N^*M)_x/R^+ \text{ and for any } p \in \dot{\pi}^{-1}(x) \cap V,$$

$$(2.3) \quad \bar{\omega}^{-1}(\text{char } \mathcal{M}) \cap \rho^{-1}\rho(\{x\} \times_M V) \subset T_M^*X.$$

Let S be a closed subset of N with $x \in \partial S$ and

$$(2.4) \quad iN_x^*(S) \subset \rho\bar{\omega}^{-1}(V)_x,$$

(in the identification $iT^*N \simeq T_N^*Y$). We then have, in a neighborhood of x ,

$$(2.5) \quad \text{Hom}_{\mathcal{D}_X}(\mathcal{M}, \Gamma_{M \setminus S}(H^0(\mathcal{B}_{M|X}^a))) \xleftarrow{\sim} \text{Hom}_{\mathcal{D}_X}(\mathcal{M}, H^0(\mathcal{B}_{M|X}^a)).$$

PROOF. We shall follow a method closely related to [Kan]. Let $\Omega = \Omega^\pm$ with $\Omega^+ \cup \Omega^- = M \setminus N$: by reasoning as in § 1, and observing that

$$\begin{aligned} R\Gamma_{\partial}(\mathcal{B}_{M|X}^a) &= R\Gamma_{(\partial \times_X T_X^*X)}(\mathcal{B}_{M|X}^2) \\ &= R\Gamma_{(\partial \times_X T_X^*X)}(\mathcal{B}_{\partial|X}^2) \\ &= R\pi_* R\Gamma_{(T_X^*X \cup \pi^{-1}(N))}(\mathcal{B}_{\partial|X}^2), \end{aligned}$$

we get a distinguished triangle:

$$(2.6) \quad \mathcal{B}_{\tilde{\Omega}|X}^a \longrightarrow R\Gamma_{\tilde{\Omega}}(\mathcal{B}_{M|X}^a) \longrightarrow R\tilde{\pi}_* R\Gamma_{\tilde{\pi}^{-1}(N)}(\mathcal{B}_{\tilde{\Omega}|X}^2) \longrightarrow {}^{+1}.$$

Let $\tilde{\Omega} = \Omega' \times Z$, $\tilde{M} = M' \times Z$, $\tilde{N} = N' \times Z$, and put $\mathcal{F} = R\mathrm{Hom}_{\mathfrak{g}_X}(\mathcal{M}, \mathcal{C}_{\tilde{\Omega}|X}^b)|_{\tilde{M} \times T^*X}$. We note that (2.2) implies $(p; -\theta) \notin SS(\mathcal{F})$ for $\tilde{\Omega} = \tilde{\Omega}^+$ (or $(p; +\theta) \notin SS(\mathcal{F})$ for $\tilde{\Omega} = \tilde{\Omega}^-$). It follows $R\Gamma_{\tilde{\pi}^{-1}(N)}(\mathcal{F}) = R\Gamma_{\tilde{\pi}^{-1}(\tilde{M} \setminus \tilde{\Omega})}(\mathcal{F}) = 0$. By applying $R\Gamma_{T^*X' \times L}[n_2]$ to the last equality (n_2 being the codimension of V), we then get, for a neighborhood U of x on N ,

$$(2.7) \quad R\Gamma_{\tilde{\pi}^{-1}(N)} R\mathrm{Hom}_{\mathfrak{g}_X}(\mathcal{M}, \mathcal{B}_{\tilde{\Omega}|X}^2)|_{U \times_M V} = 0.$$

Note now that (2.2), (2.3) imply:

$$\bar{\omega}^{-1}(\mathrm{char} \mathcal{M}) \cap \rho^{-1}\rho(U \times_M V) \subset U \times_M V,$$

which gives, combined with (2.7):

$$(2.8) \quad R\tilde{\pi}_* R\Gamma_{\tilde{\pi}^{-1}(N)} R\mathrm{Hom}_{\mathfrak{g}_X}(\mathcal{M}, \mathcal{B}_{\tilde{\Omega}|X}^2)|_U = 0.$$

By (2.6) this implies:

$$(2.9) \quad R\mathrm{Hom}_{\mathfrak{g}_X}(\mathcal{M}, \mathcal{B}_{\tilde{\Omega}|X}^a)|_U \simeq R\mathrm{Hom}_{\mathfrak{g}_X}(\mathcal{M}, R\Gamma_{\tilde{\Omega}}(\mathcal{B}_{M|X}^a))|_U.$$

Since $\mathcal{B}_{M|X}^a$ is concentrated in degree ≥ 0 , then for every $u \in \mathrm{Hom}_{\mathfrak{g}_X}(\mathcal{M}, \Gamma_{M \setminus S}(H^0(\mathcal{B}_{M|X}^a)))$, we can define $u^{\pm} = u|_{\mathcal{O}^{\pm}} \in H^0(R\mathrm{Hom}_{\mathfrak{g}_X}(\mathcal{M}, R\Gamma_{\mathcal{O}^{\pm}}(\mathcal{B}_{M|X}^a)))$. Owing to (2.9) and (1.11) we get

$$SS(\gamma(u^{\pm})) \cap \rho(U \times_M V) \subset T_Y^* Y.$$

We also clearly have

$$\mathrm{supp}(\gamma(u^+) - \gamma(u^-)) \subset S.$$

Therefore the conclusion is an immediate consequence of the following two lemmas.

LEMMA 2.2 (cf. [Ô]). *Let F be a closed set of M and let $u \in (\mathcal{B}_M)_x$, $x \in \partial F$. Then*

$$\begin{cases} SS(u) \cap N_x^*(F) \subset \{0\} \\ \mathrm{supp}(u) \subset F \end{cases} \iff u_x = 0$$

PROOF. Easy application of Kashiwara-Holmgren's theorem and of the sweeping out procedure by Bony-Schapira.

LEMMA 2.3. *Let $u \in \mathrm{Hom}(\mathcal{M}, \Gamma_{(M \setminus N)}(\mathcal{B}_M))$; then*

$$u \in \mathrm{Hom}_{\mathfrak{g}_X}(\mathcal{M}, H^0(\mathcal{B}_{M|X}^a)) \iff \begin{cases} \gamma(u^{\pm}) \in H^0(\mathcal{B}_{N|Y}^a) \\ \gamma(u^+) - \gamma(u^-) = 0. \end{cases}$$

PROOF. It is enough to recall the triangle

$$\mathcal{C}_{M|X} \longrightarrow \mathcal{C}_{\rho^+|X} \oplus \mathcal{C}_{\rho^-|X} \longrightarrow \mathcal{C}_{N|X}[1] \longrightarrow {}^{+1},$$

and the estimation

$$SS(u)|_N \subset \bigcup_{\pm} SS_{\rho^{\pm},0}^{\mathcal{A},0}(u^{\pm})|_N \subset \rho^{-1}(\bigcup_{\pm} SS(r(u^{\pm}))),$$

(cf. [S]).

COROLLARY 2.4. *In the situation of Theorem 2.1 assume in addition:*

$$(2.10) \quad \text{Hom}_{\mathcal{D}_X}(\mathcal{M}, \Gamma_{\tau^{-1}(S)}(\mathcal{C}_{M|X}))_p = 0 \quad \forall p \in \dot{T}_M^*X \setminus V, \pi(p) = x.$$

Then (for $\mathcal{A}_M = \mathcal{O}_X|_M$)

$$(2.11) \quad \text{Hom}_{\mathcal{D}_X}(\mathcal{M}, \Gamma_{(M \setminus S)}(\mathcal{A}_M))_x \simeq \text{Hom}_{\mathcal{D}_X}(\mathcal{M}, \mathcal{A}_M)_x.$$

By the argument in the proof of (2.7) and by the injectivity of $\mathcal{C}_{M|X}|_V \rightarrow \mathcal{B}_{M|X}^2$, a sufficient condition for (2.10) is that $\forall p \in (\dot{T}_M^*X \cap \text{ch } \mathcal{M}) \setminus V$, $\pi(p) = x$, there exist a (regular) involutive manifold $V_p \subset \dot{T}_M^*X$ with $V_p \ni p$, and invariant under the flow of $N \times_M \dot{T}_M^*X$, and there exist $\theta_p \in \dot{T}_x^*M$ with $S \subset \{m \in M; \langle m - x, \theta_p \rangle \geq 0\}$ such that (2.2) is satisfied by θ_p and $\tilde{V}_p$.

REMARK 2.5. It is clear from Lemma 2.2 that we can even consider in Theorem 2.1 some singular set S such that $N_{x_0}^*(S) = T_{x_0}^*N$ as in the following example:

$$\begin{aligned} M &= M' \times L \simeq R^{n_1} \times R^{n_2} \ni x = (x', x''), \quad (n_1 \geq 2), \\ N &= \{x'_1 = 0\}, \quad V = \{\gamma'' = 0\}, \quad S = \{\phi \leq 0\}, \end{aligned}$$

where ϕ is a C^0 -function on N which verifies

$$\phi(x_0) = 0, \quad \partial_{x'_1} \phi(x_0) \neq 0, \quad \partial_{x''} \phi(x_0) = 0, \quad \partial_{x'_1} \phi, \partial_{x''} \phi \in C^0.$$

REMARK 2.6. Theorem 2.1 extends the results of [Kan 1], [Ô 1]. These are obtained by choosing $L \simeq R^{n-2}$ and by taking a subset $S \subset N$ with $N^*(S) = \{dx_2\}$. Moreover condition (2.2) is replaced there by (classical) microhyperbolicity.

EXAMPLE 2.7. Let $M = M' \times L$, $N = N' \times L$, $M' = R \times N'$, $S = S' \times L$, $x_0 = 0 \in \partial S$. Take coordinates $(x_1, x', x'') \in R \times N' \times L$ and $(z, \zeta) \in T^*X$, $z = x + iy$, $\zeta = \xi + i\eta$. Let $V = \{\eta'' = 0\}$ and consider the $\mathcal{D}_X$ -module $\mathcal{M}$ associated to the equation

$$Pu = 0 \quad \text{for } P = D_1^2 - (x_1^r + x'^s)D'^2 + D'^{r/2}, \quad r, s \text{ even}, \quad r \geq 2.$$

Then (2.2)–(2.4) hold with $\pm\theta = \pm dx_1$ (cf. [S-Z 1]) and thus we get (2.5) and (2.11) (as (2.10) is trivial in the present situation).

EXAMPLE 2.8. Let $M = M' \times L$, $N = N' \times L$, $M' = R^2 \ni (x_1, x_2)$, $N' = \{0\} \times R$, $L = R^{n_2} \ni x'' = (x_3, x_4, x'')$, set $V = \{\eta'' = 0\}$, and consider the $\mathcal{D}_X$ -module $\mathcal{M}$ associated to the system

$$Pu = Qu = 0 \quad \text{for } P = D_1^3 + D_3^3 + D''^3, \quad Q = D_2(D_3^2 + D_4^2).$$

For $S = S' \times L$ with $0 \in \partial S$, we have (2.2)–(2.4) and hence also (2.5). Moreover for any $p \in V$ and for $\pm\theta_p = \pm dx_1$ or $\pm dx_2$ we have (2.2) with $\tilde{V}_p = T_M^* X$. Therefore if we let $S = \{x_1 = x_2 = 0\}$, we get (2.10) and (2.11). (This extends Example 1.1 of [Ô 1].)

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