The Iwasawa λ_p -invariants of Γ -transforms of the generating functions of the Bernoulli numbers

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Abstract

We give an elementary proof of Kida's formula on Hurwitz-type relations of cyclotomic λ_p -invariants of abelian number fields, using the theory of Γ -transforms of rational functions. Our proof will be carried out only in terms of rational functions.

§1. Introduction

Let K be a finite imaginary abelian extension of $\mathbf{Q}$. For a fixed prime number p , let $\mu_p^-(K)$ and $\lambda_p^-(K)$ denote the minus part of the Iwasawa μ_p -invariant and λ_p -invariant of the cyclotomic $\mathbf{Z}_p$ -extension of K , respectively. In [8], Sinnott studied Γ -transforms of p -adic measures for the purpose of calculating $\mu_p^-(K)$ and succeeded in giving a simple proof of the theorem of Ferrero and Washington as an application. He was able to reduce $\mu_p^-(K)$ to μ_p -invariants of certain rational functions (see [8, Proposition 4.1]). Similar reduction may be extended for the calculation of $\lambda_p^-(K)$. In [6], Kida showed a partial step of the reduction for $\lambda_p^-(K)$ (see PROPOSITION 1 below and note that this step corresponding to $\mu_p^-(K)$ is trivial). The importance of PROPOSITION 1 is that $\lambda_p^-(K)$ is represented only by the λ_p -invariants of rational functions. Thus we can calculate $\lambda_p^-(K)$ by calculating the λ_p -invariants of the generating functions of the Bernoulli numbers (in easier cases, Ferrero [2, Proposition 3.3] obtained some results, by a similar method).

In this paper, using this result of Kida, we study the λ_p -invariants of the generating functions of the Bernoulli numbers, and obtain the following (see section 3 below for details)

THEOREM 2. *If τ has a p -power order, then we have*

$$\lambda_p(H_{\theta\tau}(T)) = \begin{cases} \lambda_p(H_\theta(T)) + p^{n_\tau} & \text{if } \theta(\ell) \equiv 1 \pmod{p}, \\ \lambda_p(H_\theta(T)) & \text{if } \theta(\ell) \not\equiv 1 \pmod{p}. \end{cases}$$

This THEOREM 2 implies that the λ_p -invariant is almost independent of the

difference of the component with p -power order of Dirichlet character. As an application of THEOREM 2, we can easily deduce Kida's formula for abelian number fields from THEOREM 2, namely, a formula concerning Hurwitz-type relations of $\lambda_p^-(K)$ (in [3] and [4], Kida proved these relations for CM -fields which are called Kida's formula). In [9], Sinnott gave another proof of Kida's formula, using the theory of p -adic L -functions. Thus the present paper may be viewed as an explanation for Sinnott's proof in terms of the λ_p -invariants of certain rational functions in abelian case.

§2. Preliminaries

Let p be a fixed prime number and put $q = 4$ or p according as $p = 2$ or not, and let $\mathcal{O}$ be the integer ring of a fixed finite extension of $\mathbf{Q}_p$. Then for any p -adic $\mathcal{O}$ -valued measure α on $\mathbf{Z}_p$, there uniquely exists a $\mathcal{O}$ -coefficient power series $F(T)$ in $T - 1$ such that

$$F(T) = \int_{\mathbf{Z}_p} T^x d\alpha(x) = \sum_{n=0}^{\infty} \left\{ \int_{\mathbf{Z}_p} \binom{x}{n} d\alpha(x) \right\} (T - 1)^n.$$

We call this relation a fundamental relation of measures and power series. Conversely, $F(T)$ determines α uniquely. Hereafter we denote the n -th coefficient of $F(T)$ by α_n for $n \geq 0$. The Iwasawa invariants of α are defined to be those of $F(T)$. Let κ be a topological generator of $1 + q\mathbf{Z}_p$, and we define an isomorphism φ from $\mathbf{Z}_p$ to $1 + q\mathbf{Z}_p$ by $\varphi(x) = \kappa^x$ for $x \in \mathbf{Z}_p$. We define two new measures $\alpha \circ \varphi$ and $\alpha|_{\mathfrak{A}}$ dependent on a compact open subset $\mathfrak{A}$ of $\mathbf{Z}_p$ by

$$\alpha \circ \varphi(A) = \alpha(\varphi(A)) \quad \text{and} \quad \alpha|_{\mathfrak{A}}(A) = \alpha(A \cap \mathfrak{A}),$$

respectively, for a compact open subset A of $\mathbf{Z}_p$.

In this notation, Kida [6] showed, under the assumption of $\kappa = e^q$, the following:

PROPOSITION 1.

$$\lambda_p(\alpha \circ \varphi) = \frac{1}{q} \lambda_p(\alpha|_{1+q\mathbf{Z}_p}).$$

We note that

- 1) for studying the Iwasawa invariants of p -adic L -functions, it is sufficient to consider the case of $\kappa = e^q$,
- 2) Childress [1] showed PROPOSITION 1, under a certain strong assumption, in the case where κ is any topological generator of $1 + q\mathbf{Z}_p$, and
- 3) we give in [7] a simple proof of PROPOSITION 1 without any assumption.

In the next section, we calculate the λ_p -invariants of the generating functions of the Bernoulli numbers by using PROPOSITION 1 (see THEOREMS 1 and 2 below).

§3. Main Theorems

In this section, we explain our main THEOREMS 1 and 2 which will be useful for the proof of Kida's formula in the next section.

We fix a Dirichlet character χ which has the decomposition $\chi = \theta\tau$ satisfying

- (i) τ has a p -power order and its conductor ℓ is a prime number, and
- (ii) for all $a \in \mathbf{Z}$, $\theta\tau(a) = \theta(a)\tau(a)$.

Let f be the conductor of θ , and fix a natural number c such that $1 < c \equiv 1 \pmod{qf}$ and $(c, \ell) = 1$, and ε_χ be a periodic function on $\mathbf{Z}$ defined by

$$\varepsilon_\chi(a) = \begin{cases} \chi(a) & \text{if } c \nmid a, \\ (1-c)\chi(a) & \text{if } c \mid a. \end{cases}$$

Let $\mathcal{O}$ be the integer ring of the field generated over $\mathbf{Q}_p$ by the values of θ and τ . Then we define

$$F_\chi(T) = \sum_{a=1}^g \frac{\varepsilon_\chi(a)T^a}{T^g - 1} \in \mathcal{O}[[T-1]],$$

where g is any multiple of the period of ε_χ , and let α_χ be the measure associated to $F_\chi(T)$, and ω be the Teichmüller character mod q and decompose m in $\mathbf{Z}_p^\times$ into $m = \omega(m)(1 + m_1p^{n_m})$ with $(m_1, p) = 1$ and $n_m \in \mathbf{Z}$. If χ is odd and distinct from ω^{-1} , then the λ_p -invariant $\lambda(\chi)$ of the Iwasawa power series representing the p -adic L -function $L_p(s, \lambda\omega)$ is given by Proposition 4.1. of [8] as follows:

$$\lambda(\chi) = \lambda_p(\Gamma_{\alpha_\chi}(s)) - p^{n_c}/q, \quad (1)$$

where $\Gamma_{\alpha_\chi}(s)$ is called the Γ -transform of α_χ defined by

$$\Gamma_{\alpha_\chi}(s) = \int_{\mathbf{Z}_p^\times} \langle x \rangle^s d\alpha_\chi(x) \quad \text{for } s \in \mathbf{Z}_p$$

and $\langle x \rangle = x\omega^{-1}(x)$ for $x \in \mathbf{Z}_p^\times$. Furthermore, we define $\lambda(\rho)$ for a general Dirichlet character ρ by (1). Then THEOREM 1 is expressed as follows:

THEOREM 1. Assume the above conditions.

- (i) Suppose $\theta \neq 1$ and ω^{-1} . Then we have

$$\lambda(\theta\tau) = \begin{cases} \lambda(\theta) + p^{n_c}/q & \text{if } \theta(\ell) \equiv 1 \pmod{\wp}, \\ \lambda(\theta) & \text{if } \theta(\ell) \not\equiv 1 \pmod{\wp}, \end{cases}$$

where $\wp$ is a prime ideal of $\mathcal{O}$.

- (ii) Suppose $\theta = 1$ or ω^{-1} . Then we have

$$\lambda(\theta\tau) = p^{n_c}/q - 1.$$

We note that we use the same c which is defined for χ to define ε_θ .

We will prove the equivalence of THEOREM 2 to THEOREM 1, and explain the equivalence in the following:

For $F(T) \in \mathcal{O}[[T-1]]$, let α be the measure associated to $F(T)$. We put $\alpha^{(i)} = \alpha|_{1+q\mathbf{Z}_p}$ for $1 \leq i \leq q-1$ with $(i, p) = 1$, and denote the power series associated to $\alpha^{(i)}$ by $(F(T))^{(i)} = F^{(i)}(T)$. For $a \in \mathbf{Z}_p^\times$, we define a measure $\alpha \circ a$ by $\alpha \circ a(A) = \alpha(aA)$ for a compact open subset A of $\mathbf{Z}_p$. Then the power series associated to $\alpha \circ a$ is given by $F(T^{a^{-1}})$. Using our notation, and setting

$$H_\chi(T) = \frac{1}{2} \sum_{\substack{i=1 \\ (i,p)=1}}^{q-1} F_\chi^{(i)}(T^{\omega^{-1}(i)}) \in \mathcal{O}[[T-1]],$$

we see that the measure associated to $H_\chi(T)$ is given by

$$\frac{1}{2} \sum_{\substack{i=1 \\ (i,p)=1}}^{q-1} \alpha_\chi \circ \omega(i)|_{1+q\mathbf{Z}_p} = \frac{1}{2} \sum_{\substack{i=1 \\ (i,p)=1}}^{q-1} \alpha_\chi^{(i)} \circ \omega(i).$$

On the other hand, by [8], we obtain

$$\lambda_p(\Gamma_{\alpha_\chi}(s)) = \lambda_p \left(\sum_{\substack{i=1 \\ (i,p)=1}}^{q-1} \alpha_\chi \circ \omega(i) \circ \varphi \right).$$

Therefore, by PROPOSITION 1, THEOREM 1 is equivalent to the following:

THEOREM 2. *Assume the same conditions of THEOREM 1.*

(i) *Suppose $\theta \neq 1$ and ω^{-1} . Then we have*

$$\lambda_p(H_{\theta\tau}(T)) = \begin{cases} \lambda_p(H_\theta(T)) + p^{n_\epsilon} & \text{if } \theta(\ell) \equiv 1 \pmod{\wp}, \\ \lambda_p(H_\theta(T)) & \text{if } \theta(\ell) \not\equiv 1 \pmod{\wp}. \end{cases}$$

(ii) *Suppose $\theta = 1$ or ω^{-1} . Then we have*

$$\lambda_p(H_\theta(T)) = p^{n_c} + p^{n_\epsilon} - q.$$

To prove THEOREM 2, we need several lemmas.

LEMMA 1. *Let $m = \sum_{j=0}^\infty m_j p^j$ and $n = \sum_{j=0}^r n_j p^j$ with $0 \leq m_j, n_j < p$. Then we have*

$$\binom{m}{n} \equiv \binom{m_0}{n_0} \cdots \binom{m_r}{n_r} \pmod{p}.$$

PROOF. Since this is well-known, we omit the proof.

Hereafter we write $n = n'_0 + n'_1 q$ with $0 \leq n'_0 \leq q - 1$ for an integer n .

Let α be an $\mathcal{O}$ -valued measure on $\mathbf{Z}_p$ and $F(T)$ be the power series associated to α , and let $\wp(n)$ be the ideal of $\mathcal{O}[[T - 1]]$ generated by $\wp$ and $(T - 1)^n$ for $n \geq 0$. For $1 \leq i \leq q - 1$ with $(i, p) = 1$, we have the following:

LEMMA 2. Suppose $\mu_p(F^{(i)}(T)) = 0$ and $\lambda_p(\alpha|_{i+q\mathbf{Z}_p}) = \lambda$. Then we have

$$\alpha^{(i)}_n \equiv \binom{i}{n'_0} \alpha^{(i)}_{qn'_1} \pmod{p} \quad \text{for } n \geq 0.$$

In particular, $\lambda \equiv 0 \pmod{q}$ and $\alpha^{(i)}_{\lambda+1} \not\equiv 0 \pmod{\wp}$. $\square$

PROOF. By the fundamental relation of measures and power series, we have

$$\begin{aligned} \alpha^{(i)}_n &= \int_{i+q\mathbf{Z}_p} \binom{x}{n} d\alpha(x) \\ &\equiv \int_{i+q\mathbf{Z}_p} \binom{i}{n'_0} \binom{x}{qn'_1} d\alpha(x) \pmod{p}. \end{aligned}$$

Therefore the lemma is now clear.

LEMMA 3. Let β be a measure and $G(T)$ be the power series associated to β . Then the following conditions (i) and (ii) are equivalent:

- (i) $\alpha \equiv \beta \pmod{\wp}$
- (ii) $F(T) \equiv G(T) \pmod{\wp[[T - 1]]}$.

PROOF. The proof is easy.

LEMMA 4. For $a \in \mathbf{Z}_p$ such that $a \equiv 1 \pmod{q}$, we have

$$(F(T^a))^{(i)} = F^{(i)}(T^a). \quad (2)$$

PROOF. The measure associated to the left hand side of (2) is $\alpha \circ a^{-1}|_{i+q\mathbf{Z}_p}$, and it is obviously equal to $(\alpha|_{i+q\mathbf{Z}_p}) \circ a^{-1}$. This is nothing but the measure associated to the right hand side of (2). $\square$

LEMMA 5. If $\lambda_p(F(T)) = \lambda$, then we have

$$F(T^a) \equiv \alpha_\lambda a^\lambda (T - 1)^\lambda \pmod{\wp(\lambda + 1)} \quad \text{for } a \in \mathbf{Z}_p^\times.$$

PROOF. The proof is easy.

LEMMA 6. Furthermore if $\lambda \equiv 0 \pmod q$ and $a \equiv 1 \pmod q$, then we have

$$F(T^a) \equiv F(T) + a_1 \alpha_{\lambda+1} (T-1)^{\lambda+p^{n_a}} \pmod{\wp(\lambda+p^{n_a}+1)}.$$

PROOF. We substitute

$$T^a - 1 \equiv T - 1 + a_1(T-1)^{p^{n_a}} + a_1(T-1)^{p^{n_a}+1} \pmod{\wp(p^{n_a}+2)}$$

for

$$F(T^a) = \sum_{n=0}^{\infty} \alpha_n (T^a - 1)^n.$$

Then we obtain what we want.

We note that by LEMMA 2, we can use LEMMA 6 for $F^{(i)}(T)$ such that $\mu_p(F^{(i)}(T)) = 0$.

LEMMA 7. We set

$$\tilde{F}_\tau(T) = \sum_{a=1}^{\ell} \frac{\tau(a)T^a}{T^a - 1}.$$

Then we have

$$\tilde{F}_\tau^{(i)}(T) \equiv -i\ell_1(T-1)^{p^{n_\ell}-q} \pmod{\wp(p^{n_\ell}-q+1)}.$$

PROOF. We note that by the assumption on τ , $\tilde{F}_\tau(T)$ belongs to $\mathcal{O}[[T-1]]$ and $\ell \equiv 1 \pmod p$. By the fundamental relation of measures and power series, if $\tilde{\alpha}$ is associated to $\tilde{F}_\tau(T)$, then we have

$$\begin{aligned} \tilde{F}_\tau^{(i)}(T) &= \int_{\mathbf{Z}_p} \frac{1}{q} \sum_{\zeta_q} \zeta_q^{x-i} T^x d\tilde{\alpha}(x) \\ &= \frac{1}{q} \sum_{\zeta_q} \zeta_q^{-i} \tilde{F}_\tau(\zeta_q T) \\ &= \sum_{a=0}^{\ell-1} \frac{\tau(i+qa)T^{i+qa}}{T^{q\ell}-1}. \end{aligned}$$

Hence

$$\begin{aligned} (T^{q\ell}-1)\tilde{F}_\tau^{(i)}(T) &= \sum_{a=0}^{\ell-1} \tau(i+qa)T^{i+qa} \\ &\equiv \sum_{a=0}^{\ell-1} T^{i+qa} - T^{\ell j} \pmod{\wp[[T-1]]}, \end{aligned} \tag{3}$$

where $j = 4 - i$ or i according as $p = 2$ and $\ell \equiv 3 \pmod{4}$ or not. Let I_n be the n -th coefficient of (3) as a polynomial in $T - 1$ for $n \geq 0$. Then we have

$$\begin{aligned}
 I_n &= \sum_{a=0}^{\ell-1} \binom{i+qa}{n} - \binom{\ell j}{n} \\
 &= \binom{i}{n'_0} \sum_{a=0}^{\ell-1} \binom{a}{n'_1} - \binom{\ell j}{n} \pmod{p} \\
 &\equiv \binom{i}{n'_0} \binom{\ell}{1+n'_1} - \binom{\ell j}{n} \pmod{p} \\
 &\equiv \begin{cases} \binom{i}{n'_0} \{ \binom{1+\ell_1 p^{n_\ell}}{1+n'_1} - \binom{i \ell_1 p^{n_\ell}/q}{n'_1} \} & \pmod{p} \text{ if } i = j, \\ \binom{i}{n'_0} \{ \binom{2^{n_\ell}-1-(\ell_1+1)2^{n_\ell}}{1+n'_1} - \binom{2^{n_\ell-2}-1-\{(4-i)\ell_1+1\}2^{n_\ell-2}}{n'_1} \} & \pmod{2} \text{ if } i \neq j. \end{cases} \\
 &\equiv \begin{cases} 0 & \pmod{p} \text{ if } n < p^{n_\ell}, \\ -i\ell_1 & \pmod{p} \text{ if } n = p^{n_\ell}. \end{cases}
 \end{aligned}$$

This completes the proof of LEMMA 7.

Now we can prove THEOREM 2.

PROOF. Suppose $\theta \neq 1$ and ω^{-1} . Then

$$F_{\theta\tau}(T) \equiv F_\theta(T) - \theta(\ell)F_\theta(T^{(\ell)}) \pmod{\wp[[T-1]]}.$$

Because of $F_\theta(T^{-1}) \equiv F_\theta(T) \pmod{\wp[[T-1]]}$ if $p = 2$ and $\theta \neq 1$ and the condition $\ell \equiv 1 \pmod{p}$, we have

$$F_\theta(T^{(\ell)}) \equiv F_\theta(T^{(\ell)}) \pmod{\wp[[T-1]]}.$$

Hence, by LEMMAS 3 and 4, we have

$$F_{\theta\tau}^{(i)}(T) \equiv F_\theta^{(i)}(T) - \theta(\ell)F_\theta^{(i)}(T^{(\ell)}) \pmod{\wp[[T-1]]}.$$

Using this, the definition of $H_{\theta\tau}(T)$ implies

$$H_{\theta\tau}(T) \equiv H_\theta(T) - \theta(\ell)H_\theta(T^{(\ell)}) \pmod{\wp[[T-1]]}.$$

Because $\mu_p(H_{\theta\tau}(T)) = 0$ by Theorem 1 of [8], $H_{\theta\tau}(T)$ satisfies the conditions of LEMMAS 2 and 6. Therefore we obtain

$$\begin{aligned}
 H_{\theta\tau}(T) &\equiv (1 - \theta(\ell))H_\theta(T) + \text{"a } p\text{-adic unit"} (T-1)^{\lambda+p^{n_\ell}} \\
 &\pmod{\wp(\lambda + p^{n_\ell} + 1)},
 \end{aligned}$$

where λ is the λ_p -invariant of $H_\theta(T)$. This means that our proposition holds.

Next suppose $\theta = 1$ or ω^{-1} . Then we have

$$F_{\theta\tau}^{(i)}(T) = \theta(\ell)F_\tau^{(i)}(T).$$

By writing $F_\tau(T) = \tilde{F}_\tau(T) - c\tau(c)\tilde{F}_\tau(T^c)$, LEMMAS 3 and 4 imply

$$F_\tau^{(i)}(T) \equiv \tilde{F}_\tau^{(i)}(T) - \tilde{F}_\tau^{(i)}(T^c) \pmod{\wp[[T-1]]}.$$

Hence by LEMMAS 2, 6 and 7, we have

$$\begin{aligned} F_\tau^{(i)}(T) &\equiv \text{"a } p\text{-adic unit independent on } i" \\ &\quad \times i^2(T-1)^{p^{n_c}+p^{n_\ell}-q} \pmod{\wp(p^{n_c}+p^{n_\ell}-q+1)}. \end{aligned}$$

By LEMMA 5, (2) and (3), we obtain

$$\begin{aligned} H_{\theta\tau}(T) &\equiv \text{"a } p\text{-adic unit"} (T-1)^{p^{n_c}+p^{n_\ell}-q} \\ &\quad \pmod{\wp(p^{n_c}+p^{n_\ell}-q+1)}. \end{aligned}$$

This completes the proof of THEOREM 2. $\square$

§4. An application

We give an elementary proof of the formula of Kida for abelian number fields as an application of THEOREM 1.

Let K be a finite abelian extension of $\mathbf{Q}$. We use the following notation associated to K :

K^+ : the maximal real subfield of K .

K_∞ : the cyclotomic $\mathbf{Z}_p$ -extension of K .

f_K : the conductor of K .

$\lambda_p^-(K)$: the minus part of cyclotomic λ_p -invariant of K .

X_K (resp. X_K^-): the set of (resp. odd) Dirichlet characters associated to K .

$X_K(\ell)$ (resp. $X_K^-(\ell)$): all of elements of X_K (resp. X_K^-) whose conductors are divisible by a prime number ℓ .

χ_K : an element of X_K .

$\delta(K)$: 1 or 0 according as K contains ζ_q or not, where ζ_q is a primitive root of the unity.

In this notation, the formula of Kida is expressed as follows:

PROPOSITION 2. *Let K and F be finite imaginary abelian extensions of $\mathbf{Q}$ such that K is a p -extension of F . Then we have*

$$\lambda_p^-(K) - \delta(K) = [K_\infty : F_\infty] \{ \lambda_p^-(F) - \delta(F) \} + \sum_l (e_l - 1) - \sum_{l_+} (e_{l_+} - 1),$$

where the summation is taken over all prime ideals l of K_∞ (resp. l_+ of K_∞^+) which do not lie above p , and e_l (resp. e_{l_+}) denotes the ramification index of l (resp. l_+) in K_∞/F_∞ (resp. K_∞^+/F_∞^+).

We note that

1) it is easy to check that if PROPOSITION 2 holds for K/F and L/F , it holds for K/F , where L is an intermediate field of K/F . This allows us to reduce ourselves to the case where $[F^+ : \mathbf{Q}]$ is not divisible by p if $p > 2$,

2) we may assume without loss of generality that f_K is not divisible by qp , and

3) by the above assumption, it is easy to see

$$[K_\infty : F_\infty] = [K : F]$$

and

$$\begin{aligned} & \sum_l (e_l - 1) - \sum_{l_+} (e_{l_+} - 1) \\ &= \begin{cases} \sum_\ell p^{n_\ell-1} \#X_E(\ell) \#\{\chi_F : \text{odd}, \chi_F(\ell) = 1\} & \text{if } p > 2, \\ \sum_\ell 2^{n_\ell-2} \{\#X_E^-(\ell) - [K : F] \#X_{E \cap F}^-(\ell)\} \#\{\chi_L : \chi_L(\ell) = 1\} & \text{if } p = 2, \end{cases} \end{aligned}$$

where the summation is taken over all prime numbers ℓ distinct from p and E is the maximal p -extension of $\mathbf{Q}$ in K and L is the maximal extension of $\mathbf{Q}$ in K with $p \nmid [L : \mathbf{Q}]$.

PROOF. For $\chi_K \in X_K^-$ distinct from ω^{-1} , it is well-known that

$$\lambda_p^-(K) = \sum_{\chi_K : \text{odd}, \neq \omega^{-1}} \lambda(\chi_K).$$

By THEOREM 1, we have in the case of $p > 2$

$$\begin{aligned} \lambda_p^-(K) &= \sum_{\chi_F : \text{odd}, \chi_E \neq 1} \lambda(\chi_F \chi_E) + \sum_{\chi_F : \text{odd}, \neq \omega^{-1}} \lambda(\chi_F) \\ &= [K : F] \lambda_p^-(F) + \delta(F) \sum_{\chi_E \neq 1} \lambda(\omega^{-1} \chi_E) \\ &\quad + \sum_\ell p^{n_\ell-1} \#X_E(\ell) \#\{\chi_F : \text{odd}, \chi_F(\ell) = 1\} - \delta(F). \end{aligned}$$

Since $\delta(K) = \delta(F)$, we complete the proof in the case of $p > 2$. Next, suppose $p = 2$, and we begin with the calculation of $\lambda_2^-(E)$. By THEOREM 1, we have

$$\begin{aligned} \lambda_2^-(E) &= \sum_\ell 2^{n_\ell-2} \#X_E^-(\ell) - \{\#X_E^- - \delta(E)\} \\ &= \sum_\ell 2^{n_\ell-2} \#X_E^-(\ell) - [E^+ : \mathbf{Q}] + \delta(K). \end{aligned}$$

By a similar calculation for K , we have

$$\lambda_2^-(K) = \sum_{\chi_L \neq 1, \chi_E: \text{odd}} \lambda(\chi_L \chi_E) + \sum_{\chi_E: \text{odd}, \neq \omega^{-1}} \lambda(\chi_E).$$

Hence we obtain

$$\begin{aligned} \lambda_2^-(K) - \delta(K) &= [E^+ : \mathbf{Q}] \left(\sum_{\chi_L \neq 1} \lambda(\chi_L) - 1 \right) \\ &\quad + \sum_{\ell} 2^{n_{\ell}-2} \# X_E^-(\ell) \# \{ \chi_L : \chi_L(\ell) = 1 \}. \end{aligned} \quad (4)$$

If we set $K = F$ in the above equality, then we obtain

$$\begin{aligned} \lambda_2^-(F) - \delta(F) &= [E^+ \cap F : \mathbf{Q}] \left(\sum_{\chi_L \neq 1} \lambda(\chi_L) - 1 \right) \\ &\quad + \sum_{\ell} 2^{n_{\ell}-2} \# X_{E \cap F}^-(\ell) \# \{ \chi_L : \chi_L(\ell) = 1 \}. \end{aligned} \quad (5)$$

By (4) and (5) we obtain what we want.

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