Amalgams of quasithin solvable groups

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(Received May 7, 1990)

§1. Introduction

In this paper we study the triples (X, S, Y) of finite groups satisfying the following conditions:

- (a) S is a common Sylow 2-subgroup of X and Y ;
- (b) both X and Y are 2-irreducible;
- (c) no nonidentity subgroup of S is normal both in X and in Y ;
- (d) $C_X(O_2(X)) \subseteq O_2(X)$ and $C_Y(O_2(Y)) \subseteq O_2(Y)$.

Here, a group G is said to be H -irreducible for a subgroup H of G if H is contained in a unique maximal subgroup of G , and a finite group G is said to be 2-irreducible if G is T -irreducible for a Sylow 2-subgroup T of G . We will call the triple (X, S, Y) an *amalgam* because we often embed the groups S , X , Y into the amalgamated product of X and Y over S .

In previous papers [7] and [11], we studied such amalgams of thin solvable groups, and determined the isomorphism types (or the isomorphism classes) of possible ones. Applying the result, Gomi [3] gave an alternative short proof of Janko's theorem [9] which classifies the finite thin simple groups with solvable 2-local subgroups. The success has stimulated our study to obtain a similar alternative proof of Smith's theorem [10] which classifies the finite quasithin simple groups with solvable 2-local subgroups.

The purpose of this paper is to lay a foundation for such a proof by extending the result of [7] to amalgams of quasithin solvable groups. In fact, in their forthcoming paper [6], Gomi and Hayashi will apply our result to an alternative proof of Smith's theorem.

In order to state our result precisely, we review here the following definition.

A finite group G is said to be *thin* (resp. *quasithin*) if $e(G) \leq 1$ (resp. $e(G) \leq 2$), where the invariant $e(G)$ is defined as follows:

$$\begin{aligned} e(G) &= \max\{m_{2,p}(G) \mid 2 \neq p \in \pi(G)\}; \\ m_{2,p}(G) &= \max\{m_p(L) \mid L \text{ is a 2-local subgroup of } G\}; \\ m_p(L) &= \max\{m_p(E) \mid E \text{ is an elementary abelian } p\text{-subgroup of } L\}. \end{aligned}$$

In this terminology, the main result of [7] may be stated as follows.

THEOREM A ([7]). *Let S, X, Y be finite groups satisfying the conditions (a)–(d) and the following additional condition (e₁):*

(e₁) *both X and Y are solvable and thin.*

Then the amalgam (X, S, Y) or (Y, S, X) is a $GL_3(2)$ -amalgam, an $Sp_4(2)$ -amalgam, a $G_2(2)'$ -amalgam, a $G_2(2)$ -amalgam, an M_{12} -amalgam, an $\text{Aut}(M_{12})$ -amalgam, a ${}^2F_4(2)'$ -amalgam, or a ${}^2F_4(2)$ -amalgam.

We will give no precise definition of the above eight types of amalgams, but we will only note that the isomorphism classes of S, X, Y are completely determined. (See [7] and [11] for details.)

Now, our result may be stated as follows.

THEOREM B. *Let S, X, Y be finite groups satisfying the conditions (a)–(d) and the following additional conditions (e₂) and (g):*

(e₂) *both X and Y are solvable and quasithin;*

(g) $S = [O_2(X), O^2(X)](O_2(X) \cap O_2(Y))[O_2(Y), O^2(Y)]$.

Then both X and Y are thin.

We note that Theorem B together with Theorem A determines the isomorphism classes of S, X, Y for all amalgams (X, S, Y) satisfying the conditions (a)–(d), (e₂), and (g). Also, Theorem B really generalizes Theorem A because the amalgams considered in [7] satisfy the condition (g).

Though the invariant $e(G)$ for a finite group G has been playing an important role in the finite group theory, the new invariant $f(G)$ defined as follows seems more essential than $e(G)$ to the analysis both in [6] and in this paper:

$$f(G) = \max\{n_{2,p}(G) \mid 2 \neq p \in \pi(G)\};$$

$$n_{2,p}(G) = \max\{n_p(L) \mid L \text{ is a 2-local subgroup of } G\};$$

$$n_p(L) = \max\{m_p(V) \mid V \text{ is a chief } p\text{-factor of } L\}.$$

Indeed, it is not Theorem B but the following theorem that is proved in this paper and applied to [6].

THEOREM C. *Let S, X, Y be finite groups satisfying the conditions (a)–(d), (g), and the following condition (f₂):*

(f₂) *both X and Y are solvable and $f(X) \leq 2 \leq f(Y)$.*

Then both X and Y are thin.

Since Thompson observed in his analysis in the N -group paper [12] that $e(G) \leq 2$ implies $f(G) \leq 2$, Theorem C is stronger than Theorem B. Moreover, since [6] needs the structure of only the amalgams with the conditions (a)–(d), (f₂), and (g), our study is sufficient to an application to Smith's theorem.

It may be possible to generalize Theorem C further by eliminating the condition (g), but such a generalization does not seem to have important applications.

On the other hand, it is no doubt relevant to replace the condition (f_2) by the following weaker condition (f):

(f) both X and Y are solvable.

In fact, this kind of generalization is necessary to a similar alternative proof of a theorem of Gorenstein and Lyons [8] which completes the classification of the finite simple groups with solvable 2-local subgroups.

Theorem C is established by a graph theoretical approach of Delgado, Goldshmidt, and Stellmacher [2]. For an amalgam (X, S, Y) , we construct the amalgamated product G of X and Y over S , and define $\Gamma = \Gamma(X, S, Y)$ to be the (right) coset graph of G with respect to the subgroups X and Y . Then G acts on Γ by (right) multiplication. For a vertex ξ of Γ , we denote by G_ξ the stabilizer of ξ in G , and put $Q_\xi = O_2(G_\xi)$ and $V_\xi = \langle \Omega_1(Z(T)) \mid T \in \text{Syl}_2(G_\xi) \rangle$. Finally, we set $b(X, S, Y)$ to be the minimum distance $d(\alpha, \delta)$ of $\alpha, \delta \in V(\Gamma)$ under the condition that $V_\alpha \not\subseteq Q_\delta$.

Suppose that the amalgam (X, S, Y) satisfies the hypothesis of Theorem C. We divide the proof of Theorem C into three parts by the value of $b = b(X, S, Y)$: the small case; the even case; the odd case. First, we deal with the small case where $b \leq 2$ in Section 3. In this case, we can replace the condition (f_2) by the weaker condition (f). Next, we treat the case where b is even and $b \geq 4$ in Section 5, and the case where b is odd and $b \geq 3$ in Sections 6–8. Contrary to the small case, our present proof needs the condition (f_2) in the latter two cases.

Since most of the analysis of this paper is devoted to determination of the chief 2-factors involved within X and Y , the theory of $GF(2)$ -representations for 2-irreducible solvable groups is of fundamental importance. The necessary lemmas on $GF(2)$ -representations for such groups are collected in the next section.

We close this section with the following four lemmas.

(1.1) *Let S be a 2-group, and t a noncentral involution of S . Then the following holds.*

(1) *There exists $u \in t^S$ such that $\langle t, u \rangle \cong \mathbf{E}_{2^2}$.*

(2) *If $m(S) = 2$, then $\Omega_1(Z(S)) \subseteq \langle t, u \rangle$ and $\langle t, u \rangle$ is the unique fours subgroup of S containing t .*

PROOF. Take $s \in N_S(C_S(t)) - C_S(t)$, and put $u = t^s$. Then u has the desired property. Suppose $m(S) = 2$, and let $t \in E \subseteq S$ and $E \cong \mathbf{E}_{2^2}$. Then $\Omega_1(Z(S)) \subseteq E$ and so $E = \langle t, \Omega_1(Z(S)) \rangle$. Thus E is the unique fours subgroup of S containing t .

(1.2) *Let t be an involution inverting an abelian group P of odd order, and V a $GF(2)P\langle t \rangle$ -module with $C_V(P) = 0$. Then $|V| = |V : C_V(t)|^2$.*

PROOF. We may assume that V is faithful and indecomposable. Let C be

a nonidentity cyclic subgroup of P , and take $u \in t^C$ so that $C = \langle tu \rangle$. Then, by the $P\langle t \rangle$ -invariant decomposition $V = [V, C] \oplus C_V(C)$, we have $C_V(t) \cap C_V(u) \subseteq C_V(C) = 0$, and hence $|V| = |V : C_V(t)|^2$.

(1.3) *Let G be a finite group acting on a normal series*

$$T = T_0 \supseteq T_1 \supseteq \cdots \supseteq T_{r-1} \supseteq T_r = 1$$

of a finite group T . Then

$$|T : U| \cdot |[U, A]| \geq \prod_{i=1}^r |T_{i-1}/T_i : U_i/T_i| \cdot |[U_i/T_i, A]|$$

for a subgroup A of G and an A -invariant subgroup U of T , where $U_i = (T_{i-1} \cap U)T_i$ ($1 \leq i \leq r$).

PROOF. Define $V_i = (T_{i-1} \cap [U, A])T_i$ ($1 \leq i \leq r$). Then we have

$$[U_i/T_i, A] \subseteq V_i/T_i \quad (1 \leq i \leq r), \text{ and}$$

$$|T : U| \cdot |[U, A]| = \prod_{i=1}^r |T_{i-1}/T_i : U_i/T_i| \cdot |V_i/T_i|.$$

Hence the assertion follows.

(1.4) *Let P be a p -group for some odd prime p acting on a 2-group T . Suppose that $P = P_1 P_2 \triangleright P_i$ ($i = 1, 2$), that $T = C_T(P_1) C_T(P_2)$, and that $C_T(P) = 1$. Then*

$$T = C_T(P_1) \times C_T(P_2).$$

PROOF. It suffices to prove that $C_T(P_i) \triangleleft T$ ($i = 1, 2$). Let $\{i, j\} = \{1, 2\}$. Then, since P_j acts on $C_T(P_i)$, we have

$$C_T(P_i) = [C_T(P_i), P_j] C_{C_T(P_i)}(P_j) = [C_T(P_i) C_T(P_j), P_j] C_T(P) = [T, P_j] \triangleleft T.$$

§2. 2-Irreducible solvable groups

Throughout this section, we will assume that all groups are finite and that all $GF(2)$ -representations of groups are finite dimensional. This section is devoted to a preliminary study of 2-irreducible solvable groups.

The first lemma describes the fundamental structure of such groups. Recall that $n_p(G) = \max\{m_p(V) \mid V \text{ is a chief } p\text{-factor of } G\}$ for a group G and an odd prime p .

LEMMA 2.1 *Let G be a 2-irreducible solvable group, S a Sylow 2-subgroup of G , and M the unique maximal subgroup of G containing S . Put $Q = O_2(G)$. Then the following holds.*

- (1) $O^2(G) \neq 1$.
 - (2) *If $N \triangleleft G \neq SN$ (or, equivalently, $O^2(G) \not\subseteq N \triangleleft G$), then $S \cap N \triangleleft G$, $Q \in \text{Syl}_2(QN)$, $O_2(G/N) = QN/N$, and G/N is 2-irreducible and solvable.*
 - (3) $G = O_{2,p,2}(G)$ for some odd prime p .
- Let $P \in \text{Syl}_p(G)$.*
- (4) S/Q acts irreducibly and faithfully on $PQ/\Phi(P)Q$.
 - (5) $Z(S/Q)$ is cyclic, and an element t of S inverts $PQ/\Phi(P)Q$ if and only if $Q \neq tQ \in \Omega_1(Z(S/Q))$.
 - (6) *If $n_p(G) \leq 2$, then $m_p(P/\Phi(P)) \leq 2$ and $m_2(S/Q) \leq 2$.*
 - (7) *The following holds:*
 - (i) P is cyclic if and only if $n_p(G) = 1$;
 - (ii) if S/Q is elementary abelian, then $|S/Q| \leq 2$ and P is cyclic;
 - (iii) if P is cyclic, then S/Q is cyclic.
 - (8) $M = \Phi(P)S$, $\cap M^G = \Phi(P)Q$, and $\Phi(P)Q/Q = \Phi(O^2(G/Q))$.
 - (9) *If $O^2(G/QN)$ is cyclic and nonidentity (or, equivalently, $n_p(G/QN) = 1$) for some $N \triangleleft G$, then P is cyclic and $n_p(G) = 1$.*
 - (10) *If $Q \neq zQ \in \Omega_1(Z(S/Q))$ for an element $z \in S$, then the following four conditions for an element $x \in G$ are equivalent:*
 - (i) $x \notin M$;
 - (ii) $G = \langle S, z^x \rangle$;
 - (iii) $G = \langle S, S^x \rangle$;
 - (iv) $G = \langle S, x \rangle$.
 - (11) *If $x \in S - Q$, there exists $T \in \text{Syl}_2(G)$ such that $G = \langle T, x \rangle$.*

PROOF. Part (1) holds because $S \subseteq M \subset G$.

Let $N \triangleleft G \neq SN$. Then $G = N_G(S \cap N)SN$ by a Frattini argument. Thus the 2-irreducibility of G shows that $G = N_G(S \cap N)$, and hence $S \cap N \triangleleft G$. Since $QN \triangleleft G \neq SQN$, we also have $Q \subseteq S \cap QN \triangleleft G$, so $Q = S \cap QN \in \text{Syl}_2(QN)$. Let $\bar{G} = G/N$. Then $\bar{S} \in \text{Syl}_2(\bar{G})$ and $\bar{M}$ is the unique maximal subgroup of $\bar{G}$ containing $\bar{S}$. Suppose R is the inverse image of $O_2(\bar{G})$ in G . Then $QN \subseteq R \subseteq SN$, so $Q = S \cap R \in \text{Syl}_2(R)$ as above. Thus the index $|R : QN|$ is odd, and hence $R = QN$. This proves (2).

For the remainder of this proof, we may assume $Q = 1$, replacing G by G/Q .

The solvability of G shows that either $G = O(G)$ or $O_2(G/O(G)) \neq 1$, so $G = SO(G)$ by (2). Therefore $G = O_{2,p,2}(G)$. Let p be a prime divisor of the index $|G : M|$, and choose a Hall $\{2, p\}$ -subgroup, say H , of G containing S . Then we have $H \not\subseteq M$, and so $G = H$ by the 2-irreducibility of G . Thus (3) holds.

Now, let $P \in \text{Syl}_p(G)$ and $W = P/\Phi(P)$. Then $P = O_p(G)$ by (3), and so

$\Phi(P) \triangleleft G$ and S acts faithfully on W . Since W is completely reducible as a $GF(p)S$ -module by Maschke's theorem, the 2-irreducibility of G forces that S is irreducible on W . This proves (4), and hence $Z(S)$ is cyclic and $\Omega_1(Z(S))$ inverts W by Schur's lemma. Put $\Omega_1(Z(S)) = \langle z \rangle$, and suppose W is inverted by some element $t \in S$. Then $tz \in C_S(W) = 1$, and so $t = z$. Thus (5) holds.

In proving (6) and (7), let $m = m_p(W)$. Then we have $1 \leq m \leq n_p(G)$ because $P \neq 1$ by (1) and W is a chief factor of G by (3) and (4). Since S is isomorphic to an irreducible subgroup of $\text{Aut}(W)(\cong GL_m(p))$ by (4), we conclude that $m_2(S) \leq 2$ if $n_p(G) \leq 2$, and that S is cyclic if $m = 1$. This proves (6) and (iii) of (7). Part (i) of (7) holds because $P \triangleleft G$ by (3), and (ii) of (7) holds by (4) and (5).

Since $P = O^2(G)$ by (3), we have $\Phi(P) = \Phi(O^2(G))$. Put $L = \cap M^G$. Let $N \triangleleft G$, and suppose $O^2(G/N) \neq 1$. Then we have $G \neq SN$, so N has odd order by (2). Thus $L \subseteq P$ by setting $N = L$, and so $N \subseteq L \subseteq P \cap M \neq P$. Therefore we have $\Phi(P) = L = P \cap M$ by setting $N = \Phi(P)$, and hence $N \subseteq \Phi(P)$ and $M = S\Phi(P)$. This proves (8). Suppose that $O^2(G/QN)$ is cyclic and nonidentity. Since G/N is 2-irreducible and solvable with $O_2(G/N) = 1$ by (2), we have $O^2(G/N) = P/N$ by (3). Hence P is cyclic as $N \subseteq \Phi(P)$ by (8), and so $n_p(G) = 1$ by (7). Thus (9) holds.

To prove (10), first let $x \notin M$. Put $\overline{G} = G/\Phi(P)$, and take an element $y \in P$ so that $\overline{x} \in \overline{S}y$. Since $\overline{z}$ inverts $\overline{y}$ by (4), $\overline{y} \in \langle \overline{z}, \overline{z}^{\overline{y}} \rangle \subseteq \langle \overline{S}, \overline{z}^{\overline{y}} \rangle = \langle \overline{S}, \overline{z}^{\overline{x}} \rangle$, so $M \not\cong y \in \langle S, z^x \rangle \Phi(P)$. Therefore $\langle S, z^x \rangle \not\subseteq M$, and hence $G = \langle S, z^x \rangle$. This shows that (i) implies (ii). Clearly, (ii) implies (iii), (iii) implies (iv), and (iv) implies (i).

Finally, let $1 \neq x \in S$. Since $x \notin \Phi(P) = \cap M^G$ by (8), there exists an element $g \in G$ such that $x \notin M^g$. Thus we have $G = \langle S^g, x \rangle$ by (10). This proves (11).

In the next two propositions, we need the following definitions.

Let S be a 2-group. We will denote by $\mathcal{A}(S)$ the set of elementary abelian subgroups of S of maximal order. Put $J(S) = \langle \mathcal{A}(S) \rangle$, the Thompson subgroup of S , and $K(S) = C_S(\Omega_1(Z(J(S))))$. The characteristic subgroup $Q(S)$ of S is defined in [5].

PROPOSITION 2.2 *Let G be a 2-irreducible solvable group with $C_G(O_2(G)) \subseteq O_2(G)$, and S a Sylow 2-subgroup of G . Suppose that $\Omega_1(Z(S)) \not\subseteq Z(G)$ and that $Q(K(S)) \not\triangleleft G$. Then*

$$[O_2(G), O^2(G)] \subseteq \langle \Omega_1(Z(S))^G \rangle.$$

PROOF. Let $Z = \Omega_1(Z(S))$ and $V = \Omega_1(Z(O_2(G)))$. Then we have $Z \subseteq V$ because $C_G(O_2(G)) \subseteq O_2(G)$. Put $\overline{G} = G/C_G(V)$. We will regard V as a faithful $GF(2)\overline{G}$ -module.

Since G satisfies Hypothesis III of [4], the pair $(\overline{G}, V)$ satisfies Hypothesis II of [4] by 6.6 of [4]. Therefore 5.1, 5.3, and 5.4 of [4] force G to have a unique noncentral

chief factor within V , and then it lies within $\langle Z^G \rangle$ because $Z(G) \not\subseteq Z \subseteq V$. Since $[O_2(G), O^2(G)] \subseteq V$ by 6.7 of [4], we conclude that $[O_2(G), O^2(G)] \subseteq \langle Z^G \rangle$.

PROPOSITION 2.3 *Let G be a 2-irreducible solvable group with $C_G(O_2(G)) \subseteq O_2(G)$, S a Sylow 2-subgroup of G , and A a group of automorphisms of S of odd order. Then some nonidentity A -invariant subgroup of S is normal in G .*

PROOF. Suppose false. Then no nonidentity characteristic subgroup of S is normal in G . Thus, by the corollary of the main theorem of [4], G is described as follows:

$$G = SN;$$

$$N = \langle J(S)^G \rangle = L_1 \times \cdots \times L_m \times D_1 \times \cdots \times D_n \times E;$$

$$L_1 \cong \cdots \cong L_m \cong \Sigma_4;$$

$$D_1 \cong \cdots \cong D_n \cong \mathbf{D}_8;$$

$$E^2 = 1.$$

Let $T = J(S)$. Since $T \subseteq S \cap N = J(S \cap N)$ by the structure of N , we have $T = S \cap N \in \text{Syl}_2(N)$. Now, define $Q = O_2(N)$, $T_i = T \cap L_i$, $Z_i = T'_i$, $V_i = O_2(L_i)$, $\mathbf{E}_{2^2} \cong W_i \subseteq T_i = \langle V_i, W_i \rangle$ ($1 \leq i \leq m$), $Z = Z_1 \times \cdots \times Z_m$, $V = V_1 \times \cdots \times V_m$, $W = W_1 \times \cdots \times W_m$. Then the Krull-Remak-Schmidt theorem together with the 2-irreducibility of G yields that S permutes, by conjugation, $Z_1, \dots, Z_m, D'_1, \dots, D'_n$, and transitively permutes $L'_1, \dots, L'_m$. Thus we conclude that S transitively permutes $Z_1, \dots, Z_m$. Since $[Q, O^2(G)] \subseteq V$, we can take, by our hypothesis, $a \in A$ so that $T = T^a \supseteq V^a \not\subseteq Q = C_T(V)$. Since $S \triangleright [V, V^a] = [V_1, V^a] \times \cdots \times [V_m, V^a]$ and $[V_i, V^a] = 1$ or Z_i ($1 \leq i \leq m$), we have $[V, V^a] = Z$ by the choice of a and the above remark. Therefore $[V^{a^{-1}}, V] = Z^{a^{-1}} \neq 1$, and hence $V^{a^{-1}} \not\subseteq Q$. Thus the same argument also shows that $[V, V^{a^{-1}}] = Z$, so we have

$$Z = [V, V^a] = Z^a.$$

Let $\mathcal{A} = \mathcal{A}(S)$, and put

$$\mathcal{A}_V = \{B \in \mathcal{A} \mid V \subseteq B\} \quad \text{and} \quad \mathcal{A}_W = \{B \in \mathcal{A} \mid W \subseteq B\}.$$

Let $B \in \mathcal{A}$. Then we have

$$B \cap T_i = V_i \quad \text{or} \quad W_i \quad \text{for} \quad 1 \leq i \leq m$$

because $B = (B \cap T_1) \times \cdots \times (B \cap T_m) \times (B \cap (D_1 \times \cdots \times D_n \times E))$. Therefore

$$\mathcal{A}_V \cap \mathcal{A}_W = \phi,$$

and, for $B, C \in \mathcal{A}$ with $[B, C] \supseteq Z$, we have $B \in \mathcal{A}_V$ if and only if $C \in \mathcal{A}_W$.

Now, let $s = |\langle a \rangle|$. Pick $B \in \mathcal{A}_V$, and define $B_k = B^{a^k}$ ($k \geq 0$). Then $B_k \in \mathcal{A}$ and $[B_k, B_{k+1}] \supseteq [V^{a^k}, V^{a^{k+1}}] = [V, V^a]^{a^k} = Z^{a^k} = Z$ ($k \geq 0$), so

$$B_0 = B \in \mathcal{A}_V, B_1 \in \mathcal{A}_W, \dots, B_{s-1} \in \mathcal{A}_V, B = B_s \in \mathcal{A}_W$$

by the above remark. But this contradicts that $\mathcal{A}_V \cap \mathcal{A}_W = \phi$.

Let G be a 2-irreducible solvable group with $C_G(O_2(G)) \subseteq O_2(G)$, and let V be a noncentral chief 2-factor of G . Then the pair (G, V) satisfies the hypothesis of the following lemma, regarding V as a $GF(2)G$ -module.

LEMMA 2.4 *Let G be a 2-irreducible solvable group, S a Sylow 2-subgroup of G , and V a $GF(2)G$ -module with $Q = O_2(G) \subseteq C_G(V) \not\subseteq O^2(G)$. Put $\bar{G} = G/C_G(V)$. Then the following holds.*

- (1) $\bar{G}$ is a 2-irreducible solvable group with $O_2(\bar{G}) = 1$, $C_S(V) = Q$, and $C_G(V)/Q \subseteq \Phi(O^2(G/Q))$.
- (2) $C_G([V, O^2(G)]) = C_G(V)$.
- (3) $V = [V, O^2(G)] \oplus C_V(O^2(G))$.

PROOF. Part (1) follows from (2.1)–(2)(8). For the proof of (2) and (3), put $P = O^2(G)$ and $W = [V, P]$. Note that $O^2(\bar{G}) = \bar{P}$. We may assume $C_G(V) = 1$ replacing G by $\bar{G}$ if necessary. Since $[W, P] = W \neq 0$, we have $P \not\subseteq C_G(W)$, and hence $C_G(W)$ has odd order by (2.1)–(2). Thus $C_G(W) \subseteq O^2(C_G(V/W/1)) \subseteq C_G(V) = 1$, so (2) holds. Part (3) holds because P has odd order by (2.1)–(3).

From now on, we will use the following terminology and notation.

First, let G and H be groups. We will say that a $GF(2)G$ -module V is *induced* by a $GF(2)H$ -module W and denote by $(G, V) \ll (H, W)$ if there exist a one to one homomorphism $\sigma : \bar{G} = G/C_G(V) \rightarrow \bar{H} = H/C_H(W)$ and a $GF(2)$ -isomorphism $f : V \rightarrow W$ such that $(v\bar{g})f = (vf)(\bar{g}\sigma)$ for all $v \in V$ and $g \in G$.

Next, let G be a group acting on a group T . Then we will define

$$\begin{aligned} [T, G; 0] &= T, \quad \text{and} \\ [T, G; k] &= [[T, G; k-1], G] \quad \text{for } k \geq 1. \end{aligned}$$

In order to clarify the structure of chief 2-factors of 2-irreducible solvable groups, we often need several information on faithful $GF(2)\bar{G}$ -modules for 2-irreducible solvable groups $\bar{G}$ with $O_2(\bar{G}) = 1$ by (2.4). Thus, for a while, we will study such $GF(2)$ -representations.

The first three lemmas give precise information on certain $GF(2)$ -representations for 2-irreducible $\{2, 3\}$ -groups with Sylow 3-subgroups isomorphic to $\mathbf{E}_{3^2}$ or Esp_{27} . Here, Esp_{27} denotes the extraspecial group of order 27 and of exponent 3.

LEMMA 2.5 *Let G be a 2-irreducible $\{2, 3\}$ -group with $O_2(G)=1$, $S \in \text{Syl}_2(G)$, $P = O^2(G) \cong \mathbf{E}_{3^2}$, and V a four dimensional faithful $GF(2)G$ -module. Define*

$$V_n = [V, S; n] \quad (n \geq 0), \quad Z = \Omega_1(Z(S)), \quad \text{and} \quad E = C_S(V_2).$$

Then the following holds.

- (1) $(G, V) \ll (O_4^+(2), GF(2)^4)$, $S \cong \mathbf{Z}_4$ or $\mathbf{D}_8$, $|S : E| = 2$, and $Z \subseteq E$.
- (2) *If $S \cong \mathbf{D}_8$ (or, equivalently, $G \cong O_4^+(2)$), then there exist generators σ , τ of S , generators φ , ψ of P , and a basis $\{v_1, v_2, v_3, v_4\}$ of V such that*

$$(\mathcal{R}) \quad \begin{cases} v_1^\sigma = v_4, \quad v_2^\sigma = v_2, \quad v_3^\sigma = v_3, \\ v_1^\tau = v_2, \quad v_3^\tau = v_4, \\ v_1^\varphi = v_4, \quad v_2^\varphi = v_2, \quad v_3^\varphi = v_3, \quad v_4^\varphi = v_1 + v_4, \\ v_1^\psi = v_1, \quad v_2^\psi = v_3, \quad v_3^\psi = v_2 + v_3, \quad v_4^\psi = v_4. \end{cases}$$

- (3) $|V_{n-1} : V_n| = 2 \quad (1 \leq n \leq 4)$, and

$$V = V_0 \supseteq V_1 \supseteq V_2 \supseteq V_3 \supseteq V_4 = 0$$

is the unique S -composition series of V .

- (4) $V_3 = C_V(S)$, $V_2 = [V, T] = C_V(T)$ for $T = Z$ or E , and E is the unique quadratic fours subgroup of S if $S \cong \mathbf{D}_8$.

Let A be an elementary abelian subgroup of S .

- (5) *If $|V : C_V(A)| \leq |A|$ or $|[V, A]| \leq |A|$, then $Z \neq A \subseteq E$ and $|V : C_V(A)| = |A| = |[V, A]|$.*

- (6) *If $Z \neq A \subseteq E$ and $A \subseteq B \subseteq C_S(A)$, then $[V, B] = [V, A] + [C_V(A), B]$.*

- (7) *If $1 \neq A \subseteq E$, then $V_2 = V_3 + [V, A] = C_{V_1}(A)$.*

- (8) *Let $V_2 \subset W \subset V$. If $1 \neq A \subseteq E$, then the following holds:*

- (i) *if $[W, A] = V_2$, then $Z \neq A = E$ and $W = V_1$;*
- (ii) *if $[W, A] = V_3$, then $A = Z$ and $W = V_1$;*
- (iii) *if $[W, A] = 0$, then $Z \neq A \neq E$ and $W \neq V_1$.*

Let $T \subseteq S$.

- (9) *If $T \neq 1$, then $[V, T] \not\subseteq V_3$ and $[V_1, T] \neq 0$.*

- (10) *Suppose $|V : C_V(T)| \leq 2^2$. Then $|[V, T]| \leq 2^2$ and, moreover, $[V, T] \supseteq C_V(S)$ if $|[V, T]| = 2^2$.*

Suppose $S \cong \mathbf{D}_8$.

- (11) *Let $H = EP$ and $x \in G$. Choose $y \in P$ so that $x \in Sy$. Then the following five conditions are equivalent:*

- (i) $E \cap E^x \neq 1$;
- (ii) $H \neq \langle E, E^x \rangle$;

- (iii) $\langle y \rangle \triangleleft H$;
- (iv) $V_2 \cap V_2^x \neq 0$;
- (v) $C_V(y) \neq 0$.
- (12) Let $x \in G$. Then $S \cap S^x \neq 1$ and $E \cap E^x = E \cap S^x = S \cap E^x$.
- (13) Let $x \in G$. Then $S \cap S^x \neq 1$ and $E \cap E^x = E \cap S^x = S \cap E^x$.

PROOF. Embed $G = SP$ into $L = GL(V)$, and let $N = N_L(P)$. Then $G \subseteq N \cong O_4^+(2)$ and N is a Sylow 3-normalizer of L . Choose $R \in \text{Syl}_2(N)$ so that $S \subseteq R$. Then $N = RP$, $S = R \cap G$, and $R \cong \mathbf{D}_8$. Therefore $S \cong \mathbf{D}_8$ or $\mathbf{Z}_4$ by (2.1)–(7) because P is not cyclic.

Let q be the nondegenerate quadratic form on V so that

$$N = \{x \in GL(V) \mid q(v^x) = q(v) \text{ for all } v \in V\}.$$

We can take a basis $\{v_1, v_2, v_3, v_4\}$ of V so that

$$(\mathcal{Q}) \quad \begin{cases} q(v_i) = 1 & (1 \leq i \leq 4), \\ f(v_i, v_j) = \begin{cases} 0 & (i+j \neq 5), \\ 1 & (i+j = 5) \end{cases} \end{cases}$$

where f is the bilinear form on V associated with q . Define the elements $\sigma, \tau, \varphi, \psi$ of L by $(\mathcal{R})$. Then we have $\sigma, \tau, \varphi, \psi \in N$ by $(\mathcal{Q})$. Since $\sigma^2 = \tau^2 = (\sigma\tau)^4 = 1$ and $\varphi^3 = \psi^3 = [\varphi, \psi] = 1$ by $(\mathcal{R})$, $\langle \sigma, \tau \rangle$ is a Sylow 2-subgroup of N and $\langle \varphi, \psi \rangle = O^2(N) = P$. Thus, replacing R by its N -conjugate if necessary, we have

$$R = \langle \sigma, \tau \rangle, \quad S \supseteq \langle \sigma\tau \rangle, \quad Z = \langle (\sigma\tau)^2 \rangle.$$

Note that $V_{n-1} \neq V_n$ if $V_{n-1} \neq 0$. Therefore we have

$$(\mathcal{V}) \quad \begin{cases} V_1 = \langle v_1 + v_3, v_1 + v_2, v_3 + v_4 \rangle, \\ V_2 = \langle v_1 + v_4, v_2 + v_3 \rangle, \\ V_3 = \langle v_1 + v_2 + v_3 + v_4 \rangle, \\ V_4 = 0 \end{cases}$$

by $(\mathcal{R})$, so $|V_{n-1} : V_n| = 2$ ($1 \leq n \leq 4$). Thus

$$V = V_0 \supseteq V_1 \supseteq V_2 \supseteq V_3 \supseteq V_4 = 0$$

is the unique S -composition series of V , and, in particular, $V_3 = C_V(S)$. Let $D = \langle \sigma, Z \rangle$ and $C = \langle \tau, Z \rangle$. Then $[V_2, D] = 0 \neq [V_2, C]$ by $(\mathcal{R})$ and $(\mathcal{V})$, so $D = C_R(V_2)$ and $E = S \cap D \neq S$. Thus $|S : E| = |R : D| = 2$, and so $Z \subseteq E$. Moreover, $(\mathcal{R})$ and $(\mathcal{V})$ show that $V_2 = [V, U] = C_V(U)$ for $U = Z$ or D , and that $[V, C, C] = V_3$.

Therefore we conclude that D is the unique quadratic fours subgroup of R . This proves (1)–(4).

Let A be an elementary abelian subgroup of S .

To prove (5), suppose $|V : C_V(A)| \leq |A|$ or $||[V, A]| \leq |A|$. We may assume $A \neq 1$. If $A \subseteq C$, then $|V : C_V(A)| > |A| < |[V, A]|$ by (R), a contradiction. Therefore we have $Z \not\subseteq A \subseteq D$, and hence $|V : C_V(A)| = |A| = |[V, A]|$ by (R).

For the proof of (6) and (7), we may assume $Z \not\subseteq A \subseteq E$, and so $S \cong D_8$ and $C_S(A) = E$. Hence we may also assume $B = E$ in proving (6). Computation using (R) and (V) shows that $V_3 \not\subseteq [V, A] \subseteq V_2 \subseteq C_{V_1}(A) \subset V_1$ and that $[C_V(A), E] \not\subseteq [V, A]$ if $A \neq E$. Thus

$$V_2 = V_3 + [V, A] = C_{V_1}(A) = [V, A] + [C_V(A), E].$$

In proving (8), let $V_2 \subset W \subset V$. Computation shows that $[V, A] \not\subseteq V_3$ if $Z \not\subseteq A$ and that $V_3 \not\subseteq [W, D] = [W, Z] \subset V_2$ if $W \neq V_1$.

Suppose first that $[W, A] \supseteq V_3$. Then the above remark shows that $Z \subseteq A$, and then it also shows that $W = V_1$. Hence (i) and (ii) hold because $[V_1, D] = V_2$ and $[V_1, Z] = V_3$.

Suppose $[W, A] = 0$. Then $W \neq V_1$ by (7), so $[W, Z] \neq 0$ by the above. Hence we have $Z \not\subseteq A$, so (iii) holds.

Let $T \subseteq S$. If $T \subseteq E$, then (9) holds by (7), and (10) holds because $[V, T] \subseteq V_2$ also by (7). Computation shows that $C_V(\sigma\tau) = V_3 = C_V(C)$ and that $[V, t] = C_V(t) \supseteq V_3$ for $t \in C - Z$. Thus (9) and (10) also hold if $T \not\subseteq E$.

Suppose $S \cong D_8$. Let $H = EP$, and put $K = \langle E, E^x \rangle$ for $x \in P$. Note that $H \cong \Sigma_3 \times \Sigma_3$, and so $Z(H) = 1$.

First, we will prove that $K = \langle E, x \rangle$, so we may assume $K \neq H$. Then $K = E(K \cap P) = E^x(K \cap P) \cong D_{12}$, so $1 \neq Z(K) = C_E(K \cap P) = C_{E^x}(K \cap P) = Z(K)^x$. Therefore $x \in C_H(Z(K)) = K$, and hence $K = \langle E, x \rangle$.

Now, we prove (11). We may assume $x = y$ because $E^x = E^y$ and $V_2^x = V_2^y$. Let $1 \neq t \in E \cap E^x$. Then $K \subseteq C_H(t) \subset H$ and $0 \neq [V, t] \subseteq V_2 \cap V_2^x$. This shows that (i) implies (ii) and (iv). If $H \neq K$, then $\langle x \rangle = K \cap P \triangleleft K^x P = H$, and if $\langle x \rangle \triangleleft H$, then $1 \neq C_E(x) \subseteq E \cap E^x$. This shows that (ii) implies (iii) and that (iii) implies (i). Note that $[V_2 \cap V_2^x, K] = 0$. Thus we have $H \neq K$ and $C_V(x) \neq 0$ if $V_2 \cap V_2^x \neq 0$. This shows that (iv) implies (ii) and (v). Suppose $C_V(x) \neq 0$. Then we have $C_{V_2}(x) = C_{C_V(x)}(Z) \neq 0$ because $C_V(Z) = V_2$ and Z inverts x . Thus $V_2 \cap V_2^x \neq 0$. This shows that (v) implies (iv).

To prove (12), we may assume $x \in P$ as in (11). Let Δ be the set of subgroups of P of order 3, and put $\Delta_1 = \{X \in \Delta \mid X \triangleleft H\}$ and $\Delta_2 = \{X \in \Delta \mid X \not\triangleleft H\}$. Then $|\Delta_1| = |\Delta_2| = 2$ by the structure of H , and G leaves Δ_1 and Δ_2 invariant. Thus we have $|G : N_G(\langle x \rangle)| \leq 2$, and so $C_S(x) \neq 1$ because $|S| = 2^3$ and $|N_G(\langle x \rangle) : C_G(x)| \leq 2$. Since $G = \langle S, S^x \rangle$ and $O_2(G) = 1$, we have $|S \cap S^x| \leq 2$, so $S \cap S^x =$

$C_S(x)$. Consequently, if $C_S(x) \subseteq E$ then $C_S(x) = E \cap E^x = E \cap S^x = S \cap E^x$, and if $C_S(x) \not\subseteq E$ then $1 = E \cap E^x = E \cap S^x = S \cap E^x$.

LEMMA 2.6 *Let $G \cong O_4^+(2)$, and let V be an at most five dimensional $GF(2)G$ -module. Suppose $(G, \bar{V}) \ll (O_4^+(2), GF(2)^4)$ for $\bar{V} = V/C_V(G)$. Denote by $\mathcal{J}$ the set of subgroups J of G of order 3 with $C_{\bar{V}}(J) = 0$. Also, denote by $\mathcal{W}$ the set of S -invariant subspaces W with $\bar{W} = C_{\bar{V}}(E)$ and $V = \langle W^G \rangle$, where S ranges over the set $\text{Syl}_2(G)$ and E is the fours subgroup of S acting quadratically on V . Then, for a given hyperplane H of V , there exist $J \in \mathcal{J}$ and $W \in \mathcal{W}$ such that H contains at most one member of $W^J = \{W^x \mid x \in J\}$.*

PROOF. Suppose false for some hyperplane H of V . Let $W \in \mathcal{W}$. Then $0 \neq C_V(G) \not\subseteq W$ because $\bar{V} = \bar{W} + \bar{W}^x$ for all $x \in O^2(G)$ with $C_{\bar{V}}(x) = 0$ by (2.5)–(11). Moreover, replacing W by its conjugate, if necessary, we may assume $W \subseteq H$.

Let $\{v_0, v_1, v_2, v_3, v_4\}$ be a basis of V so that $C_V(G) = \langle v_0 \rangle$, that $[V, G] = \langle v_1, v_2, v_3, v_4 \rangle$, and that the representation of G on $[V, G]$ is given in (2.5)–(2). We will use the notation introduced there. Then

$$S = \langle \sigma, \tau \rangle \quad \text{and} \quad E = \langle \sigma, (\sigma\tau)^2 \rangle.$$

Since $W \cap [V, G] = C_{[V, G]}(S) = \langle v_1 + v_2 + v_3 + v_4 \rangle$, we have

$$W = \langle v_0 + v_1 + v_4, v_1 + v_2 + v_3 + v_4 \rangle,$$

and so

$$\begin{aligned} W^{\varphi\psi} &= \langle v_0 + v_1, v_1 + v_2 \rangle, & W^{\varphi\psi^{-1}} &= \langle v_0 + v_1, v_1 + v_3 \rangle \\ W^{\varphi^{-1}\psi} &= \langle v_0 + v_4, v_2 + v_4 \rangle, & W^{\varphi^{-1}\psi^{-1}} &= \langle v_0 + v_4, v_3 + v_4 \rangle \end{aligned}$$

by calculation using $(\mathcal{R})$ in (2.5)–(2). Our hypothesis forces that either $W^{\varphi\psi} \subseteq H$ or $W^{\varphi^{-1}\psi^{-1}} \subseteq H$, and that either $W^{\varphi^{-1}\psi} \subseteq H$ or $W^{\varphi\psi^{-1}} \subseteq H$. Then we have $H = V$ for all of the possible four cases, a contradiction.

LEMMA 2.7 *Let G be a 2-irreducible $\{2, 3\}$ -group with $O_2(G) = 1$, $S \in \text{Syl}_2(G)$, $P = O^2(G) \cong \text{Esp}_{27}$, and V a six dimensional faithful $GF(2)G$ -module. Put $\langle t \rangle = \Omega_1(Z(S))$, $Z = \Omega_1(Z(P))$, and $T = C_S(Z)$. Then the following holds.*

- (1) $|S : T| \leq 2$, $t \in T$, $C_V(Z) = 0$, and P is irreducible on V .
- (2) $|V/C_V(t)| = |C_V(t)/[V, t]| = |[V, t]| = 2^2$.
- (3) If $W = V/C_V(t)$, $C_V(t)/[V, t]$, or $[V, t]$, then $C_{SZ}(W) = T$ and $(SZ, W) \ll (GL_2(2), GF(2)^2)$.
- (4) S (resp. T) is embedded in SD_{16} (resp. $\mathbf{Q}_8$), $|T| \geq 4$, all the elements of S of order 4 are contained in T , and $[V, U] = C_V(t)$ and $[C_V(t), U] = [V, t]$ for all subgroups U with $\langle t \rangle \subset U \subseteq T$.

- (5) If $S \neq T$, then $|V : C_V(S)| = |C_V(S)| = 2$, $[V, S, t] = C_V(S)$, $[[V, S, S]] > 2^2$, $[V, S, S, S] \neq 0$, and $S \cong \mathbf{Z}_8$, $\mathbf{D}_8$, or SD_{16} .
- (6) Let A be an elementary abelian subgroup of S .
- (i) If $[V, t, A] = 0$, then $A \subseteq \langle t \rangle$.
- (ii) If $|V : C_V(A)| \leq 2^3$, then $|A| \leq 2$.

PROOF. First, since $Z \cong \mathbf{Z}_3$, we have $|S : T| \leq 2$. Also, (2.1)–(7) shows that $|S| \geq 4$ because $P/Z \cong \mathbf{E}_{3^2}$. Thus $T \neq 1$, and so $t \in T$. Since $GL_5(2)$ has abelian Sylow 3-subgroups, P is irreducible on V , and hence $C_V(Z) = 0$.

Let $H = TP$ and $F = GF(2^2)$. Then V is regarded as a three dimensional FH -module by Schur's lemma because $Z \subseteq Z(H)$. Thus t induces an F -transvection on V , and so $|V/C_V(t)| = |C_V(t)/[V, t]| = |[V, t]| = 2^2$. Let $W = V/C_V(t)$, $C_V(t)/[V, t]$, or $[V, t]$. Then we have $(SZ, W) \ll (GL_2(2), GF(2)^2)$. Since W is considered to be a one dimensional FT -module, we have $[W, T] = 0$, and hence $C_{SZ}(W) = T$. This proves (1)–(3).

Let $\bar{P} = P/Z$. Then S is embedded into $GL(\bar{P})$ because S is faithful on $\bar{P}$. Thus $T = S \cap Sp(\bar{P}) \subseteq SL(\bar{P})$. Therefore S (resp. T) is isomorphic to a subgroup of SD_{16} (resp. $\mathbf{Q}_8$), and hence all the elements of S of order 4 are contained in T . Suppose $|T| = 2$. Then $S \cong \mathbf{Z}_4$ by (1) and (2.1)–(7). Thus $S = T$ by the above, a contradiction. This shows that $|T| \geq 4$. Let $\langle t \rangle \subset U \subseteq T$, and suppose either $[V, U] \subset C_V(t)$ or $[C_V(t), U] \subset [V, t]$. Since V is an FU -module, we have $\dim_F([V, U]) = 1$ or $\dim_F(C_V(U)) = 2$, and so $[V, U^2] \subseteq [V, U, U] = 0$ in either case. Hence $U^2 = 1$, a contradiction. Thus $[V, U] = C_V(t)$ and $[C_V(t), U] = [V, t]$. This proves (4).

Suppose $S \neq T$. Pick $u \in S - T$, and let $W = [V, T]$ or $C_V(T)$. Then $|W : C_W(u)| = |C_W(u)| = 2$ by (3). Therefore $[V, S] = [V, T] + [V, u]$ and $C_V(S) = C_{C_V(T)}(u)$, and then $|V : [V, S]| = |C_V(S)| = 2$. Thus, as $[V, S, t]$ is S -invariant and of order 2, $[V, S, t] = C_V(S)$. Since $[V, S, T] \supset C_V(T) \supset C_V(S)$, we also have $[V, S, S, S] \neq 0$ and $[[V, S, S]] > 2^2$. Since $|S| \geq 8$ and S is not generated by elements of order 4 by (1) and (4), we conclude that $S \cong \mathbf{Z}_8$, $\mathbf{D}_8$, or SD_{16} , so (5) holds.

For the proof of (6), note that $A \cap T \subseteq \langle t \rangle$ by (4). If $[V, t, A] = 0$, then $A \subseteq C_S([V, t]) = T$ by (3), so (i) holds. Suppose $|A| > 2$. Then $A \not\subseteq T$, and so A contains an involution u so that $A = \langle t, u \rangle$ and that u inverts Z . Hence $|V : C_V(A)| = |V : C_V(t)| \cdot |C_V(t) : C_{C_V(t)}(u)| = 2^2 \cdot 2^2 > 2^3$ by (2) and (1.2) because $C_{C_V(t)}(Z) = 0$ by (1). This proves (ii).

Now, we determine the possibility of the pairs of 2-irreducible solvable groups and their $GF(2)$ -representations under the hypothesis that some 2-subgroup has a high dimensional centralizer.

LEMMA 2.8 *Let t be an involution acting on a p -group P for some odd prime*

p with $P = [P, t]$, and V a faithful $GF(2)P\langle t \rangle$ -module with $C_V(P) = 0$. Suppose $|V : C_V(t)| \leq 2^3$. Then one of the thirteen cases listed in the following table occurs.

	$n(P\langle t \rangle, V)$	P	$m(V)$	$m(V/C_V(t))$
(1)	1	$\mathbf{Z}_3$	2	1
(2)	1	$\mathbf{Z}_5$	4	2
(3)	1	$\mathbf{Z}_7$	6	3
(4)	1	$\mathbf{Z}_9$	6	3
(5)	1	Esp_{27}	6	2
(6)	1	$\mathbf{Z}_3 \text{ wr } \mathbf{Z}_3$	6	3
(7)	2	$\mathbf{Z}_3$	4	2
(8)	2	$\mathbf{E}_{3^2}$	4	2
(9)	2	Esp_{27}	8	3
(10)	2	$\mathbf{Z}_3 \times Esp_{27}$	8	3

	$n(P\langle t \rangle, V)$	P	$m(V)$	$m(V/C_V(t))$
(11)	3	$\mathbf{Z}_3$	6	3
(12)	3	$\mathbf{E}_{3^2}$	6	3
(13)	3	$\mathbf{E}_{3^3}$	6	3

Moreover, $C_V(\Phi(P)) \neq 0$ in the cases (9) and (10).

PROOF. First, we deal with the case where $P\langle t \rangle$ is irreducible on V . Put $Z = Z(P)$.

Assume first that $[Z, t] \neq 1$. Put $Z_0 = [Z, t]$. Since $C_V(Z_0) = 0$ and t inverts Z_0 , we have $|V| = |V : C_V(t)|^2 = 2^{2n}$ ($n \leq 3$) by (1.2). Let W be an irreducible $GF(2)Z_0$ -submodule of V . Then $Z_0/C_{Z_0}(W)$ is cyclic by Schur's lemma. Since $C_V(C_{Z_0}(W))$ is P -invariant, we have $C_V(C_{Z_0}(W)) = V$, so $C_{Z_0}(W) = 1$. Hence Z_0 is cyclic.

Suppose P is abelian. Then $Z_0 = P$, and so, inspecting irreducible dihedral subgroups of $GL_{2n}(2)$, we have $P \cong \mathbf{Z}_3$ if $n = 1$, $\mathbf{Z}_5$ if $n = 2$, $\mathbf{Z}_7$ or $\mathbf{Z}_9$ if $n = 3$.

Suppose P is nonabelian. Then, inspecting nonabelian p -subgroups of $GL_6(2)$, we have $|V| = 2^6$ and $p = 3$. Suppose P is an extraspecial group of order 27, and let $\bar{P} = P/Z$. Since t inverts $\bar{P}$, we have $t \in SL(\bar{P}) = Sp(\bar{P})$, and so $[Z, t] = 1$, a contradiction. Thus we have $P \cong \mathbf{Z}_3 \text{ wr } \mathbf{Z}_3$ (the wreath product of $\mathbf{Z}_3$ by $\mathbf{Z}_3$).

Assume $[Z, t] = 1$. Then P is nonabelian, and, moreover, Z is cyclic by Schur's lemma because $Z \subseteq Z(P\langle t \rangle)$. Let $Z \subseteq L \subseteq P$ so that $L \triangleleft P\langle t \rangle$ and $|L : Z| = p$, and put $K = \Omega_1(L)$. Since $P \neq C_P(L)$, L is noncyclic abelian, so $L = KZ$ and $K \cong \mathbf{E}_{p^2}$. Thus $C_P(K) = C_P(L) \neq P = [P, t]$, and hence $[K, t] \neq 1$. Let Δ be the set of subgroups of K of order p distinct from $K \cap Z$. Then $V = \bigoplus_{X \in \Delta} C_V(X)$ by (2.1) of [1] because $C_V(K \cap Z) = 0$. Since $|\Delta| = p$ and P acts transitively on

Δ , $|V| = |C_V(X)|^p$, and so $||[V, X]| = |V : C_V(X)| = |C_V(X)|^{p-1}$ for all $X \in \Delta$. Now, put $X = [K, t]$. Since t inverts X and $X \in \Delta$, $||[V, X]| = |[V, X] : C_{[V, X]}(t)|^2 \leq |V : C_V(t)|^2 \leq 2^6$, and so $|C_V(X)|^{p-1} \leq 2^6$. Note that $|C_V(X)| \geq 2^2$ because $C_{C_V(X)}(Z) = 0$. Therefore we have $p = 3$ and $|C_V(X)| = 2^2$, and so $|V| = 2^6$. Hence $P\langle t \rangle$ is embedded in $GL_3(2^2)$ by Schur's lemma because $Z \subseteq Z(P\langle t \rangle)$. Thus $P = [P, t]$ is embedded in $GL_3(2^2)' = SL_3(2^2)$, and so $P \cong Esp_{27}$.

Now, let $\{W_1, \dots, W_n\}$ ($n > 1$) be the set of $P\langle t \rangle$ -composition factors of V . Then $\prod_{i=1}^n |W_i : C_{W_i}(t)| \leq |V : C_V(t)| \leq 2^3$ by (1.3), so $n \leq 3$ and we may assume $|W_1 : C_{W_1}(t)| = 2$. Therefore we have $P/C_P(W_1) \cong \mathbf{Z}_3$ and $|W_1| = 2^2$ by a result for the irreducible case, and hence P is a 3-group.

Suppose $n = 2$. Then, since $|W_2 : C_{W_2}(t)| \leq 2^2$, $P/C_P(W_2) \cong \mathbf{Z}_3$ and $|W_2| = 2^2$, or $P/C_P(W_2) \cong Esp_{27}$ and $|W_2| = 2^6$. Therefore we have $P \cong \mathbf{Z}_3$ or $\mathbf{E}_{3^2}$ and $|V| = 2^4$, or $P \cong Esp_{27}$ or $\mathbf{Z}_3 \times Esp_{27}$ and $|V| = 2^8$.

Suppose $n = 3$. Then, since $|W_2 : C_{W_2}(t)| = |W_3 : C_{W_3}(t)| = 2$, $P/C_P(W_2) \cong P/C_P(W_3) \cong \mathbf{Z}_3$ and $|W_2| = |W_3| = 2^2$. Therefore we have $P \cong \mathbf{Z}_3$, $\mathbf{E}_{3^2}$, or $\mathbf{E}_{3^3}$ and $|V| = 2^6$.

Finally, we determine the index $|V : C_V(t)|$. If P is abelian, then $|V : C_V(t)| = |V|^{1/2}$ by (1.2). If P is nonabelian, then $n = 2$ and $1 \neq \Phi(P) \subseteq C_P(W_1)$, and so $V = C_V(\Phi(P)) \oplus [V, \Phi(P)] \cong W_1 \oplus W_2$. Therefore $|V : C_V(t)| = |W_1 : C_{W_1}(t)| \cdot |W_2 : C_{W_2}(t)|$.

Now, for a group G , define

$$n_*(G) = \max \{n_p(G) \mid 2 \neq p \in \pi(G)\}.$$

Suppose $O_2(G) \neq 1$. Then G is itself a 2-local subgroup of G , and hence

$$n_*(G) \leq f(G).$$

Suppose, moreover, that G is 2-irreducible and solvable. Then (2.1)–(3) shows that $\pi(G) = \{2, p\}$ for some odd prime p , and then we have $n_*(G) = n_p(G)$. Therefore G is thin if and only if $n_*(G) = 1$ by (2.1)–(7).

PROPOSITION 2.9 *Let G be a 2-irreducible solvable group, S a Sylow 2-subgroup of G , V an irreducible faithful $GF(2)G$ -module, and t an involution of S .*

(1) *If $t \in Z(S)$ and $|V : C_V(t)| \leq 2^3$, then one of the six cases listed in the following table occurs.*

	$O^2(G)$	$m(V)$	$m(V/C_V(t))$
(1–1)	$\mathbf{Z}_3$	2	1
(1–2)	$\mathbf{Z}_5$	4	2
(1–3)	$\mathbf{Z}_7$	6	3
(1–4)	$\mathbf{Z}_9$	6	3
(1–5)	Esp_{27}	6	2
(1–6)	$\mathbf{E}_{3^2}$	4	2

(2) If $t \notin Z(S)$, $|V : C_V(t)| \leq 2^2$, and $n_*(G) \leq 2$, then one of the three cases listed in the following table occurs.

	$O^2(G)$	$m(V)$	$m(V/C_V(t))$
(2-1)	$\mathbf{E}_{3^2}$	4	1
(2-2)	$\mathbf{E}_{3^2}$	4	2
(2-3)	$\mathbf{E}_{5^2}$	8	2

PROOF. Let $P = O^2(G)$. Since $P \neq 1$ by (2.1)–(1) and G is irreducible and faithful on V , $C_V(P) = 0$ and $O_2(G) = 1$, and then P is a Sylow p -subgroup of G for some odd prime p by (2.1)–(3).

Suppose first that $t \in Z(S)$ and $|V : C_V(t)| \leq 2^3$. Then $P = [P, t]$ by (2.1)–(5), and hence P and V are described as in (2.8). We have $P \not\cong \mathbf{Z}_3 \text{wr} \mathbf{Z}_3$ by the 2-irreducibility of G , and $m(V) \neq 8$ because G is irreducible on V . If $P \cong \mathbf{Z}_3$, then $G \cong \Sigma_3$ and $|V| = 2^2$ also because G is irreducible on V . Suppose $P \cong \mathbf{E}_{3^2}$ or $\mathbf{E}_{3^3}$, and regard V as a $GF(2)P$ -module. Then Clifford's theorem shows that $V = W_1 \oplus \cdots \oplus W_s$ where $W_1, \dots, W_s$ are the homogenous components of V . Since $C_V(P) = 0$, we have $|W_i| \geq 2^2$ ($1 \leq i \leq s$), so $2^{2s} \leq |V| \leq 2^6$ by (1.2). This shows that $|V| = 2^4$ because $s > 1$ and s is a power of 2, and hence $P \cong \mathbf{E}_{3^2}$.

Suppose next that $t \notin Z(S)$, that $|V : C_V(t)| \leq 2^2$, and that $n_p(G) = n_*(G) \leq 2$. Then $|P/\Phi(P)| = p^2$ and $m(S) = 2$ by (2.1)–(6). We can take $u \in t^S$ such that $\Omega_1(Z(S)) \subseteq \langle t, u \rangle \cong \mathbf{E}_{2^2}$ by (1.1). Define

$$P_t = [P, t], \quad P_u = [P, u], \quad V_t = [V, P_t], \quad \text{and} \quad V_u = [V, P_u].$$

Then P_t and V_t are described as in (2.8), and, moreover, $P = [P, \Omega_1(Z(S))] = P_t P_u$, and so $V = [V, P] = V_t + V_u$. Let $\bar{P} = P/(P_t \cap P_u)$. Then $\bar{P}_t$ is cyclic because $|P/\Phi(P)| = p^2$ and $\bar{P}_t \cong \bar{P}_u$. Thus, noting that $(P_t)^p = 1$ for all of the possible cases listed in (2.8), we have $|\bar{P}_t| = p$, and so $P_t \cap P_u = \Phi(P)$.

Suppose $P_t \cong \mathbf{E}_{3^2}$. Then $P_t \cong \text{Esp}_{27}$ and t inverts $Z(P) = \Phi(P)$. Therefore $|V| = |V : C_V(t)|^2 \leq 2^4$ by (1.2) as $C_V(Z(P)) = 0$, which is a contradiction because $GL_4(2)$ has abelian Sylow 3-subgroups. Suppose $P_t \cong \text{Esp}_{27}$. Then, as $Z(P) \neq \Phi(P)$ and $|\Phi(P)| = 9$, $\Phi(P) \subset C_P(\Phi(P)) \subset P$, which contradicts the 2-irreducibility of G . Therefore, we have $P_t \cong P_u \cong \mathbf{Z}_3$ or $\mathbf{Z}_5$, $|V_t| = |V_u| = 2^2$ or 2^4 , and $P = P_t \times P_u$. Since $|V| \leq |V_t| \cdot |V_u|$ and P is faithful on V , we have $|V| = 2^8$ if $P \cong \mathbf{E}_{5^2}$, and $2^4 \leq |V| \leq 2^8$ if $P \cong \mathbf{E}_{3^2}$.

Suppose $P \cong \mathbf{E}_{3^2}$. Then $P_t \langle t \rangle \cong P_u \langle u \rangle \cong \Sigma_3$, and it follows from the argument in the second paragraph that $|V| = 2^4$ or 2^8 . Assume $|V| = 2^8$. Then we have $V = V_t \oplus V_u$, so $C_P(V_t) = P_u$ and $C_P(V_u) = P_t$. Therefore, regarded as a $GF(2)P$ -module, V_t and V_u are the homogenous components of V , and hence G permutes P_t and P_u . Thus $|G : P_t \langle t \rangle \times P_u \langle u \rangle| = 2$ because $\text{Aut}(\Sigma_3) \cong \Sigma_3$ and $O_2(G) = 1$. Now, let W be an irreducible $GF(2)(P_t \langle t \rangle \times P_u \langle u \rangle)$ -submodule of V_t . Then $|W| = 2^2$ as

$[V_t, P_u \langle u \rangle] = 0$, so $|\langle W^G \rangle| \leq |W|^2 = (2^2)^2 < |V|$, which contradicts the fact that G is irreducible on V . Thus we have $|V| = 2^4$.

LEMMA 2.10 *Let G be a 2-irreducible solvable group with $O_2(G) = 1$, S a Sylow 2-subgroup of G , and V a faithful $GF(2)G$ -module. Put $W = [V, O^2(G)]$, and let A be an elementary abelian subgroup of S .*

(1) *Assume $|V : C_V(A)| \leq |A| \neq 1$ and that $n_*(G) \leq 2$. Then W is irreducible, and*

$$\begin{aligned} O^2(G) \cong \mathbf{Z}_3 \quad \text{and} \quad |W| = 2^2 \quad \text{if} \quad A \subseteq \Omega_1(Z(S)), \quad \text{and} \\ O^2(G) \cong \mathbf{E}_{3^2} \quad \text{and} \quad |W| = 2^4 \quad \text{if} \quad A \not\subseteq \Omega_1(Z(S)). \end{aligned}$$

Moreover, $|V : C_V(A)| = |A|$.

(2) *Assume $\Omega_1(Z(S)) \subset A$, and that one of the following holds:*

- (i) $|V : C_V(A)| \leq 2^3$;
- (ii) $||[V, A]| \leq 2^3$;
- (iii) *there exists an A -invariant subspace U of V such that*
 $|V : U| \cdot |[U, A]| \leq 2^2$.

Then W is irreducible, and

$$O^2(G) \cong \mathbf{E}_{3^2} \quad \text{and} \quad |W| = 2^4.$$

Moreover, $|V : C_V(A)| > 2 < |[V, A]|$.

PROOF. Let $P = O^2(G)$. Throughout this proof, assume that the hypothesis of (1) or (2) is satisfied for the triple (G, V, A) .

First, we will assume that V is irreducible.

Suppose $\Omega_1(Z(S)) \not\subset A$. Then we have $|V : C_V(A)| \leq |A| \neq 1$, and, in particular, $|A| = 2$ by (2.1)–(6). Thus (2.9) shows that

$$\begin{aligned} P \cong \mathbf{Z}_3 \quad \text{and} \quad |V| = 2^2 \quad \text{if} \quad A \subseteq \Omega_1(Z(S)), \quad \text{and} \\ P \cong \mathbf{E}_{3^2} \quad \text{and} \quad |V| = 2^4 \quad \text{if} \quad A \not\subseteq \Omega_1(Z(S)). \end{aligned}$$

Therefore we have $|V : C_V(A)| = |A|$ by (2.5)–(4).

Suppose $\Omega_1(Z(S)) \subset A$. Then we may assume that the hypothesis of (2) is satisfied. Since $m(S) \geq 2$, P is not cyclic by (2.1)–(7).

Suppose first that the condition (i) holds. Let $1 \neq t \in \Omega_1(Z(S))$. Then we have $|V : C_V(t)| \leq 2^3$, and hence either

$$P \cong \mathbf{E}_{3^2} \quad \text{and} \quad |V| = 2^4, \quad \text{or} \quad P \cong \text{Esp}_{27} \quad \text{and} \quad |V| = 2^6$$

by (2.9). If $P \cong \text{Esp}_{27}$ and $|V| = 2^6$, then $|A| \leq 2$ by (2.7)–(6), a contradiction. Thus we have $P \cong \mathbf{E}_{3^2}$ and $|V| = 2^4$, and hence $|V : C_V(A)| > 2 < |[V, A]|$ by (2.5)–(5).

Suppose next that the condition (ii) holds. Then the condition (i) holds for the dual module V^* of V , and the conclusion is valid for V^* . Thus the conclusion also holds for V because $|V| = |V^*|$, $|V : C_V(A)| = |[V^*, A]|$, and $|[V, A]| = |V^* : C_{V^*}(A)|$.

Finally, suppose that the condition (iii) holds. We may assume that $|V : U| = |[U, A]| = 2$. Pick $s \in A - \{t\}$, and put $B = \langle t, s \rangle$. Then, as both t and s induce transvections on U , $|U : C_U(t)| \leq 2 \geq |U : C_U(s)|$, so $|U : C_U(B)| \leq 2^2$. Thus we have $|V : C_V(B)| \leq 2^3$. Hence the condition (i) holds for B , so $P \cong E_{3^2}$ and $|V| = 2^4$ and $|V : C_V(A)| \geq |V : C_V(B)| > 2 < |[V, B]| \leq |[V, A]|$.

We have shown that (2.10) holds if V is irreducible. Since W is faithful by (2.4)–(2), it suffices to prove that W is irreducible. This is done after the proof of (2.11).

Throughout the remainder of this paper, we will use the following terminology and notation.

Let G be a group, V a $GF(2)G$ -module, and A a subgroup of G . We will say that A induces a *transvection* (resp. a *quasitransvection*) on V if $|V : C_V(A)| \leq 2$ (resp. $|V : C_V(A)| \leq 2^2$). Put $\bar{G} = G/C_G(V)$. We will say that A induces an *offending action* on V if $|V : C_V(\bar{A})| \leq |\bar{A}|$. Moreover, we will say that A induces a *semioffending action* on V if either A induces an offending action on V , or the triple $(\bar{G}, V, \bar{A})$ satisfies the hypothesis of (2.10)–(2).

Next, let G be a group acting on a 2-group T . We will denote by $n(G, T)$ the number of the nontrivial G -composition factors of T . A subgroup A of G induces a *factorwise transvection*, a *factorwise quasitransvection*, a *factorwise offending action*, or a *factorwise semioffending action* on T if A induces a transvection, a quasitransvection, an offending action, or a semioffending action, respectively, on each nontrivial G -composition factor of T regarded as a $GF(2)G$ -module. Also, A induces a *factorwise sesquitransvection* on T if A induces a factorwise quasitransvection on T and a transvection on at least one nontrivial G -composition factor of T . Let H be another group. We will say that T is *induced by a $GF(2)H$ -module W* and denote by $(G, T) \ll (H, W)$ if each G -composition factor of T is either trivial or induced by W .

LEMMA 2.11 *Let G be a 2-irreducible solvable group, and S a Sylow 2-subgroup of G . Suppose G acts on a 2-group T with $[T, O^2(G)] \neq 1$. Put $Q = O_2(G)$, and take a subgroup A of S so that AQ/Q is elementary abelian.*

(1) *Assume $AQ/Q \neq 1$ and that one of the following holds:*

- (i) *A induces a factorwise quasitransvection on T , G is a $\{2, 3\}$ -group, and G has cyclic Sylow 3-subgroups if $AQ/Q \subseteq \Omega_1(Z(S/Q))$;*
- (ii) *A induces a factorwise sesquitransvection on T ;*
- (iii) *A induces a factorwise semioffending action on T .*

Then we have

$$\begin{aligned} C_G(W) &= QC_G(T), \\ O^2(G/C_G(W)) &\cong \mathbf{Z}_3 \quad \text{and} \quad |W| = 2^2 \quad \text{if} \quad AQ/Q \subseteq \Omega_1(Z(S/Q)), \\ O^2(G/C_G(W)) &\cong \mathbf{E}_{3^2} \quad \text{and} \quad |W| = 2^4 \quad \text{if} \quad AQ/Q \not\subseteq \Omega_1(Z(S/Q)), \end{aligned}$$

where W is an arbitrary nontrivial G -composition factor of T . Moreover, $|AQ/Q| = 2$ if (ii) holds.

(2) Assume $|T : U| \cdot |[U, A]| \leq 2^s$ for some $s \geq 0$ and some A -invariant subgroup U of T . Then $n(G, T) \leq s$ if $A \not\subseteq Q$. Moreover, $|AQ/Q| \leq 2$ and A induces a factorwise sesquitransvection on T if $s \leq n'$, where

$$n' = \begin{cases} n(G, T) + 1 & \text{if } n(G, T) > 1, \\ 1 & \text{if } n(G, T) = 1. \end{cases}$$

(3) Assume $\Omega_1(Z(S/Q)) \subset AQ/Q$ and that one of the following holds:

- (i) $|T : C_T(A)| \leq 2^{2s+1}$ for some $s \geq 0$;
- (ii) $|T : U| \cdot |[U, A]| \leq 2^{2s}$ for some $s \geq 0$ and some A -invariant subgroup U of T .

Then $n(G, T) \leq s$. Moreover, A induces a factorwise semioffending action on T if $n(G, T) = s$.

PROOF. By (2.1)–(3), we have $G = O_{2,p,2}(G)$ for some odd prime p . Let P be a Sylow p -subgroup of G , and let W be an arbitrary nontrivial G -composition factor of T . Put $\bar{G} = G/C_G(W)$, and regard W as an irreducible faithful $GF(2)\bar{G}$ -module. Since $Q \subseteq C_G(W) \not\subseteq O^2(G)$, we have $C_G(W) \subseteq \Phi(P)Q$, $|\bar{A}| = |AQ/Q|$, and $\bar{A} \subseteq \Omega_1(Z(\bar{S}))$ if and only if $AQ/Q \subseteq \Omega_1(Z(S/Q))$ by (2.4)–(1). We remark, moreover, that (2.10) holds for the triple $(\bar{G}, W, \bar{A})$.

To prove (1), assume first that $|AQ/Q| = 2$. The definitions of a factorwise semioffending action and a factorwise sesquitransvection show that (iii) implies (ii). Suppose that (ii) holds, and choose W on which A induces a transvection. Then we have

$$(*) \quad \begin{cases} \bar{P} \cong \mathbf{Z}_3 & \text{and} \quad |W| = 2^2 & \text{if } AQ/Q \subseteq \Omega_1(Z(S/Q)), \\ \bar{P} \cong \mathbf{E}_{3^2} & \text{and} \quad |W| = 2^4 & \text{if } AQ/Q \not\subseteq \Omega_1(Z(S/Q)). \end{cases}$$

by (2.9). Therefore G is a $\{2, 3\}$ -group in either case, and P is cyclic if $AQ/Q \subseteq \Omega_1(Z(S/Q))$, so (i) holds. Now, suppose (i) holds, and let W be arbitrary. Then (2.9) also yields (*). Thus $C_G(W) = \Phi(P)Q$ because $\bar{P}$ is elementary abelian, and hence $[T, \Phi(P)] = 1$ because W is arbitrary. This shows that $C_G(W) = QC_G(T)$.

Assume next that $|AQ/Q| = 2^2$. The definitions of a factorwise quasitransvection and a factorwise semioffending action show that (i) implies (iii). The condition (ii) never holds because A does not induce a transvection on W by (2.10)–(1). Now,

suppose (iii) holds. Since $AQ/Q \not\subseteq \Omega_1(Z(S/Q))$ by (2.1)–(4), we have $\overline{P} \cong E_{3^2}$ and $|W| = 2^4$ by (2.10)–(2). Therefore, similarly as in the case where $|AQ/Q| = 2$, we have $C_G(W) = QC_G(T)$. This proves (1).

For the proof of (2) and (3), let

$$T = T_0 \supseteq T_1 \supseteq \cdots \supseteq T_{r-1} \supseteq T_r = 1$$

be a G -composition series of T , and let U be an A -invariant subgroup of T . Put $U_i = (T_{i-1} \cap U)T_i$ ($1 \leq i \leq r$). Suppose $W_1 = T_{i_1-1}/T_{i_1}, \dots, W_n = T_{i_n-1}/T_{i_n}$ ($n = n(G, T)$) are the nontrivial G -composition factors, and set $V_k = U_{i_k}/T_{i_k}$ ($1 \leq k \leq n$). Then (1.3) shows that

$$\begin{aligned} |T : U| \cdot |[U, A]| &\geq \prod_{i=1}^r |T_{i-1}/T_i : U_i/T_i| \cdot |[U_i/T_i, A]| \\ &\geq \prod_{k=1}^n |W_k : V_k| \cdot |[V_k, A]|. \end{aligned} \quad \cdots (*)$$

Let $|T : U| \cdot |[U, A]| \leq 2^s$.

Suppose $A \not\subseteq Q$. Since A centralizes none of the nontrivial composition factor of T , we have $|W_k : V_k| \cdot |[V_k, A]| \geq 2$ for all k ($1 \leq k \leq n$), so

$$2^n \leq \prod_{k=1}^n |W_k : V_k| \cdot |[V_k, A]| \leq |T : U| \cdot |[U, A]| \leq 2^s.$$

This shows that $n \leq s$. Suppose $s \leq n'$. Then we have

$$\begin{aligned} |W_k : V_k| \cdot |[V_k, A]| &\leq 2^2 \quad \text{for all } k \ (1 \leq k \leq n), \quad \text{and} \\ |W_m : V_m| \cdot |[V_m, A]| &\leq 2 \quad \text{for at least one } m \ (1 \leq m \leq n). \end{aligned}$$

Hence A induces a transvection on this W_m or its dual module W_m^* , and so $|AQ/Q| = 2$ by (2.10)–(1). Therefore A induces a quasitransvection on each W_k , and a transvection on W_m . Thus (2) holds.

Let $\Omega_1(Z(S/Q)) \subset AQ/Q$.

Suppose (i) of (3) holds. Then, letting $U = C_T(A)$ in (*), we have

$$\prod_{k=1}^n |W_k : C_{W_k}(A)| \leq |T : C_T(A)| = 2^{2s+1}.$$

Therefore $n \leq s$, and, moreover, $|W_k : C_{W_k}(A)| \leq 2^3$ for all k ($1 \leq k \leq n$) if $n = s$, because $|W_k : C_{W_k}(A)| \geq 2^2$ for all k ($1 \leq k \leq n$) by (2.10)–(2).

Suppose (ii) of (3) holds. Then, by (*), we also have $n \leq s$ and $|W_k : V_k| \cdot |[V_k, A]| = 2^2$ for all k ($1 \leq k \leq n$) if $n = s$, because $|W_k : V_k| \cdot |[V_k, A]| \geq 2^2$ for all k ($1 \leq k \leq n$) by (2.10)–(2). Thus (3) holds.

Now, we return to the proof of (2.10).

As stated before, it suffices to prove that $W = [V, O^2(G)]$ is irreducible under the hypothesis of (2.10).

Suppose the hypothesis of (1) or (2) of (2.10) holds. Then there exists a unique nontrivial composition factor within V by (2.11)–(2)(3). Thus W is indeed irreducible because $W = [W, O^2(G)]$ and $C_W(O^2(G)) = 0$ by (2.4)–(3). This completes the proof of (2.10).

COROLLARY 2.12 *Let G be a 2-irreducible solvable group of even order with $O_2(G) = 1$, S a Sylow 2-subgroup of G , and V a faithful $GF(2)G$ -module. Put $W = [V, O^2(G)]$.*

(1) *Suppose $|[V, S, S]| \leq 2^2$. Then one of the following holds.*

- (i) *G is dihedral and $[V, S, S] = 0$.*
- (ii) *$O^2(G) \cong \mathbf{Z}_5$, $|W| = 2^4$, and $|[W, S, S]| = 2^2$.*
- (iii) *$O^2(G) \cong \mathbf{E}_{3^2}$, $|W| = 2^4$, and $|[W, S, S]| = 2^2$.*
- (iv) *$O^2(G) \cong \text{Esp}_{27}$, $|W| = 2^6$, $|[W, S, S]| = 2^2$, and $m(S) = 1$.*

Moreover, if $[V, S, S] \neq 0 = [V, S, S, S]$ then $O^2(G) \cong \text{Esp}_{27}$.

(2) *Suppose $|V : C_V(S)| \leq 2^2$. Then*

$$G \cong \mathbf{D}_6 \quad \text{or} \quad \mathbf{D}_{10}, \quad \text{and} \quad |W| = 2^2 \quad \text{or} \quad 2^4.$$

PROOF. Let $P = O^2(G)$. Then P is a Sylow p -subgroup for some odd prime p by (2.1)–(3).

Suppose $|[W, S, S]| \leq |[V, S, S]| \leq 2^2$. If $S^2 = 1$, then $|S| = 2$ and P is cyclic by (2.1)–(7), so (i) holds. Assume $S^2 \neq 1$, and let $t \in \Omega_1(Z(S)) \cap S^2$. Then t induces a quasitransvection on W because $[W, t] \subseteq [W, S^2] \subseteq [W, S, S]$. Since $C_W(P) = 0$ and $C_G(W) = 1$ by (2.4)–(2)(3), we have $P \cong \mathbf{Z}_3$ and $|W| = 2^2$ or 2^4 , $P \cong \mathbf{Z}_5$ and $|W| = 2^4$, $P \cong \text{Esp}_{27}$ and $|W| = 2^6$, or $P \cong \mathbf{E}_{3^2}$ and $|W| = 2^4$ by (2.8). Since $S^2 \neq 1$, we have $P \not\cong \mathbf{Z}_3$, and then t does not induce a transvection on W . Hence $|[W, t]| = |[W, S, S]| = 2^2$. If $P \cong \text{Esp}_{27}$, then, by (2.7)–(4)(5), we have $m(S) = 1$. The last remark is shown after the proof of (2).

Suppose $|W : C_W(S)| \leq |V : C_V(S)| \leq 2^2$. Then $|[W^*, S, S]| \leq 2$ for the dual module W^* of W , so G is a dihedral group by (1). Since $C_W(G) \subseteq C_W(P) = 0$ by (2.4)–(3) and $G = \langle S, S^x \rangle$ for some element $x \in G$ by (2.1)–(10), we have $|W| \leq 2^4$. Thus, inspecting dihedral subgroups of $GL_4(2)$, $G \cong \mathbf{D}_6$ or $\mathbf{D}_{10}$ and $|W| = 2^2$ or 2^4 . This proves (2).

Finally, suppose $0 < |[V, S, S]| \leq 2^2$ and $[V, S, S, S] = 0$. Then one of the three cases (ii), (iii), (iv) of (1) occurs. Suppose $|W| = 2^4$. Then $|W : C_W(S)| \leq$

$|W : [W, S, S]| = 2^2$, so G is dihedral and $[V, S, S] = 0$ by (2), a contradiction. Therefore $|W| \neq 2^4$, and hence $P \cong \text{Esp}_{27}$.

COROLLARY 2.13 *Let G be a 2-irreducible solvable group with $O_2(G) = 1$, S a Sylow 2-subgroup of G , V a faithful $GF(2)G$ -module, and $1 \neq t \in E \subseteq S$ and $E^2 = 1$. Suppose that $n_*(G) \leq 2$, that $|V : C_V(t)| \leq 2^2$ and, moreover, that $[V, t, E] = 0$ if $t \in Z(S)$. Then one of the following holds.*

(1) G has a nonidentity elementary abelian 2-subgroup inducing an offending action on $[V, O^2(G)]$.

(2) $[V, E] = [V, t] + [C_V(t), E]$, and, moreover, $E = \langle t \rangle$ if $t \in Z(S)$.

PROOF. Let $P = O^2(G)$ and $W = [V, P]$. Assume that (1) does not hold. Then we have

$$|W : C_W(t)| = 2^2 \quad \text{and} \quad [C_V(P), t] = 0.$$

We will prove that (2) holds, and we remark that (2) trivially holds if $E = \langle t \rangle$.

First, suppose $n(G, V) \geq 2$. Then (2.11)–(2) shows that t induces a factorwise sesquitransvection on V . Therefore we have

$$O^2(G) \cong \mathbf{Z}_3 \quad \text{if } t \in Z(S), \quad \text{and} \quad O^2(G) \cong \mathbf{E}_{3^2} \quad \text{if } t \notin Z(S).$$

Now, let $t \in Z(S)$ and $[V, t, E] = 0$. It suffices to show that $E = \langle t \rangle$, so we may assume that P is not cyclic. Thus we have $n(G, V) = 1$ by the above, and then G is irreducible on W , and t induces a quasitransvection but no transvection on W . Therefore either $P \cong \mathbf{E}_{3^2}$ and $|W| = 2^4$, or $P \cong \text{Esp}_{27}$ and $|W| = 2^6$ by (2.9) and our hypothesis. If $P \cong \mathbf{E}_{3^2}$, then $S \cong \mathbf{Z}_4$ by (2.5)–(4) because (1) does not hold, so $E = \langle t \rangle$. If $P \cong \text{Esp}_{27}$, then $E = \langle t \rangle$ by (2.7)–(6).

Next, let $t \notin Z(S)$. We may assume $E \neq \langle t \rangle$, and then $\Omega_1(Z(S)) \subseteq \langle t, u \rangle = E$ for some $u \in t^S$ by (1.1). Therefore we also have

$$|W : C_W(u)| = 2^2 \quad \text{and} \quad [C_V(P), u] = 0,$$

and hence $[V, E] = [W, E]$, $[V, t] = [W, t]$, and $[C_V(t), E] = [C_W(t), E]$. Thus we may assume $V = W$, and then $C_V(P) = 0$. As t induces a quasitransvection on V , (2.9) shows that $P \cong \mathbf{E}_{3^2}$ or $\mathbf{E}_{5^2}$ if $n(G, V) = 1$, and the above remark shows that $P \cong \mathbf{E}_{3^2}$ if $n(G, V) \geq 2$.

Define $P_t = [P, t]$ and $P_u = [P, u]$, and regard V as a $GF(2)P$ -module. Let W be an arbitrary homogenous component of V . Suppose $W \neq W^t$, and put $U = W \oplus W^t$. Then we have $2^2 \leq |W| \leq |U : C_U(t)| \leq |V : C_V(t)| \leq 2^2$, and so $|W| = 2^2$ and $U \subseteq [V, P_t]$. Hence $V = [V, P_t] + [V, P_u] \subseteq U$ as $W^t = W^u$. Thus we have $P \cong \mathbf{E}_{3^2}$ and $|V| = 2^4$, and then (1) holds by (2.5)–(5), a contradiction. Therefore $W = W^t$, and hence $C_P(W) = P_t$ or P_u . This shows that $V = C_V(P_t) \oplus C_V(P_u)$.

If $|C_V(P_u) : C_{C_V(P_u)}(t)| = 2$, then $|C_V(P_t)| = |C_V(P_u)| = 2^2$, and so $P \cong \mathbf{E}_{3^2}$ and $|V| = 2^4$, also a contradiction. Thus we have $|C_V(P_u) : C_{C_V(P_u)}(t)| = 2^2$,

and then $[C_V(P_t), t] = 0$ and, by symmetry, $[C_V(P_u), u] = 0$. Hence $C_V(t) = C_V(P_t) + C_{C_V(P_u)}(t)$, and then

$$[C_V(t), E] = [C_V(t), u] = [C_V(P_t), u].$$

Therefore it follows that

$$[V, E] = [V, t] + [V, u] = [V, t] + [C_V(P_t), u] = [V, t] + [C_V(t), E].$$

The next two lemmas give information on the number of noncentral chief 2-factors.

LEMMA 2.14 *Let G be a group, S a Sylow 2-subgroup of G , $Q = O_2(G) \supseteq U \triangleleft S$, $T = \langle U^G \rangle$, and $W = \cap U^G$. Then the following holds.*

- (1) $[T, Q] = \langle (U \cap [T, Q])^G \rangle = \langle [U, Q]^G \rangle$, and if $T \neq U$ then $[T, O^2(G)] \not\subseteq [T, Q]$.
- (2) If $n(G, T/W) \leq k$, then $[T, Q; k] \subseteq U$.
- (3) There exists a normal subgroup N of G such that $W \subseteq N \subseteq T$ and T/N is a noncentral chief factor of G .

PROOF. Let $L = [T, Q]$ and $M = \langle [U, Q]^G \rangle$. Then $M \subseteq \langle (U \cap L)^G \rangle \subseteq L$ because $[U, Q] \subseteq L \cap M$. Since $[UM, Q] \subseteq M$ and $\langle (UM)^G \rangle = T$, we have $L = [T, Q] \subseteq M$, so $L = \langle (U \cap L)^G \rangle = M$. Suppose $[T, O^2(G)] \subseteq L$. Then, since $UL \triangleleft S$, $T = \langle U^G \rangle \subseteq UL$, so we have $T = UL$. This shows that $T = U$, so (1) holds.

Part (2) holds for $k = 0$ by (1). Suppose $k \geq 1$, and define $T_0 = LW$ and $U_0 = U \cap T_0$. Then $T_0 = \langle U_0^G \rangle$ and $n(G, T/T_0) \geq 1$ by (1), and so $n(G, T_0/W) \leq k-1$. Thus $[T_0, Q; k-1] \subseteq U$ by the inductive hypothesis, and hence $[T, Q; k] \subseteq U$.

To prove (3), let $\tilde{T} = T/T_0$ and regard $\tilde{T}$ as a $GF(2)G$ -module with $Q \subseteq C_G(\tilde{T}) \not\subseteq O^2(G)$. Then we have $\tilde{T} = [\tilde{T}, O^2(G)] \times C_{\tilde{T}}(O^2(G)) \neq C_{\tilde{T}}(O^2(G))$ by (2.4)–(3). Choose a subgroup N so that $T_0 \subseteq N \subseteq T$ and $\tilde{N}$ is a maximal submodule of $\tilde{T}$ containing $C_{\tilde{T}}(O^2(G))$. Then N has the desired property.

LEMMA 2.15 *Let G be a 2-irreducible solvable group, S a Sylow 2-subgroup, $Q = O_2(G) \supseteq U \triangleleft S$, $T = \langle U^G \rangle$, and $W = \cap U^G$. Suppose $n(G, T/W) = r$ and that $|C_V(S)| = 2$ for all noncentral chief factors V of G within T/W . Then $|C_{U/W}(R)| \leq 2^r$ for all normal subgroups R of S with $S = QR$.*

PROOF. We may assume that $[U, R] \subseteq W$. Define the subgroups K_m and L_m ($m \geq 0$) as follows:

$$K_0 = L_0 = T;$$

if $L_{m-1} \neq W$, then choose K_m so that $W \subseteq K_m \subseteq L_{m-1}$ and that L_{m-1}/K_m is a noncentral chief factor of G (this is possible by (2.14)–(3)), and put $L_m = \langle (U \cap K_m)^G \rangle$;

if $L_{m-1} = W$, then $K_m = L_m = K_{m+1} = L_{m+1} = \cdots = W$.

Let n be a positive integer so that $L_{n-1} \neq W = L_n$. Then we have $n \leq r$. Therefore, since $|U \cap L_{m-1} : U \cap L_m| = 2$ ($1 \leq m \leq n$) by our hypothesis, we have $|U : W| = 2^n \leq 2^r$.

§3. Amalgams of solvable groups

In this section, we study the amalgams (X, S, Y) satisfying the following conditions:

- (a) S is a common Sylow 2-subgroup of X and Y ;
- (b) both X and Y are 2-irreducible;
- (c) no nonidentity subgroup of S is normal both in X and in Y ;
- (d) $C_X(O_2(X)) \subseteq O_2(X)$ and $C_Y(O_2(Y)) \subseteq O_2(Y)$;
- (f) both X and Y are solvable;
- (g) $S = [O_2(X), O^2(X)][O_2(X) \cap O_2(Y)][O_2(Y), O^2(Y)]$.

We will use the following notation throughout this section:

$$\begin{aligned} Q &= O_2(X); & R &= O_2(Y); \\ Z &= \Omega_1(Z(S)); & V &= \langle Z^X \rangle; & W &= \langle Z^Y \rangle; \\ \pi(X) &= \{2, q\}; & \pi(Y) &= \{2, r\}. \end{aligned}$$

(3.1) *If a subgroup U of S is normalized both by a Sylow q -subgroup K of X and by a Sylow r -subgroup L of Y , then $U = 1$.*

PROOF. Since $X = SK = KS$ and $Y = SL = LS$, $\langle U^S \rangle$ is normal both in X and in Y . Thus $U \subseteq \langle U^S \rangle = 1$ by (c).

(3.2) *The following holds.*

- (1) $S = QR \neq Q = [Q, O^2(X)](Q \cap R)$.
- (2) $[Q, O^2(X)] \not\subseteq Q \cap R \not\triangleleft X$.
- (3) $Q = (R^x \cap Q)(Q \cap R)$ and $X = \langle R^x, R \rangle = \langle R, x \rangle$ if $X = \langle S, x \rangle$.
- (4) *The statements in (1)–(3) remain true if we replace X, Q, R by Y, R, Q , respectively.*

PROOF. First, (4) holds by symmetry of X and Y . We have $S = QR$ and $Q = [Q, O^2(X)](Q \cap R)$ by (g), and $Q \neq S \neq R$ by (2.3). Therefore $Q \neq Q \cap R$, and hence $[Q, O^2(X)] \not\subseteq Q \cap R$.

Choose $x \in X$ so that $X = \langle S, x \rangle$. Then $X = \langle S^x, S \rangle = \langle (QR)^x, QR \rangle = \langle R^x, Q, R \rangle$ by (2.1)–(10). Let $T = (R^x \cap Q)(R \cap Q)$. Since $[Q, R^x] \subseteq Q \cap R^x \subseteq T \subseteq$

$Q \cap R \supseteq [Q, R], [Q, X] \subseteq [Q, Q]T$, and hence $Q = [Q, O^2(X)](Q \cap R) \subseteq [Q, Q]T$. This yields that $Q = T$, and so $Q \cap R \not\triangleleft X$ because $Q \neq Q \cap R$. Therefore $X = \langle R^x, Q, R \rangle = \langle R^x, (R^x \cap Q)(Q \cap R), R \rangle = \langle R^x, R \rangle = \langle R, x \rangle$.

(3.3) *Let $Y \triangleright U \subseteq Q$. Suppose that there exists a normal elementary abelian subgroup T/U of S/U not contained in R/U . Then $n_*(X) = 1$ and $|S : Q| = 2$.*

PROOF. Take $y \in Y$ so that $Y = \langle S, y \rangle$. Since $R \supseteq T^y \triangleleft S^y$, $\Omega_1(Z(S^y/R)) \subseteq T^y R/R$, so $Y = \langle S, T^y \rangle$ by (2.1)–(10). Thus $Y = \langle Q, T^y \rangle$ by (3.2)–(3), and so, $[R, O^2(Y)] \subseteq [R, Y] \subseteq [R, Q][R, T^y] \subseteq (Q \cap R)(R \cap T^y)$. Therefore $S = Q[R, O^2(Y)] = Q(R \cap T^y)$, and hence S/Q is elementary abelian. Thus (2.1)–(7) shows that $n_*(X) = 1$ and $|S : Q| = 2$.

(3.4) *Suppose that either $V \not\subseteq R$ or $W \not\subseteq Q$. Then $n_*(X) = 1 = n_*(Y)$.*

PROOF. We may assume, by symmetry, that $V \not\subseteq R$. Then $n_*(X) = 1$ and $|R : Q \cap R| = |S : Q| = 2$ by (3.3). Thus V induces a factorwise transvection on R because $[Q \cap R, V] = 1$. Since $S = QR \subseteq C_Y(V)R$, $VR/R \subseteq \Omega_1(Z(S/R))$, and hence (2.11)–(1) shows that $O^2(Y/R) = O^2(Y/RC_Y(R)) \cong \mathbf{Z}_3$. Therefore $n_*(Y) = 1$ by (2.1)–(9).

(3.5) *If $\langle V, W \rangle \subseteq Q \cap R$, then either $Z \subseteq Z(X)$ or $Z \subseteq Z(Y)$.*

PROOF. Suppose $\langle V, W \rangle \subseteq Q \cap R$, and that $Z(X) \not\subseteq Z \not\subseteq Z(Y)$. By symmetry, we may assume $Q(K(S)) \not\triangleleft X$. Then (2.2) shows that $[Q, O^2(X)] \subseteq V \subseteq Q \cap R$, contrary to (3.2)–(2).

(3.6) *If $Z \subseteq Z(Y)$, then $V = [V, O^2(X)]$ and $C_V(X) = 1$.*

PROOF. Since $C_V(S) \cap C_V(O^2(X)) = C_Z(X) = 1$ by (c), we have $C_V(X) \subseteq C_V(O^2(X)) = 1$. Thus we have $V = [V, O^2(X)]$ by (2.4)–(3), regarding V as a $GF(2)X$ -module.

(3.7) *Suppose $Z \subseteq Z(Y)$. If $\langle V^Y \rangle$ is not abelian, then $n_*(X) = 1 = n_*(Y)$.*

PROOF. We may assume $V \subseteq R$ by (3.4), and so $[Q/V, O^2(X)] \neq 1$ by (3.2)–(2). Take $y \in Y$ so that $[V, V^y] \neq 1$. We may assume $|V : V \cap Q^y| \leq |V : V \cap Q^{y^{-1}}| = |V^y : V^y \cap Q|$, replacing y by y^{-1} if necessary. Let $\bar{X} = X/C_X(V)$, and regard V as a faithful $GF(2)\bar{X}$ -module. Then we have

$$C_V(\bar{S}) \not\subseteq C_V(\bar{X})$$

by (3.6). Moreover, since $Q \subseteq C_X(V) \not\subseteq O^2(X)$, $\bar{X}$ is a 2-irreducible solvable group with $O_2(\bar{X}) = 1$ and $V^y \cap Q = C_{V^y}(V)$ by (2.4)–(1). Similarly, regarding V^y as a $GF(2)X^y$ -module, we have $V \cap Q^y = C_V(V^y) = C_V(\bar{V}^y)$, and hence $|V : C_V(\bar{V}^y)| \leq |\bar{V}^y| \neq 1$.

Let $\bar{A}$ be a nonidentity elementary abelian 2-subgroup of $\bar{X}$ that is minimal under the condition $|V : C_V(\bar{A})| \leq |\bar{A}| \neq 1$. We will prove that

$$|\bar{A}| \cdot |C_V(\bar{A})| \geq |\bar{B}| \cdot |C_V(\bar{B})| \quad \text{for all subgroups } \bar{B} \text{ of } \bar{A}.$$

This is true if $\bar{B} = 1$ or $\bar{A}$. Suppose $1 \neq \bar{B} \neq \bar{A}$. Then the minimality of $\bar{A}$ forces that $|V : C_V(\bar{B})| > |\bar{B}|$, and hence $|\bar{A}| \cdot |C_V(\bar{A})| \geq |V| > |\bar{B}| \cdot |C_V(\bar{B})|$.

We have shown that the pair $(\bar{X}, V)$ satisfies Hypothesis II of [5]. Therefore there exists, by 5.1 of [5], a normal subgroup $\bar{N}$ of $\bar{X}$ such that $\bar{X} = \bar{S}\bar{N}$ and $\bar{N} \cong \Sigma_3 \times \cdots \times \Sigma_3$ (k copies), and, moreover, $V = [V, \bar{N}]$ by (3.6). Since $V^y \triangleleft R$ and $S = QR$, we can choose $t \in V^y - Q$ so that $tQ \in Z(S/Q)$. Thus we have

$$2^k = |V : C_V(\bar{t})| \leq |V : C_V(\overline{V^y})| \leq |\overline{V^y}|$$

by 5.6 of [5]. Since $[V, V^y, V^y] = 1$, 5.7 of [5] shows that $|\overline{V^y}| = 2^k$ and that $C_{\bar{X}}(\overline{V^y}) = \overline{V^y}$, and then 5.8 of [5] shows that $\overline{V^y} = \bar{S} \cap \bar{N}$. Therefore $|V : V \cap Q^y| = |V^y : V^y \cap Q| = |V : V \cap Q^{y^{-1}}|$. This shows that $\bar{V}^z = \bar{S} \cap \bar{N}$ for all $z \in Y$ with $[V, V^z] \neq 1$, and hence $\bar{U} = \bar{S} \cap \bar{N}$ and $U = (U \cap Q)V^y = (U \cap Q)V^{y^{-1}} = (U \cap Q^y)V$, where $U = \langle V^Y \rangle$. Therefore $[U, V^y] = [V, V^y] \subseteq V$, and so $Q/V \not\subseteq VV^y/V \subseteq \Omega_1(Z(U/V)) \triangleleft S/V$. Thus $n_*(Y) = 1$ and $|Q^y : R \cap Q^y| = |S^y : R| = 2$ by (3.3).

Note that $V \subseteq Q \cap R \subseteq C_{S^y \bmod Q^y}(V) = VQ^y$ as above. Thus $Q \cap R = V(Q \cap R \cap Q^y)$, and hence t induces a factorwise transvection on Q/V . Therefore $O^2(X/QC_X(Q/V)) \cong \mathbf{Z}_3$ by (2.11)–(1), and so $n_*(X) = 1$ by (2.1)–(9).

(3.8) Let $X \triangleright A \subseteq R$, and suppose $n(Y, B/C) = 1$ where $B = \langle A^Y \rangle$ and $C = \cap A^Y$. Let $D \triangleleft Y$ and $E \triangleleft X$ so that

$$[D, O^2(Y)] \subseteq B \subseteq D \subseteq R, \quad A \subseteq E \subseteq Q, \quad \text{and} \quad B \not\subseteq E.$$

Then $[E \cap D, [R, O^2(Y)]] \subseteq A$. In particular, $[E, O^2(X)] \subseteq A$ if $E \subseteq D$.

PROOF. We may assume $D = (E \cap D)B = \langle (E \cap D)^Y \rangle$. Let $P = [R, O^2(Y)]$, $F = [D, O^2(Y)]C$, and $J = [D, P]C$. Since $J \subseteq [D, R]C$, we have $[D, O^2(Y)] \not\subseteq J$ by (2.14)–(1), and so $J \subset F \subseteq B$ and $[J, O^2(Y)] \subseteq C$ because $n(Y, D/C) = 1$. Since $[B, R] \subseteq C$ by (2.14)–(2), we have $B/C = F/C \times C_{B/C}(O^2(Y))$ by (2.4)–(3), regarding B/C as a $GF(2)Y$ -module. Thus $C_{F/C}(O^2(Y)) = 1$ and so $J \subseteq C$. This shows that $[E \cap D, P] \subseteq J \subseteq C \subseteq A$. Suppose $E \subseteq D$. Since $P \not\subseteq Q$ by (3.2)–(3), X has no noncentral chief factor within E/A , and hence $[E, O^2(X)] \subseteq A$.

(3.9) Let $X \triangleright A \subseteq R$, and suppose $n(Y, B/C) = 1$ and $(Y, B/C) \ll (O_4^+(2), GF(2)^4)$ where $B = \langle A^Y \rangle$ and $C = \cap A^Y$. Assume $B \subseteq Q$, and define $E = \cap B^X$, $D = \langle B^X \rangle$, and $D^{(n)} = [D, Q; n]A$ for $n \geq 0$. Then the following holds.

- (1) $2^4 \leq |B : C| \leq 2^4 \cdot |A : C|$.
- (2) If $|B : A| = 2^n$, then $|B \cap D^{(k-1)} : B \cap D^{(k)}| = 2$ for $1 \leq k \leq n$.
- (3) $|E : A| \leq 2$, and $D^{(m-1)} \neq E = D^{(m)}$ if $|B : E| = 2^m$.

PROOF. Let $M = [B, O^2(Y)]C$. Then $B = AM$ and, moreover, $[B, R] \subseteq C$ by (2.14)–(2) as $n(Y, B/C) = 1$. Thus we have $B/C = M/C \times C_{B/C}(O^2(Y))$ by (2.4)–(3), and hence $(Y, M/C) \ll (O_4^+(2), GF(2)^4)$, and $|B : C| = |B : M| \cdot |M : C| = |A : A \cap M| \cdot |M : C| \leq |A : A \cap M| \cdot 2^4$.

Let $|B : A| = 2^n$ and $|B : E| = 2^m$. Since B/A is isomorphic, as $GF(2)S$ -modules, to a factor module of M/C , we have

$$|[B, Q; k-1]A : [B, Q; k]A| = 2 \quad (1 \leq k \leq n),$$

by (2.5)–(3). Thus $|B \cap D^{(k-1)} : B \cap D^{(k)}| = 2$ ($1 \leq k \leq n$), and, in particular, $D^{(m-1)} \neq D^{(m)}$. Since $[E, R] \subseteq A$ and $X = \langle R^X \rangle$, $[E, S] \subseteq [E, X] \subseteq A$, and so $|E : A| \leq 2$.

(3.10) Let $X \triangleright A \neq 1$ and $Y \triangleright B, C$ with $C \subseteq A \subseteq B \subseteq R$. Suppose $O^2(Y/RC_Y(B/C))$ is elementary abelian and nonidentity. Then the following holds.

(1) Let T be a 2-subgroup of Y . If $Y \neq \langle S, T \rangle$, then $T \subseteq S^y$ and $A = A^y$ for some $y \in Y$.

(2) Let $z \in Y$. Then $A \neq A^z$ if and only if $Y = \langle S, S^z \rangle$.

PROOF. Let $M = SC_Y(B/C)$. Then M is the unique maximal subgroup of Y containing S by (2.1)–(8).

Suppose $Y \neq \langle S, T \rangle$ for a 2-subgroup T of Y . Then $T \subseteq M$, so there exists an element $y \in M$ such that $T \subseteq S^y$. We have $A = A^y$ by the definition of M . Hence (1) holds.

Let $z \in Y$. If $A = A^z$, then $A \triangleleft \langle X, S^z \rangle$, so $Y \neq \langle S, S^z \rangle$ by (c). Suppose $Y \neq \langle S, S^z \rangle$. Then $S^z = S^y$ and $A = A^y$ for some $y \in Y$ by (1). Since $S \not\trianglelefteq Y$ by (3.2)–(1), we have $zy^{-1} \in N_Y(S) \subseteq M$, and so $A^{zy^{-1}} = A$. Thus $A^z = A^y = A$, proving (2).

§4. The coset graph

Throughout the remainder of this paper, suppose the amalgam (X, S, Y) satisfies the hypothesis of Theorem C.

First, we will define the coset graph $\Gamma = \Gamma(X, S, Y)$ associated with the amalgam (X, S, Y) . Let G be the amalgamated product of X and Y over S , and embed S, X, Y into G . Note that $S = X \cap Y$ in G by the uniqueness of the standard forms of elements in an amalgamated product. The set $V(\Gamma)$ of vertices of Γ consists of the right cosets of X and Y in G , and two vertices are adjacent if and only if they are distinct and have a nonempty intersection. Then G acts on Γ by right multipli-

cation. We call a set of two adjacent vertices an *edge*, and denote by $E(\Gamma)$ the set of edges of Γ .

(4.1) *The group G acts transitively on $E(\Gamma)$.*

PROOF. Let $\{\xi, \eta\} \in E(\Gamma)$. We may assume $\xi = Xg$ and $\eta = Yg'$ for some elements $g, g' \in G$. Since $Xg \cap Yg' \neq \phi$, there exists an element $h \in G$ such that $(Xg \cap Yg')h \cap S \neq \phi$, and hence $Xgh \cap S \neq \phi \neq Yg'h \cap S$. Therefore we have $Xgh = X$ and $Yg'h = Y$, and so $\{\xi, \eta\}^h = \{X, Y\}$.

We denote by G_ξ the stabilizer of ξ in G for all $\xi \in V(\Gamma)$. Then (4.1) shows that the amalgam (X, S, Y) is isomorphic to an amalgam $(G_\xi, G_{\xi\eta}, G_\eta)$ for all edges $\{\xi, \eta\} \in E(\Gamma)$ where $G_{\xi\eta} = G_\xi \cap G_\eta$, and hence (f_2) forces that $n_*(G_\xi) \leq f(G_\xi) \leq 2 \geq f(G_\eta) \geq n_*(G_\eta)$.

For $\xi \in V(\Gamma)$, let $\Delta(\xi) = \Delta_\Gamma(\xi)$ be the set of vertices of Γ adjacent to ξ , and put $Q_\xi = O_2(G_\xi)$ and $V_\xi = \langle \Omega_1(Z(T)) \mid T \in \text{Syl}_2(G_\xi) \rangle$. Then G_ξ is transitive on $\Delta(\xi)$ by (4.1), and hence $\text{Syl}_2(G_\xi) = \{G_{\xi\eta} \mid \eta \in \Delta(\xi)\}$. Thus we have

$$V_\xi = \langle \Omega_1(Z(G_{\xi\eta})) \mid \eta \in \Delta(\xi) \rangle. \quad \dots (*)$$

A *path of length n* in Γ is a sequence $S = (\xi_0, \xi_1, \dots, \xi_n)$ of $n+1$ vertices of Γ such that $\{\xi_{i-1}, \xi_i\} \in E(\Gamma)$ for $1 \leq i \leq n$.

(4.2) *The graph Γ is connected, that is, every pair of vertices is joined by a path.*

PROOF. Let Γ_0 be a connected component of Γ containing the edge $\{X, Y\}$. Then G leaves Γ_0 invariant because $G = \langle X, Y \rangle$. Thus every edge of Γ is contained in Γ_0 by (4.1), and hence $\Gamma = \Gamma_0$.

Let $\lambda, \mu \in V(\Gamma)$. The *distance* $d(\lambda, \mu) = d_\Gamma(\lambda, \mu)$ of $\lambda, \mu \in V(\Gamma)$ is the minimum length of the paths from λ through μ in Γ . We will define $b = b(X, S, Y)$ to be the minimum distance $d(\alpha, \delta)$ of $\alpha, \delta \in V(\Gamma)$ under the condition that $V_\alpha \not\subseteq Q_\delta$.

(4.3) *We have $1 \leq b < \infty$.*

PROOF. Note that $\bigcap_{\xi \in V(\Gamma)} Q_\xi \subseteq \bigcap_{\xi \in V(\Gamma)} G_\xi = 1$ by (c). Thus there exists a pair (α, δ) of vertices of Γ such that $V_\alpha \not\subseteq Q_\delta$. Hence b is well-defined and $b < \infty$ because $d(\alpha, \delta) < \infty$ by (4.2). Also, we have $1 \leq b$ because $V_\xi = \langle \Omega_1(Z(G_{\xi\eta})) \mid \eta \in \Delta(\xi) \rangle$ by (*) and $\Omega_1(Z(G_{\xi\eta})) \subseteq C_{G_\xi}(Q_\xi) \subseteq Q_\xi$ by (d).

Now, define

$$\mathcal{A} = \{\alpha \in V(\Gamma) \mid V_\alpha \not\subseteq Z(G_\alpha)\},$$

$$\mathcal{B} = V(\Gamma) - \mathcal{A}, \quad \text{and}$$

$$\mathcal{C} = \{(\alpha, \delta) \mid \alpha, \delta \in V(\Gamma), V_\alpha \not\subseteq Q_\delta, d(\alpha, \delta) = b\}.$$

(4.4) Let $\{\xi, \eta\} \in E(\Gamma)$, and suppose $\eta \in \mathcal{B}$. Then $\xi \in \mathcal{A}$, $V_\eta = \Omega_1(Z(G_{\xi\eta}))$, and $V_\xi = \langle V_\eta^{G_\xi} \rangle$.

PROOF. We have $V_\eta = \langle \Omega_1(Z(G_{\xi\eta}))^{G_\eta} \rangle = \Omega_1(Z(G_{\xi\eta}))$ by (*). Hence $\xi \in \mathcal{A}$ by (c), and $V_\xi = \langle \Omega_1(Z(G_{\xi\eta}))^{G_\xi} \rangle = \langle V_\eta^{G_\xi} \rangle$ by (*) and the above.

(4.5) If $(\alpha, \delta) \in \mathcal{C}$, then $\alpha \in \mathcal{A}$.

PROOF. Let $(\alpha, \beta, \dots, \delta)$ be a path of length b from α through δ . Suppose $\alpha \in \mathcal{B}$. Then $V_\alpha \subseteq V_\beta$ by (4.4), so $V_\beta \not\subseteq Q_\delta$ and $d(\beta, \delta) < b$, contrary to the minimality of b .

(4.6) Suppose $b \geq 2$. Then either

$$\xi \in \mathcal{A} \quad \text{and} \quad \eta \in \mathcal{B}, \quad \text{or} \quad \eta \in \mathcal{A} \quad \text{and} \quad \xi \in \mathcal{B}$$

for all $\{\xi, \eta\} \in E(\Gamma)$.

PROOF. Let $\{\xi, \eta\} \in E(\Gamma)$. Since $b \geq 2$, we have $\langle V_\xi, V_\eta \rangle \subseteq Q_\xi \cap Q_\eta$, so either $\Omega_1(Z(G_{\xi\eta})) \subseteq Z(G_\xi)$ or $\Omega_1(Z(G_{\xi\eta})) \subseteq Z(G_\eta)$ by (3.5). This yields that either $\xi \in \mathcal{B}$ or $\eta \in \mathcal{B}$, and so the assertion follows from (4.4).

For the proof of Theorem C, it suffices to prove that $n_*(G_\xi) = 1 = n_*(G_\eta)$ for some $\{\xi, \eta\} \in E(\Gamma)$ by (4.1). We now obtain the following theorem.

THEOREM 4.7 Suppose $b \leq 2$. Then $n_*(G_\xi) = 1 = n_*(G_\eta)$ for some $\{\xi, \eta\} \in E(\Gamma)$.

PROOF. Let $(\alpha, \delta) \in \mathcal{C}$. We have $V_\alpha \not\subseteq Q_\delta$ and $d(\alpha, \delta) = b$ by the definition of $\mathcal{C}$.

Suppose first that $b = 1$. Then $\{\alpha, \delta\} \in E(\Gamma)$, and hence $n_*(G_\alpha) = 1 = n_*(G_\delta)$ by (3.4).

Suppose next that $b = 2$, and take $\beta \in V(\Gamma)$ so that $\beta \in \Delta(\alpha) \cap \Delta(\delta)$. Then (4.3) and (4.4) show that $\alpha \in \mathcal{A} \ni \delta$ and $\beta \in \mathcal{B}$. Since $V_\alpha \subseteq Q_\beta$ by the minimality of b and $C_{G_{\beta\delta}}(V_\delta) = Q_\delta$ by (2.4)–(1), we have $[V_\alpha, V_\delta] \neq 1$, and so $\langle V_\alpha^{G_\beta} \rangle$ is nonabelian. Thus $n_*(G_\alpha) = 1 = n_*(G_\beta)$ by (3.7).

In the remaining four sections, we treat the case where $b \geq 3$. By (4.6), we can identify an edge with an ordered pair $\langle \xi, \eta \rangle$ of adjacent vertices with $\xi \in \mathcal{A}$ and $\eta \in \mathcal{B}$. Since $G_\xi \not\cong G_\eta$ for all $\xi \in \mathcal{A}$ and $\eta \in \mathcal{B}$, (4.1) shows that G acts transitively on $\mathcal{A}$ and $\mathcal{B}$, respectively. Therefore, for the proof of Theorem C, it suffices to prove that $n_*(G_\xi) = 1 = n_*(G_\eta)$ for some $\xi \in \mathcal{A}$ and some $\eta \in \mathcal{B}$.

§5. Even case

In this section, we consider the case where b is even and $b \geq 4$.

THEOREM 5.1 *Let $\xi \in \mathcal{A}$ and $\eta \in \mathcal{B}$. If b is even and $b \geq 4$, then $n_*(G_\xi) = 1 = n_*(G_\eta)$.*

PROOF. Pick $(\alpha, \delta) \in \mathcal{C}$, and denote by $(\alpha, \alpha+1, \alpha+2, \dots, \delta-1, \delta)$ a path of length b from α through δ . Then $\alpha \in \mathcal{A}$, $\alpha+1 \in \mathcal{B}$, $\alpha+2 \in \mathcal{A}$, $\dots$, $\delta-1 \in \mathcal{B}$, $\delta \in \mathcal{A}$ by (4.5) and (4.6). Also, we have $V_\delta \subseteq Q_{\alpha+1} \subseteq G_{\alpha, \alpha+1}$ (resp. $V_\alpha \subseteq Q_{\delta-1} \subseteq G_{\delta-1, \delta}$) by the minimality of b , and so $V_\delta \cap Q_\alpha = C_{V_\delta}(V_\alpha)$ (resp. $V_\alpha \cap Q_\delta = C_{V_\alpha}(V_\delta)$) by (2.4)–(1), regarding V_α (resp. V_δ) as a $GF(2)G_{\alpha^-}$ (resp. $GF(2)G_{\delta^-}$) module. Since $V_\alpha \not\subseteq Q_\delta$ by the definition of $\mathcal{C}$, we have $[V_\alpha, V_\delta] \neq 1$, so $V_\delta \not\subseteq Q_\alpha$. Now, by symmetry of α and δ , we may assume $|V_\alpha : V_\alpha \cap Q_\delta| \leq |V_\delta : V_\delta \cap Q_\alpha| \neq 1$, and hence

$$(*) \quad |V_\alpha : C_{V_\alpha}(\bar{V}_\delta)| \leq |\bar{V}_\delta| \neq 1,$$

where $\bar{G}_\alpha = G_\alpha / C_{G_\alpha}(V_\alpha)$. We will prove that $n_*(G_\alpha) = 1 = n_*(G_{\alpha+1})$ in a series of lemmas.

For a while, regard V_α as a faithful $GF(2)\bar{G}_\alpha$ -module.

$$(1) \quad O^2(\bar{G}_\alpha) \cong \mathbf{E}_{3^k} \text{ and } |V_\alpha| = 2^{2k}, \text{ where } k = 1 \text{ or } 2.$$

PROOF. By (2.4)–(1) and (3.6), $\bar{G}_\alpha$ is a 2-irreducible solvable group with $O_2(\bar{G}_\alpha) = 1$ and $V_\alpha = [V_\alpha, O^2(\bar{G}_\alpha)]$. Since $n_*(\bar{G}_\alpha) \leq n_*(G_\alpha) \leq f(G_\alpha) \leq 2$, the assertion follows from (*) together with (2.10)–(1).

Throughout the remainder of this proof of (5.1), let the letter k have the same meaning as in (1).

Now, we will introduce the following notation.

NOTATION. Let $\langle \xi, \eta \rangle \in E(\Gamma)$.

$$\begin{aligned} Z_{\xi\eta} &= \Omega_1(Z(G_{\xi\eta} \bmod Q_\xi)), \\ E_{\xi\eta} &= C_{G_{\xi\eta} \bmod Q_\xi}(C_{V_\xi}(Z_{\xi\eta})), \\ U_\eta &= \langle V_\xi^{G_\eta} \rangle, \\ U_\eta^{(n)} &= [U_\eta, Q_\eta; n] \quad \text{for } n \geq 0, \\ W_\xi &= \cap U_\eta^{G_\xi}, \quad \text{and} \\ T_\xi &= \langle U_\eta^{G_\xi} \rangle. \end{aligned}$$

(2) *If $k = 2$, then $Z_{\alpha, \alpha+1} \not\subseteq V_\delta \subseteq E_{\alpha, \alpha+1}$, and $\bar{E}_{\alpha, \alpha+1}$ is the unique fours subgroup of $\bar{G}_{\alpha, \alpha+1}$ acting quadratically on V_α .*

PROOF. This follows from (2.5)–(4)(5).

(3) *Let $\langle \xi, \eta \rangle \in E(\Gamma)$. Then*

$$U_\eta^{(2k-1)} = V_\eta = \cap V_\xi^{G_\eta}.$$

PROOF. Let $Z = \cap V_\xi^{G_\eta}$. Since $G_\eta = \langle Q_\xi^{G_\eta} \rangle$ by (3.2)–(3), we have $V_\eta \subseteq Z \subseteq C_{V_\xi}(G_\eta) \subseteq V_\eta$.

Suppose first that $k = 1$. Then we have $|V_\xi : V_\eta| = 2$ by (1), and hence $[V_\xi, Q_\eta] = V_\eta$ because $Q_\eta \not\subseteq Q_\xi = C_{G_\xi}(V_\xi)$ by (3.2)–(1) and (2.4)–(1). Suppose next that $k = 2$. Then (2.5)–(4) shows that $[V_\xi, Q_\eta; 3] = [V_\xi, G_\xi; 3] = V_\eta$. Therefore we have $[V_\xi, Q_\eta; 2k-1] = V_\eta$, and so $[U_\eta, Q_\eta; 2k-1] = V_\eta$ by (2.14)–(1).

Let $\lambda \in \Delta(\delta)$ so that $G_\delta = \langle V_\alpha, G_{\delta\lambda} \rangle$. We have $G_\delta = \langle V_\alpha, Q_\delta \rangle$ by (3.2)–(3).

(4) *The following holds.*

- (i) $U_{\alpha+1} \cap V_\lambda = 1$.
- (ii) $G_{\alpha+1} \supseteq Q_{\alpha+2} \supseteq U_\lambda^{(2k-2)} \not\subseteq G_{\alpha, \alpha+1}$.
- (iii) If $k = 1$, then $V_{\alpha+1} = V_{\delta-1}$, $W_{\delta-1} = U_\delta \cap Q_{\alpha+1}$, and $[W_{\alpha+2}, U_\lambda] = 1$.

PROOF. Since $U_{\alpha+1} \cap V_\lambda$ is centralized by G_δ and G_λ , we have $U_{\alpha+1} \cap V_\lambda = 1$, so (i) holds.

Since $V_{\alpha+2} \subseteq Q_\lambda$ and $U_\lambda^{(2k-2)} \subseteq U_\lambda \subseteq Q_{\alpha+3}$ by the minimality of b , $[V_{\alpha+2}, U_\lambda^{(2k-2)}] \subseteq V_{\alpha+2} \cap U_\lambda^{(2k-1)} \subseteq U_{\alpha+1} \cap V_\lambda = 1$ by (i), so $U_\lambda^{(2)} \subseteq C_{G_{\alpha+2, \alpha+3}}(V_{\alpha+2}) = Q_{\alpha+2} \subseteq G_{\alpha+1}$. Let $W = V_\delta(U_\lambda^{(2k-2)} \cap G_{\alpha, \alpha+1})$. Then we have $[W, Q_\lambda] \subseteq V_\delta \subseteq W$ because $[U_\lambda^{(2k-2)}, Q_\lambda] = V_\lambda \subseteq V_\delta$ by (3).

Suppose first that $k = 2$. Since $[W, V_\delta] = 1$ by the minimality of b ,

$$[V_\alpha, W] = [V_\alpha, V_\delta][V_\alpha \cap Q_\delta, W] \subseteq W$$

by (2) and (2.5)–(6). Thus we have $W \triangleleft \langle V_\alpha, Q_\lambda \rangle = G_\delta$, and hence $[W, G_\delta] = [W, \langle Q_\lambda^{G_\delta} \rangle] \subseteq V_\delta$. Therefore (3.8) shows that $U_\lambda^{(2)} \not\subseteq W$ because $[U_\lambda^{(2)}, O^2(G_\lambda)] \not\subseteq V_\lambda$ by (2.14)–(1).

Suppose next that $k = 1$. Then $|V_\alpha : V_{\alpha+1}| = 2 = |V_\delta : V_{\delta-1}|$, so $V_{\alpha+1} = [V_\alpha, V_\delta] = V_{\delta-1}$. Thus we have $[V_\alpha, W] \subseteq V_{\alpha+1} = V_{\delta-1} \subseteq W$, and so $W \triangleleft G_\delta$. This shows that $W = W_\delta \not\subseteq U_\lambda$. Since $W_{\alpha+2} \subseteq U_{\alpha+3} \subseteq Q_\lambda$ and $U_\lambda \subseteq Q_{\alpha+3}$ by the minimality of b , we have $[W_{\alpha+2}, U_\lambda] \subseteq W_{\alpha+2} \cap V_\lambda \subseteq U_{\alpha+1} \cap V_\lambda = 1$ by (i). This proves (ii) and (iii).

Now, we will assume $k = 2$, and derive a contradiction.

$$(5) \quad O^2(G_{\alpha+1}/Q_{\alpha+1}C_{G_{\alpha+1}}(U_{\alpha+1})) \cong \mathbf{Z}_3 \text{ or } \mathbf{E}_{3^2}.$$

PROOF. By (3) and (4-i), we have $[U_{\alpha+1} \cap Q_\lambda, U_\lambda^{(2)}] \subseteq U_{\alpha+1} \cap V_\lambda = 1$. Since $m(G_{\delta-1, \delta}/Q_\delta) \leq 2 \geq m(G_{\delta\lambda}/Q_\lambda)$ by (2.1)–(6),

$$\begin{aligned} |U_{\alpha+1} : C_{U_{\alpha+1}}(U_\lambda^{(2)})| &\leq |U_{\alpha+1} : U_{\alpha+1} \cap Q_\lambda| \\ &\leq |U_{\alpha+1} : U_{\alpha+1} \cap Q_\delta| \cdot |U_{\alpha+1} \cap Q_\delta : U_{\alpha+1} \cap Q_\delta \cap Q_\lambda| \\ &\leq 2^2 \cdot 2^2 = 2^4. \end{aligned}$$

Since $n(G_{\alpha+1}, U_{\alpha+1}) \geq 3$ by (3) and (2.14)–(2), $U_\lambda^{(2)}$ induces a factorwise sesquitransvection on $U_{\alpha+1}$ by (2.11)–(2). Hence the assertion follows from (2.11)–(1).

(6) *A Contradiction.*

PROOF. Since $V_{\alpha+1}[V_\alpha, V_\delta] \neq V_{\alpha+1}$ by (2.5)–(7) and $U_\lambda^{(2)}$ centralizes $V_{\alpha+1}[V_\alpha, V_\delta]$ by (4-ii) and the minimality of b , we have $G_{\alpha+1} \neq \langle G_{\alpha, \alpha+1}, U_\lambda^{(2)} \rangle$. Therefore there exists $x \in G_{\alpha+1}$ such that $U_\lambda^{(2)} \subseteq G_{\alpha, \alpha+1}^x = G_{\alpha^x, \alpha+1}$ and $V_{\alpha^x} = V_\alpha^* = V_\alpha \not\subseteq Q_\delta$, by (5) and (3.10)–(1). Thus we have $(\alpha^x, \delta) \in \mathcal{C}$, but then this contradicts (4-ii).

We have shown that $k = 1$. Thus $n_*(G_\alpha) = 1$ by (2.1)–(9).

Now, put $\hat{U}_{\alpha+1} = U_{\alpha+1}/V_{\alpha+1}$, $C_{\alpha+1} = C_{G_{\alpha+1}}(\hat{U}_{\alpha+1})$, and $\tilde{G}_{\alpha+1} = G_{\alpha+1}/C_{\alpha+1}$. We have $O^2(G_{\alpha+1}) \not\subseteq C_{\alpha+1}$ as $V_{\alpha+1} \subseteq V_\alpha \subseteq U_{\alpha+1}$, and $Q_{\alpha+1} \subseteq C_{\alpha+1}$ by (3). Therefore $\tilde{G}_{\alpha+1}$ is a 2-irreducible solvable group with $O_2(\tilde{G}_{\alpha+1}) = 1$ by (2.4)–(1). If $n_*(\tilde{G}_{\alpha+1}) = 1$, then $O^2(\tilde{G}_{\alpha+1})$ is cyclic, and so $n_*(G_{\alpha+1}) = 1$ by (2.1)–(9). Since $n_*(\tilde{G}_{\alpha+1}) \leq n_*(G_{\alpha+1}) \leq f(G_{\alpha+1}) \leq 2$, we will assume $n_*(\tilde{G}_{\alpha+1}) = 2$, and argue for a contradiction.

(7) $|W_\alpha| = 2^3$, $|U_{\alpha+1}| = 2^5$, $[U_{\alpha+1}, U_\lambda]V_{\alpha+1} = W_{\alpha+2} = C_{U_{\alpha+1} \bmod V_{\alpha+1}}(U_\lambda)$, and $n(G_\alpha, T_\alpha/W_\alpha) \geq 2$.

PROOF. We will regard $\hat{U}_{\alpha+1}$ as a faithful $GF(2)\tilde{G}_{\alpha+1}$ -module. By (4-iii) and (2.4)–(1), we have

$$|\hat{U}_{\alpha+1} : C_{\hat{U}_{\alpha+1}}(\tilde{U}_\lambda)| \leq |U_{\alpha+1} : W_{\alpha+2}| = |U_\lambda : W_\delta| = |\tilde{U}_\lambda|.$$

Since $n_*(\tilde{G}_{\alpha+1}) = 2$ by our hypothesis, we have $O^2(\tilde{G}_{\alpha+1}) \cong \mathbf{E}_{3^2}$ and $|\hat{U}_{\alpha+1}| \geq |[\hat{U}_{\alpha+1}, O^2(\tilde{G}_{\alpha+1})]| = 2^4$ by (2.10)–(1), and hence $(G_{\alpha+1}, \hat{U}_{\alpha+1}) \ll (O_4^+(2), GF(2)^4)$ by (2.5)–(1). Thus (3.8) shows that $|W_\alpha : V_\alpha| \leq 2$, and then $|W_\alpha : V_\alpha| = 2$ and $|U_{\alpha+1} : W_{\alpha+2}| = 2^2$. Consequently, we have $|W_\alpha| = 2^3$, $|U_{\alpha+1}| = 2^5$, and $|\tilde{U}_\lambda| = 2^2$. Hence, by (2.5)–(3)(4)(5), $\widehat{W}_{\alpha+2} = [\hat{U}_{\alpha+1}, \tilde{G}_{\alpha+1, \alpha+2}; 2]$ and $\tilde{U}_\lambda$ is the unique quadratic fours subgroup of $\tilde{G}_{\alpha+1, \alpha+2}$. Therefore $[\hat{U}_{\alpha+1}, \tilde{U}_\lambda] = \widehat{W}_{\alpha+2} = C_{\hat{U}_{\alpha+1}}(\tilde{U}_\lambda)$ by (2.5)–(4), and $T_\alpha^{(1)} \neq W_\alpha$ by (3.9), and then $n(G_\alpha, T_\alpha/W_\alpha) \geq 2$ by (2.14)–(2).

$$(8) \quad T_\delta \subseteq Q_{\alpha+2} \text{ and } T_\delta^2 \subseteq Q_{\alpha+1}.$$

PROOF. Note that $T_\delta \subseteq Q_{\alpha+3} \cap Q_\lambda$ by the minimality of b . As $[T_\delta, U_\lambda] \subseteq V_\lambda \subseteq W_\delta$ and $T_\delta = \langle U_\lambda^{G_\delta}, [T_\delta, T_\delta] \rangle \subseteq W_\delta$, and so $T_\delta^2 \subseteq W_\delta \subseteq Q_{\alpha+1}$. By (4-iii), we have

$[V_\alpha, T_\delta \cap Q_{\alpha+1}] \subseteq V_{\alpha+1} = V_{\delta-1} \subseteq W_\delta$. Therefore

$$\begin{aligned} |T_\delta/W_\delta : C_{T_\delta/W_\delta}(V_\alpha)| &\leq |T_\delta : T_\delta \cap Q_{\alpha+1}| \\ &\leq |T_\delta : T_\delta \cap Q_{\alpha+2}| \cdot |T_\delta \cap Q_{\alpha+2} : T_\delta \cap Q_{\alpha+2} \cap Q_{\alpha+1}| \\ &\leq 2 \cdot 2^2 = 2^3 \end{aligned}$$

because $m(G_{\alpha+2, \alpha+3}/Q_{\alpha+2}) \leq 1$ and $m(G_{\alpha+1, \alpha+2}/Q_{\alpha+1}) \leq 2$. Thus V_α induces a factorwise sesquitransvection on T_δ/W_δ by (7) and (2.11)–(2). Since G_δ is a $\{2, 3\}$ -group with cyclic Sylow 3-subgroups, we conclude that $O^2(G_\delta/Q_\delta C_{G_\delta}(T_\delta/W_\delta)) \cong \mathbf{Z}_3$ by (2.11)–(1), and hence $T_\delta = \langle U_{\delta-1}, U_\lambda, U_\lambda^x \rangle$ for some $x \in V_\alpha - Q_\delta$. This shows that $T_\delta \subseteq Q_{\alpha+2}$.

(9) *A contradiction.*

PROOF. Since $|U_{\alpha+1} : U_{\alpha+1} \cap Q_\delta| = 2$, we have $W_{\alpha+2} \subset U_{\alpha+1} \cap Q_\delta$, so $V_{\alpha+1} \subset [U_{\alpha+1} \cap Q_\delta, U_\lambda]V_{\alpha+1} \subseteq W_{\alpha+2} \cap U_\lambda \subseteq W_{\alpha+2} \cap W_\delta$. Hence

$$|W_{\alpha+2}W_\delta : W_\delta| = |W_{\alpha+2} : W_{\alpha+2} \cap W_\delta| \leq 2.$$

Since $|\tilde{U}_\lambda| = |U_\lambda : W_\delta| = 2^2$ by (7) and $\tilde{T}_\delta$ is an elementary abelian subgroup of $\tilde{G}_{\alpha+1, \alpha+2}$ containing $\tilde{U}_\lambda$ by (8), we have $\tilde{T}_\delta = \tilde{U}_\lambda$, and so

$$[U_{\alpha+1}, T_\delta] \subseteq [U_{\alpha+1}, U_\lambda]V_{\alpha+1} = W_{\alpha+2} \subseteq W_{\alpha+2}W_\delta$$

by (7). Therefore we have $n(G_\alpha, T_\alpha/W_\alpha) \leq 1$ by (2.11)–(2), which contradicts (7). This completes the proof of (5.1).

§6. Odd case

In the remainder of this paper, we will assume that b is odd and that $b \geq 3$. The main aim of this section is to prove that $n_*(G_\xi) = 1$ for some (and hence all) $\xi \in \mathcal{A}$.

Define

$$\begin{aligned} U_\eta &= \langle V_\xi \mid \xi \in \Delta(\eta) \rangle \quad \text{for } \eta \in \mathcal{B}, \quad \text{and} \\ \mathcal{C}' &= \{(\beta, \delta) \mid \beta, \delta \in \mathcal{B}, U_\beta \not\subseteq Q_\delta, \quad \text{and} \quad d(\beta, \delta) = b-1\}. \end{aligned}$$

Suppose $(\alpha, \delta) \in \mathcal{C}$, and that $(\alpha, \beta, \dots, \delta)$ is a path of length b from α through δ . Then we have $\beta \in \mathcal{B} \ni \delta$ by (4.5) and (4.6) as in Section 4, and, moreover, $U_\beta \not\subseteq Q_\delta$ and $d(\beta, \delta) = b-1$ by the definition of $\mathcal{C}$. This shows that $\mathcal{C}' \neq \emptyset$.

LEMMA 6.1 *Let $\beta, \delta \in \mathcal{B}$, and suppose $d(\beta, \delta) = b-1$. Then*

$$|V_\alpha : V_\alpha \cap Q_\delta| \leq 2 \quad \text{for all } \alpha \in \Delta(\beta).$$

PROOF. We will assume that $|V_\alpha : V_\alpha \cap Q_\delta| \geq 2^2$ for some $\alpha \in \Delta(\beta)$, and argue for a contradiction.

Let $(\beta, \beta+1, \dots, \delta-1, \delta)$ be a path of length $b-1$ from β through δ . Then we have $U_\beta \subseteq Q_{\delta-1}$ and $U_\delta \subseteq Q_{\beta+1}$ by the minimality of b . Since $m(G_{\delta-1, \delta}/Q_\delta) \leq 2$ by (2.1)–(6), we have $\Omega_1(Z(G_{\delta-1, \delta}/Q_\delta)) \subset V_\alpha Q_\delta/Q_\delta$. Let $\widehat{U}_\delta = U_\delta/V_\delta$. Then $n(G_\delta, \widehat{U}_\delta) \geq 1$ and

$$\begin{aligned} |\widehat{U}_\delta : C_{\widehat{U}_\delta}(V_\alpha)| &\leq |U_\delta : U_\delta \cap Q_\alpha| \\ &\leq |U_\delta : U_\delta \cap Q_\beta| \cdot |U_\delta \cap Q_\beta : U_\delta \cap Q_\beta \cap Q_\alpha|. \end{aligned}$$

Suppose first that $n_*(G_{\delta-1}) = 1$. Then $G_\alpha\beta/Q_\alpha$ is cyclic by (2.1)–(7), and so $|\widehat{U}_\delta : C_{\widehat{U}_\delta}(V_\alpha)| \leq 2^2 \cdot 2 = 2^3$. Hence $n(G_\delta, \widehat{U}_\delta) = 1$ and V_α induces a factorwise semioffending action on $\widehat{U}_\delta$ by (2.11)–(3). Thus $(G_\delta, \widehat{U}_\delta) \ll (O_4^+(2), GF(2)^4)$ by (2.11)–(1) and (2.5)–(2), and so $|V_{\delta-1, \delta} : V_\delta| \leq 2$ by (2.15). This shows that $|V_\alpha : V_\alpha \cap Q_\delta| \leq |V_\alpha : V_\beta| \leq 2$, a contradiction.

Suppose next that $n_*(G_{\delta-1}) = 2$. Put $\overline{G}_{\delta-1} = G_{\delta-1}/C_{G_{\delta-1}}(V_{\delta-1})$ and regard $V_{\delta-1}$ as a faithful $GF(2)\overline{G}_{\delta-1}$ -module. Since $n_*(\overline{G}_{\delta-1}) = 2$ by (2.1)–(9), we have $[V_{\delta-1}, Q_\delta; 2] = [V_{\delta-1}, G_{\delta-1, \delta}; 2] \neq 1$ by (2.12)–(1). Hence $[U_\delta, Q_\delta] \not\subseteq V_\delta^{G_{\delta-1}} = V_{\delta-1}$, and so $n(G_\delta, \widehat{U}_\delta) \geq 2$. Thus, by (2.11)–(3), $n(G_\delta, \widehat{U}_\delta) = 2$ and V_α induces a factorwise semioffending action on $\widehat{U}_\delta$ because $|\widehat{U}_\delta : C_{\widehat{U}_\delta}(V_\alpha)| \leq 2^2 \cdot 2^2 = 2^4$. Hence we have $|V_{\delta-1, \delta} : V_\delta| \leq 2^2$ as before by (2.11)–(1), (2.5)–(2), and (2.15). But then $O^2(\overline{G}_{\delta-1})$ is cyclic and nonidentity by (2.12)–(2), and so $n_*(G_{\delta-1}) = 1$ by (2.1)–(9), a contradiction.

THEOREM 6.2 *Let $\xi \in \mathcal{A}$ and $\eta \in \mathcal{B}$. If b is odd and $b \geq 3$, then $n_*(G_\xi) = 1$. More precisely,*

$$G_\xi/C_{G_\xi}(V_\xi) \cong \mathbf{D}_6 \quad \text{or} \quad \mathbf{D}_{10}, \quad \text{and} \quad |V_\xi| = |V_\eta|^2 = 2^2 \quad \text{or} \quad 2^4.$$

PROOF. Let $(\beta, \delta) \in \mathcal{C}'$. Take $\alpha \in \Delta(\beta)$ and $\lambda \in \Delta(\delta)$ so that $G_\delta = \langle V_\alpha, G_{\delta\lambda} \rangle = \langle V_\alpha, Q_\lambda \rangle$. Note that $V_\delta = V_\lambda \cap Q_\alpha$. We may assume $\xi = \lambda$ and $\eta = \delta$.

Suppose that $|V_\lambda : V_\delta| \geq 2^3$. Then we have $|V_\lambda : V_\lambda \cap Q_\beta| = 2$, $|V_\lambda \cap Q_\beta : V_\lambda \cap Q_\beta \cap Q_\alpha| = |V_\lambda \cap Q_\beta : V_\delta| = 2^2$, and $|V_\alpha : C_{V_\alpha}(V_\lambda \cap Q_\beta)| \leq |V_\alpha : V_\alpha \cap Q_\lambda| \leq 2^3$ by (6.1). Thus $\Omega_1(Z(G_{\alpha\beta}/Q_\alpha)) \subset (V_\lambda \cap Q_\beta)Q_\alpha/Q_\alpha$, so $O^2(G_\alpha/C_{G_\alpha}(V_\alpha)) \cong \mathbf{E}_{3^2}$ and $|V_\alpha| = 2^4$ by (2.10)–(2) and (3.6). Therefore we have $[U_\beta, Q_\beta; 3] = V_\beta = \cap V_\alpha^{G_\beta}$ similarly as in (3) of (5.1), and hence $n(G_\beta, U_\beta/V_\beta) \geq 3$ by (2.14)–(2). Since $|U_\beta : C_{U_\beta}(V_\lambda)| \leq |U_\beta : U_\beta \cap Q_\lambda| \leq 2^4$, $O^2(G_\beta/Q_\beta C_{G_\beta}(U_\beta/V_\beta)) \cong \mathbf{Z}_3$ or $\mathbf{E}_{3^2}$ similarly as in (5) of (5.1). Since $V_\beta \neq [V_\alpha, V_\lambda \cap Q_\beta] \subseteq V_\alpha \cap U_\delta$, we have $G_\beta \neq \langle Q_\alpha, U_\delta \rangle$, so $G_\beta \neq \langle G_{\alpha\beta}, U_\delta \rangle$ by (3.2)–(3). Thus there exists an element $x \in G_\beta$ such that $U_\delta \subseteq G_{\alpha^*, \beta}$ and $V_{\alpha^*} = V_\alpha$ by (3.10)–(1). Therefore $V_\delta = V_\lambda \cap Q_{\alpha^*}$, and so

$|V_\lambda : V_\delta| = |V_\lambda Q_{\alpha^*} : Q_{\alpha^*}| \leq 2^2$, a contradiction. This shows that $|V_\lambda : V_\delta| \leq 2^2$. Let $\overline{G}_\lambda = G_\lambda / C_{G_\lambda}(V_\lambda)$, and regard V_λ as a faithful $GF(2)\overline{G}_\lambda$ -module. Then we have $|V_\lambda : C_{V_\lambda}(\overline{G}_{\delta\lambda})| \leq 2^2$, and hence the assertions follow from (3.6), (2.12)–(2), and (2.1)–(7).

COROLLARY 6.3 *Let $\eta \in \mathcal{B}$. Then U_η/V_η is elementary abelian, $O^2(G_\eta) \not\subseteq C_{G_\eta}(U_\eta/V_\eta)$, and $[U_\eta, Q_\eta] = V_\eta$.*

PROOF. Let $\eta \in \mathcal{B}$ and $\xi \in \Delta(\eta)$. Note that U_η is elementary abelian as $b \geq 3$. We have $O^2(G_\eta) \not\subseteq C_{G_\eta}(U_\eta/V_\eta)$ because $V_\eta \subseteq V_\xi \subseteq U_\eta$. Since $G_{\xi\eta} = Q_\xi Q_\eta$ by (3.2)–(1), we have $[V_\xi, Q_\eta] = [V_\xi, G_{\xi\eta}] = V_\eta$ by (6.2), so $[U_\eta, Q_\eta] = [\langle V_\xi^{G_\eta} \rangle, Q_\eta] = \langle [V_\xi, Q_\eta]^{G_\eta} \rangle = V_\eta$.

COROLLARY 6.4 *Let $(\beta, \delta) \in \mathcal{C}'$. Then*

$$|V_\beta V_\delta : V_\beta| = |V_\beta V_\delta : V_\delta| \leq 2.$$

In particular, $V_\beta \cap V_\delta \neq 1$ if $|V_\xi| = 2^4$ for $\xi \in \mathcal{A}$.

PROOF. By (6.2), we may assume $|V_\beta| = |V_\delta| = 2^2$, and hence it suffices to prove that $V_\beta \cap V_\delta \neq 1$. Take $\alpha \in \Delta(\beta)$ and $\lambda \in \Delta(\delta)$ so that $G_\delta = \langle V_\alpha, G_{\delta\lambda} \rangle = \langle V_\alpha, Q_\lambda \rangle$. Then $V_\delta = V_\lambda \cap Q_\alpha$. Since $|V_\alpha : V_\beta| = |V_\lambda : V_\delta| = 2^2$, we have $V_\beta \subset V_\alpha \cap Q_\delta$ and $V_\delta \subset V_\lambda \cap Q_\beta$ by (6.1). Hence $1 \neq [V_\alpha \cap Q_\delta, V_\lambda \cap Q_\beta] \subseteq V_\beta \cap V_\delta$.

LEMMA 6.5 *One of the following holds.*

- (1) *If $(\beta, \delta) \in \mathcal{C}'$, then*
 - (i) $|U_\beta : U_\beta \cap Q_\delta| = |U_\delta : U_\delta \cap Q_\beta| = 2$,
 - (ii) U_β (resp. U_δ) induces a quasitransvection but no transvection on U_δ/V_δ (resp. U_β/V_β),
 - (iii) $C_{U_\delta \bmod V_\delta}(U_\beta) \subseteq Q_\beta$ and $C_{U_\beta \bmod V_\beta}(U_\delta) \subseteq Q_\delta$,
 - (iv) $V_\beta V_\delta \subseteq [U_\beta, U_\delta] \subseteq U_\beta \cap U_\delta$, and
 - (v) $|V_\beta V_\delta : V_\beta| = |V_\beta V_\delta : V_\delta| = |[U_\beta, U_\delta] : V_\beta V_\delta| = 2$.
- (2) *Let $\eta \in \mathcal{B}$, and put $\tilde{G}_\eta = G_\eta / C_{G_\eta}(U_\eta/V_\eta)$ and $\hat{U}_\eta = [U_\eta/V_\eta, O^2(\tilde{G}_\eta)]$.*

Then

$$O^2(\tilde{G}_\eta) \cong \mathbf{E}_{3^k} \quad \text{and} \quad |\hat{U}_\eta| = 2^{2k},$$

where $k = 1$ or 2 . Moreover, $m_2(\tilde{G}_\eta) \geq 2$ if $k = 2$.

PROOF. Suppose (2) does not hold. Let $(\beta, \delta) \in \mathcal{C}'$, and let $(\beta, \beta+1, \dots, \delta)$ be a path of length b from β through δ .

We will first prove (i) of (1). Assume $U_\delta \subseteq Q_\beta$. Since $[U_\beta, U_\delta] \subseteq V_\beta \subseteq V_\beta V_\delta$ and $|V_\beta V_\delta : V_\delta| \leq 2$ by (6.3) and (6.4), U_β induces a transvection on U_δ/V_δ . Thus (2) holds for δ by (2.10)–(1), a contradiction. Now, by symmetry, we may assume $|U_\delta : U_\delta \cap Q_\beta| = |U_\delta Q_\beta : Q_\beta| = 2^2$. Define $\tilde{G}_\beta$ and $\hat{U}_\beta$ as in (2). Note that $m_2(\tilde{G}_\beta) \geq 2$,

and so $\Omega_1(Z(\tilde{G}_{\beta, \beta+1})) \subset \tilde{U}_\delta$ and $n_*(G_\beta) = 2$ by (2.1)–(5)(7). Take $z \in U_\delta - Q_\beta$ so that $\tilde{z} \in \Omega_1(Z(\tilde{G}_{\beta, \beta+1}))$. Then $[U_\beta \cap Q_\delta, z] \subseteq [U_\beta \cap Q_\delta, U_\delta] \subseteq V_\delta \subseteq V_\beta V_\delta$, and so $|\hat{U}_\beta : C_{\hat{U}_\beta}(z)| \leq |\hat{U}_\beta : C_{\hat{U}_\beta}(z)| \leq 2^3$. If $n(G_\beta, \hat{U}_\beta) \geq 2$, then z induces a factorwise sesquitransvection on $\hat{U}_\beta$ by (2.11)–(2), and so $n_*(G_\beta) = 1$ by (2.11)–(1) and (2.1)–(9), a contradiction. If $n(G_\beta, \hat{U}_\beta) = 1$, then $\hat{U}_\beta$ is irreducible and faithful as a $GF(2)\tilde{G}_\beta$ -module by (2.4)–(2)(3), and hence either $O^2(\tilde{G}_\beta) \cong \mathbf{E}_{3^2}$ and $|\hat{U}_\beta| = 2^4$, or $O^2(\tilde{G}_\beta) \cong \text{Esp}_{27}$ and $|\hat{U}_\beta| = 2^6$. However, since $[U_\beta, z, U_\delta] \subseteq [U_\beta, U_\delta, U_\delta] = 1$, we have $O^2(\tilde{G}_\beta) \not\cong \text{Esp}_{27}$ by (2.7)–(6), and hence (2) holds for β . This proves (i) of (1).

Take $\alpha \in \Delta(\beta)$ and $\lambda \in \Delta(\delta)$ so that $V_\alpha \not\subseteq Q_\delta$ and $V_\lambda \not\subseteq Q_\beta$. Since $U_\beta = V_\alpha(U_\beta \cap Q_\delta)$ by (i) and $|U_\delta : U_\delta \cap Q_\alpha| \leq 2^2$, U_β induces a quasitransvection on U_δ/V_δ , and, by symmetry, U_δ also induces a quasitransvection on U_β/V_β . Moreover, neither of them is a transvection because (2) does not hold. Thus we have $U_\delta \cap Q_\beta \not\subseteq Q_\alpha$ and $U_\beta \cap Q_\delta \not\subseteq Q_\lambda$, and hence $C_{U_\delta \bmod V_\delta}(U_\beta) = U_\delta \cap Q_\beta \cap Q_\alpha \subset U_\delta \cap Q_\beta$ and $C_{U_\beta \bmod V_\beta}(U_\delta) = U_\beta \cap Q_\delta \cap Q_\lambda \subset U_\beta \cap Q_\delta$. This shows that

$$V_\beta V_\delta = [U_\beta, U_\delta \cap Q_\beta][U_\beta \cap Q_\delta, U_\delta] \subseteq [U_\beta, U_\delta] \subseteq U_\beta \cap U_\delta,$$

and so $|V_\beta V_\delta : V_\beta| = |V_\beta V_\delta : V_\delta| = |[U_\beta, U_\delta] : V_\beta V_\delta| = 2$ by (6.4). This proves (ii)–(v) of (1).

§7. Odd case with $b = 3$

In this section, we will consider the case where $b = 3$.

THEOREM 7.1 *Let $\eta \in \mathcal{B}$. If $b = 3$, then $n_*(G_\eta) = 1$.*

PROOF. Suppose false. Let $(\beta, \delta) \in \mathcal{C}'$ and $\gamma \in \Delta(\beta) \cap \Delta(\delta)$.

(1) $V_\beta \neq V_\delta$.

PROOF. Suppose false. Then we have $[U_\beta, U_\delta \cap Q_\beta] \subseteq V_\beta = V_\delta$ by (6.3), so $U_\delta \not\subseteq Q_\beta$ because $U_\beta \not\subseteq Q_\delta = C_{G_{\delta-1, \delta}}(U_\delta/V_\delta)$ by (6.3) and (2.4)–(1). Thus, by symmetry, we may assume $|U_\beta : U_\beta \cap Q_\delta| \leq |U_\delta : U_\delta \cap Q_\beta|$. Then U_δ induces an offending action on U_β/V_β . Note that $U_\delta Q_\beta \supseteq Z_{\beta\gamma}$ because $G_{\beta\gamma} = Q_\beta Q_\gamma$ and $Q_\gamma \triangleright U_\delta$. Since $n_*(G_\beta/C_{G_\beta}(U_\beta/V_\beta)) > 1$, we have $|U_\delta : U_\delta \cap Q_\beta| = 2^2$ and $O^2(G_\beta/C_{G_\beta}(U_\beta/V_\beta)) \cong \mathbf{E}_{3^2}$ by (2.10)–(1), and so $G_{\beta\gamma}/Q_\beta \cong \mathbf{D}_8$ and $|U_\beta : U_\beta \cap Q_\delta| = 2^2$ by (2.5)–(1)(5). Since $[U_\beta, Q_\beta \cap Q_\gamma] \subseteq V_\beta \subseteq Q_\delta$ and $Q_\gamma/(Q_\gamma \cap Q_\delta) \cong G_\gamma \delta/Q_\delta \cong \mathbf{D}_8$, we have $Q_\gamma \neq (Q_\beta \cap Q_\gamma)(Q_\gamma \cap Q_\delta)$, so $(Q_\beta \cap Q_\gamma)(Q_\gamma \cap Q_\delta) = U_\beta(Q_\gamma \cap Q_\delta)$. Hence $Q_\beta \cap Q_\gamma = U_\beta(Q_\beta \cap Q_\gamma \cap Q_\delta)$, and so $[Q_\beta \cap Q_\gamma, U_\delta] \subseteq U_\beta$. This shows that U_δ induces a factorwise transvection on Q_β/U_β , which contradicts (2.11)–(1).

(2) $G_\gamma = \langle G_{\beta\gamma}, G_{\gamma\delta} \rangle = \langle Q_\beta, Q_\delta \rangle$ and $Q_\gamma = (Q_\beta \cap Q_\gamma)(Q_\gamma \cap Q_\delta)$.

PROOF. Since $O^2(G_\gamma/C_{G_\gamma}(V_\gamma))$ has prime order by (6.2), we have $G_\gamma = \langle G_{\beta\gamma}, G_{\gamma\delta} \rangle$ by (1) and (3.10)–(2). The remaining assertions follow from (3.2)–(3).

NOTATION. Let $\langle \xi, \eta \rangle \in E(\Gamma)$.

$$\begin{aligned} Z_{\xi\eta} &= \Omega_1(Z(G_{\xi\eta} \bmod Q_\eta)), \\ W_\xi &= \langle [U_\eta, Z_{\xi\eta}]^{G_\xi} \rangle, \\ C_\xi &= C_{G_\xi}(W_\xi), \\ T_\xi &= \langle U_\eta^{G_\xi} \rangle, \quad \text{and} \\ T_\xi^{(1)} &= [T_\xi, Q_\xi]. \end{aligned}$$

(3) $U_\beta Q_\delta = Z_{\gamma\delta}$, $U_\delta Q_\beta = Z_{\beta\gamma}$, and U_β (resp. U_δ) induces a quasitransvection on U_δ/V_δ (resp. U_β/V_β), but no factorwise sesquitransvection on Q_δ (resp. Q_β).

PROOF. Since $[U_\beta, Q_\beta \cap Q_\gamma] \subseteq V_\beta \subseteq Q_\delta \not\subseteq U_\beta$, we have $U_\beta Q_\delta = Z_{\gamma\delta}$, so U_β induces no factorwise sesquitransvection on Q_δ by (2.11)–(1) because $n_*(G_\delta) > 1$. If $U_\delta \subseteq Q_\beta$, then $[U_\beta, U_\delta] \subseteq V_\beta$, so U_β induces a transvection on U_δ/V_δ , a contradiction. Hence $U_\delta \not\subseteq Q_\beta$, so, by symmetry, $U_\delta Q_\beta = Z_{\beta\gamma}$ and U_δ induces no factorwise sesquitransvection on Q_β . Since $|U_\beta : U_\beta \cap Q_\delta| = |U_\delta : U_\delta \cap Q_\beta| = 2$, we conclude that U_β (resp. U_δ) induces a quasitransvection on U_δ/V_δ (resp. U_β/V_β) as in the proof of (6.5).

(4) Let $\langle \xi, \eta \rangle \in E(\Gamma)$. Then the following holds.

- (i) $G_\xi/C_{G_\xi}(V_\xi) \cong \mathbf{D}_6$, $|V_\xi| = 2^2$, $|W_\xi : V_\xi| = 2$, and $|Q_\xi : C_\xi| = 2^2$.
- (ii) Either

$$\begin{aligned} O^2(G_\eta/C_{G_\eta}(U_\eta/V_\eta)) &\cong \mathbf{E}_{3^2} \quad \text{and} \quad |U_\eta/V_\eta| = 2^4, \quad \text{or} \\ O^2(G_\eta/C_{G_\eta}(U_\eta/V_\eta)) &\cong \text{Esp}_{27} \quad \text{and} \quad |U_\eta/V_\eta| = 2^6. \end{aligned}$$

- (iii) $W_\xi = [U_\eta, Z_{\xi\eta}] \subseteq T_\xi$, $C_{G_\xi}(W_\xi/V_\eta) = C_\xi Q_\eta$, $|G_{\xi\eta} : C_\xi Q_\eta| = 2$, and $|Q_\eta : C_\xi \cap Q_\eta| = 2^2$.

PROOF. We may assume $\xi = \gamma$ and $\eta = \beta$. Since $V_\beta \cap V_\delta$ is centralised both by G_β and by G_γ by (2), we have $V_\beta \cap V_\delta = 1$, and so $G_\gamma/C_{G_\gamma}(V_\gamma) \cong \mathbf{D}_6$ and $|V_\gamma| = 2^2$ by (6.2) and (6.4).

By (3), we have $V_\gamma = V_\beta V_\delta \subseteq [U_\beta, U_\delta]$ and $|[U_\beta, U_\delta] : V_\gamma| = 2$ as in the proof of (6.5). Thus $[U_\beta, Z_{\beta\gamma}] = [U_\beta, U_\delta] \triangleleft \langle Q_\beta, Q_\delta \rangle = G_\gamma$ by (2). Therefore $W_\gamma = [U_\beta, Z_{\beta\gamma}] \subseteq U_\beta \subseteq Q_\gamma$ and $|W_\gamma : V_\gamma| = 2$, and hence $T_\gamma \subseteq C_{Q_\gamma}(W_\gamma) = C_\gamma$ and $|Q_\gamma : C_\gamma| \leq 2^2$.

Let $E_{\gamma\delta} = C_{G_{\gamma\delta}}(W_\gamma/V_\delta)$. Then $C_\gamma \subseteq E_{\beta\gamma} \cap E_{\gamma\delta}$, and $Q_\gamma = (Q_\beta \cap Q_\gamma) \cdot (Q_\gamma \cap Q_\delta) = (E_{\beta\gamma} \cap Q_\gamma)(Q_\gamma \cap E_{\gamma\delta})$ by (2). Note that $V_\gamma = \Omega_1(Z(Q_\gamma))$ because

$V_\beta = C_{\Omega_1(Z(Q_\gamma))}(G_{\beta\gamma})$. Thus $Q_\gamma \not\subseteq E_{\gamma\delta}$, and so we have $|Q_\gamma : C_\gamma| = 2^2$ and $C_\gamma = E_{\beta\gamma} \cap E_{\gamma\delta} = (Q_\beta \cap C_\gamma)(C_\gamma \cap Q_\delta)$. Therefore $E_{\gamma\delta} = C_\gamma Q_\delta$ and $|Q_\delta : C_\gamma \cap Q_\delta| = 2^2$. Hence (i) and (iii) hold.

Let $\tilde{G}_\beta = G_\beta / C_{G_\beta}(U_\beta / V_\beta)$. Then

$$U_\beta / V_\beta = [U_\beta / V_\beta, O^2(\tilde{G}_\beta)] \times C_{U_\beta / V_\beta}(O^2(\tilde{G}_\beta))$$

by (2.4)–(3) and (6.3). Note that U_δ induces a quasitransvection but no factorwise transvection on $[U_\beta / V_\beta, O^2(\tilde{G}_\beta)]$ by (3). Hence $[U_\beta / V_\beta, O^2(\tilde{G}_\beta)]$ is irreducible as a $GF(2)\tilde{G}_\beta$ -module and $[U_\beta / V_\beta, U_\delta] \subseteq [U_\beta / V_\beta, O^2(\tilde{G}_\beta)]$. Thus $V_\gamma \subseteq [U_\beta, U_\delta] \subseteq V_\beta[U_\beta, O^2(\tilde{G}_\beta)]$, so $U_\beta = \langle V_\gamma^{G_\beta} \rangle = V_\gamma V_\beta[U_\beta, O^2(\tilde{G}_\beta)] = V_\beta[U_\beta, O^2(\tilde{G}_\beta)]$. Since U_β / V_β is a faithful $GF(2)\tilde{G}_\beta$ -module by (2.4)–(2), we have either $O^2(\tilde{G}_\beta) \cong \mathbf{E}_{3^2}$ and $|U_\beta / V_\beta| = 2^4$, or $O^2(\tilde{G}_\beta) \cong \text{Esp}_{27}$ and $|U_\beta / V_\beta| = 2^6$ by (2.9). Hence (ii) holds.

Now, we divide the proof into two parts.

Case I. $O^2(G_\eta / C_{G_\eta}(U_\eta / V_\eta)) \cong \text{Esp}_{27}$ and $|U_\eta / V_\eta| = 2^6$ for $\eta \in \mathcal{B}$.

NOTATION. Let $\langle \xi, \eta \rangle \in E(\Gamma)$.

$$\begin{aligned} X_{\xi\eta} &= C_{U_\eta \bmod V_\eta}(Z_{\xi\eta}), \quad \text{and} \\ T_\xi^* &= \langle X_{\xi\eta}^{G_\xi} \rangle. \end{aligned}$$

(I-1) Let $\langle \xi, \eta \rangle \in E(\Gamma)$. Then

$$\begin{aligned} G_{\xi\eta} / Q_\eta &\cong \mathbf{Z}_8, \mathbf{D}_8, \text{ or } SD_{16}, \text{ and } C_\xi Q_\eta / Q_\eta \cong \mathbf{Z}_4 \quad \text{or } \mathbf{Q}_8, \\ [U_\eta, G_{\xi\eta}] &\supset X_{\xi\eta} \supset W_\xi, \\ |U_\eta : [U_\eta, G_{\xi\eta}]| &= |[U_\eta, G_{\xi\eta}] : X_{\xi\eta}| = 2, \\ [[U_\eta, G_{\xi\eta}], C_\xi] &\not\subseteq W_\xi \supseteq [X_{\xi\eta}, C_\xi], \quad \text{and} \\ [[U_\eta, G_{\xi\eta}], Z_{\xi\eta}] &\subseteq V_\xi \not\supseteq [X_{\xi\eta}, G_{\xi\eta}], \end{aligned}$$

PROOF. We see that this follows from (4-ii,iii) together with (2.7)–(4)(5), regarding U_η / V_η as a six dimensional $GF(2)(G_\eta / C_{G_\eta}(U_\eta / V_\eta))$ -module.

(I-2) Let $\langle \xi, \eta \rangle \in E(\Gamma)$. Then $W_\xi \subseteq T_\xi^* \subseteq T_\xi^{(1)} \subseteq Q_\eta$, $U_\eta \cap T_\xi^{(1)} = [U_\eta, G_{\xi\eta}]$, and $[T_\xi^*, C_\xi] \subseteq W_\xi \supseteq (T_\xi^*)^2$.

PROOF. We may assume $\xi = \gamma$ and $\eta = \delta$. Let $\lambda \in \Delta(\gamma)$. If $U_\lambda \not\subseteq Q_\delta$, then $(\lambda, \delta) \in \mathcal{C}'$, so $U_\lambda \subseteq Z_{\gamma\delta}$ by (3). Thus $T_\gamma = \langle U_\lambda \mid \lambda \in \Delta(\gamma) \rangle \subseteq Z_{\gamma\delta}$, and so $T_\gamma^{(1)} = [T_\gamma, Q_\gamma] \subseteq [Z_{\gamma\delta}, G_{\gamma\delta}] \subseteq Q_\delta$. Since $W_\gamma = [U_\beta, U_\delta]$ as in the proof of (4), we have $T_\gamma^{(1)} \supseteq [U_\beta, U_\delta] = W_\gamma \supseteq V_\delta$, and so $[U_\delta, G_{\gamma\delta}] \subseteq [U_\delta, Q_\gamma]V_\delta$

$\subseteq U_\delta \cap T_\gamma^{(1)} \subset U_\delta$. Thus $U_\delta \cap T_\gamma^{(1)} = [U_\delta, G_{\gamma\delta}] \supseteq X_{\gamma\delta} \supseteq W_\gamma$ by (I-1), and hence $T_\gamma^{(1)} = \langle (U_\delta \cap T_\gamma^{(1)})^{G_\gamma} \rangle = \langle X_{\gamma\delta}^{G_\gamma} \rangle = T_\gamma^*$ by (2.14)–(1). The third assertion also holds because $[T_\gamma^*, T_\gamma^*] \subseteq [T_\gamma^*, C_\gamma] = \langle [X_{\gamma\delta}, C_\gamma]^{G_\gamma} \rangle \subseteq W_\gamma$ and $T_\gamma^*/W_\gamma = \langle (X_{\gamma\delta}/W_\gamma)^{G_\gamma} \rangle = \Omega_1(T_\gamma^*/W_\gamma)$.

$$(I-3) \quad U_\delta \cap Q_\beta = [U_\delta, G_{\gamma\delta}].$$

PROOF. Since $G_{\gamma\delta} = Q_\gamma Q_\delta = (Q_\beta \cap Q_\gamma)Q_\delta$ by (2), $[U_\delta, G_{\gamma\delta}] = [U_\delta, (Q_\beta \cap Q_\gamma)Q_\delta] = [U_\delta, Q_\beta \cap Q_\gamma]W_\gamma \subseteq U_\delta \cap Q_\beta$. Thus $U_\delta \cap Q_\beta = [U_\delta, G_{\gamma\delta}]$ because $|U_\delta : [U_\delta, G_{\gamma\delta}]| = 2 = |U_\delta : U_\delta \cap Q_\beta|$ by (3) and (I-1).

(I-4) Let $\xi \in \mathcal{A}$. Then

$$[T_\xi^{(1)}, O^2(G_\xi)] \not\subseteq T_\xi^* \quad \text{and} \quad [T_\xi^*, O^2(G_\xi)] \not\subseteq W_\xi.$$

PROOF. We may assume $\xi = \gamma$. Suppose first that $[T_\gamma^{(1)}, O^2(G_\gamma)] \subseteq T_\gamma^*$. Then we have $T_\gamma^{(1)} = \langle (U_\delta \cap Q_\beta)^{G_\gamma} \rangle = (U_\delta \cap Q_\beta)T_\gamma^*$ by (I-2), (I-3), and (2.14)–(1). Since $[U_\delta \cap Q_\beta, C_\gamma \cap Q_\delta] \subseteq V_\delta \subseteq W_\gamma \supseteq [T_\gamma^*, C_\gamma \cap Q_\delta]$ by (I-2), we have $[T_\gamma^{(1)}, C_\gamma \cap Q_\delta] \subseteq W_\gamma$, and hence $[[U_\beta, G_{\beta\gamma}], C_\gamma] \subseteq [U_\beta \cap T_\gamma^{(1)}, (Q_\beta \cap C_\gamma)(C_\gamma \cap Q_\delta)] \subseteq W_\gamma$, contrary to (I-1).

Suppose next that $[T_\gamma^*, O^2(G_\gamma)] \subseteq W_\gamma$. Then we have $X_{\beta\gamma} = X_{\gamma\delta} \subseteq U_\beta \cap U_\delta$, and so $[X_{\gamma\delta}, G_{\gamma\delta}] \subseteq [U_\beta \cap U_\delta, \langle Q_\beta, Q_\delta \rangle] \subseteq V_\beta V_\delta = V_\gamma$. This also contradicts (I-1).

Now, take $\alpha \in \Delta(\beta)$ so that $V_\alpha \not\subseteq Q_\delta$.

(I-5) The following holds.

- (i) $G_\beta = \langle Z_{\alpha\beta}, G_{\beta\gamma} \rangle$,
- (ii) $U_\delta \cap Q_\beta \not\subseteq Q_\alpha$,
- (iii) $[T_\alpha^* \cap Z_{\gamma\delta}, U_\delta \cap Q_\beta] \subseteq W_\alpha$, and
- (iv) $T_\alpha^* \not\subseteq Z_{\gamma\delta}$.

PROOF. Let $\tilde{G}_\beta = G_\beta / C_{G_\beta}(U_\beta/V_\beta)$. Since $X_{\alpha\beta} \not\subseteq Q_\delta \supseteq X_{\beta\gamma}$, we have $\tilde{Z}_{\alpha\beta} \neq \tilde{Z}_{\beta\gamma}$, so $\tilde{G}_{\alpha\beta} \not\subseteq Z(O_3(\tilde{G}_\beta))\tilde{G}_{\beta\gamma}$ because $[\tilde{Z}_{\alpha\beta}, Z(O_3(\tilde{G}_\beta))] = 1$ by (2.7)–(1). Thus we have $\tilde{G}_\beta = \langle \tilde{G}_{\beta\gamma}, \tilde{G}_{\gamma\delta} \rangle$ because $Z(O_3(\tilde{G}_\beta))\tilde{G}_{\beta\gamma}$ is the unique maximal subgroup of $\tilde{G}_\beta$ containing $\tilde{G}_{\beta\gamma}$ by (2.1)–(8). Hence $G_\beta = \langle G_{\beta\gamma}, G_{\gamma\delta} \rangle$ by the 2-irreducibility of G_β , and then (i) holds by (2.1)–(10).

Since $U_\beta = V_\alpha(U_\beta \cap Q_\delta)$ by the choice of α , V_α does not induce a transvection on U_δ/V_δ by (3), and hence (ii) holds also by (3).

Note that $X_{\alpha\beta} \subseteq U_\beta \cap T_\alpha^* \subseteq U_\beta \cap T_\alpha^{(1)} = [U_\beta, G_{\beta\gamma}]$. Therefore, if $U_\beta \cap T_\alpha^* \neq X_{\alpha\beta}$ then $U_\beta \cap T_\alpha^* = U_\beta \cap T_\alpha^{(1)}$ by (I-1), so $T_\alpha^* = \langle (U_\beta \cap T_\alpha^{(1)})^{G_\alpha} \rangle = T_\alpha^{(1)}$, a contradiction. This shows that $X_{\alpha\beta} = U_\beta \cap T_\alpha^*$, and hence

$$[T_\alpha^* \cap Z_{\gamma\delta}, U_\delta \cap Q_\beta] \subseteq T_\alpha^* \cap V_\gamma = T_\alpha^* \cap U_\beta \cap V_\gamma = X_{\alpha\beta} \cap V_\gamma = V_\beta \subseteq W_\alpha$$

by (i) and (I-4). Thus, in particular, $T_\alpha^* \not\subseteq Z_{\gamma\delta}$ by (ii). This proves (iii) and (iv).

(I-6) Let $\xi \in \mathcal{A}$. Then $O^2(G_\xi/C_{G_\xi}(T_\xi^*/W_\xi)) \cong \mathbf{Z}_3$.

PROOF. Suppose first that $T_\alpha^* \subseteq Q_\gamma$. Since $(T_\alpha^*)^2 \subseteq W_\alpha \subseteq Z_{\gamma\delta}$ and $G_{\gamma\delta}/Z_{\gamma\delta}$ is isomorphic to a subgroup of $\mathbf{D}_8$ by (I-1) and (I-2), we have

$$|T_\alpha^* : T_\alpha^* \cap Z_{\gamma\delta}| \leq 2^2,$$

and so $U_\delta \cap Q_\beta$ induces a quasitransvection on T_α^*/W_α .

Suppose next that $T_\alpha^* \not\subseteq Q_\gamma$. Then

$$[T_\alpha^*, T_\gamma^{(1)} \cap C_\alpha] \subseteq W_\alpha \cap T_\gamma^{(1)} \subseteq W_\alpha \cap Q_\delta \neq W_\alpha$$

by (I-2) and the choice of α , so $|[T_\alpha^*, T_\gamma^{(1)} \cap C_\alpha]W_\gamma : W_\gamma| \leq 2$. Also, $|T_\gamma^{(1)} : T_\gamma^{(1)} \cap C_\alpha| \leq |Q_\beta : C_\alpha \cap Q_\beta| = 2^2$ by (4-iii) and (I-2), and hence

$$|T_\gamma^{(1)} : T_\gamma^{(1)} \cap C_\alpha| \cdot |[T_\alpha^*, T_\gamma^{(1)} \cap C_\alpha]W_\gamma : W_\gamma| \leq 2^3.$$

Therefore T_α^* induces a factorwise quasitransvection on $T_\gamma^{(1)}/W_\gamma$ by (I-4).

Thus, letting $\xi = \alpha$ or γ , we have $O^2(G_\xi/Q_\xi C_{G_\xi}(T_\xi^*/W_\xi)) \cong \mathbf{Z}_3$ in either case by (2.11)–(1) because G_ξ is a $\{2, 3\}$ -group with cyclic Sylow 3-subgroups.

(I-7) A contradiction.

PROOF. By (I-6), $T_\gamma^* = \langle X_{\beta\gamma}, X_{\gamma\delta}, X_{\gamma\delta}^t \rangle$ for some $t \in G_{\beta\gamma} - Q_\gamma$. Note that $[U_\beta, X_{\beta\gamma}] = 1$, that $[U_\beta, X_{\gamma\delta}] \subseteq V_\beta \cap V_\delta = 1$, and that $[U_\beta, X_{\beta\gamma}^t] = [U_\beta, X_{\beta\gamma}]^t = 1$. Thus we have $[U_\beta, T_\gamma^*] = 1$, and hence $[T_\gamma, T_\gamma^*] = [\langle U_\beta^{G_\gamma} \rangle, T_\gamma^*] = 1$, and, in particular, T_γ^* is elementary abelian. Since $T_\alpha^* \subseteq C_{Q_\beta}(U_\beta) \subseteq C_{Q_\gamma}(W_\gamma) \subseteq C_\gamma$, we have $T_\alpha^* \subseteq \Omega_1(C_\gamma) \subseteq Z_{\gamma\delta}$ by (I-1), which contradicts (iv) of (I-5).

Case II. $O^2(G_\eta/C_{G_\eta}(U_\eta/V_\eta)) \cong \mathbf{E}_{3^2}$ and $|U_\eta/V_\eta| = 2^4$ for $\eta \in \mathcal{B}$.

$$(II-1) \quad U_\beta \cap Q_\delta = [U_\beta, G_{\beta\gamma}] = [U_\beta, Q_\gamma] = [U_\beta, Q_\gamma \cap Q_\delta].$$

PROOF. We have $|U_\beta : [U_\beta, G_{\beta\gamma}]| = 2$ by (2.5)–(3), regarding U_β/V_β as a four dimensional $GF(2)(G_\beta/C_{G_\beta}(U_\beta/V_\beta))$ -module, so the assertion follows as in the proof of (I-3).

(II-2) Let $\eta \in \mathcal{B}$. Then $O^2(G_\eta/C_{G_\eta}(Q_\eta/C_{Q_\eta}(U_\eta))) \cong \mathbf{E}_{3^2}$, $Q_\eta/C_{Q_\eta}(U_\eta) \cong \mathbf{E}_{2^4}$, and $[C_{Q_\eta}(U_\eta), O^2(G_\eta)] \subseteq U_\eta$.

PROOF. We may assume $\eta = \delta$. Take $\lambda \in \Delta(\delta)$ so that $U_\delta = W_\gamma W_\lambda$. (This is possible by (2.5)–(11).) Then $C_{Q_\delta}(U_\delta) = (C_\gamma \cap Q_\delta) \cap (Q_\delta \cap C_\lambda) \subseteq Q_\gamma$, so $|Q_\delta/C_{Q_\delta}(U_\delta)| \leq 2^4$ by (4-iii) and $[Q_\delta, O^2(G_\delta)] \not\subseteq C_{Q_\delta}(U_\delta)$ by (3.2)–(2). Since $[U_\beta, C_\gamma \cap Q_\delta]V_\beta \subseteq W_\gamma \subseteq U_\delta \subseteq C_{Q_\delta}(U_\delta)$ by (2.5)–(4), $[U_\beta, C_{Q_\delta}(U_\delta)] \subseteq U_\delta$ and

$|Q_\delta/C_{Q_\delta}(U_\delta) : C_{Q_\delta/C_{Q_\delta}(U_\delta)}(U_\beta)| \leq 2^2$ because U_β does not induce a factorwise transvection on Q_δ by (3). Therefore $[C_{Q_\delta}(U_\delta), O^2(G_\delta)] \subseteq U_\delta$, and (2.9) shows that $Q_\delta/C_{Q_\delta}(U_\delta) \cong \mathbf{E}_{2^4}$ and that $O^2(G_\delta/C_{G_\delta}(Q_\delta/C_{Q_\delta}(U_\delta))) \cong \mathbf{E}_{3^2}$.

Now, define $R_\eta = [Q_\eta, O^2(G_\eta)]U_\eta$ and $L_\eta = C_{R_\eta}(O^2(G_\eta))$ for $\eta \in \mathcal{B}$.

$$(II-3) \quad R_\delta \supseteq U_\beta \cap Q_\delta \not\subseteq C_{R_\delta}(U_\delta).$$

PROOF. Since $[U_\beta, C_{Q_\delta}(U_\delta)] \subseteq [U_\beta, C_\gamma] \subseteq W_\gamma \subseteq U_\delta \subseteq R_\delta$ by (2.5)–(4), we have $[U_\beta, Q_\delta] = [U_\beta, C_{Q_\delta}(U_\delta)R_\delta] \subseteq R_\delta$. Therefore $U_\beta \cap Q_\delta = [U_\beta, Q_\gamma \cap Q_\delta] \subseteq R_\delta$ by (II-1) and $[U_\beta \cap Q_\delta, U_\delta] \not\subseteq V_\beta$ by (3).

Let $P \in \text{Syl}_3(G_\delta)$. Then

$$\begin{aligned} \Phi(P)Q_\delta &= C_{G_\delta}(U_\delta/V_\delta) = C_{G_\delta}(Q_\delta/C_{Q_\delta}(U_\delta)) = Q_\delta C_{G_\delta}(Q_\delta) \quad \text{and} \\ P/\Phi(P) &\cong \mathbf{E}_{3^2}. \end{aligned}$$

Take two subgroups P_1 and P_2 of P so that

$$\begin{aligned} P/\Phi(P) &= P_1\Phi(P)/\Phi(P) \times P_2\Phi(P)/\Phi(P) \quad \text{and} \\ |C_{Q_\delta/C_{Q_\delta}(U_\delta)}(P_1)| &= |C_{Q_\delta/C_{Q_\delta}(U_\delta)}(P_2)| = 2^2. \end{aligned}$$

$$(II-4) \quad |C_{U_\delta/V_\delta}(P_i)| = 2^2 \quad (i = 1, 2).$$

PROOF. Suppose false. Since $P_1\Phi(P)Q_\delta$ and $P_2\Phi(P)Q_\delta$ are conjugate in G_δ by the 2-irreducibility of G_δ , we have $U_\delta = [U_\delta, P]V_\delta = [U_\delta, P_i]V_\delta$ ($i = 1, 2$). Also, since $[P_i, C_{Q_\delta}(P_i), U_\delta] = 1 = [V_\delta, P_i] \supseteq [C_{Q_\delta}(P_i), U_\delta, P_i]$ ($i = 1, 2$), the three-subgroup-lemma shows that $[U_\delta, P_i, C_{Q_\delta}(P_i)] = 1$ ($i = 1, 2$). Therefore, since $Q_\delta = \langle C_{Q_\delta}(U_\delta), C_{Q_\delta}(P_1), C_{Q_\delta}(P_2) \rangle$ by the choice of P_1 and P_2 , we have $[U_\delta, Q_\delta] = 1$, a contradiction.

(II-5) Let $\eta \in \mathcal{B}$. Then $R_\eta/U_\eta \cong \mathbf{E}_{2^4}$ or $\mathbf{Q}_8 * \mathbf{Q}_8$. (We denote by $*$ the central product.)

PROOF. We may assume $\eta = \delta$. Since $G_\delta = Q_\delta P N_{G_\delta}(P)$ by a Frattini argument, we have $[Q_\delta, P]U_\delta \triangleleft G_\delta$, so $R_\delta = [Q_\delta, P]U_\delta$.

Now, let $\overline{Q}_\delta = Q_\delta/U_\delta$. Note that $C_{\overline{Q}_\delta}(P) = \overline{C_{Q_\delta}(U_\delta)} \triangleleft \overline{Q}_\delta$ by (II-2), and hence $[P, \overline{C_{Q_\delta}(U_\delta)}, \overline{Q}_\delta] = 1 = [\overline{C_{Q_\delta}(U_\delta)}, \overline{Q}_\delta, P]$. Therefore the three-subgroup-lemma shows that $[\overline{R}_\delta, \overline{C_{Q_\delta}(U_\delta)}] = [\overline{Q}_\delta, P, \overline{C_{Q_\delta}(U_\delta)}] = 1$ so $\overline{Q}_\delta = \overline{R}_\delta * \overline{C_{Q_\delta}(U_\delta)}$. Moreover, as $C_{\overline{R}_\delta}(P_i) \triangleleft \overline{R}_\delta$ ($i = 1, 2$), the same argument also shows that $\overline{R}_\delta = [\overline{R}_\delta, P_i] * C_{\overline{R}_\delta}(P_i)$ ($i = 1, 2$). Since $C_{\overline{R}_\delta}(P_i) = C_{\overline{R}_\delta}(P) * [\overline{R}_\delta, P_{3-i}]$ ($i = 1, 2$), we conclude that $\overline{R}_\delta = C_{\overline{R}_\delta}(P)[\overline{R}_\delta, P_1][\overline{R}_\delta, P_2] = C_{\overline{R}_\delta}(P_1) * C_{\overline{R}_\delta}(P_2)$. Since $\overline{C_{R_\delta}(U_\delta)} \subseteq Z(\overline{R}_\delta)$ and $|C_{\overline{R}_\delta}(P_i) : \overline{C_{R_\delta}(U_\delta)}| = 2^2$ ($i = 1, 2$), we have $||C_{\overline{R}_\delta}(P_i), C_{\overline{R}_\delta}(P_i)|| \leq 2$

($i = 1, 2$), and hence $C_{\bar{R}_\delta}(P_i) \cong \mathbf{E}_{2^2}$ or $\mathbf{Q}_8$. Consequently, we have $\bar{R}_\delta \cong \mathbf{E}_{2^4}$ or $\mathbf{Q}_8 * \mathbf{Q}_8$ because $\Omega_1(\bar{R}_\delta) \supseteq \overline{U_\beta \cap Q_\delta} \not\subseteq \overline{C_{R_\delta}(U_\delta)} = [\bar{R}_\delta, \bar{R}_\delta] = \Phi(\bar{R}_\delta)$ by (II-1) and the definition of R_δ .

(II-6) *Let $\eta \in \mathcal{B}$. Then $R_\eta/L_\eta \cong \mathbf{E}_{2^8}$.*

PROOF. We may assume $\eta = \delta$. Note that $G_\delta = Q_\delta N_{G_\delta}(P) = R_\delta N_{G_\delta}(P)$. Since $U_\delta/V_\delta \subseteq Z(R_\delta/V_\delta)$ and $R_\delta = \langle X \mid C_{R_\delta}(U_\delta) \subseteq X \text{ and } X/U_\delta \text{ is cyclic} \rangle$, R_δ is generated by a set of abelian subgroups containing $C_{R_\delta}(U_\delta)$, and hence $C_{R_\delta}(U_\delta)/V_\delta \subseteq Z(R_\delta/V_\delta)$. Since $V_\delta \subseteq C_{R_\delta}(P) \subseteq C_{R_\delta}(U_\delta)$, we have $C_{R_\delta}(P) \triangleleft G_\delta$, and so $L_\delta = C_{R_\delta}(P)$. Hence $|R_\delta/L_\delta| = 2^8$.

Now, let $\hat{R}_\delta = R_\delta/L_\delta$ and $\tilde{P} = P/\Phi(P)$. Then $\hat{R}_\delta = C_{\hat{R}_\delta}(\tilde{P}_1)C_{\hat{R}_\delta}(\tilde{P}_2)$ and $\tilde{P} = \tilde{P}_1 \times \tilde{P}_2$. Since $C_{\hat{R}_\delta}(\tilde{P}_1) \cap C_{\hat{R}_\delta}(\tilde{P}_2) = C_{\hat{R}_\delta}(\tilde{P}) = 1$, we have $\hat{R}_\delta = C_{\hat{R}_\delta}(\tilde{P}_1) \times C_{\hat{R}_\delta}(\tilde{P}_2)$ by (1.4), so $C_{\hat{R}_\delta}(\tilde{P}_i) \cong \mathbf{E}_{2^4}$ or $\mathbf{Z}_4 \times \mathbf{Z}_4$ ($i = 1, 2$). Thus $\hat{R}_\delta \cong \mathbf{E}_{2^8}$ or $\mathbf{Z}_4 \times \mathbf{Z}_4 \times \mathbf{Z}_4 \times \mathbf{Z}_4$. In the latter case, we have $\Omega_1(R_\delta) \subseteq C_{R_\delta}(U_\delta)$, contrary to (II-3). Therefore $R_\delta/L_\delta \cong \mathbf{E}_{2^8}$.

(II-7) $O^2(G_\gamma/C_{G_\gamma}(T_\gamma^{(1)}/W_\gamma)) \cong \mathbf{Z}_3$, $T_\gamma^{(1)} \subseteq R_\delta$, and $|T_\gamma^{(1)}/W_\gamma| = 2^3$.

PROOF. Since $U_\beta \cap T_\gamma^{(1)} = U_\beta \cap Q_\delta$ by (II-1), $[T_\gamma^{(1)}, U_\delta] \neq 1$, and hence we can choose $\lambda \in \Delta(\delta)$ and $\mu \in \Delta(\lambda)$ so that $T_\gamma^{(1)} \not\subseteq Q_\lambda$ and $(\delta, \mu) \in \mathcal{C}'$. Then $[T_\gamma^{(1)}, T_\lambda^{(1)} \cap Q_\gamma] \subseteq W_\gamma \cap T_\lambda^{(1)} \subseteq U_\delta \cap T_\lambda^{(1)}$ and $|T_\lambda^{(1)} : T_\lambda^{(1)} \cap Q_\gamma| \leq |G_{\gamma\delta} : Q_\gamma| = 2 = |U_\delta \cap T_\lambda^{(1)} : W_\lambda|$, so $T_\lambda^{(1)}$ induces a quasitransvection on $T_\lambda^{(1)}/W_\lambda$. Thus $O^2(G_\lambda/C_{G_\lambda}(T_\lambda^{(1)}/W_\lambda)) \cong \mathbf{Z}_3$, and hence $T_\gamma^{(1)} = \langle (U_\beta \cap T_\gamma^{(1)})^{G_\gamma} \rangle = \langle (U_\beta \cap Q_\delta)^{G_{\gamma\delta}}, U_\delta \cap T_\gamma^{(1)} \rangle \subseteq R_\delta$ by (II-3).

If $L_\delta = V_\delta$, then $R_\delta' \subseteq V_\delta \subseteq W_\gamma$, so $[T_\gamma^{(1)}, R_\delta] \subseteq W_\gamma$, a contradiction. Therefore $L_\delta \neq V_\delta$, and so $R_\delta/U_\delta \cong \mathbf{Q}_8 * \mathbf{Q}_8$. Now, suppose $|T_\gamma^{(1)}/W_\gamma| = 2^2$. Then $|T_\gamma^{(1)}U_\delta/U_\delta| = 2$, so $T_\gamma^{(1)}U_\delta/U_\delta \subseteq Z(R_\delta/U_\delta) = C_{R_\delta}(U_\delta)/U_\delta$. Therefore $U_\beta \cap Q_\delta \subseteq T_\gamma^{(1)} \subseteq C_{R_\delta}(U_\delta)$, also a contradiction. Thus $|T_\gamma^{(1)}/W_\gamma| = 2^3$.

(II-8) *A contradiction.*

PROOF. Since $[T_\gamma^{(1)}, Q_\gamma] \subseteq W_\gamma \subseteq C_{R_\delta}(U_\delta)$, we have $|T_\gamma^{(1)} : T_\gamma^{(1)} \cap C_{R_\delta}(U_\delta)| = |T_\gamma^{(1)}C_{R_\delta}(U_\delta) : C_{R_\delta}(U_\delta)| = 2$, and so $|C_{R_\delta}(U_\delta) : T_\gamma^{(1)} \cap C_{R_\delta}(U_\delta)| = 2$. Let $N_\gamma = C_{R_\delta}(U_\beta) \cap T_\gamma^{(1)} \cap C_{R_\delta}(U_\delta)$. Then $|T_\gamma^{(1)}/N_\gamma| = 2^2 = |C_{R_\delta}(U_\delta)N_\gamma|$. Choose $\lambda \in \Delta(\delta)$ and $\mu \in \Delta(\lambda)$ so that $(\delta, \mu) \in \mathcal{C}'$ and $\langle U_\beta, U_\mu \rangle$ has a cyclic 3-subgroup X with $C_{C_{R_\delta}(U_\delta)}(X) = L_\delta$. (This is possible by (II-4) and the choice of P_1 and P_2). Then $N_\gamma \cap N_\lambda \subseteq C_{C_{R_\delta}(U_\delta)}(X) = L_\delta$ and $|C_{R_\delta}(U_\delta) : N_\gamma \cap N_\lambda| \leq |C_{R_\delta}(U_\delta) : L_\delta| = 2^4$. Therefore $L_\delta = N_\gamma \cap N_\lambda \subseteq N_\gamma$, so, by symmetry, $L_\beta \subseteq N_\gamma$. Thus, since $[N_\gamma, Q_\beta] \subseteq L_\beta \subseteq N_\gamma \supseteq L_\delta \supseteq [N_\gamma, Q_\delta]$, $N_\gamma \triangleleft \langle Q_\beta, Q_\delta \rangle = G_\gamma$, and so $[T_\gamma^{(1)}, O^2(G_\gamma)] \not\subseteq N_\gamma$. However, by (II-6) and (II-7), we have $[T_\gamma^{(1)}, R_\delta] \subseteq L_\delta \subseteq$

N_γ , a contradiction.

§8. Odd case with $b \geq 5$

In this final section, we will argue for a contradiction under the following hypothesis.

HYPOTHESIS 8.1

- (1) The parameter b is odd and $b \geq 5$.
- (2) $n_*(G_\eta) > 1$ for $\eta \in \mathcal{B}$.

NOTATION. Let $\beta, \delta \in V(\Gamma)$. We denote by $(\beta, \beta+1, \dots, \delta-1, \delta)$ a path of length $d(\beta, \delta)$ from β through δ .

We will first prove the following.

PROPOSITION 8.2 *Part (2) of (6.5) holds.*

PROOF. We will assume otherwise, and derive a contradiction.

- (1) Let $(\beta, \delta) \in \mathcal{C}'$ and $\alpha \in \Delta(\beta)$. If $V_\alpha \not\subseteq Q_\delta$, then $G_\beta = \langle G_{\alpha\beta}, U_\delta \rangle$.

PROOF. Take $(\beta, \delta) \in \mathcal{C}'$ and $\alpha \in \Delta(\beta)$ so that $V_\alpha \not\subseteq Q_\delta$. Let $D_\beta = C_{G_\beta}(U_\beta/V_\beta)$, $\tilde{G}_\beta = G_\beta/D_\beta$, and $\tilde{U}_\beta = [U_\beta/V_\beta, O^2(\tilde{G}_\beta)]$. Note that $n_*(\tilde{G}_\beta) > 1$ by (2.1)–(9) and (8.1)–(2). If $n(G_\beta, \tilde{U}_\beta) \geq 2$, then $\tilde{U}_\delta$ induces a factorwise transvection on $\tilde{U}_\beta$ by (6.5)–(1), and so $O^2(\tilde{G}_\beta) \cong \mathbf{E}_{3^2}$ by (2.11)–(1). Suppose $n(G_\beta, \tilde{U}_\beta) = 1$. Then $\tilde{U}_\delta$ induces a quasitransvection on the irreducible faithful $GF(2)\tilde{G}_\beta$ -module $\tilde{U}_\beta$ by (2.4)–(2)(3) and (6.5)–(1). Thus we have $O^2(\tilde{G}_\beta) \cong \mathbf{E}_{3^2}, \mathbf{E}_{5^2}$, or Esp_{27} by (2.9), and, in particular, $\tilde{U}_\delta = \Omega_1(Z(\tilde{G}_{\beta, \beta+1}))$ if $O^2(\tilde{G}_\beta) \cong Esp_{27}$.

Suppose $G_\beta \neq \langle G_{\alpha\beta}, U_\delta \rangle$. Then $\tilde{G}_\beta \neq \langle \tilde{G}_{\alpha\beta}, \tilde{U}_\delta \rangle$ by the 2-irreducibility of G_β . Note that the unique maximal subgroup of $\tilde{G}_\beta$ containing $\tilde{G}_{\alpha\beta}$ is

$$\begin{aligned} &\tilde{G}_{\alpha\beta} \text{ if } O^2(\tilde{G}_\beta) \cong \mathbf{E}_{3^2} \text{ or } \mathbf{E}_{5^2}, \text{ and} \\ &\tilde{G}_{\alpha\beta}Z(O^2(\tilde{G}_\beta)) \text{ if } O^2(\tilde{G}_\beta) \cong Esp_{27} \end{aligned}$$

by (2.1)–(8). Thus we have $\tilde{U}_\delta \subseteq \tilde{G}_{\alpha\beta}$ if $O^2(\tilde{G}_\beta) \cong \mathbf{E}_{3^2}$ or $\mathbf{E}_{5^2}$. Suppose $O^2(\tilde{G}_\beta) \cong Esp_{27}$. Then, as $[\tilde{U}_\delta, Z(O^2(\tilde{G}_\beta))] = 1$ by (2.7)–(1), we have $\tilde{U}_\delta \subseteq \tilde{G}_{\alpha\beta}$ also in this case. This shows that $U_\delta \subseteq G_{\alpha\beta}D_\beta$. Therefore $U_\delta \subseteq G_{\alpha\beta}^x$ and $V_\alpha = V_\alpha^x$ for some $x \in G_\beta$ as in the proof of (1) of (3.10). Thus we have $U_\beta = V_\alpha(U_\beta \cap Q_\delta) = V_\alpha^x(U_\beta \cap Q_\delta)$ and $|U_\delta : U_\delta \cap Q_\alpha^x| \leq |G_{\alpha\beta}^x : Q_\alpha^x| \leq 2$, and then U_β induces a transvection on U_δ/V_δ , contrary to (6.5)–(1).

Define $T_\xi = \langle U_\eta \mid \eta \in \Delta(\xi) \rangle$ for $\xi \in \mathcal{A}$.

Note that T_ξ is elementary abelian by the minimality of b .

(2) *There exist $(\beta, \delta) \in \mathcal{C}'$ and $\alpha \in \Delta(\beta)$ such that $V_\alpha \not\subseteq Q_\delta$ and $T_\alpha \subseteq Q_{\delta-2}$.*

PROOF. Suppose false. Let $(\beta, \delta) \in \mathcal{C}'$ and $\alpha \in \Delta(\beta)$ so that $V_\alpha \not\subseteq Q_\delta$. Then $T_\alpha \not\subseteq Q_{\delta-2}$ by our hypothesis. Take $\theta \in \Delta(\alpha)$ so that $U_\theta \not\subseteq Q_{\delta-2}$. Then $(\theta, \delta-2) \in \mathcal{C}'$, and so $V_\theta \subseteq U_{\delta-2} \subseteq T_{\delta-1}$ by (6.5)–(1). Since $G_\beta = \langle G_{\alpha\beta}, U_\delta \rangle = \langle G_{\alpha\beta}, T_{\delta-1} \rangle = \langle Q_\alpha, T_{\delta-1} \rangle$ by (1) and (3.2)–(3), we have $V_\theta \subseteq Z(G_\beta)$, and so $V_\theta = V_\beta$. Therefore, after repeated application of this “shift” operation, we can take $(\beta, \delta) \in \mathcal{C}'$ such that $V_\beta = V_{\beta+2} = \cdots = V_{\delta-2} = V_\delta$, contrary to (6.5)–(1).

Now, take $(\beta, \delta) \in \mathcal{C}'$ and $\alpha \in \Delta(\beta)$ so that

$$V_\alpha \not\subseteq Q_\delta, \quad G_\beta = \langle G_{\alpha\beta}, U_\delta \rangle, \quad \text{and} \quad T_\alpha \subseteq Q_{\delta-2}.$$

(3) *$U_\delta \cap Q_\beta \not\subseteq Q_\alpha$ and $T_\alpha \not\subseteq Q_{\delta-1}$.*

PROOF. Since V_α does not induce a transvection on U_δ/V_δ and $|U_\delta : U_\delta \cap Q_\beta| = 2$ by (6.5)–(1), we have $U_\delta \cap Q_\beta \not\subseteq Q_\alpha$. Suppose $T_\alpha \subseteq Q_{\delta-1}$. If (2.13)–(1) holds, then (6.5)–(2) holds by (2.10)–(1), contrary to our hypothesis. Hence we have

$$[T_\alpha, U_\delta] \subseteq [U_\beta, U_\delta][T_\alpha, C_{U_\delta \bmod V_\delta}(U_\beta)]V_\delta \subseteq T_\alpha$$

by (2.13)–(2) and (6.5)–(1). Therefore $T_\alpha \triangleleft \langle G_{\alpha\beta}, U_\delta \rangle = G_\beta$, a contradiction.

(4) $O^2(G_\alpha/C_{G_\alpha}(T_\alpha/V_\alpha)) \cong \mathbf{Z}_3$.

PROOF. It suffices to show that $U_\delta \cap Q_\beta$ induces a factorwise quasitransvection on T_α/V_α by (2.11)–(1) because G_α is a $\{2, 3\}$ -group with cyclic Sylow 3-subgroups.

Since $[U_\beta, U_\delta \cap Q_\beta] \subseteq V_\beta \subseteq V_\alpha$ and $[T_\alpha \cap Q_\delta, U_\delta] \subseteq V_\delta$, we have $[U_\beta(T_\alpha \cap Q_\delta), U_\delta \cap Q_\beta] \subseteq V_\alpha V_\delta$. Thus

$$\begin{aligned} & |T_\alpha/V_\alpha : [U_\beta(T_\alpha \cap Q_{\delta-1} \cap Q_\delta)/V_\alpha, U_\delta \cap Q_\beta]| \\ & \leq |T_\alpha : T_\alpha \cap Q_{\delta-1}| \cdot |T_\alpha \cap Q_{\delta-1} : U_\beta(T_\alpha \cap Q_{\delta-1} \cap Q_\delta)| \cdot |V_\alpha V_\delta : V_\alpha| \\ & \leq 2 \cdot 2 \cdot 2. \end{aligned}$$

If $n(G_\alpha, T_\alpha/V_\alpha) > 1$, then $U_\delta \cap Q_\beta$ induces a factorwise sesquitransvection on T_α/V_α by (2.11)–(2).

Suppose $n(G_\alpha, T_\alpha/V_\alpha) = 1$. Then we have $[T_\alpha, Q_\alpha] \subseteq U_\beta$ by (2.14)–(2), and so $[T_\alpha, Q_\alpha, Q_\beta] \subseteq V_\beta \subseteq V_\alpha$. Thus $[T_\alpha, Q_\alpha, G_\alpha] \subseteq [T_\alpha, Q_\alpha, \langle Q_\beta^{G_\alpha} \rangle] \subseteq \langle [T_\alpha, Q_\alpha, Q_\beta]^{G_\alpha} \rangle \subseteq V_\alpha$, and hence $[U_\beta, G_{\alpha\beta}; 2] = [U_\beta, Q_\alpha, G_{\alpha\beta}] \subseteq V_\alpha$. Thus $[U_\beta, G_{\alpha\beta}; 3] \subseteq V_\beta$ and $|[U_\beta, G_{\alpha\beta}; 2]V_\beta : V_\beta| \leq 2^2$ by (6.2), so $O^2(G_\beta/C_{G_\beta}(U_\beta/V_\beta))$

$\cong Esp_{27}$ and $[[U_\beta/V_\beta, O^2(G_\beta)]] = 2^6$ by (2.12) and (8.1)–(2). Since $[U_\delta, U_\beta, T_\alpha \cap Q_{\delta-1}] = 1$, we have $[U_\delta/V_\delta, Z_{\delta-1, \delta}, (T_\alpha \cap Q_{\delta-1})Q_\delta] = 1$ by (6.5)–(1), and so $T_\alpha \cap Q_{\delta-1} = Z_{\delta-1, \delta} = U_\beta(T_\alpha \cap Q_{\delta-1} \cap Q_\delta)$ by (2.7)–(6). This shows that

$$|T_\alpha/V_\alpha : [U_\beta(T_\alpha \cap Q_{\delta-1} \cap Q_\delta)/V_\alpha, U_\delta \cap Q_\beta]| \leq 2^2$$

and so $U_\delta \cap Q_\beta$ induces a factorwise quasitransvection on T_α/V_α .

(5) *A contradiction.*

PROOF. Since $U_{\delta-2} \neq U_\delta$ by the minimality of b , we have $G_{\delta-1} = \langle G_{\delta-2, \delta-1}, G_{\delta-1, \delta} \rangle$ by (4) and (3.10)–(2), so $G_{\delta-1} = \langle T_\alpha, G_{\delta-1, \delta} \rangle$. Therefore $V_\delta \subseteq Z(G_{\delta-1})$ because $V_\delta \subseteq U_\beta \subseteq T_\alpha$ by (6.5)–(1), a contradiction.

(8.3) *Let $\xi \in \mathcal{A}$ and $\eta \in \mathcal{B}$. Then the following holds.*

- (1) $G_\xi/C_{G_\xi}(V_\xi) \cong \mathbf{D}_6$ and $|V_\xi| = 2^2$.
- (2) $(G_\eta, U_\eta/V_\eta) \ll (O_4^+(2), GF(2)^4)$, $G_\eta/C_{G_\eta}(U_\eta/V_\eta) \cong O_4^+(2)$, and $n(G_\eta, U_\eta/V_\eta) = 1$.

PROOF. We may assume $\langle \xi, \eta \rangle \in E(\Gamma)$. Then (2) follows from (8.1)–(2), (8.2), and (2.5)–(1). Hence $|V_\xi : V_\eta| \leq 2$ by (2.5)–(3) and (2.15), and so (1) holds by (6.2).

NOTATION. Let $\langle \xi, \eta \rangle \in E(\Gamma)$.

$$\begin{aligned} W_\xi &= \cap U_\eta^{G_\xi}. \\ T_\xi &= \langle U_\eta^{G_\xi} \rangle. \\ T_\xi^{(n)} &= [T_\xi, Q_\xi; n]V_\xi \quad \text{for } n \geq 0. \\ K_\eta &= \langle T_\xi^{G_\eta} \rangle. \\ Z_{\xi\eta} &= Z(G_{\xi\eta} \bmod Q_\eta). \\ E_{\xi\eta} &= C_{G_{\xi\eta}}(C_{U_\eta \bmod V_\eta}(Z_{\xi\eta})). \end{aligned}$$

(8.4) *Let $\xi \in \mathcal{A}$ and $\eta \in \mathcal{B}$.*

(1) *One of the following holds.*

- (i) $|W_\xi| = 2^3$, $|U_\eta| = 2^5$, $W_\xi = T_\xi^{(2)} \neq V_\xi$, $n(G_\xi, T_\xi/W_\xi) \geq 2$.
- (ii) $|W_\xi| = 2^2$, $|U_\eta| = 2^5$, $W_\xi = T_\xi^{(3)} = V_\xi$, $n(G_\xi, T_\xi/W_\xi) \geq 3$.
- (iii) $|W_\xi| = 2^2$, $|U_\eta| = 2^6$, $W_\xi = T_\xi^{(4)} = V_\xi$, $n(G_\xi, T_\xi/W_\xi) \geq 4$.
- (2) *Let $M = [U_\eta, O^2(G_\eta)]V_\eta$ and $L = C_{U_\eta}(O^2(G_\eta))$. Then*

$$U_\eta/V_\eta = M/V_\eta \times L/V_\eta \quad \text{and} \quad |L/V_\eta| \leq 2.$$

PROOF. We may assume $\langle \xi, \eta \rangle \in E(\Gamma)$. By (3.9), we have

$$\begin{aligned} 2^5 &\leq |U_\eta| \leq 2^6, \quad |W_\xi : V_\xi| \leq 2, \quad \text{and} \\ T_\xi^{(m-1)} &\neq W_\xi = T_\xi^{(m)} \quad \text{if} \quad |U_\eta : W_\xi| = 2^m. \end{aligned}$$

Therefore, to prove (1), we will assume $|U_\eta| = 2^6$ and $|W_\xi| = 2^3$, and argue for a contradiction. Let $L = C_{U_\eta}(O^2(G_\eta))$ and $D = C_{Q_\eta}(L)$. As $|L| = 2^2$, $|Q_\eta : D| \leq 2$, so $D \not\subseteq Q_\xi$ by (3.2)–(2). Suppose $L \not\subseteq W_\xi$. Then, since $|U_\eta : LW_\xi| = 2^2$ and $[LW_\xi, Q_\xi] \subseteq LV_\xi \cap W_\xi = V_\xi$, we have $T_\xi^{(3)} \subseteq V_\xi \subset W_\xi$, a contradiction. Thus $L \subseteq W_\xi$, and so $|W_\xi : L| = 2$. Now, take $x \in G_\xi$ so that $G_\xi = \langle G_{\xi\eta}, x \rangle$. Then $L \cap L^x \neq 1$. Since $\langle D, D^x \rangle$ contains a Sylow 3-subgroup of G_ξ , $L \cap L^x$ is centralized both by a Sylow 3-subgroup of G_ξ and by a Sylow 3-subgroup of G_η . Therefore $L \cap L^x = 1$ by (3.1), a contradiction. Thus (1) holds. Part (2) is a consequence of (6.3), (2.4)–(3), (8.3)–(2), and (1).

(8.5) Let $\langle \xi, \eta \rangle \in E(\Gamma)$, and suppose $|W_\xi| = 2^3$. Then

$$C_{G_{\xi\eta}}(W_\xi) \subseteq Q_\xi \cap E_{\xi\eta}, [T_\xi, C_{G_{\xi\eta}}(W_\xi)] \subseteq W_\xi, \text{ and } [K_\eta, C_{G_{\xi\eta}}(U_\eta)] \subseteq U_\eta.$$

PROOF. We have $E_{\xi\eta} = C_{G_{\xi\eta}}(W_\xi/V_\eta) \supseteq C_{G_{\xi\eta}}(W_\xi) \subseteq C_{G_{\xi\eta}}(V_\xi) = Q_\xi$, $[U_\eta, E_{\xi\eta}] = W_\xi$ and $C_{G_{\xi\eta}}(U_\eta) \subseteq C_{G_{\xi\eta}}(U_\eta/V_\eta) = Q_\eta$ by (2.5)–(3)(4), (2.4)–(1), and (6.3), and so $[U_\eta, C_{Q_\xi}(W_\xi)] \subseteq W_\xi$. Therefore $[T_\xi, C_{Q_\eta}(U_\eta)] \subseteq [T_\xi, C_{G_{\xi\eta}}(W_\xi)] = [T_\xi, C_{Q_\xi}(W_\xi)] = \langle [U_\eta, C_{Q_\xi}(W_\xi)]^{G_\xi} \rangle \subseteq W_\xi \subseteq U_\eta$, and hence $[K_\eta, C_{G_{\xi\eta}}(U_\eta)] = [K_\eta, C_{Q_\eta}(U_\eta)] = \langle [T_\xi, C_{Q_\eta}(U_\eta)]^{G_\eta} \rangle \subseteq U_\eta$.

(8.6) Let $\beta, \delta \in \mathcal{B}$ and $d(\beta, \delta) = b-1$. Let $\alpha \in \Delta(\beta)$ and $\theta \in \Delta(\alpha)$, and assume

$$U_\delta \cap Q_\beta \not\subseteq Q_\alpha \text{ and } G_\alpha = \langle G_{\theta\alpha}, G_{\alpha\beta} \rangle.$$

Then $C_{U_\theta}(U_\delta) \subseteq W_\alpha$.

PROOF. Let $F = C_{U_\theta}(U_\delta)$. Note that $G_\alpha = \langle Q_\theta, U_\delta \cap Q_\beta \rangle$ by (3.2)–(3). Since $V_\alpha F$ is normalized by Q_θ and $U_\delta \cap Q_\beta$, we have $V_\alpha F \triangleleft G_\alpha$, and hence $F \subseteq V_\alpha F \subseteq \cap U_\theta G_\alpha = W_\alpha$.

(8.7) Let $\beta, \delta \in \mathcal{B}$ and $d(\beta, \delta) = b-1$, and $\alpha \in \Delta(\beta)$. Suppose $|W_\alpha| = 2^3$ and $[V_\alpha, T_{\delta-1}] \neq 1$, and define $B = [T_\alpha, U_{\delta-2}]$. Then the following holds.

(1) $B \subseteq W_\alpha \cap W_{\delta-3}$ and $|V_\beta B| \leq 2^2$.

(2) Assume $Z_{\delta-3, \delta-2} \subseteq T_\alpha Q_{\delta-2}$. Then $C_{U_{\delta-2}}(T_\alpha) = W_{\delta-3}$, $B \cap V_{\delta-2} = 1$, $V_\beta \subseteq B$, and $|W_\alpha \cap W_{\delta-3}| \geq 2^2$.

(3) Assume $V_\beta \not\subseteq B \not\subseteq V_{\delta-2}$ and $|K_{\delta-2} : K_{\delta-2} \cap G_{\alpha\beta}| \leq 2$. Then

$$|K_{\delta-2} : K_{\delta-2} \cap Q_\alpha| \cdot |[T_\alpha, K_{\delta-2} \cap Q_\alpha]U_{\delta-2} : U_{\delta-2}| \leq 2^a$$

where $a = \max\{2, 8-b\}$.

PROOF. Since $[W_\alpha, U_{\delta-2}] \subseteq [U_\beta, U_{\delta-2}] = 1$ and $[T_\alpha, W_{\delta-3}] \subseteq [T_\alpha, U_{\delta-4}] = 1$ by the minimality of b , we have $B \subseteq W_\alpha \cap W_{\delta-3}$ and $T_\alpha \subseteq E_{\delta-3, \delta-2}$ by (8.5). Also, since $V_\beta B \subseteq C_{W_\alpha}(T_{\delta-1}) \not\subseteq V_\alpha$, we have $|V_\beta B| \leq 2^2$.

Suppose $Z_{\delta-3, \delta-2} \subseteq T_\alpha Q_{\delta-2}$. Then we have $C_{U_{\delta-2}}(T_\alpha) = W_{\delta-3} = BV_{\delta-2}$ by (2.5)–(4). Therefore $B \cap V_{\delta-2} = 1$ and $|W_\alpha \cap W_{\delta-3}| \geq |B| = 2^2$, and hence $V_\beta \subseteq B$.

Now, suppose $V_\beta \not\subseteq B \not\subseteq V_{\delta-2}$ and $|K_{\delta-2} : K_{\delta-2} \cap G_{\alpha\beta}| \leq 2$. Then $C_{W_\alpha}(T_{\delta-1}) = V_\beta B$ and $|W_\alpha : V_\beta B| = 2$, so $W_\alpha = V_\alpha B$. Since $[B, K_{\delta-2}] \subseteq [U_{\delta-2}, K_{\delta-2}] = 1$ by the minimality of b , we have $[W_\alpha, K_{\delta-2} \cap Q_\alpha] = [V_\beta B, K_{\delta-2} \cap Q_\alpha] = 1$. Hence $[T_\alpha, K_{\delta-2} \cap Q_\alpha] \subseteq W_\alpha \cap K_{\delta-2}$ by (8.5), and, in particular, $[T_\alpha, K_{\delta-2} \cap Q_\alpha] \subseteq V_\beta B$ if $b \geq 7$ as $K_{\delta-2}$ is abelian in that case. Note that, if $K_{\delta-2} \cap G_{\alpha\beta} \not\subseteq Q_\alpha$, $V_\beta = [V_\alpha, K_{\delta-2} \cap G_{\alpha\beta}] \subseteq [U_\beta, K_{\delta-2}] \subseteq U_{\delta-2}$ by (8.5). Therefore, we have

$$|K_{\delta-2} : K_{\delta-2} \cap Q_\alpha| \cdot |[T_\alpha, K_{\delta-2} \cap Q_\alpha]U_{\delta-2} : U_{\delta-2}| \leq 2^a$$

both in the case $K_{\delta-2} \cap G_{\alpha\beta} \subseteq Q_\alpha$ and in the case $K_{\delta-2} \cap G_{\alpha\beta} \not\subseteq Q_\alpha$.

(8.8) Let $\alpha \in \mathcal{A}$. Suppose $b = 5$ and $|W_\alpha| = 2^3$. Take $\theta, \beta \in \Delta(\alpha)$ so that $G_\alpha = \langle G_{\theta\alpha}, G_{\alpha\beta} \rangle$. Then the following holds.

- (1) $K_\beta Q_\theta = Q_\theta, Z_{\theta\alpha},$ or $E_{\theta\alpha}$.
- (2) If $K_\beta Q_\theta = E_{\theta\alpha}$, then $n(G_\beta, K_\beta/U_\beta) \geq 2$.

PROOF. Since $[W_\alpha, K_\beta] \subseteq [U_\beta, K_\beta] = 1$, we have $K_\beta \subseteq E_{\theta\alpha}$ by (8.5). Also, since $G_\alpha = \langle G_{\theta\alpha}, G_{\alpha\beta} \rangle$, we have $G_{\theta\alpha} = Q_\theta(Q_\alpha \cap Q_\beta)$ by (3.2)–(3), and so $K_\beta Q_\theta \triangleleft G_{\theta\alpha}$. Therefore $K_\beta Q_\theta = Q_\theta, Z_{\theta\alpha},$ or $E_{\theta\alpha}$ because $G_{\theta\alpha}/Q_\theta \cong \mathbf{D}_8$.

Suppose $K_\beta Q_\theta = E_{\theta\alpha}$. Then $[K_\beta, G_{\theta\alpha}]Q_\theta = [K_\beta, Q_\beta \cap Q_\alpha]Q_\theta \neq Q_\theta$, and so $[K_\beta, Q_\beta] \not\subseteq T_\alpha$. Thus $n(G_\beta, K_\beta/U_\beta) \geq 2$ by (2.14)–(2).

(8.9) Let $\beta \in \mathcal{B}$ and $\alpha, \xi \in \Delta(\beta)$, and suppose $|W_\alpha| = 2^3$. Then the following holds.

- (1) If $W_\alpha \cap W_\xi = V_\beta$ (or, equivalently, $U_\beta = W_\alpha W_\xi$), then $G_\beta = \langle G_{\alpha\beta}, x \rangle = \langle Q_\alpha, x \rangle$ for all $x \in E_{\beta\xi} - Q_\beta$.
- (2) If $W_\alpha \neq W_\alpha \cap W_\xi \neq V_\beta$ (or, equivalently, $W_\alpha \neq W_\alpha W_\xi \neq U_\beta$), then $W_\alpha = V_\alpha(W_\alpha \cap W_\xi)$ and $G_{\alpha\beta} \cap G_{\beta\xi} = E_{\alpha\beta} \cap E_{\beta\xi}$.

PROOF. Put $D_\beta = C_{G_\beta}(U_\beta/V_\beta)$. Then $D_\beta \cap G_{\beta\xi} = Q_\beta$ by (6.3) and (2.4)–(1).

Suppose first that $W_\alpha \cap W_\xi = V_\beta$, and let $x \in E_{\beta\xi} - Q_\beta$. Then $G_{\alpha\beta}D_\beta \cap E_{\beta\xi} = E_{\alpha\beta}D_\beta \cap E_{\beta\xi} = D_\beta \cap G_{\beta\xi} = Q_\beta$ by (2.5)–(12), so $x \notin G_{\alpha\beta}D_\beta$. Thus $G_\beta = \langle G_{\alpha\beta}, x \rangle$ because $G_{\alpha\beta}D_\beta$ is the unique maximal subgroup of G_β containing $G_{\alpha\beta}$, and hence $G_\beta = \langle Q_\alpha, x \rangle$ by (3.2)–(3).

Suppose next that $W_\alpha \neq W_\alpha \cap W_\xi \neq V_\beta$. Since $W_\alpha \neq W_\xi$, we have $G_\beta = \langle G_{\alpha\beta}, G_{\beta\xi} \rangle$ by (8.2) and (3.10)–(2), and hence $G_\beta = \langle G_{\alpha\beta}, Z_{\beta\xi} \rangle$ by (2.1)–(10). Thus $V_\alpha \not\subseteq C_{U_\beta \text{ mod } V_\beta}(Z_{\beta\xi}) = W_\xi$ by (2.5)–(4), and so $W_\alpha = V_\alpha(W_\alpha \cap W_\xi)$. Put $R = E_{\alpha\beta}D_\beta \cap E_{\beta\xi}$. Then $R \not\subseteq Q_\beta$ by (2.5)–(11). If $G_{\alpha\beta} \cap G_{\beta\xi} \not\subseteq R$, then $Z_{\beta\xi} \subseteq \langle G_{\alpha\beta} \cap G_{\beta\xi}, R \rangle \subseteq G_{\alpha\beta}$ because $G_{\beta\xi}/Q_\beta \cong \mathbf{D}_8$, a contradiction. Hence we have $G_{\alpha\beta} \cap G_{\beta\xi} \subseteq R \subseteq E_{\beta\xi}$, and so, by symmetry, $G_{\alpha\beta} \cap G_{\beta\xi} \subseteq E_{\alpha\beta} \cap E_{\beta\xi}$.

PROPOSITION 8.10 *Let $\beta, \delta \in \mathcal{B}$ and $d(\beta, \delta) = b-1$. Then*

$$[U_\beta, U_\delta \cap Q_\beta] = 1.$$

PROOF. Suppose false. Then $[U_\beta, U_\delta \cap Q_\beta] = V_\beta$. Define $D_\eta = C_{G_\eta}(U_\eta/V_\eta)$ for all $\eta \in \mathcal{B}$.

(1) *There exists $\alpha \in \Delta(\beta)$ such that*

$$[V_\alpha, U_\delta \cap Q_\beta] = V_\beta \quad \text{and} \quad G_{\alpha\beta} \cap G_{\beta, \beta+1} \neq Q_\beta.$$

PROOF. Choose $\alpha \in \Delta(\beta)$ so that $[V_\alpha, U_\delta \cap Q_\beta] = V_\beta$, and put $R = G_{\alpha\beta}D_\beta \cap G_{\beta, \beta+1}$. Then $R \neq Q_\beta$ by (8.3)–(2) and (2.5)–(12). Since $G_\beta \neq \langle G_{\alpha\beta}, R \rangle$, there exists $x \in G_\beta$ such that $R \subseteq G_{\alpha\beta}x$ and $V_{\alpha^x} = V_\alpha$ by (8.2) and (3.10)–(1). Thus α^x has the desired properties.

Now, let $\alpha \in \Delta(\beta)$ be as in (1). Define

$$B = [T_\alpha, U_{\delta-2}] \quad \text{and} \quad D = [T_\alpha \cap Q_{\delta-1}, U_\delta \cap Q_\beta].$$

(2) *The following holds.*

- (i) $U_\delta \cap Q_\beta \not\subseteq Q_\alpha$.
- (ii) $V_\beta \subseteq D \subseteq T_\alpha \cap [T_\alpha \cap Q_{\delta-1}, U_\delta] \subseteq T_\alpha \cap U_\delta$.
- (iii) $|D| \leq 2^3$ and $|W_\alpha D : W_\alpha| \leq |D : V_\beta| \leq 2^2$.
- (iv) If $|D| = 2^3$ and $|U_\delta| = 2^5$, then $D \supseteq V_{\delta-1}$.

PROOF. Parts (i) and (ii) follow from the choice of α and the definition of D . Now, suppose $|U_\delta : D| \leq 2^2$. Then $T_\alpha \cap Q_{\delta-1}$ induces a quasitransvection on U_δ/V_δ , and so $|D| \leq |[T_\alpha \cap Q_{\delta-1}, U_\delta]V_\delta| = |[T_\alpha \cap Q_{\delta-1}, M]V_\delta| \leq 2^3$ by (2.5)–(10) and (8.4)–(2), where $M = [U_\delta, O^2(G_\delta)]V_\delta$. Thus, since $|U_\delta| \leq 2^6$, we have $|D| \leq 2^3$, and so $|W_\alpha D : W_\alpha| \leq |D : V_\beta| \leq 2^2$. Finally, let $|D| = 2^3$ and $|U_\delta| = 2^5$. Then $D = [T_\alpha \cap Q_{\delta-1}, U_\delta]V_\delta \supseteq V_{\delta-1}$ by the above remark and (2.5)–(10).

Now, let $n = n(G_\xi, T_\xi/W_\xi)$ for $\xi \in \mathcal{A}$. Note that $n \geq 2$ by (8.4).

(3) $O^2(G_\xi/C_{G_\xi}(T_\xi/W_\xi)) \cong \mathbf{Z}_3$ for $\xi \in \mathcal{A}$.

PROOF. We may assume $\xi = \alpha$, and it is sufficient to prove that $U_\delta \cap Q_\beta$ induces a factorwise quasitransvection on T_α/W_α as in the proof of (4) of (8.2).

Since $T_\alpha \subseteq Q_{\delta-3}$ by the minimality of b ,

$$\begin{aligned} & |T_\alpha : T_\alpha \cap Q_{\delta-1}| \cdot |W_\alpha D : W_\alpha| \\ & \leq |T_\alpha : T_\alpha \cap Q_{\delta-2}| \cdot |T_\alpha \cap Q_{\delta-2} : T_\alpha \cap Q_{\delta-2} \cap Q_{\delta-1}| \cdot |D : V_\beta| \\ & \leq 2^2 \cdot 2 \cdot 2^2 \end{aligned}$$

by (2-iii), so we may assume $n \leq 3$, and hence $|U_\beta| = 2^5$ by (8.4). If $|D| = 2^3$, then $V_{\delta-1} \subseteq D \subseteq T_\alpha$ by (2-ii,iv), so $T_\alpha \cap Q_{\delta-2} \subseteq C_{G_{\delta-2,\delta-1}}(V_{\delta-1}) = Q_{\delta-1}$. Hence we have either

$$(*) \quad |W_\alpha D : W_\alpha| \leq |D : V_\beta| \leq 2 \quad \text{or} \quad T_\alpha \cap Q_{\delta-2} \subseteq Q_{\delta-1}.$$

Therefore

$$|T_\alpha : T_\alpha \cap Q_{\delta-1}| \cdot |W_\alpha D : W_\alpha| \leq 2^4,$$

so we may assume $n = 2$, and then $|W_\alpha| = 2^3$ by (8.4).

Now, suppose

$$|T_\alpha : T_\alpha \cap Q_{\delta-1}| \cdot |W_\alpha D : W_\alpha| = 2^4.$$

Then $|T_\alpha : T_\alpha \cap Q_{\delta-2}| = 2^2 \leq |T_\alpha : T_\alpha \cap Q_{\delta-1}|$ by (*), and hence $C_{U_{\delta-2}}(T_\alpha) = W_{\delta-3}$, $B \cap V_{\delta-2} = 1$, and $|W_\alpha \cap W_{\delta-3}| \geq 2^2$ by (8.7)-(2). Since $[T_\alpha \cap Q_{\delta-2}, W_{\delta-1}] \subseteq [T_\alpha \cap Q_{\delta-2}, U_{\delta-2}] \subseteq B \cap V_{\delta-2} = 1$, we have $T_\alpha \cap Q_{\delta-2} \subseteq Q_{\delta-1}$, so $T_\alpha \cap Q_{\delta-1} = T_\alpha \cap Q_{\delta-2}$. Thus $D \subseteq [T_\alpha \cap Q_{\delta-1}, U_\delta] \subseteq W_{\delta-1} \subseteq U_{\delta-2}$, and so $D \subseteq C_{U_{\delta-2}}(T_\alpha) = W_{\delta-3}$. Therefore $|W_\alpha D : W_\alpha| \leq |W_\alpha W_{\delta-3} : W_\alpha| \leq 2$, and hence

$$|T_\alpha : T_\alpha \cap Q_{\delta-1}| \cdot |W_\alpha D : W_\alpha| \leq 2^3,$$

which is a contradiction as required.

$$(4) \quad T_\alpha \not\subseteq Q_{\delta-1}.$$

PROOF. Suppose false. Since $|W_\alpha D : W_\alpha| \leq 2^2$ by (2-iii), we have $n = 2$ by (2.11)-(2). Therefore

$$(i) \quad |W_\alpha| = 2^3 \quad \text{and} \quad |U_\beta| = 2^5$$

by (8.4), and so

$$(ii) \quad |D| = 2^3 \quad \text{and} \quad W_\alpha \cap D = V_\beta.$$

Here, we will prove that

$$(iii) \quad [T_\alpha, U_\delta] \subseteq D \quad \text{and} \quad |U_\delta : U_\delta \cap Q_\alpha| = 2.$$

To show this, we may assume $U_\delta \not\subseteq Q_\beta$. As $|U_\delta \cap Q_\beta : D| \leq 2$ and $|D : V_\delta| = 2^2$ by (i) and (ii), $|T_\alpha : T_\alpha \cap Q_\delta| = 2^2 = |U_\delta : D| \geq |U_\delta : C_{U_\delta}(T_\alpha)|$, so $T_\alpha Q_\delta = E_{\delta-1,\delta}$ and $D = W_{\delta-1}$ by (2.5)-(4)(5). Therefore, $[T_\alpha, U_\delta] \subseteq W_{\delta-1} = D \subseteq T_\alpha$ by (2.5)-(4) and (2-ii), and hence $G_\beta \neq \langle G_{\alpha\beta}, U_\delta \rangle$. If $U_\delta \not\subseteq G_{\alpha\beta}$, then $Z_{\beta,\beta+1} \subseteq \langle G_{\alpha\beta} \cap G_{\beta,\beta+1}, U_\delta \rangle$, so $G_\beta = \langle G_{\alpha\beta}, Z_{\beta,\beta+1} \rangle = \langle G_{\alpha\beta}, U_\delta \rangle$, a contradiction. Thus we have $U_\delta \subseteq G_{\alpha\beta}$, and hence $|U_\delta : U_\delta \cap Q_\alpha| \leq |G_{\alpha\beta} : Q_\alpha| = 2$, proving (iii).

Now, take $\theta \in \Delta(\alpha)$ so that $G_\alpha = \langle G_{\theta\alpha}, G_{\alpha\beta} \rangle$. Then $G_\alpha = \langle G_{\theta\alpha}, U_\delta \cap Q_\beta \rangle = \langle Q_\theta, U_\delta \cap Q_\beta \rangle$ by (2-i) and (3.2)-(3). Since $[U_\theta, U_\delta \cap Q_\alpha] \subseteq C_{U_\theta}(U_\delta) \subseteq W_\alpha$ by (8.6), we have $[U_\theta, U_\delta \cap Q_\alpha] \subseteq W_\alpha \cap D = V_\beta \subseteq V_\alpha$ by (ii) and (iii), so $U_\delta \cap Q_\alpha \subseteq Q_\theta$ by (2.5)-(9). Therefore $[U_\theta, U_\delta \cap Q_\alpha] \subseteq C_{V_\theta}(U_\delta) = 1$ by the choice of θ , and hence U_θ induces a transvection on U_δ/V_δ by (iii). Thus $||[U_\theta, U_\delta]| \leq 2^2$, and so

$|U_\theta[U_\theta, U_\delta] : U_\theta| \leq |[U_\theta, U_\delta] : V_\beta| \leq 2$ as $V_\beta = [V_\alpha, U_\delta \cap Q_\beta] \subseteq [U_\theta, U_\delta]$. Now, take $x \in U_\delta \cap Q_\beta - Q_\alpha$. Then $\cap U_\theta^{G_\alpha} = W_\alpha \subset U_\theta \cap U_\theta^x \triangleleft \langle Q_\theta, x \rangle = G_\alpha$, a contradiction.

(5) $T_\alpha \not\subseteq Q_{\delta-2}$, $|W_\alpha| = 2^3$, $|U_\beta| = 2^5$, $W_{\delta-1} \supseteq V_\beta \not\subseteq B \not\subseteq V_{\delta-2}$, and $|K_{\delta-2} : K_{\delta-2} \cap G_{\alpha\beta}| \leq 2$.

PROOF. Take $\theta \in \Delta(\alpha)$ so that $U_\theta \not\subseteq Q_{\delta-1}$. Then $U_\theta \neq U_\beta$, so $G_\alpha = \langle G_{\theta\alpha}, G_{\alpha\beta} \rangle$ by (3) together with (3.10)–(2), and then $V_\theta \neq V_\beta$ by (8.3)–(1) and (3.10)–(2). Let $F = [U_\theta, U_{\delta-2}]$. Then $B \supseteq F \subseteq C_{U_\theta}(U_\delta) \subseteq W_\alpha$ by (8.6). Suppose $F \subseteq V_\alpha$. Then $U_{\delta-2} \subseteq Q_\theta$ by (2.5)–(9), and so $F \subseteq C_{V_\theta}(U_\delta) = 1$. Thus $U_\theta \subseteq C_{G_{\delta-3, \delta-2}}(U_{\delta-2}) \subseteq C_{Q_{\delta-2}}(V_{\delta-1}) \subseteq Q_{\delta-1}$ by (8.5), a contradiction. Therefore $F \not\subseteq V_\alpha$, so

$$(i) \quad B \not\subseteq V_\beta \subseteq V_\alpha \neq V_\alpha F \subseteq W_\alpha.$$

Hence we have

$$(ii) \quad |W_\alpha| = 2^3 \text{ and } |U_\beta| = 2^5$$

by (8.4), and then, as $[U_\beta, W_{\delta-1}] \subseteq [U_\beta, U_{\delta-2}] = 1$ by the minimality of b , $V_\beta \subseteq [U_\beta, U_\delta] \subseteq W_{\delta-1}$ by (8.5).

If $W_\alpha \cap W_{\beta+1} \neq V_\beta$, then $E_{\alpha\beta} \cap E_{\beta, \beta+1} \neq Q_\beta$ by the choice of α and (8.9)–(2), and so

$$|K_{\delta-2} : K_{\delta-2} \cap G_{\alpha\beta}| \leq |E_{\beta, \beta+1} : E_{\alpha\beta} \cap E_{\beta, \beta+1}| \leq 2.$$

If $W_\alpha \cap W_{\beta+1} = V_\beta$, then $B \not\subseteq W_{\beta+1}$ as $B \subseteq W_\alpha$ by (8.7)–(1), so $K_{\delta-2}$ centralizes a hyperplane $BW_{\beta+1}$ of U_β . Thus

$$|K_{\delta-2} : K_{\delta-2} \cap G_{\alpha\beta}| \leq |K_{\delta-2} : K_{\delta-2} \cap Q_\beta| \leq 2$$

by (2.5)–(5).

Finally, suppose $T_\alpha \subseteq Q_{\delta-2}$. If $|D| = 2^3$, then $V_{\delta-1} \subseteq D \subseteq T_\alpha$ by (2-ii, iv), and so $T_\alpha \subseteq C_{Q_{\delta-2}}(V_{\delta-1}) = Q_{\delta-1}$, which contradicts (4). Therefore $|D| \leq 2^2$, and hence $|T_\alpha : T_\alpha \cap Q_{\delta-1}| \cdot |W_\alpha D : W_\alpha| \leq 2 \cdot 2 = 2^2$. Thus we have $n = 2$ by (8.4), and so

$$(iii) \quad |D| = 2^2 \text{ and } D \not\subseteq W_\alpha.$$

Here, we will prove that

$$(iv) \quad D \supseteq V_{\delta-2}.$$

Note that $D \supseteq [T_\alpha \cap Q_{\delta-1}, W_{\delta-1}] \subseteq [T_\alpha, U_{\delta-2}] \subseteq V_{\delta-2}$. Thus we may assume that $[T_\alpha \cap Q_{\delta-1}, W_{\delta-1}] = 1$, so $D \subseteq [T_\alpha \cap Q_{\delta-1}, U_\delta] \subseteq W_{\delta-1}$ by (8.5). Therefore $D \cap V_{\delta-1} \neq 1$ by (ii) and (iii), and hence $D \cap V_{\delta-1} = C_{V_{\delta-1}}(T_\alpha) = V_{\delta-2}$, proving (iv).

Now, since $V_{\delta-2} = F \not\subseteq V_\alpha$ by (i), $V_\beta \neq V_{\delta-2}$, so $D = V_\beta V_{\delta-2} = V_\beta F \subseteq W_\alpha$ by (iv) and (i), contrary to (iii). Thus $T_\alpha \not\subseteq Q_{\delta-2}$, and hence $B \not\subseteq V_{\delta-2}$.

(6) If $T_\alpha Q_{\delta-2} \neq Z_{\delta-3, \delta-2}$, then $b \geq 7$, $G_{\delta-2} = \langle G_{\delta-3, \delta-2}, G_{\delta-2, \delta-1} \rangle$, and $O^2(G_\eta / Q_\eta C_{G_\eta}(K_\eta)) \cong \mathbf{E}_{3^2}$ for $\eta \in \mathcal{B}$.

PROOF. We may assume $\eta = \delta - 2$. Let $\overline{G}_{\delta-2} = G_{\delta-2}/Q_{\delta-2}C_{G_{\delta-2}}(K_{\delta-2}/U_{\delta-2})$. By (5) together with (8.7)–(3), we have

$$|K_{\delta-2} : K_{\delta-2} \cap Q_{\alpha}| \cdot |[T_{\alpha}, K_{\delta-2} \cap Q_{\alpha}]U_{\delta-2} : U_{\delta-2}| \leq 2^a$$

where $a = \max\{2, 8-b\}$.

Now, suppose $b = 5$. As $Q_{\delta-2} \neq K_{\beta}Q_{\delta-2} \neq Z_{\delta-3, \delta-2}$, $n(G_{\beta}, K_{\beta}/U_{\beta}) \geq 2$ by (8.8)–(2). Thus the above inequality shows that T_{α} induces a factorwise sesquitransvection on $K_{\delta-2}/U_{\delta-2}$ by (2.11)–(2), and hence $O^2(\overline{G}_{\delta-2}) \cong \mathbf{E}_{3^2}$ by (2.11)–(1). Therefore $G_{\delta-2} = \langle G_{\delta-3, \delta-2}, G_{\delta-2, \delta-1} \rangle$ by (3.10)–(2) because $T_{\delta-3} \neq T_{\delta-1}$ by the minimality of b , and so $G_{\delta-2} = \langle G_{\delta-3, \delta-2}, Z_{\delta-2, \delta-1} \rangle$ by (2.11)–(11). Hence $V_{\delta-3} \triangleleft G_{\delta-2}$ because $[W_{\delta-1}, Z_{\delta-2, \delta-1}] \subseteq V_{\delta-2} \subseteq V_{\delta-3} = V_{\beta}V_{\delta-2} \subseteq W_{\delta-1}$ by (3) and (5) together with (2.5)–(4), a contradiction. Thus we have $b \geq 7$ so $a = 2$, and hence T_{α} induces a factorwise quasitransvection on $K_{\delta-2}/U_{\delta-2}$. Since $T_{\alpha}Q_{\delta-2} \neq Z_{\delta-3, \delta-2}$ and $O^2(\overline{G}_{\delta-2})$ is a 3-group, we have $O^2(\overline{G}_{\delta-2}) \cong \mathbf{E}_{3^2}$ by (2.11)–(1). This shows that $D_{\delta-2} = Q_{\delta-2}C_{G_{\delta-2}}(K_{\delta-2}/U_{\delta-2})$, and hence $D_{\delta-2} = Q_{\delta-2}C_{G_{\delta-2}}(K_{\delta-2})$.

(7) We have $b \geq 7$, $G_{\delta-2} = \langle G_{\delta-3, \delta-2}, G_{\delta-2, \delta-1} \rangle$, $|T_{\alpha}Q_{\delta-2} : Q_{\delta-2}| = 2$, $T_{\alpha}Q_{\delta-2} \neq Z_{\delta-3, \delta-2}$, and $O^2(G_{\eta}/G_{\eta}C_{G_{\eta}}(K_{\eta})) \cong \mathbf{E}_{3^2}$ for $\eta \in B$.

PROOF. In view of (6), it suffices to show that $Z_{\delta-3, \delta-2} \not\subseteq T_{\alpha}Q_{\delta-2}$. Suppose $Z_{\delta-3, \delta-2} \subseteq T_{\alpha}Q_{\delta-2}$. Then we have $V_{\beta} \subseteq B \subseteq W_{\delta-3}$ and $B \cap V_{\delta-2} = 1$ by (8.7)–(1)(2), and so

$$(i) \quad W_{\delta-3} \cap W_{\delta-1} \supseteq V_{\beta}V_{\delta-2} \neq V_{\delta-2}.$$

Take $\theta \in \Delta(\alpha)$ so that $U_{\theta} \not\subseteq Q_{\delta-2}$. Then $U_{\theta} \neq U_{\beta}$, so $G_{\alpha} = \langle G_{\theta\alpha}, G_{\alpha\beta} \rangle$ by (3) and (3.10)–(2), and, moreover, $T_{\alpha} = \langle U_{\theta}, U_{\theta'}, U_{\beta} \rangle$ where $\theta' = \theta^x$ for some $x \in U_{\delta} \cap Q_{\beta} - Q_{\alpha}$ by (2-i) and (3). Therefore $B = [U_{\theta}, U_{\delta-2}][U_{\theta'}, U_{\delta-2}]$ and $[U_{\theta}, U_{\delta-2}] = [U_{\theta}, U_{\delta-2}]^x = [U_{\theta'}, U_{\delta-2}]$, so $B = [U_{\theta}, U_{\delta-2}]$. Hence we have $Z_{\delta-3, \delta-2} \subseteq U_{\theta}Q_{\delta-2}$, and so $C_{U_{\delta-2}}(U_{\theta}) = W_{\delta-3}$ by (2.5)–(4). As $[U_{\theta}, U_{\delta-2} \cap Q_{\theta}] \subseteq V_{\theta} \cap B = 1$, $U_{\delta-2} \cap Q_{\theta} \subseteq C_{U_{\delta-2}}(U_{\theta}) = W_{\delta-3}$. Thus $|U_{\delta-2}Q_{\theta} : Q_{\theta}| = 2^2$ and so

$$(ii) \quad U_{\delta-2}Q_{\theta} = E_{\theta\alpha}.$$

Similarly, since $[U_{\theta} \cap Q_{\delta-2}, U_{\delta-2}] \subseteq B \cap V_{\delta-2} = 1$, we have $U_{\theta} \cap Q_{\delta-2} \subseteq C_{U_{\theta}}(U_{\delta-2}) = W_{\alpha}$ by (ii), so $|U_{\theta} : U_{\theta} \cap Q_{\delta-2}| = 2^2$. Thus $T_{\alpha}Q_{\delta-2} \supseteq U_{\theta}Q_{\delta-2} = E_{\delta-3, \delta-2}$, and hence $O^2(G_{\delta-2}/Q_{\delta-2}C_{G_{\delta-2}}(K_{\delta-2})) \cong \mathbf{E}_{3^2}$ by (6). Therefore we have $G_{\delta-2} = \langle G_{\delta-3, \delta-2}, G_{\delta-2, \delta-1} \rangle$ and

$$(iii) \quad W_{\delta-3} \cap W_{\delta-1} = V_{\beta}V_{\delta-2} \neq V_{\delta-2}$$

by (i) and (3.10)–(2) because $T_{\delta-3} \neq T_{\delta-1}$ by the minimality of b .

Now, put $R = U_{\theta} \cap E_{\delta-2, \delta-1}D_{\delta-2}$. Since $R \not\subseteq Q_{\delta-2}$ by (iii) and (2.5)–(11), we have $|U_{\theta} : R| = |R : W_{\alpha}| = 2$. However, $[R, U_{\delta-2}] \subseteq B \cap W_{\delta-3} \cap W_{\delta-1} = V_{\beta} \subseteq V_{\alpha}$, which is impossible under the condition (ii) by (2.5)–(8).

$$(8) \quad V_\beta \neq V_{\delta-2}.$$

PROOF. Suppose false. As $U_{\delta-2} \neq U_\delta$, $G_{\delta-1} = \langle G_{\delta-2, \delta-1}, G_{\delta-1, \delta} \rangle$ by (3) and (3.10)–(2), so

$$[U_\beta, U_\delta \cap Q_\beta] = V_\beta = V_{\delta-2} \neq V_\delta.$$

Therefore $V_{\delta-1} \supseteq [U_\beta, U_\delta \cap Q_\beta] \not\subseteq V_\delta$, and hence $U_\beta \not\subseteq Q_\delta$. If $U_\delta \subseteq Q_\beta$, then $[U_\beta, U_\delta] = V_{\delta-2} \subseteq V_{\delta-1}$, so $U_\beta \subseteq Q_\delta$ by (2.5)–(9), a contradiction. Thus $U_\delta \not\subseteq Q_\beta$, and hence

$$(*) \quad U_\beta Q_\delta = Z_{\delta-1, \delta}$$

by (2.5)–(8). As $U_\beta \neq U_{\beta+2}$, we have $G_{\beta+1} = \langle G_{\beta, \beta+1}, G_{\beta+1, \beta+2} \rangle$ and $V_\beta \neq V_{\beta+2}$ by (3.10)–(2) as above. Therefore,

$$W_{\beta+1} = V_{\beta+1}[U_\beta, U_\delta] = V_\beta V_{\beta+2}[U_\beta, U_\delta] = V_{\delta-2} V_{\beta+2}[U_\beta, U_\delta],$$

so $[W_{\beta+1}, K_\delta] = 1$, and hence $G_{\beta+2} \neq \langle G_{\beta+1, \beta+2}, K_\delta \rangle$. Thus, by (7) and (3.10)–(1), there exists $\xi \in \Delta(\beta+2)$ such that $K_\delta \subseteq G_{\xi, \beta+2}$, $T_\xi = T_{\beta+1}$, and $W_\xi = W_{\beta+1}$. Hence $[U_\beta, K_\delta] \subseteq [T_\xi, K_\delta] \subseteq W_\xi$ by (8.5), and so U_β induces a factorwise sesquitransvection on K_δ because $n(G_\delta, K_\delta) \geq 2$ and $|W_\xi| = 2^3$. Thus $O^2(G_\delta/Q_\delta C_{G_\delta}(K_\delta)) \cong \mathbf{Z}_3$ by (*) and (2.11)–(1), which contradicts (7).

$$(9) \quad T_\alpha \cap Q_{\delta-2} = T_\alpha \cap Q_{\delta-1} \text{ and } D \subseteq W_{\delta-1}.$$

PROOF. Suppose first that $B \cap V_{\delta-2} = 1$. Then, since $[T_\alpha \cap Q_{\delta-2}, U_{\delta-2}] \subseteq B \cap V_{\delta-2} = 1$, $T_\alpha \cap Q_{\delta-2} = T_\alpha \cap Q_{\delta-1}$ by (4) and (7), so $D \subseteq [T_\alpha \cap Q_{\delta-1}, U_\delta] \subseteq W_{\delta-1}$. Therefore it is sufficient to prove that $B \cap V_{\delta-2} = 1$.

Assume otherwise. Then $B \supset V_{\delta-2}$ by (4), so

$$W_\alpha \supseteq B = V_\beta V_{\delta-2} = W_{\delta-3} \cap W_{\delta-1} \neq V_{\delta-2} \quad \text{and} \quad W_{\delta-1} = B V_{\delta-1}$$

by (8), (8.7)–(1), and (8.9)–(2). Therefore, by (3.10)–(1), there exists $\lambda \in \Delta(\delta-2)$ such that $T_\alpha \subseteq G_{\delta-2, \lambda}$, $T_{\delta-1} = T_\lambda$, $W_{\delta-1} = W_\lambda$, and $V_{\delta-1} = V_\lambda$. Thus $|T_\alpha : T_\alpha \cap Q_\lambda| \leq |G_{\delta-2, \lambda} : Q_\lambda| \leq 2 = |W_\alpha W_\lambda : W_\alpha|$ and $W_\lambda = B V_\lambda$.

Let $F = [T_\alpha \cap Q_\lambda, U_\delta \cap Q_\beta]$. Since $[T_\alpha \cap Q_\lambda, W_\lambda] = [T_\alpha \cap Q_\lambda, B V_\lambda] = 1$, $F \subseteq [T_\alpha \cap Q_\lambda, T_\lambda \cap Q_\beta] \subseteq T_\alpha \cap W_\lambda$. If $F \not\subseteq B$, then $V_\lambda \subseteq W_\lambda = B F \subseteq T_\alpha$, so $T_\alpha \subseteq Q_\lambda$. Therefore, $|T_\alpha : T_\alpha \cap Q_\lambda| \cdot |W_\alpha F : W_\alpha| \leq 2$ both in the case $F \not\subseteq B$ and in the case $F \subseteq B$, so we have $n \leq 1$, a contradiction.

$$(10) \quad W_{\delta-1} \subseteq T_\alpha.$$

PROOF. Since $|T_\alpha : T_\alpha \cap Q_{\delta-1}| \cdot |W_\alpha D : W_\alpha| \geq 2^n \geq 2^2$ by (2.11)–(2),

$$(i) \quad W_\alpha \not\subseteq D$$

by (7) and (9).

Suppose $W_{\delta-1} \not\subseteq T_\alpha$. Then

$$V_\beta \subset D = T_\alpha \cap W_{\delta-1} \subset W_{\delta-1} \quad \text{and} \quad |D| = 2^2.$$

First, we will prove that

$$(ii) \quad W_{\delta-1} = DV_\delta \text{ and } V_\beta V_\delta \not\subseteq V_{\delta-1}.$$

Assume otherwise. Then $D \cap V_{\delta-1} \neq V_{\delta-2}$, so $V_{\delta-1} = V_{\delta-2}(D \cap V_{\delta-1})$. Thus $[T_\alpha, V_{\delta-1}] = 1$. If $U_{\delta-2} = W_{\delta-3}W_{\delta-1}$, then $G_{\delta-2} = \langle T_\alpha, Q_{\delta-1} \rangle$ by (8.9)–(1), and hence $V_{\delta-1} \subseteq Z(G_{\delta-2})$, a contradiction. Therefore $U_{\delta-2} \neq W_{\delta-3}W_{\delta-1} = W_{\delta-3}V_{\delta-1}$ by (8.9)–(2), and so $[T_\alpha, W_{\delta-1}] = 1$. Thus, by (3.10)–(1), there exists $\xi \in \Delta(\delta-2)$ such that $T_\alpha \subseteq G_{\delta-2, \xi}$, $T_\xi = T_{\delta-1}$, and $W_\xi = W_{\delta-1}$, so we have $B = [T_\alpha, U_{\delta-2}] \subseteq T_\alpha \cap W_\xi = D$. Hence $D = V_\beta B \subseteq W_\alpha$, contrary to (i). This proves (ii).

Now, we will derive a contradiction. Suppose first that $U_\delta \not\subseteq Q_\beta$. Then, since $V_\beta \subset [U_\beta, U_\delta] \subseteq T_\alpha \cap W_{\delta-1} = D$, we have $[U_\beta, U_\delta] = D$. Therefore, by (ii),

$$(iii) \quad Z_{\delta-1, \delta} \subseteq U_\beta Q_\delta \subseteq (T_\alpha \cap Q_{\delta-1})Q_\delta \subseteq E_{\delta-1, \delta} \text{ and}$$

$$(iv) \quad W_{\delta-1} \supset V_\beta V_\delta = [U_\beta, U_\delta \cap Q_\beta]V_\delta \not\subseteq V_{\delta-1}.$$

Thus (iii) together with (2.5)–(8) shows that

$$U_\delta \cap Q_\beta = [U_\delta, G_{\delta-1, \delta}]V_\delta,$$

but then (iv) is impossible also by (2.5)–(8).

Suppose next that $U_\delta \subseteq Q_\beta$. Then $U_\beta = V_\alpha(U_\beta \cap Q_\delta)$ by the choice of α . Let $H = U_\delta \cap Q_\alpha$. Then $H = C_{U_\delta \bmod V_\delta}(U_\beta)$, so

$$DV_\delta = [U_\beta, U_\delta][T_\alpha \cap Q_{\delta-1}, H]V_\delta \quad \text{and}$$

$$[U_\beta, U_\delta][T_\alpha \cap Q_{\delta-1}, H] \subseteq D.$$

Thus $D = [U_\beta, U_\delta][T_\alpha \cap Q_{\delta-1}, H] = V_\beta[T_\alpha \cap Q_{\delta-1}, H] \subseteq W_\alpha$, contrary to (i).

(11) *A contradiction.*

PROOF. Take $\theta \in \Delta(\alpha)$ so that $G_\alpha = \langle G_{\theta\alpha}, U_\delta \cap Q_\beta \rangle$. Let $x \in O^2(G_\theta)$ so that $C_{U_\theta/V_\theta}(x) = 1$, let $t \in Z_{\alpha^*, \theta} - Q_\theta$, and put $\kappa = \alpha^t$. Then $W_\kappa \cap W_\alpha = V_\theta$ by (2.5)–(11), and t permutes T_κ and T_α because t inverts $\langle x \rangle Q_\theta / Q_\theta$. Since $[W_\kappa, U_{\delta-4}] \subseteq [U_\theta, U_{\delta-4}] = 1$, we have $[T_\kappa, U_{\delta-4}] \subseteq C_{W_\kappa}(U_\delta) \subseteq W_\alpha$ by (8.5), so $[T_\kappa, U_{\delta-4}] \subseteq C_{W_\kappa \cap W_\alpha}(U_\delta) = C_{V_\theta}(U_\delta) = 1$. Thus $T_\kappa \subseteq C_{Q_{\delta-5}}(U_{\delta-4}) \subseteq Q_{\delta-3}$.

Now, let $K = T_\kappa T_\alpha$. Note that K is abelian as $b \geq 7$. Since $[K, W_{\delta-1}] = 1$ by (10), we have $G_{\delta-3} \neq \langle K, G_{\delta-2, \delta-1} \rangle$, so there exists $\xi \in \Delta(\delta-2)$ such that $K \subseteq G_{\delta-2, \xi}$, $T_\xi = T_{\delta-1}$, and $W_\xi = W_{\delta-1}$, and hence $[K, T_\xi] \subseteq W_\xi \subseteq K$. Therefore $K \triangleleft \langle Q_\theta, U_\delta \cap Q_\beta \rangle = G_\alpha$, and so $K \triangleleft G_\alpha^t = G_\kappa$. This shows that $K \triangleleft \langle G_\kappa, G_\alpha \rangle = \langle G_\theta, G_\alpha \rangle$, a contradiction. This completes the proof of (8.10).

Now, let $\mathcal{C}_0' = \{(\beta, \delta) \in \mathcal{C}' \mid U_\delta \not\subseteq Q_\beta\}$.

COROLLARY 8.11 *Let $\beta, \delta \in \mathcal{B}$ and $d(\beta, \delta) \leq b-1$. Then the following holds.*

- (1) *Either $(\beta, \delta) \in \mathcal{C}_0'$ or $[U_\beta, U_\delta] = 1$.*
- (2) $U_\delta \cap Q_\beta = C_{U_\delta}(U_\beta) = C_{U_\delta \bmod V_\delta}(U_\beta)$.
- (3) $|U_\beta : U_\beta \cap Q_\delta| = |U_\delta : U_\delta \cap Q_\beta|$.
- (4) $Z_{\delta-1, \delta} \neq U_\beta Q_\delta \subseteq E_{\delta-1, \delta}$.
- (5) *If $U_\beta \subseteq M \subseteq E_{\delta-1, \delta}$, then $[M, U_\delta] \subseteq [U_\beta, U_\delta][M, U_\delta \cap Q_\beta]V_\delta$.*
- (6) *If $(\beta, \delta) \in \mathcal{C}_0'$, then $K_\beta \cap G_{\delta-1, \delta} \subseteq E_{\delta-1, \delta}$.*

PROOF. Suppose $[U_\beta, U_\delta] \neq 1$. Then we have $U_\beta \not\subseteq Q_\delta$ and $U_\delta \not\subseteq Q_\beta$ by (8.10), and then $d(\beta, \delta) \geq b-1$ by the minimality of b . This shows that $(\beta, \delta) \in \mathcal{C}_0'$, so (1) holds.

In proving (2)–(6), we may assume that $[U_\beta, U_\delta] \neq 1$, and, by symmetry, that $|U_\beta : U_\beta \cap Q_\delta| \geq |U_\delta : U_\delta \cap Q_\beta|$. Let $\tilde{G}_\delta = G_\delta / C_{G_\delta}(U_\delta / V_\delta)$ and $\tilde{U}_\delta = U_\delta / V_\delta$. Then we have

$$|\hat{U}_\delta : C_{\hat{U}_\delta}(\tilde{U}_\beta)| \leq |U_\delta : U_\delta \cap Q_\beta| \leq |U_\beta : U_\beta \cap Q_\delta| = |\tilde{U}_\beta|$$

by (6.3), (2.4)–(1), and (8.10). Hence $|\tilde{U}_\beta| = |\hat{U}_\delta : C_{\hat{U}_\delta}(\tilde{U}_\beta)|$ by (2.10)–(1), and so (2) and (3) holds. Since $(G_\delta, [\hat{U}_\delta, O^2(G_\delta)]) \ll (O_4^+(2), GF(2)^4)$ by (8.3)–(2), (4) and (5) follow from (5) and (6) of (2.5), respectively. Let $K = K_\beta \cap G_{\delta-1, \delta}$. Since K_β centralizes U_β by the minimality of b , $\tilde{K} \subseteq C_{\tilde{G}_{\delta-1, \delta}}(\tilde{U}_\beta) \subseteq \tilde{E}_{\delta-1, \delta}$ by (4), so (6) holds.

LEMMA 8.12 *Let $(\beta, \delta) \in \mathcal{C}_0'$, and suppose there exist $\alpha \in \Delta(\beta)$ and $M \triangleleft G_{\alpha\beta}$ such that*

$$M \not\triangleleft G_\beta = \langle G_{\alpha\beta}, U_\delta \rangle \quad \text{and} \quad U_\beta \subseteq M \subseteq T_\alpha \cap Q_{\delta-1}.$$

Define $K = \langle M^{G_\beta} \rangle$. Then the following holds.

- (1) $M \not\subseteq V_\delta \subseteq [M, U_\delta] \subseteq MV_\delta \subseteq K$.
- (2) *If $U_\beta \subseteq K^* \subseteq K$ and $V_\delta \subseteq K^* \triangleleft G_\beta$, then $[K, O^2(G_\beta)] \subseteq K^*$.*

PROOF. Let $N = [U_\beta, U_\delta][M, U_\delta \cap Q_\beta]$. Then $M \supseteq N \subseteq [M, U_\delta] \subseteq NV_\delta$ by (8.11)–(5)(6), and $|NV_\delta : N| \leq 2$. Therefore, since U_δ does not normalize M , $[M, U_\delta] = NV_\delta$ and $V_\delta \not\subseteq M$, so $V_\delta \subseteq [M, U_\delta] \subseteq MV_\delta \subseteq K$. Thus (1) holds.

Let $K^* \subseteq K' \subseteq K$ and assume $K' \triangleleft G_{\alpha\beta}$. Since $[M, U_\delta] \subseteq MV_\delta \subseteq MK^*$, $K = MK^*$, so $K' = (M \cap K')K^*$. Thus we have $[K', U_\delta] \subseteq [M \cap K', U_\delta]K^* \subseteq (M \cap K')V_\delta K^* = K'$. Therefore all $G_{\alpha\beta}$ -invariant subgroups between K^* and K are normal in G_β , and so $[K, O^2(G_\beta)] \subseteq K^*$. Hence (2) holds.

PROPOSITION 8.13 *Let $\xi \in \mathcal{A}$ and $\eta \in \mathcal{B}$. Then*

$$|W_\xi| = 2^3 \quad \text{and} \quad |U_\eta| = 2^5.$$

PROOF. We may assume $\langle \xi, \eta \rangle \in E(\Gamma)$. Define $K_\eta^* = \langle T_\xi^{(2)^G} \eta \rangle$.

(1) Let $\beta, \mu \in \mathcal{B}$ and $d(\beta, \mu) \leq b - 3$. Then

$$[K_\beta^*, U_\mu] = 1.$$

PROOF. Let $\alpha \in \Delta(\beta)$ and $\theta \in \Delta(\alpha)$. Then $d(\theta, \mu) \leq b - 1$, so we have $[U_\theta, Q_\alpha; 2] \subseteq C_{U_\theta \text{ mod } V_\theta}(U_\mu) = C_{U_\theta}(U_\mu)$ by (8.11)–(1)(4) and (2.5)–(4). Since $K_\beta^* = \langle [U_\theta, Q_\alpha; 2] \mid \alpha \in \Delta(\beta), \theta \in \Delta(\alpha) \rangle$, we have $[K_\beta^*, U_\mu] = 1$.

(2) One of the following holds.

- (i) There exist $(\beta, \delta) \in \mathcal{C}_0'$ and $\alpha \in \Delta(\beta)$ such that $G_\beta = \langle G_{\alpha\beta}, U_\delta \rangle$ and $[T_\alpha, U_{\delta-2}] = 1$.
- (ii) There exist $(\beta, \delta) \in \mathcal{C}_0'$, $\alpha \in \Delta(\beta)$ and $\theta \in \Delta(\alpha)$ such that $G_\beta = \langle G_{\alpha\beta}, U_\delta \rangle$, $G_\alpha = \langle G_{\theta\alpha}, G_{\alpha\beta} \rangle$, and $U_\theta \not\subseteq Q_{\delta-2}$.
- (iii) $|W_\xi| = 2^3$ for $\xi \in \mathcal{A}$.

PROOF. Suppose that neither (i) nor (ii) holds.

Let $(\beta, \delta) \in \mathcal{C}_0'$, and take $\alpha \in \Delta(\beta)$ so that $G_\beta = \langle G_{\alpha\beta}, U_\delta \rangle$. Then $T_\alpha \not\subseteq Q_{\delta-2}$ by our hypothesis. Take $\theta \in \Delta(\alpha)$ so that $U_\theta \not\subseteq Q_{\delta-2}$. Then $(\theta, \delta - 2) \in \mathcal{C}_0'$, and $G_\alpha \neq \langle G_{\theta\alpha}, G_{\alpha\beta} \rangle$ also by our hypothesis, and so $V_\theta = V_\beta$ by (3.10)–(2). Thus, changing notation, there exists $(\beta, \delta) \in \mathcal{C}_0'$ such that $V_\beta = V_{\beta+2} = \dots = V_{\delta-2} = V_\delta$.

Now, let $(\beta, \delta) \in \mathcal{C}_0'$ and $\alpha \in \Delta(\beta)$ so that $G_\beta = \langle G_{\alpha\beta}, U_\delta \rangle$ and $U_\beta \supseteq V_\delta$. Then $T_\alpha \subseteq K_\beta^* \subseteq C_{Q_{\delta-3}}(U_{\delta-2}) \subseteq Q_{\delta-1}$ by (1), and so $T_\alpha^{(2)}U_\beta \triangleleft G_\beta$ by (8.12)–(1). Since $[T_\alpha^{(2)}, Q_\alpha, Q_\alpha] \subseteq W_\alpha$ by (8.4) and $n_*(G_\beta) > 1$, G_β has no noncentral chief factor within $T_\alpha^{(2)}U_\beta/U_\beta$ by (2.12)–(1), and so $[T_\alpha^{(2)}U_\beta, O^2(G_\beta)] \subseteq U_\beta$. Thus $[T_\alpha^{(2)}, O^2(G_\alpha)] \subseteq W_\alpha$ by (3.8). Therefore $T_\alpha^{(2)} = W_\alpha$ by (2.14)–(1), so $|W_\alpha| = 2^3$ by (8.4). Hence (iii) holds.

(3) Part (iii) of (2) holds.

PROOF. Suppose false. Then either (i) or (ii) of (2) holds.

First, let $(\beta, \delta) \in \mathcal{C}_0'$ and $\alpha \in \Delta(\beta)$ as in (i) of (2). Since $[T_\alpha^{(2)}, Q_\alpha, Q_\alpha] \subseteq W_\alpha \not\subseteq [T_\alpha^{(2)}, O^2(G_\alpha)]$, $T_\alpha^{(2)}U_\beta \not\triangleleft G_\beta$ by (3.8). Thus $V_\delta \subseteq K_\beta^*$ by (8.12)–(1), and hence $[K_\beta, O^2(G_\beta)] \subseteq K_\beta^*$ by (8.12)–(2). Therefore $[U_\beta, U_\delta] \subseteq [K_{\beta+2}, U_\delta] = [K_{\beta+2}^*T_{\beta+3}, U_\delta] = 1$ by (1) and the minimality of b , a contradiction.

Next, let $(\beta, \delta) \in \mathcal{C}_0'$, $\alpha \in \Delta(\beta)$, and $\theta \in \Delta(\alpha)$ as in (ii) of (2). Then $G_{\alpha\beta} = \langle Q_\theta \cap Q_\alpha, Q_\beta \rangle$ by (3.2)–(3), and so $\langle Q_\theta \cap Q_\alpha, U_\delta \rangle$ contains a Sylow 3-subgroup, say P , of G_β . Define $F = [U_\theta, U_{\delta-2}]$ and $L = \cap (T_\alpha^{(2)}U_\beta)^{G_\beta}$. Then, as $F \subseteq [U_\theta, Q_\alpha; 2] \subseteq T_\alpha^{(2)}$ and $U_\theta \supseteq V_\alpha F \supseteq V_\theta \subseteq U_\beta, [U_\beta F, Q_\theta \cap Q_\alpha] \subseteq U_\beta \supseteq [U_\beta F, U_\delta]$, so $U_\beta F \subseteq \cap (T_\alpha^{(2)}U_\beta)^P = \cap (T_\alpha^{(2)}U_\beta)^{G_\beta} = L$. Moreover, since $[L, Q_\beta, Q_\beta] \subseteq U_\beta$ and $n_*(G_\beta) > 1$, we have also $[L, O^2(G_\beta)] \subseteq U_\beta$. Let $H = [Q_\beta, O^2(G_\beta)]$. Then

$[F, H] \subseteq [T_\alpha^{(2)} \cap L, H] \subseteq W_\alpha$ by (3.8), and so $[F, \langle Q_\theta, H \rangle] \subseteq W_\alpha$. Therefore, since $\langle Q_\theta, H \rangle$ contains a Sylow 3-subgroup, say R , of G_α , $V_\alpha \neq V_\alpha F \subseteq W_\alpha F \subseteq \cap U_\theta^R \subseteq \cap U_\theta^{G_\alpha} = W_\alpha$, a contradiction.

(8.14) *Let $(\beta, \delta) \in \mathcal{C}_0'$ and suppose $K_\beta \subseteq Q_{\delta-1}$. Then $|T_{\delta-1} : T_{\delta-1} \cap Q_\beta| = 2$, and $(G_\eta, K_\eta/U_\eta) \ll (O_4^+(2), GF(2)^4)$ and $n(G_\eta, K_\eta/U_\eta) = 1$ for $\eta \in \mathcal{B}$.*

PROOF. We may assume $\eta = \beta$. Since $W_{\delta-1} = V_{\delta-1}[U_\beta, U_\delta]$ by (2.5)–(7) and (8.11)–(4), K_β centralizes $W_{\delta-1}$, and so

$$(i) \quad [K_\beta, T_{\delta-1}] \subseteq W_{\delta-1}$$

by (8.5). Since K_β centralizes U_β by the minimality of b ,

$$(ii) \quad [K_\beta, U_\delta] \subseteq [U_\beta, U_\delta][K_\beta, U_\delta \cap Q_\beta]V_\delta \subseteq U_\beta V_\delta$$

by (2.5)–(6) and (8.11)–(2). Hence (ii) shows that $n(G_\beta, K_\beta/U_\beta) = 1$, and so $n(G_\beta, K_\beta) = 2$. Thus $T_{\delta-1}$ induces a factorwise sesquitransvection on K_β by (i) as $|W_{\delta-1}| = 2^3$, and hence $(G_\beta, K_\beta) \ll (O_4^+(2), GF(2)^4)$ and $|T_{\delta-1} : T_{\delta-1} \cap Q_\beta| = 2$.

(8.15) *Let $(\beta, \delta) \in \mathcal{C}_0'$ and $\alpha \in \Delta(\beta)$ so that $W_\alpha \cap W_{\beta+1} = V_\beta$. If $|K_{\delta-2} : K_{\delta-2} \cap Q_\beta| = 2^2$, then $[T_\alpha, U_{\delta-2}] = 1$.*

PROOF. Suppose $|K_{\delta-2} : K_{\delta-2} \cap Q_\beta| = 2^2$. Then, as $C_{U_\beta}(K_{\delta-2}) = W_{\beta+1}$, $[T_\alpha, U_{\delta-2}] \subseteq C_{W_\alpha}(K_{\delta-2}) = W_\alpha \cap W_{\beta+1} = V_\beta \subseteq V_\alpha$. Thus $U_{\delta-2} \subseteq Q_\theta$ for all $\theta \in \Delta(\alpha)$ by (2.5)–(9), and hence $[T_\alpha, U_{\delta-2}] = 1$ by (8.11)–(1).

(8.16) *Let $(\beta, \delta) \in \mathcal{C}_0'$. Then*

$$|T_{\beta+1} : T_{\beta+1} \cap Q_\delta| = |T_{\delta-1} : T_{\delta-1} \cap Q_\beta| = 2.$$

PROOF. Suppose false. Take $(\beta, \delta) \in \mathcal{C}_0'$ and $\alpha \in \Delta(\beta)$ so that $|T_{\delta-1} : T_{\delta-1} \cap Q_\beta| = 2^2$ and $W_\alpha \cap W_{\beta+1} = V_\beta$. Define

$$M = T_\alpha^{(1)}U_\beta, \quad L = \langle M^{T_{\delta-1}}, \quad \text{and} \quad K = \langle M^{G_\beta} \rangle.$$

$$(1) \quad |L : M| \leq 2.$$

PROOF. Since $[M, Q_\alpha] \subseteq U_\beta$ and $[T_\alpha^{(1)}, O^2(G_\alpha)] \not\subseteq W_\alpha$, we have $M \not\trianglelefteq G_\beta$ by (3.8). Moreover, $G_\beta = \langle G_{\alpha\beta}, U_\delta \rangle$ by the choice of α and (8.9)–(1), and $M \subseteq T_\alpha \subseteq C_{Q_{\delta-3}}(U_{\delta-2}) \subseteq Q_{\delta-1}$ by (8.5) and (8.15). Thus (8.12)–(1) shows that $V_\delta \not\subseteq M$, so $V_\delta \not\subseteq W_{\beta+1}$. Since $W_{\beta+1} = V_\beta[U_\beta, T_{\delta-1}]$ and $[U_\beta, T_{\delta-1}] \subseteq W_{\beta+1} \cap W_{\delta-1}$, we have $|W_{\delta-1} : W_{\beta+1} \cap W_{\delta-1}| \leq 2$, and hence $W_{\delta-1} = (W_{\beta+1} \cap W_{\delta-1})V_\delta$. Therefore $\langle M^{T_{\delta-1}} \rangle = M[M, T_{\delta-1}] \subseteq MW_{\delta-1} = MV_\delta$, and so $|L : M| \leq 2$.

(2) *A contradiction.*

PROOF. Let K/N be a noncentral chief factor of G_β within K/U_β . Since $1 \neq MN/N \subseteq C_{K/N}(G_{\alpha\beta}) \neq K/N$ and $G_\beta = \langle G_{\alpha\beta}, T_{\delta-1} \rangle$, we have $LN \neq MN$, so $|LN : MN| = 2$ and $|LN/N| \geq 2^2$ by (1). Let F/N be a $T_{\delta-1}$ -invariant fours subgroup of L/N , and set $C = C_{T_{\delta-1}}(F/N)$. Since $|T_{\delta-1} : T_{\delta-1} \cap Q_\beta| = 2^2$, we have $C \not\subseteq Q_\beta$, and hence $G_\beta = \langle G_{\alpha\beta}, C \rangle$ by the choice of α and (8.9)–(1). Thus we have $1 \neq MN/N \cap F/N \subseteq C_{K_\beta/N}(G_\beta)$, a contradiction.

(8.17) We have $b \geq 7$ (and hence K_η is elementary abelian for $\eta \in \mathcal{B}$).

PROOF. Suppose $b = 5$, and let $(\beta, \delta) \in \mathcal{C}_0'$. Since $Q_{\delta-3} \triangleright K_{\delta-2} \not\subseteq Q_\beta$, we have $K_{\delta-2}Q_\beta = E_{\beta, \delta-3}$. Take $\alpha \in \Delta(\beta)$ so that $W_\alpha \cap W_{\beta+1} = V_\beta$. Then $[T_\alpha, U_{\delta-2}] = 1$ by (8.15), and hence $V_\delta \not\subseteq U_\beta$ by (8.12)–(1). Thus $V_{\delta-2} \neq V_\delta$, and so $G_{\delta-1} = \langle G_{\delta-2, \delta-1}, G_{\delta-1, \delta} \rangle$ by (8.3)–(1) and (3.10)–(2), so $G_{\delta-1, \delta} = (Q_{\delta-2} \cap Q_{\delta-1})Q_\delta$ by (3.2)–(3). Therefore $T_{\delta-3}Q_\delta \triangleleft G_{\delta-1, \delta}$, and hence $|T_{\delta-1} : T_{\delta-1} \cap Q_\beta| = 2^2$, contrary to (8.16).

PROPOSITION 8.18 Let $\eta \in \mathcal{B}$. Then $n(G_\eta, K_\eta/U_\eta) \leq 2$ and $(G_\eta, K_\eta/U_\eta) \ll (O_4^+(2), GF(2)^4)$.

PROOF. Suppose false.

(1) $K_\beta \not\subseteq Q_{\delta-2}$ for all $(\beta, \delta) \in \mathcal{C}_0'$.

PROOF. Suppose $K_\beta \subseteq Q_{\delta-2}$ for some $(\beta, \delta) \in \mathcal{C}_0'$. Then $K_\beta \subseteq C_{Q_{\delta-2}}(U_{\delta-2}) \subseteq Q_{\delta-1}$ by (8.11)–(1) and (8.5). Hence (8.18) holds by (8.14).

(2) $|K_\beta : H| \cdot |[H, U_\delta]U_\beta : U_\beta| \geq 2^3$ for all $(\beta, \delta) \in \mathcal{C}_0'$ and all U_δ -invariant subgroups H between U_β and K_β .

PROOF. Suppose false. Then $n(G_\beta, K_\beta/U_\beta) \leq 2$ by (2.11)–(2), and U_δ induces a factorwise quasitransvection on K_β/U_β . Thus (8.18) holds by (2.11)–(1) because G_β is a $\{2, 3\}$ -group and $U_\delta Q_\beta \subseteq Z_{\beta, \beta+1}$ by (8.11)–(4).

(3) There exist $(\beta, \delta) \in \mathcal{C}_0'$ and $\alpha \in \Delta(\beta)$ such that

$$T_\alpha \not\subseteq Q_{\delta-2} \quad \text{and} \quad G_\beta = \langle G_{\alpha\beta}, G_{\beta, \beta+1} \rangle.$$

PROOF. Suppose false. Let $(\beta, \delta) \in \mathcal{C}_0'$. We can take, by (1), $\alpha \in \Delta(\beta)$ so that $T_\alpha \not\subseteq Q_{\delta-2}$. Then we have $G_\beta \neq \langle G_{\alpha\beta}, G_{\beta, \beta+1} \rangle$ by our present hypothesis, and so $V_\alpha = V_{\beta+1}$ by (8.3)–(2) and (3.10)–(2). Take $\theta \in \Delta(\alpha)$ so that $U_\theta \not\subseteq Q_{\delta-2}$. Then $(\theta, \delta-2) \in \mathcal{C}_0'$. Therefore, changing notation, there exists $(\beta, \delta) \in \mathcal{C}_0'$ such that

$$U_\beta \supseteq V_{\beta+1} = V_{\beta+3} = \cdots = V_{\delta-3} = V_{\delta-1} \supseteq V_\delta.$$

Now, let $(\beta, \delta) \in \mathcal{C}_0'$ and $\alpha \in \Delta(\beta)$ so that $U_\beta \supseteq V_\delta$ and $G_\beta = \langle G_{\alpha\beta}, U_\delta \rangle$.

Then $T_\alpha \subseteq Q_{\delta-2}$ by our hypothesis, and hence $T_\alpha \subseteq C_{Q_{\delta-2}}(U_{\delta-2}) \subseteq Q_{\delta-1}$ by the minimality of b and (8.11)–(1). Thus $T_\alpha \triangleleft G_\beta$ by (8.12)–(1), a contradiction.

Now, take $\alpha \in \Delta(\beta)$ and $\theta \in \Delta(\alpha)$ so that

$$G_\beta = \langle G_{\alpha\beta}, G_{\beta, \beta+1} \rangle \quad \text{and} \quad U_\theta \not\subseteq Q_{\delta-2}.$$

$$(4) \quad |K_{\delta-2} : K_{\delta-2} \cap Q_\beta| = 2^2.$$

PROOF. Suppose $|K_{\delta-2} : K_{\delta-2} \cap Q_\beta| = 2$. Since $V_\beta \not\subseteq [T_\alpha, U_{\delta-2}] \subseteq W_\alpha \cap U_{\delta-2}$, and since $V_\beta \subseteq U_{\delta-2}$ if $K_{\delta-2} \cap Q_\beta \not\subseteq Q_\alpha$, we have

$$|K_{\delta-2} : K_{\delta-2} \cap Q_\alpha| \cdot |W_\alpha : W_\alpha \cap U_{\delta-2}| \leq 2^3.$$

Moreover, we have $[U_\theta, K_{\delta-2} \cap Q_\alpha] \subseteq W_\alpha$ because $K_{\delta-2} \cap Q_\alpha \subseteq E_{\theta\alpha}$ by (8.11)–(6), so

$$V_\alpha \subseteq W_\alpha = [U_\theta, K_{\delta-2} \cap Q_\alpha](W_\alpha \cap U_{\delta-2}) \subseteq K_{\delta-2},$$

by (2) and the above inequality. Thus $K_{\delta-2} \cap Q_\beta \subseteq Q_\alpha$, and hence $[U_\theta, K_{\delta-2} \cap Q_\beta] \subseteq [U_\theta, U_{\delta-2}][U_\theta \cap Q_{\delta-2}, K_{\delta-2} \cap Q_\beta]V_\theta \subseteq U_{\delta-2}V_\theta$.

Consequently, we have

$$|K_{\delta-2} : K_{\delta-2} \cap Q_\beta| \cdot |[U_\theta, K_{\delta-2} \cap Q_\beta]U_{\delta-2} : U_{\delta-2}| \leq 2 \cdot 2 = 2^2,$$

contrary to (2).

Let $L_\eta = \cap T_\xi^{G_\eta}$ for $\langle \xi, \eta \rangle \in E(\Gamma)$.

$$(5) \quad |T_\xi^{(1)}L_\eta : L_\eta| \leq 2.$$

PROOF. We may assume $\eta = \beta$, and $W_\xi \cap W_{\beta+1} = V_\beta$. Let $x \in U_\delta - Q_\beta$ and put $M = T_\xi^{(1)}L_\beta$. By (4) and (8.15), $[M, U_{\delta-2}] \subseteq [T_\xi, U_{\delta-2}] = 1$, and so $M[M, U_\delta] \subseteq MV_\delta$ by (8.12)–(1). Therefore we have $|M : M \cap M^x| \leq 2$. Since $[T_\xi^{(1)}, Q_\xi] \subseteq W_\xi \subseteq U_\beta$ and $G_\beta = \langle G_{\xi\beta}, x \rangle = \langle Q_\xi, Q_\xi^x \rangle$ by (8.9)–(1) and (3.2)–(3), we have $[M \cap M^x, G_\beta] \subseteq U_\beta$, and so $M \cap M^x \subseteq L_\beta$. Hence we have $|T_\xi^{(1)}L_\beta : L_\beta| \leq 2$.

$$(6) \quad O^2(G_\xi/C_{G_\xi}(T_\xi^{(1)}/W_\xi)) \cong \mathbf{Z}_3 \text{ for } \xi \in \mathcal{A}.$$

PROOF. This follows from (2.11)–(1) because G_ξ is a $\{2, 3\}$ -group with cyclic Sylow 3-subgroups and $[Q_\eta, O^2(G_\eta)]$ induces a transvection on $T_\xi^{(1)}/W_\xi$ by (5) and (3.8), where $\eta \in \Delta(\xi)$.

$$(7) \quad |K_\beta : K_\beta \cap Q_{\delta-2}| = 2.$$

PROOF. Suppose $U_\theta \cap T_\alpha^{(1)} = U_\beta \cap T_\alpha^{(1)}$. Then we have $[[U_\theta, G_{\theta\alpha}], U_{\delta-2}] = [U_\theta \cap T_\alpha^{(1)}, U_{\delta-2}] \subseteq [U_\beta, U_{\delta-2}] = 1$ by the minimality of b . Hence $U_{\delta-2} \subseteq Q_\theta$ by

(2.5)–(9), and so $U_\theta \subseteq C_{Q_{\delta-3}}(U_{\delta-2}) \subseteq Q_{\delta-2}$, a contradiction. Thus $U_\theta \cap T_\alpha^{(1)} \neq U_\beta \cap T_\alpha^{(1)}$, and so $G_\alpha = \langle G_{\theta\alpha}, G_{\alpha\beta} \rangle$ by (6) and (3.10)–(2), and then $V_\alpha = V_\theta V_\beta$.

Now, suppose $|K_\beta : K_\beta \cap Q_{\delta-2}| = 2^2$. Let $\lambda \in \Delta(\delta-2)$ so that $W_{\delta-3} \cap W_\lambda = V_{\delta-2}$. Then, by (8.15), $[U_\beta, T_\lambda] = 1$, and so $T_\lambda \subseteq Q_\alpha$. Since $T_\lambda \not\leq G_{\delta-2} = \langle U_\theta, G_{\delta-2, \lambda} \rangle$, we have $V_\theta \subseteq K_{\delta-2}$ by (8.12)–(1), and so $[V_\alpha, K_{\delta-2}] = 1$. As $G_\beta = \langle G_{\alpha\beta}, G_{\beta, \beta+1} \rangle = \langle Q_\alpha, K_{\delta-2} \rangle$ by (4) and the choice of α , $V_\alpha \subseteq Z(G_\beta)$, a contradiction.

(8) *A contradiction.*

PROOF. If $G_{\delta-2} = \langle G_{\delta-3, \delta-2}, G_{\delta-2, \delta-1} \rangle$, then, for $(\delta-2, \theta) \in \mathcal{C}_0'$ and $\delta-1 \in \Delta(\delta-2)$, we have $|K_\beta : K_\beta \cap Q_{\delta-2}| = 2^2$ by (4), contrary to (7). Thus $G_{\delta-2} \neq \langle G_{\delta-3, \delta-2}, G_{\delta-2, \delta-1} \rangle$, and hence $W_{\delta-3} = W_{\delta-1}$ by (3.10)–(1). Therefore $[K_\beta \cap Q_{\delta-2}, W_{\delta-1}] = 1$, and so

$$[K_\beta \cap Q_{\delta-2}, U_\delta] \subseteq [U_\beta, U_\delta][K_\beta \cap Q_{\delta-2}, U_\delta \cap Q_\beta]V_\delta \subseteq U_\beta V_\delta.$$

Thus $|K_\beta : K_\beta \cap Q_{\delta-2}| \cdot |[K_\beta \cap Q_{\delta-2}, U_\delta]U_\beta : U_\beta| \leq 2 \cdot 2 = 2^2$, contrary to (2).

COROLLARY 8.19 *Let $\xi \in \mathcal{A}$. Then $O^2(G_\xi/C_{G_\xi}(T_\xi^{(1)}/W_\xi)) \cong \mathbf{Z}_3$.*

PROOF. Define $L_\eta = \cap T_\xi^{G_\eta}$ for $\eta \in \Delta(\xi)$. Then, by (8.18) and (2.15), we have

$$|T_\xi^{(1)}L_\eta : L_\eta| \leq |C_{K_\eta/L_\eta}(Q_\xi)| \leq 2^2,$$

so $[Q_\eta, O^2(G_\eta)]$ induces a factorwise quasitransvection on $T_\xi^{(1)}/W_\xi$ by (3.8). Hence the assertion follows from (2.11)–(1).

(8.20) *Let $(\beta, \delta) \in \mathcal{C}_0'$ and suppose $|K_\beta : K_\beta \cap Q_{\delta-2}| = 2$. Then either $V_\delta \subseteq K_\beta$ or $V_{\delta-2} \subseteq U_\beta$.*

PROOF. Take $\alpha, \alpha' \in \Delta(\beta)$ so that

$$W_\alpha \cap W_{\beta+1} = W_{\alpha'} \cap W_{\beta+1} = W_\alpha \cap W_{\alpha'} = V_\beta.$$

If $T_\alpha \subseteq Q_{\delta-2}$, then, since $G_\beta = \langle G_{\alpha\beta}, U_\delta \rangle \not\leq T_\alpha \subseteq Q_{\delta-1}$, $V_\delta \subseteq K_\beta$ by (8.12)–(1). Thus we may assume that $T_\alpha \not\subseteq Q_{\delta-2} \not\subseteq T_{\alpha'}$. Hence

$$W_\alpha \supseteq [T_\alpha, U_{\delta-2}] \not\subseteq V_\beta \not\subseteq [T_{\alpha'}, U_{\delta-2}] \subseteq W_{\alpha'},$$

so $[T_\alpha, U_{\delta-2}] \neq [T_{\alpha'}, U_{\delta-2}]$. Since $K_\beta Q_{\delta-2} = U_\theta Q_{\delta-2}$ for $\theta \in \Delta(\alpha)$ with $U_\theta \not\subseteq Q_{\delta-2}$, we have $|[K_\beta, U_{\delta-2}]| \leq |[U_\theta, U_{\delta-2}]V_{\delta-2}| = |U_\theta : U_\theta \cap Q_{\delta-2}| \cdot |V_{\delta-2}| = |K_\theta : K_\theta \cap Q_{\delta-2}| \cdot |V_{\delta-2}| \leq 2^2$ by (8.11)–(2), so $V_{\delta-2} \subseteq [K_\beta, U_{\delta-2}] = [T_\alpha, U_{\delta-2}][T_{\alpha'}, U_{\delta-2}] \subseteq W_\alpha W_{\alpha'} = U_\beta$.

PROPOSITION 8.21 *Let $\eta \in \mathcal{B}$. Then $n(G_\eta, K_\eta/U_\eta) = 1$ and $(G_\eta, K_\eta/U_\eta) \ll (O_4^+(2), GF(2)^4)$.*

PROOF. It suffices to prove that $n(G_\eta, K_\eta/U_\eta) = 1$ by (8.18). Assume otherwise.

(1) $K_\beta \not\subseteq Q_{\delta-2}$ for all $(\beta, \delta) \in \mathcal{C}_0'$.

PROOF. This follows from (8.14) as in the proof of (1) of (8.18).

(2) $K_\beta \not\subseteq V_\delta$ for all $(\beta, \delta) \in \mathcal{C}_0'$.

PROOF. Suppose $K_\beta \supseteq V_\delta$ for some $(\beta, \delta) \in \mathcal{C}_0'$. Since $U_{\delta-2} \cap T_{\delta-1}^{(1)} \neq U_\delta \cap T_{\delta-1}^{(1)}$ as in the proof of (7) of (8.18), $G_{\delta-1} = \langle G_{\delta-2, \delta-1}, G_{\delta-1, \delta} \rangle$ by (8.19) and (3.10)–(2), and so $V_{\delta-1} = V_{\delta-2} V_\delta$. Therefore K_β centralizes $W_{\delta-1} = V_{\delta-2} V_\delta [U_\beta, U_\delta]$, so $G_{\delta-2} \neq \langle K_\beta, G_{\delta-2, \delta-1} \rangle$. Thus, by (8.18) and (3.10)–(1), there exists $\lambda \in \Delta(\delta-2)$ such that $K_\beta \subseteq G_{\delta-2, \lambda}$, $W_\lambda = W_{\delta-1}$, and $T_\lambda = T_{\delta-1} \not\subseteq Q_\beta$. Take $\mu \in \Delta(\lambda)$ so that $U_\mu \not\subseteq Q_\beta$. Since $K_\beta \subseteq C_{G_{\delta-2}}(W_\lambda) \subseteq Q_\lambda$, we have $n(G_\beta, K_\beta/U_\beta) = 1$ by (8.14), a contradiction.

(3) *A contradiction.*

PROOF. Take $(\beta, \delta) \in \mathcal{C}_0'$, $\alpha \in \Delta(\beta)$, and $\theta \in \Delta(\alpha)$ so that $(\theta, \delta-2) \in \mathcal{C}_0'$.

Suppose $|K_\beta : K_\beta \cap Q_{\delta-2}| = 2^2$, and choose $\lambda \in \Delta(\delta-2)$ so that $W_{\delta-3} \cap W_\lambda = V_{\delta-2}$. Then $[U_\beta, T_\lambda] = 1$ by (8.15), so $V_\theta \subseteq K_{\delta-2}$ by (8.12)–(1), which contradicts (2).

Therefore we have $|K_\beta : K_\beta \cap Q_{\delta-2}| = 2$, and hence either $V_\delta \subseteq K_\beta$ or $V_{\delta-2} \subseteq U_\beta \subseteq K_\theta$ by (8.20), which also contradicts (2).

Define $L_\eta = \cap T_\xi^{G_\eta}$ and $N_\eta = C_{K_\eta \bmod L_\eta}(O^2(G_\eta))$ for $\langle \xi, \eta \rangle \in E(\Gamma)$.

COROLLARY 8.22 *The following holds.*

- (1) $L_\eta = T_\xi \cap N_\eta \supseteq [K_\eta, Q_\eta]$.
- (2) $2^4 = |K_\eta/N_\eta| \leq |K_\eta/L_\eta| \leq 2^6$.
- (3) $T_\xi^{(1)} N_\eta/N_\eta = C_{K_\eta/N_\eta}(G_{\xi\eta})$ and $|T_\xi^{(1)} N_\eta/N_\eta| = 2$.
- (4) $T_\xi N_\eta/N_\eta = [K_\eta/N_\eta, E_{\xi\eta}] = C_{K_\eta/N_\eta}(E_{\xi\eta})$ and $|T_\xi N_\eta/N_\eta| = 2^2$.
- (5) $O^2(G_\xi/Q_\xi C_{G_\xi}(T_\xi)) \cong \mathbf{Z}_3$.
- (6) *The following four conditions for $x \in O^2(G_\eta)$ are equivalent.*
 - (i) $C_{U_\eta/V_\eta}(x) = 1$.
 - (ii) $C_{K_\eta/U_\eta}(x) = 1$.
 - (iii) $W_\xi \cap W_\xi^x = V_\eta$.
 - (iv) $T_\xi N_\eta \cap T_\xi^x N_\eta = N_\eta$.
- (7) *Let $x \in O^2(G_\eta)$. If $C_{K_\eta/U_\eta}(x) = 1$, then $K_\eta = \langle T_\xi^{(x)} \rangle$.*

PROOF. Part (1) follows from (8.21), (2.14)–(1), and the definition of L_η and

N_η , and then (8.21) and (2.4)–(3) show that $|K_\eta/N_\eta| = 2^4$. Since $T_\xi^{(2)} \subseteq W_\xi \subseteq L_\eta \not\subseteq T_\xi^{(1)}$ by (8.13), (8.4), and (3.8), $T_\xi^{(1)}N_\eta/N_\eta = C_{K_\eta/N_\eta}(G_{\xi\eta})$, $|T_\xi^{(1)}N_\eta/N_\eta| = 2$, and $|T_\xi/L_\eta| = |T_\xi N_\eta/N_\eta| = 2^2$ by (2.5)–(3). Therefore $|K_\eta/L_\eta| \leq 2^4 \cdot |T_\xi/L_\eta| = 2^6$ by (3.9)–(1). Since $E_{\xi\eta} = C_{Q_\xi}(W_\xi)Q_\eta$ and $[T_\xi, C_{Q_\xi}(W_\xi)] \subseteq W_\xi \subseteq N_\eta$ by (8.5), (2.5)–(4) shows that $T_\xi N_\eta/N_\eta = [K_\eta/N_\eta, E_{\xi\eta}] = C_{K_\eta/N_\eta}(E_{\xi\eta})$, proving (2)–(4).

Since $|T_\xi : T_\xi^{(1)}L_\eta| = |T_\xi^{(1)}U_\eta : T_\xi^{(1)}| = 2$ and $[L_\eta, [Q_\eta, O^2(G_\eta)]] \subseteq U_\eta$, $[Q_\eta, O^2(G_\eta)]$ induces a quasitransvection on $T_\xi/T_\xi^{(1)}$, and hence $O^2(G_\xi/C_{G_\xi}(T_\xi/T_\xi^{(1)})) \cong \mathbf{Z}_3$. Thus, together with (8.19) and (8.2)–(1), we have $O^2(G_\xi/C_{G_\xi}(T_\xi)) \cong \mathbf{Z}_3$, so (5) holds.

Now, let $x \in O^2(G_\eta)$. Define $D = C_{G_\eta}(U_\eta/V_\eta) = C_{G_\eta}(K_\eta/U_\eta)$ and $H = E_{\xi\eta}O^2(G_\eta)$. Then any of the four conditions in (6) is equivalent to the condition that $\langle x \rangle D \not\triangleleft H$ by (2.5)–(11), so (6) follows. Suppose $C_{K_\eta/N_\eta}(x) = 1$, and let $M = \langle T_\xi^{(x)} \rangle$. Since $K_\eta = MN_\eta$ and $\langle x \rangle D \triangleleft O^2(G_\eta)D$, $[K_\eta, O^2(G_\eta)]L_\eta = [K_\eta, x]L_\eta \subseteq M$, so $K_\eta = T_\xi[K_\eta, O^2(G_\eta)] = M$, proving (7).

We will now derive a final contradiction.

(1) *There exists $(\beta, \delta) \in \mathcal{C}_0'$ such that*

$$W_{\beta+1} \cap W_{\beta+3} = V_{\beta+2}, \quad W_{\delta-3} \cap W_{\delta-1} = V_{\delta-2}, \quad \text{and} \\ \text{either } [K_\beta, U_{\delta-2}] = 1 \quad \text{or} \quad [K_\delta, U_{\beta+2}] = 1.$$

PROOF. Let $(\beta, \delta) \in \mathcal{C}_0'$. Since $K_{\beta+2} \not\subseteq Q_\delta$, there exists $\xi \in \Delta(\beta+2)$ such that $T_\xi \not\subseteq Q_\delta$ and $W_\xi \cap W_{\beta+3} = V_{\beta+2}$ by (8.22)–(6)(7). Take $\eta \in \Delta(\xi)$ so that $U_\eta \not\subseteq Q_\delta$. Then $(\eta, \delta) \in \mathcal{C}_0'$. Similarly, we can take $\lambda \in \Delta(\delta-2)$ and $\mu \in \Delta(\lambda)$ such that $W_{\delta-3} \cap W_\lambda = V_{\delta-2}$ and $(\eta, \mu) \in \mathcal{C}_0'$.

Changing notation, let $(\beta, \delta) \in \mathcal{C}_0'$ so that $W_{\beta+1} \cap W_{\beta+3} = V_{\beta+2}$ and $W_{\delta-3} \cap W_{\delta-1} = V_{\delta-2}$. Suppose

$$[K_\beta, U_{\delta-2}] \neq 1 \neq [K_\delta, U_{\beta+2}].$$

Since $K_\delta \not\subseteq Q_{\beta+2}$ by (8.11)–(1), we can take, as above, $\lambda \in \Delta(\delta)$ and $\mu \in \Delta(\lambda)$ so that $W_{\delta-1} \cap W_\lambda = V_\delta$ and $(\beta+2, \mu) \in \mathcal{C}_0'$. Therefore $G_{\delta-2} = \langle G_{\delta-3, \delta-2}, G_{\delta-2, \delta-1} \rangle = \langle K_\beta, Q_{\delta-1} \rangle$ by (8.9)–(1) because $Q_{\delta-2} \not\subseteq K_\beta \subseteq E_{\delta-3, \delta-2}$. Note that $V_{\delta-1} = V_{\delta-2}V_\delta$ by (3.10)–(2) and (8.22)–(5) as $U_{\delta-2} \neq U_\delta$. If $V_\delta \subseteq K_\beta$, then $[K_\beta, V_{\delta-1}] = 1$, so $[G_{\delta-2}, V_{\delta-1}] = [K_\beta, Q_{\delta-1}, V_{\delta-1}] = 1$, a contradiction. Thus $V_\delta \not\subseteq K_\beta$. If $[U_{\beta+2}, U_\mu] \subseteq W_{\delta-1}$, then $[U_{\delta+2}, U_\mu] \subseteq W_{\delta-1} \cap W_\lambda = V_\delta \subseteq V_\lambda$, which contradicts (2.5)–(9). Thus $|U_\delta : W_{\delta-1}[U_{\beta+2}, U_\mu]| \leq 2$. Since $K_{\beta+2}$ centralizes $W_{\delta-1}[U_{\beta+2}, U_\mu]$, we have $|K_{\beta+2} : K_{\beta+2} \cap Q_\delta| \leq 2$, and hence $K_{\beta+2} \subseteq Q_\delta$ by (8.20) applied to $(\beta+2, \mu) \in \mathcal{C}_0'$. Therefore $[K_{\beta+2}, U_\delta] = 1$ by (8.11)–(1).

Now, let $(\beta, \delta) \in \mathcal{C}_0'$ so that

$$W_{\beta+1} \cap W_{\beta+3} = V_{\beta+2}, \quad W_{\delta-3} \cap W_{\delta-1} = V_{\delta-2}, \quad \text{and} \quad [K_\beta, U_{\delta-2}] = 1.$$

$$(2) \quad T_{\beta+1} \not\subseteq V_\delta \subseteq K_\beta.$$

PROOF. By (8.12)–(1), we have $V_\delta \subseteq K_\beta$. Suppose $V_\delta \subseteq T_{\beta+1}$. Then $[K_\beta, U_\delta] \subseteq [U_\beta, U_\delta][K_\beta, U_\delta \cap Q_\beta]V_\delta \subseteq T_{\beta+1}$, and so $[K_\beta, E_{\beta, \beta+1}] \subseteq [K_\beta, \langle U_\delta^{G_{\beta, \beta+1}} \rangle Q_\beta] \subseteq T_{\beta+1}$ by (8.11)–(4), (8.21), and (3.14)–(2). Thus $K_{\beta+2} = T_{\beta+1}T_{\beta+3} = T_\lambda T_{\beta+3}$ for $\lambda \in \Delta(\beta+2)$ with $W_{\beta+1} \cap W_\lambda = W_\lambda \cap W_{\beta+3} = V_{\beta+2}$ because $\langle E_{\beta+1, \beta+2}, E_{\beta+2, \beta+3} \rangle \supseteq O^2(G_{\beta+2}) \subseteq \langle E_{\lambda, \beta+2}, E_{\beta+2, \beta+3} \rangle$ by (2.5)–(11). Therefore $[U_\beta, U_\delta] \subseteq [K_{\beta+2}, U_\delta] = [T_{\beta+1}, U_\delta] = [T_\lambda, U_\delta] \subseteq W_{\beta+1} \cap W_\lambda = V_{\beta+2} \subseteq V_{\beta+1}$, so $[U_\beta, U_\delta] = 1$, a contradiction.

$$(3) \quad [U_\beta, K_{\delta-2} \cap Q_\beta] = 1.$$

PROOF. Suppose false. Then $V_\beta = [U_\beta, K_{\delta-2} \cap Q_\beta] \subseteq U_{\delta-2} \subseteq T_{\delta-1}$. Thus $[K_\delta, V_{\beta+1}] = [K_\delta, V_\beta V_{\beta+2}] = 1$, and so $K_\delta \subseteq Q_{\beta+2}$ by (8.9)–(1), and then $[K_\delta, U_{\beta+2}] = 1$ by (8.11)–(1). However, applying (2) to $(\delta, \beta) \in \mathcal{C}_0'$, we have $V_\beta \not\subseteq T_{\delta-1}$, a contradiction.

$$(4) \quad \text{Let } \lambda \in \Delta(\delta-2), \text{ and assume } T_\lambda \not\subseteq Q_\beta. \text{ Then}$$

$$[K_\beta, T_\lambda]U_\beta = U_\beta W_\lambda \quad \text{and} \quad |U_\beta W_\lambda : U_\beta| = 2.$$

PROOF. Let $\mu, \mu' \in \Delta(\lambda)$ so that $T_\lambda = \langle U_{\delta-2}, U_\mu, U_{\mu'} \rangle$. Since $[K_{\delta-2}, Q_{\delta-2}] \subseteq L_{\delta-2} \subseteq T_\lambda \cap Q_\beta$, and U_μ and $U_{\mu'}$ are conjugate in $Q_{\delta-2}$, $U_\mu \not\subseteq Q_\beta \not\subseteq U_{\mu'}$. Thus $T_\lambda = (T_\lambda \cap Q_\beta)U_\mu = (T_\lambda \cap Q_\beta)U_{\mu'}$ by (8.16), so

$$[K_\beta, T_\lambda]U_\beta = [K_\beta, U_\mu]U_\beta = [K_\beta, U_{\mu'}]U_\beta$$

by (3) and (8.5). Therefore,

$$\begin{aligned} [K_\beta, U_\mu]U_\beta &= U_\beta [U_\beta, U_\mu][K_\beta, U_\mu \cap Q_\beta]V_\mu = U_\beta V_\mu \quad \text{and, similarly,} \\ [K_\beta, U_{\mu'}]U_\beta &= U_\beta V_{\mu'}. \end{aligned}$$

Hence $[K_\beta, T_\lambda]U_\beta = U_\beta V_\mu V_{\mu'} = U_\beta [U_\beta, U_\mu]V_\lambda = U_\beta W_\lambda$ by (8.9)–(1) and $|U_\beta W_\lambda : U_\beta| = |U_\beta V_\mu : U_\beta| = 2$.

$$(5) \quad \text{Let } \lambda_1, \lambda_2, \lambda_3 \in \Delta(\delta-2) \text{ so that}$$

$$W_{\lambda_i} \cap W_{\lambda_j} = V_{\delta-2} \quad (i \neq j).$$

Then precisely two of $T_{\lambda_1}, T_{\lambda_2}, T_{\lambda_3}$ are contained in Q_β . In particular, $|K_{\delta-2} : K_{\delta-2} \cap Q_\beta| = 2$.

PROOF. Note that $K_{\delta-2} = \langle T_{\lambda_1}, T_{\lambda_2}, T_{\lambda_3} \rangle$, and suppose $T_{\lambda_1} \not\subseteq Q_\beta \not\subseteq T_{\lambda_2}$. We may assume that $K_{\delta-2} = (K_{\delta-2} \cap Q_\beta)T_{\lambda_1}T_{\lambda_2}$. Hence we have

$$U_\beta U_{\delta-2} \supseteq [K_\beta, K_{\delta-2}]U_\beta \supseteq [K_\beta, T_{\lambda_1}][K_\beta, T_{\lambda_2}]U_\beta = U_\beta W_{\lambda_1} W_{\lambda_2} = U_\beta U_{\delta-2}$$

by (3).

If $|K_{\delta-2} : K_{\delta-2} \cap Q_\beta| = 2^2$, then $|[K_\beta, K_{\delta-2}]U_\beta : U_\beta| \leq 2^2$ by (4), so $|U_\beta : U_\beta \cap U_{\delta-2}| = |U_\beta U_{\delta-2} : U_\beta| \leq 2^2$. If $|K_{\delta-2} : K_{\delta-2} \cap Q_\beta| = 2$, then $|[K_\beta, K_{\delta-2}]U_\beta : U_\beta| \leq 2$ by (3) and (4) as $K_{\delta-2} = (K_{\delta-2} \cap Q_\beta)T_{\lambda_1}$, so $|U_\beta : U_\beta \cap U_{\delta-2}| = |U_\beta U_{\delta-2} : U_\beta| \leq 2$. Therefore $|U_\beta/V_\beta : C_{U_\beta/V_\beta}(K_{\delta-2})| \leq |U_\beta : U_\beta \cap Q_{\delta-2}| \leq |K_{\delta-2} : K_{\delta-2} \cap Q_\beta|$, so $V_\beta \subseteq W_{\beta+1} \subseteq C_{U_\beta}(K_{\delta-2}) = U_\beta \cap U_{\delta-2} \subseteq T_{\delta-1}$ by (2.5)–(5), which is a contradiction as in (3).

Now, choose $\lambda \in \Delta(\delta-2)$ so that

$$W_{\delta-3} \cap W_\lambda = W_\lambda \cap W_{\delta-1} = W_{\delta-3} \cap W_{\delta-1} = V_{\delta-2}.$$

Then, since $|T_{\delta-1} : T_{\delta-1} \cap Q_\beta| = 2$ by (8.16), we have $T_\lambda \subseteq Q_\beta$, so $|K_{\delta-2} : K_{\delta-2} \cap Q_\beta| = 2$.

$$(6) \quad |K_{\delta-2} : L_{\delta-2}| = 2^6 \text{ and } [N_{\delta-2}, G_{\delta-2}]L_{\delta-2} \not\subseteq Q_\beta.$$

PROOF. Suppose false. Let $H = K_{\delta-2} \cap Q_\beta$ and $\tilde{K}_{\delta-2} = K_{\delta-2}/[N_{\delta-2}, G_{\delta-2}]L_{\delta-2}$. Then $|\tilde{K}_{\delta-2}| \leq 2^5$ and $|\tilde{K}_{\delta-2} : \tilde{H}| = 2$. We see that this contradicts (5), applying (2.6) for $\tilde{K}_{\delta-2}$ and $\tilde{H}$.

Let $M_\xi = \cap K_\eta^{G_\xi}$ and $F_\xi = \langle K_\eta^{G_\xi} \rangle$ for $\langle \xi, \eta \rangle \in E(\Gamma)$.

$$(7) \quad M_\xi = T_\xi \text{ for } \xi \in \mathcal{A}.$$

PROOF. We may assume $\xi = \delta-3$. Suppose $M_{\delta-3} \neq T_{\delta-3}$. Then $|M_{\delta-3} : T_{\delta-3}| = 2$ by (3.9)–(3), and hence $[N_{\delta-2}, G_{\delta-2}]L_{\delta-2} = M_{\delta-3} \cap N_{\delta-2} \subseteq Q_\beta$, which contradicts (6).

$$(8) \quad \text{A contradiction.}$$

PROOF. Let $\eta \in \Delta(\beta+1)$ so that $U_\beta \neq U_\eta \neq U_{\beta+2}$. Since U_β and U_η are conjugate in $Q_{\beta+2}$ and $[K_{\beta+2}, Q_{\beta+2}] \subseteq L_{\beta+2} \subseteq Q_\delta$, we have $U_\eta \not\subseteq Q_\delta$, and, moreover, $T_{\beta+1} = K_\beta \cap K_{\beta+2} = K_\beta \cap K_\eta$ by (7) and (8.22)–(1)(5).

Suppose $b = 7$. Then $T_{\beta+1} \not\subseteq V_\delta \subseteq K_\beta \cap K_{\beta+2} = T_{\beta+1}$ by (2), a contradiction.

Suppose $b \geq 9$. Then $F_{\beta+1}$ is abelian, so $[K_\eta, V_\delta] \subseteq [F_{\beta+1}, V_\delta] = 1$. Thus $[K_\eta, U_{\delta-2}] = 1$ by (8.9)–(1) and (8.11)–(1). Therefore, applying (2) for $(\eta, \delta) \in \mathcal{C}_0'$, we have $V_\delta \subseteq K_\eta$, so $T_{\beta+1} \not\subseteq V_\delta \subseteq K_\beta \cap K_\eta = T_{\beta+1}$, also a contradiction.

ACKNOWLEDGEMENT. I would like to express my thanks to Professor Kensaku Gomi for his valuable advices and for showing me his unpublished results in Section 3 and in the second case of Section 7.

References

- [1] M. Aschbacher, $GF(2)$ -representation of finite groups, *Amer. J. Math.*, **104** (1982), 683–771.
- [2] A. Delgado, D.M. Goldshmidt, and B. Stellmacher, *Groups and Graphs: New Results and Methods*, Birkhäuser Verlag, Basel, Boston, Stuttgart, 1985.
- [3] K. Gomi, A note on the thin finite simple groups, *J. Algebra*, **102** (1986), 194–198.
- [4] K. Gomi, A pushing up theorem for groups of characteristic 2 type, *Sci. Papers College Arts Sci. Univ. Tokyo*, **37** (1987), 73–102.
- [5] K. Gomi, Characteristic pairs for 2-groups, *J. Algebra*, **94** (1985), 488–510.
- [6] K. Gomi and M. Hayashi, A pushing-up approach to the quasithin simple finite groups with solvable 2-local subgroups, preprint.
- [7] K. Gomi and Y. Tanaka, On pairs of groups having a common 2-subgroup of odd indices I–II, *Sci. Papers College Arts Sci. Univ. Tokyo*, **35** (1985), 11–30; **40** (1990), 37–47.
- [8] D. Gorenstein and R. Lyons, Nonsolvable finite groups with solvable 2-local subgroups, *J. Algebra*, **38** (1976), 453–522.
- [9] Z. Janko, Nonsolvable finite groups all of whose 2-local subgroups are solvable, *J. Algebra*, **21** (1972), 458–517.
- [10] F. Smith, Finite simple groups all of whose 2-local subgroups are solvable, *J. Algebra*, **34** (1975), 481–520.
- [11] Y. Tanaka, Certain pairs of finite groups with a common 2-subgroup, *J. Fac. Sci. Univ. Tokyo Sect. IA Math.*, **33** (1986), 247–272.
- [12] J.G. Thompson, Nonsolvable finite groups all of whose local subgroups are solvable I–VI, *Bull. Amer. Math. Soc.*, **74** (1968), 387–483; *Pacific J. Math.*, **33** (1970), 431–536; **39** (1971), 483–534; **48** (1973), 511–592; **50** (1974), 215–297; **51** (1974), 573–630.

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