On projective surfaces arising from an adjunction process

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Introduction

We will consider the set \( S = \{(S, L) \mid S \text{ is a smooth projective surface and } L \in \text{Pic}(S) \text{ is very ample}\} \). An adjunction process on the pair \((S, L)\) in \( S \) is the construction of a new pair \((S, H)\) in the following way: first contract all the \(-1\) rational curves \( C \) on \( S \) with \( L \cdot C = 1 \) in a new surface \( S', \pi : S' \to S \) (this is called reduction); then consider on \( S \) the line bundle \( H = K_S \otimes L \), where \( L = \pi_*L \). The pair \((S, L)\) is called the reduction of \((S, L)\).

If \((S, L)\) is not in a particular subset \( E \) of \( S \) then \((S, H)\) is again in \( S \); this theorem, with a complete description of the pairs in \( E \), is proved in [SV] and in [Se1] (see also [Io2]).

Therefore there is a map \( \mathcal{R}A : S \to E \) (where \( \mathcal{R}A \) stands for reduction + adjunction) which associates to a pair \((S, L)\) the pair \((S, H) = (S, K_S \otimes L)\).

The map \( \mathcal{R}A \) is not a surjection: since \( L \) is ample, by the Kodaira vanishing theorem, if \((S, H) \in \text{Im} \mathcal{R}A \), it must be that \( h^i(H) = 0 \), for \( i = 1, 2 \). The natural question is then to characterize the pairs in \( S \) which are in the image of \( \mathcal{R}A \).

In this paper we consider some particular subsets of \( S \) and we determine which pairs in these subsets are "adjunction pairs" in the above sense. We give a satisfactory answer for the subset \( T = \{(S, H)\} \subset S \) consisting of:

a) surfaces in \( \mathbb{P}^4 \), i.e. \( h^0(H) \leq 5 \), (section 1),

b) surfaces of degree less than or equal to 9, i.e. \( H^2 \leq 9 \) (section 3),

c) surfaces for which \( K_S^{S-1} \) is nef (section 2).

In case b) our results are complete for \( H^2 \leq 7 \) but there remain some doubtful cases for \( H^2 = 8, 9 \).

Our techniques heavily rely on the adjunction theory and the related classification results of surfaces of small sectional genus. In many instances they allow us to determine all the pairs \((S, L)\) from which a pair \((S, H) \in \text{Im} \mathcal{R}A \) comes via adjunction process.

We are indebted to the referee for his useful observations.
§0. Notation and preliminaries

0.1. We consider in this paper projective surfaces $S$, which means that $S$ is a smooth, irreducible, projective scheme of dimension 2, defined over the field of complex numbers.

We will use the standard symbols in algebraic geometry. In particular, if $D$ is a divisor on a surface $S$ we let:

- $O_S(D) =: \text{the invertible sheaf associated to } D$
- $h^i(D) = h^i(O_S(D)) =: \dim H^i(O_S(D))$
- $\chi(D) = \chi(O_S(D)) =: \sum_{i=0,1,2}(-1)^i h^i O_S(D)$
- $D^2 =: D \cdot D$ the self intersection of $D$
- $[D] =: \text{the line bundle associated to an effective divisor } D$
- $q(S) =: h^1(O_S)$, the irregularity of $S$
- $g(D) =: 1/2 (D^2 + D \cdot K_S) + 1$, the arithmetic genus of $D$
- $p_g(S) =: \dim H^0(O_S(K_S))$, the geometric genus of $S$
- $K_S =: \text{a canonical divisor on } S$
- $\kappa(S) =: \text{tr. deg}(\bigoplus_{n\geq 0} H^0(S, O_S(nK_S))) - 1$, the Kodaira dimension of $S$.

We do not distinguish rotationally between a divisor and its associated line bundle.

0.2. We briefly introduce some well known classes of surfaces which are frequently used in the paper.

We denote by $\mathbb{P}^n$ the $n$-dimensional projective space and by $O_{\mathbb{P}^n}(r)$ the $r$ th-power of the hyperplane bundle.

A surface $S$ is called a geometrically ruled surface if $S$ is a holomorphic $\mathbb{P}^1$ bundle, $p : S \to R$, over a non singular curve $R$. Equivalently, $p : S = \mathbb{P}(\mathcal{E}) \to R$ for some rank 2 locally free sheaf on $R$. We can choose $\mathcal{E}$ in such a way that there exists a section, $\sigma$, of $p$ such that: $\text{Num}(S) \approx \mathbb{Z}[\sigma] \oplus \mathbb{Z}[f]$, where $f$ is a fibre of $p$, and $f^2 = 0$, $f \cdot \sigma = 1$ and $\sigma^2 = -e = \deg \mathcal{E}$. ($\approx$ denotes numerical equivalence). For $r \geq 0$ we denote as $F_r$ the geometrically ruled surface $\mathbb{P}(O_{\mathbb{P}^1} \oplus O_{\mathbb{P}^1}(-r))$. $F_r$ is the unique holomorphic $\mathbb{P}^1$-bundle over $\mathbb{P}^1$ with a section $\sigma$ satisfying $\sigma^2 = -r$. For $r \geq 1$, $\sigma$ is unique.

A surface $S$ is called ruled if it is birational to a geometrically ruled one.

A polarized surface $(S, L)$ is called a scroll (respectively a conic bundle) if $S$ is a ruled surface and $O(L)_f = O_{\mathbb{P}^1}(1)$ for a general fibre $f$ (respectively if $O(L)_f = O_{\mathbb{P}^1}(2)$).

0.3. Given a surface $S$ we denote by $B(S, p_1, \ldots, p_n)$ the surface obtained by blowing up the points $p_i$ on $S$. Let $\pi : B(S, p_1, \ldots, p_n) \to S$ be the blowing-up.

Let $L$ be a line bundle on $S$. We denote by $|\pi^* L - \sum t_i p^{k_i}|$ the linear system
on $B(S, p_1, \ldots, p_n)$ whose elements are in a one to one correspondence with the elements of the linear system $|L|$ which pass through $t_i$ points of multiplicity $k_i$. Of course $\pi^*L - \sum t_ip_i^{k_i}$ will denote the line bundle associated to the above linear system.

Frequently the points $p_1, \ldots, p_n$ will be assumed to satisfy some general position assumption in order to insure the very ampleness of $\pi^*L - \sum t_ip_i^{k_i}$. In this case we let $B_n(S) = B(S, p_1, \ldots, p_n)$ for shortness. In case $S = \mathbb{F}_r$ or $\mathbb{P}^2$ for the explicit condition expressing the general position assumption with respect to a linear system see [Li, sec. 0].

0.4. As discussed in the introduction our aim is to study the image of the "adjunction process" in some particular case: namely given $(S, H)$ in $T \subset S$ we ask whether $H = KS \otimes L$ for some (ample) line bundle $L$ on $S$ such that $(S, L)$ is the reduction of a pair $(\tilde{S}, \tilde{L}) \in \mathcal{S}$; we will say that $(S, H)$ is in $\text{Im} \mathcal{R}A$. In particular if $L$ itself is very ample, i.e. $(S, L) \in \mathcal{S}$ then we will say that $(S, H)$ is in $\text{Im} \mathcal{A}$.

Apart from some special cases [So, (2.4)] it is not yet known whether $L$ is very ample and even if it is spanned by its global sections. So in our notation, it is not known if $\text{Im} \mathcal{R}A = \text{Im} \mathcal{A}$.

REMARK 0.4.1. We have two obvious necessary condition for $(S, H)$ to be in $\text{Im} \mathcal{R}A$. The first one is that

i) $L = H - KS$ is ample.

The second one is that, since the reduction is a birational map,

ii) there exists a pair $(\tilde{S}, \tilde{L}) \in \mathcal{S}$ such that $g(\tilde{L}) = g(L)$ and $S$ is birational to $\tilde{S}$.

Note that i) implies $h^i(H) = 0$, for $i = 1, 2$.

In the above hypothesis we will shorten our notation letting $\chi = \chi(\mathcal{O}_S)$, $p_g = p_g(S)$, $q = q(S)$, $g = g(L)$, $d = L^2$, ... and using the $'$ for the corresponding numerical invariants related to the line bundle $H(g(H) = g', H^2 = d', \ldots)$.

0.4.2. We recall the following relations between the characters of $L$ and of $H$, which can be proved by using the adjunction formula, the Riemann-Roch theorem and the Kodaira vanishing theorem:

i) $d' - g' + 2 = g$,

ii) $n + 1 =: h^0(H) = p_g + g - q$,

iii) $d' + d = KS^2 + 4(g - 1)$.

0.5. We will frequently use the following well known theorem.

CASTELNUOVO'S LEMMA. (see [G-H]). If $C$ is an irreducible curve embedded in $\mathbb{P}^{l-1}$ and $C$ belongs to no linear hyperplane $\mathbb{P}^{l-2}$, then, with $d'$ the degree of $C$ and $g'$ the genus of $C$:
§1. Some examples and surfaces in $\mathbb{P}^4$

1.1. Let $(S, H)$ be a polarized pair in $S$. The fact that

$$h^i(H) = 0 \text{ for } i = 1, 2$$

is a necessary but not sufficient condition for the pair to be in the image of $R\mathcal{A}$ as the following example shows.

**Example A.** Let $(S, H)$ be the minimal elliptic surface of degree 7 of $\mathbb{P}^4$ (e.g. see [La, §6] or [Io1, Prop. 8.7]). Since $S$ is regular and $|K_S|$ consists of a linear pencil of plane cubics with respect to $H$, we have $p_g = 2, q = 0$ and $K_S \cdot H = 3$. Therefore

$$x(H) = x + H \cdot (H - K_S)/2 = 3 + (7 - 3)/2 = 5.$$  

Since $h^0(H) = 5$, this means that $h^1(H) = h^2(H)$. On the other hand, by Serre duality, $h^2(H) = h^0(K_S - H) = 0$, since $(K_S - H) \cdot H < 0$. So (1.1.1) is satisfied. However $(S, H) \notin R\mathcal{A}$. Actually, let $L = H - K_S$; then we have $L^2 = (H - K_S)^2 = 1$. Assume that $(S, H) \in R\mathcal{A}$; since $1 \leq L^2 < 1$ we have that $(S, L) = (\mathbb{P}^2, O_{\mathbb{P}^2}(1))$, contradicting the fact that $S$ is an elliptic surface.

**Example B.** Let $S$ be a geometrically ruled surface polarized by a very ample line bundle $H$ with $q = 0, 1$ or 2.

If $q = 0$, by [Ha, p. 380], we have that

$$1.2. \text{ The line bundle } H \approx [a \sigma + b f] \text{ on } F_e \text{ is very ample iff } a > 0 \text{ and } b > ae; \text{ so } (F_e, H) \in R\mathcal{A} \text{ iff it is in } \mathcal{A}, \text{ iff } a > 0 \text{ and } b > ae + (e - 2).$$

In particular this shows that the quadric surface of $\mathbb{P}^3$ and the rational cubic scroll of $\mathbb{P}^4$ are in $R\mathcal{A}$.

When $q = 1$ and 2 we restrict ourselves to the case of scrolls. It is easy to prove the following.

**Proposition 1.3.** Let $(S, H \approx [\sigma + bf])$ be a scroll over a smooth curve of genus $q = 1$ or 2. $H$ is very ample and $(S, H)$ is in $\mathcal{A}$ unless in the following cases

$$q = 1 \quad e > 0 \quad \text{ and } \quad b = 3 + e, \ldots, 3 + (2e - 1),$$

$$q = 2 \quad e \geq 0 \quad \text{ and } \quad b = 5 + e, \ldots, 5 + (2e + 1),$$

$$e = -1 \quad \text{ and } \quad b = 4,$$

$$e = -2 \quad \text{ and } \quad b = 3, 4.$$
Moreover, except possibly for the last two cases, i.e. \( e = -2 \) and \( b = 3, 4 \), these pairs are not even in \( \text{Im}\, R.A \).

**Proof.** The first part follows immediately by applying corollaries 1.3.3 and 1.4.2 of [Bi] to the line bundles \( H \) and \( L = H - K_S \); to prove the second part, suppose that \( H \) is very ample and that \( L \) is the direct image through the reduction \( \pi : \tilde{S} \to S \) of a very ample line bundle \( \tilde{L} \). Then, if \( C \subset \tilde{S} \) is the proper transform of the section \( \sigma \), we must have \( \tilde{L} \cdot C \geq 2q + 1 \), for \( q = 1 \) or 2. This necessary condition is never satisfied, apart from possibly the last two cases.

**Example C.** Let \( S = B_6(P^2) \) be the plane \( P^2 \) blown-up at six points in general position. Then \( \text{Pic}(S) \) is generated over \( \mathbb{Z} \) by the seven elements \( 1, e_1, \ldots, e_6 \) corresponding to the inverse image of a general line and to the 6(-1) curves introduced by the blowing-up respectively. They satisfy \( i^2 = 1 \), \( e_i e_j = -\delta_{ij} \), \( e_i 1 = 0 \); moreover a line bundle \( H = [a \sum b_i e_i] \) is ample iff is very ample iff (see [Ha, p. 405])

\[
\begin{align*}
& b_i > 0 \text{ for every } i, \quad a > b_i + b_j \text{ for every } i, j, \quad 2a > \sum_{i \neq j} b_i, \text{ for every } j.
\end{align*}
\]

Recalling that \( K_S = [-3l + \sum e_i] \) it is easy to verify that \( (S, H) \in \text{Im}A \) for any very ample line bundle \( H \). In particular by taking \( H = -K_S \) we see that \( (S, H) \), the cubic surface of \( P^3 \) is in \( \text{Im}\, R.A \).

As a corollary of the above discussion we easily get the following.

**Proposition 1.4.** Let \((S, H)\) be a surface of \( P^3 \). Then \((S, H) \in \text{Im}A \) iff \((S, H) \in \text{Im}R.A \) iff \( d' = H^2 \leq 4 \).

1.5. We now consider the pairs \((S, H) \in S\) such that \( h^0(H) = 5 \). If \((S, H) \in \text{Im}R.A \), by 0.4.1, i) we can suppose \( H = K_S + L \) with \( L \) ample. As a first thing note that, using the equality 0.4.2, ii), we get \( 5 = h^0(H) = p_g + g - q = \chi - 1 + g \), therefore

\[
\chi + g = 6.
\]

Suppose \( \chi < 0 \); then \( S \) is a ruled surface with \( q \geq 2 \). In this case \( \chi = 1 - q \) and then (1.5.1) gives \( g = q + 5 \).

Let \( C \in |L| \) be a general element. Note that, since \( H \) is very ample, \((S, L)\) can be neither a scroll nor a conic bundle; hence the ruling projection of \( S \) gives a morphism of degree \( r \geq 3 \) from \( C \) to the base curve of \( S \), which has genus \( q \) and then, from the Riemann-Hurwitz formula we get

\[
2g - 2 = r(2q - 2) + b, \quad \text{with } b \geq 0,
\]

which, combined with \( g = q + 5 \), gives \( r \leq (q + 4)/(q - 1) \). Due to the condition \( r \geq 3 \) we thus immediately see that \( q = 2 \) or 3.
Now look at the double point formula for surfaces in $\mathbb{P}^4$ [Ha, p. 434]:

\[(1.5.2) \quad d'(d' - 10) + 12\chi = 2Ks^2 + 5H \cdot Ks.\]

Note that we have $H \cdot Ks = Ks \cdot (Ks + L) = g' - g = d' + 2 - 2g = d' - 8 - 2q$, using the equality in 0.4.2, i). On the other hand we have $Ks^2 = 8(1-q) - s$, where $s \geq 0$ is the number of blowing-ups we need to get $S$ from a minimal model. Taking into account these relations, (1.5.2) gives the equation

\[d'^2 - 15d' + 14q + 36 + 2s = 0,\]

whose discriminant $\Delta = 81 - 56q - 8s$ is negative when $q = 2$ or 3. This shows that case $\chi < 0$ cannot occur. In other words, recalling (1.5.1), this gives

**Proposition 1.6.** Every surface of $\mathbb{P}^4$ which lies in $\text{Im} R\mathcal{A}$ satisfies $\chi \geq 0$ and $g = g(L) \leq 6$.

Assume therefore that $\chi \geq 0$. If $g \leq 4$ then, by (1.5.1), $\chi \geq 2$ and thus $S$ is not ruled. Therefore, if $(S, H) \in \text{Im} R\mathcal{A}$, then there exists a pair $(\widetilde{S}, \widetilde{L})$ with $\widetilde{S}$ nonruled and $g(\widetilde{L}) \leq 4$ (see 0.4.1, ii)). So looking at the classification of non ruled surfaces of small sectional genus (see [Li]) we have the quartic $K3$ surface of $\mathbb{P}^3$ for $g = 3$, and the $K3$ surface given by the complete intersection of type $(2, 3)$ of $\mathbb{P}^4$ for $g = 4$ (and $\chi = 2$). Of course in these cases $(S, H) = (S, L) \in \text{Im} \mathcal{A} \subset \text{Im} R\mathcal{A}$.

Consider now the surfaces with $g = 5$. From (1.5.1) we have $\chi = 1$. $S$ cannot have Kodaira dimension $\geq 0$ (see [Li]). So $S$ is rational and then, using the classification again [Li], we see that $(S, L)$ (respectively $(\widetilde{S}, \widetilde{L})$) is one of the following pairs (see 0.2 and 0.3 for the notation):

- $S = B_{10}(\mathbb{P}^2)$, $L = \pi^*O_{\mathbb{P}^2}(7) - 10p^2$, $(\widetilde{S} = B_{10+t}(\mathbb{P}^2)$, $\widetilde{L} = \pi^*O_{\mathbb{P}^2}(7) - 10p^2 - tp$, with $t = 0, 1$),
- $S = B_t(\mathbb{F}_e)$, $e \leq 2$, $L = \pi^*\{4\sigma + (2e + 5)f\} - 7p^2$, $(\widetilde{S} = B_{t+t}(\mathbb{F}_e)$, $\widetilde{L} = \pi^*\{4\sigma + (2e + 5)f\} - 7p^2 - tp$, with $t = 0, 1, 2, 3$),
- $S = B_5(\mathbb{P}^2)$, $L = \pi^*O_{\mathbb{P}^2}(6) - 5p^2$, $(\widetilde{S} = B_{5+t}(\mathbb{P}^2)$, $\widetilde{L} = \pi^*O_{\mathbb{P}^2}(6) - 5p^2 - tp$, with $t = 0, \ldots, 7$),
- $S = \mathbb{F}_t$, $L = \{3\sigma + 5f\}$, $(\widetilde{S} = B_t(\mathbb{F}_t)$, $\widetilde{L} = \pi^*\{3\sigma + 5f\} - tp$, with $t = 0, \ldots, 12$).

(Note that for all the above pairs $(\widetilde{S}, \widetilde{L})$, $\widetilde{L}$ is actually very ample provided that the points $p$ satisfy a general position condition, except in the last two cases for $t = 7$ and 12 respectively).

For all surfaces listed above we have in fact $h^0(Ks + L) = 5$. Note that $(S, H)$ is the Bordiga sextic surface in the first case, the Castelnuovo quintic surface (a conic bundle), in the second one, the Del Pezzo surface, complete intersection of two quadrics in the third case and the cubic scroll in the fourth case.
Finally consider the surfaces with $g = 6$. From 1.5.1 we have $\chi = 0$, so that either $S$ is an elliptic geometrically ruled surface or it has nonnegative Kodaira dimension.

In the last case, from [Li] we see that $S$ can only be one of the following:

- an abelian surface of degree 10 in $\mathbb{P}^4$;
- a hyperelliptic surface of degree 10 in $\mathbb{P}^4$.

The first case corresponds to a well known surface, discovered by Commensatti, related to the Horrocks-Mumford bundle and studied by many people (see for instance [Hu]). Surfaces of the second type have been recently discovered by Serrano [Se2]. In the second case $(S, H) = (S, L)$, while in the third one $H - L = K_S$ is of order 3.

Now assume that $S$ is an elliptic ruled surface; since we have $H \in KS = H \in (H - L) = d' - (K_S + L) \cdot L = d' - (2g - 2) = d' - 10$ and $K_S^2 \geq 0$, then (1.5.2) gives $5 \leq d' \leq 10$.

In view of known results on the classification of surfaces of $\mathbb{P}^4$ (e.g. see [Ra]), we conclude that $(S, H)$ is the elliptic quintic scroll, or equivalently

- $S$ is the elliptic $\mathbb{P}^1$-bundle of invariant $e = 1$, $L = [3\sigma + f]$.

We can therefore summarize what we have proved in the following:

**Theorem 1.7.** Let $(S, H)$ be a polarized surface such that $h^0(H) \leq 5$. Then $L = H - K_S$ is very ample (i.e. $(S, H) \in \text{Im}A$) if and only $(S, H)$ is in $\text{Im}A$ if and only if it is one of the pairs in italic above or a surface of degree less than 5 in $\mathbb{P}^3$.

Recalling the classification of surfaces of small degree, they are: all surfaces of degree $\leq 6$ in $\mathbb{P}^4$, except the quintic and the sextic surfaces in $\mathbb{P}^3$, plus the abelian and the hyperelliptic surfaces of degree 10.

**§2. Surfaces with numerically effective anticanonical bundle**

The aim of this section is to prove the following theorem and some of its corollaries (for a study of the surfaces with nef anticanonical bundle see for instance [Sa]).

**Theorem 2.1.** Let $S$ be a surface for which $-K_S$ is nef. Then for every very ample line bundle $H$ on $S$ (or just ample and spanned with $H^2 \geq 10$), we have that $H - K_S$ is very ample, i.e. $(S, H) \in \text{Im}A$.

**Proof.** By the Nakai criterion the line bundle $L_m := H - mK_S$ is ample for all $m \geq 0$. Our theorem is equivalent to the fact that $L = L_1$ is very ample: to prove it we apply Reider’s theorem to the line bundle $L_2$ (see [Re]).
By the Hodge Index theorem, the ampleness of $H$ and of $(H - 2K_S)$ and the nefness of $-K_S$, we have that either $-K_S \cdot H \geq 1$ and $-K_S \cdot (H - 2K_S) \geq 1$, or $K_S$ is numerically equivalent to zero. Therefore for $H^2 \geq 6$ or, if $K_S \approx 0$, for $H^2 \geq 10$, we have that

$$L_2^2 = H^2 + (-2K_S) \cdot H + (-2K_S) \cdot (H - 2K_S) \geq 10.$$ 

In this hypothesis, by Reider's theorem [Re, theorem 1], $L$ is very ample unless there exists an effective divisor $E \subset S$ such that

1) $L_2 \cdot E = 0$ and $E^2 = -1$ or $-2$
2) $L_2 \cdot E = 1$ and $E^2 = -1$ or $0$
3) $L_2 \cdot E = 2$ and $E^2 = 0$.

The first case is impossible since $L_2$ is ample; in the two remaining cases, by the nefness of $-K_S$, we have that, respectively, $H \cdot E = 1$ or $2$ and $K_S \cdot E = 0$. In both cases $g(E) = 0$ and this combined with the adjunction formula and the selfintersection imposed to $E$ gives a contradiction.

To conclude the proof we consider now the cases $H^2 \leq 5$ or $K_S \approx 0$ and $H^2 \leq 9$.

In the last case, since $g' = g(H) = 1 + d'/2$, we have that $g' \leq 5$. In view of the classification of surfaces with small sectional genus (see [Li]), we have that all surfaces with numerically trivial $K_S$ and $g' \leq 5$ are $K3's$, for which our result is trivial.

Finally we have the following:

**Remark 2.2.** Polarized pairs $(S, H)$ with $d' \leq 5$ are in $\text{ImA}$ except for $(F_3, [\sigma + 4f])$ and the quintic surface of $\mathbf{P}^3$ which are not even in $\text{ImRA}$. (Note that for both these surfaces $-K_S$ is not nef.)

**Proof.** Assume that $|H|$ embeds $S$ in $\mathbf{P}^n$. Then $n \leq d + 1$. For $n = 3, 4$ the assertion follows from section 1. So let $n = 5$. Then $d = 4$ or 5. If $d = 5$, $(S, H)$ is a Del Pezzo pair, hence $L = H - K_S = -K_S$ is very ample, hence $(S, H) \in \text{ImRA}$. If $d = 4$, $(S, H)$ is one of the following rational scrolls: $(F_0, [\sigma + 2f])$, $(F_2, [\sigma + 3f])$. Similarly, if $n = 6$, $(S, H)$ in one of the following rational scrolls: $(F_1, [\sigma + 3f])$, $(F_3, [\sigma + 4f])$. Then the assertions follow from example B.1.2 and from 1.4.

The theorem implies some corollaries, the first of which is immediate.

**Corollary 2.3.** If $S$ is a minimal surface with $k(S) = 0$, then for every very ample line bundle $H$ we have that $(S, H)$ is in $\text{ImA}$.

**Proposition 2.4.** Let $p_1, \ldots, p_s \in F_e$, $e \leq 2$, be $s \leq 8$ points in general position with respect to $-K_{F_e}$ and let $S = B_s(F_e)$. Then $-K_S$ is nef.
PROOF. The assertion is obvious if \( e \leq 1 \), since then \( B_e(\mathbb{F}_e) = B_{e+1}(\mathbb{P}^2) \).

So, look at case \( e = 2 \). Let \( \pi : S \to \mathbb{F}_2 \) denote the blowing up, set \( E_i = \pi^{-1}(p_i) \), \( \mathcal{E} = E_1 + \cdots + E_s \) and consider any irreducible curve \( C \) on \( S \). If \( C = E_i \) we have \(-K_S \cdot C = 1\); on the other hand, if \( C \) is not contracted by \( \pi \), then \( C = \pi^* C' - \sum \nu_i E_i \), \( C' \) being an irreducible curve on \( \mathbb{F}_2 \) with \( \text{mult}_{p_i}(C') = \nu_i \), and then

\[
(2.4.1) \quad -K_S \cdot C = (\pi^*[2\sigma + 4f] - \mathcal{E}) \cdot (\pi^* C' - \sum \nu_i E_i) = -K_{\mathbb{F}_2} \cdot C' - \sum \nu_i.
\]

Note that \( h^0(-K_{\mathbb{F}_2}) = 9 \), as the Mumford-Kawamata vanishing and the Riemann-Roch theorems immediately show. So, since \( s \leq 8 \), there is a divisor \( D \in | -K_{\mathbb{F}_2} - p_1 - \cdots - p_s | \). So, if \( C' \) is not a fixed component of the linear system \( S = | -K_{\mathbb{F}_2} - p_1 - \cdots - p_s | \), we have

\[
-K_{\mathbb{F}_2} \cdot C' = D \cdot C' \geq \sum \text{mult}_{p_i}(D) \cdot \text{mult}_{p_i}(C') \geq \sum \nu_i.
\]

Now assume that \( C' \) is a fixed component of \( S \). Then, due to the irreducibility of \( C' \) and the effectiveness of \(-K_{\mathbb{F}_2} - C' \), we have \( C' = a\sigma + b\mathcal{E} \), where either \((a, b) = (1, 0), (0, 1) \) or \((1, b)\), with \( 2 \leq b \leq 4 \) \([Ha, p. 380]\). Moreover

\[
\sum \nu_i(\nu_i + 1)/2 \leq h^0(C') - 1,
\]

in view of the general position assumption \([Li, (0.19)]\). As \( h^0(C') = 1, 2 \) or \( b + 2 \), according to the three cases above, we have \( \sum \nu_i(\nu_i + 1) = 0, 2 \) or \( 2b + 2 \) and therefore \((2.4.1)\) shows that \(-K_S \cdot C \geq 0\).

REMARK 2.4.2. Note that for \( e = 2, s \leq 4 \), the general position assumption on the \( p_i \)'s can be simply expressed by saying that they are: no one on \( \sigma \) and no two on the same fibre.

COROLLARY 2.5. Let \( S \) be a rational surface polarized by a very ample line bundle \( \mathcal{L} \) with \( g(\mathcal{L}) = 1 \) or \( 2 \). Then \( (S, \mathcal{L}) \) is in \( \text{Im}A \).

PROOF. If \( g(\mathcal{L}) = 1 \) then \(-K_S \) is ample, while if \( g(\mathcal{L}) = 2 \) it follows from the classification theory \((\text{e.g. see } [Li, \text{p. 169}, \text{case 3}])\) that \( S = B_s(\mathbb{F}_s), e \leq 2, s \leq 7, \mathcal{L} = \pi^*[2\sigma + (e + 3)f] - sp \). Note that \( h^0(K_{\mathbb{F}_s} + \pi \mathcal{L}) = g(\mathcal{L}) > 0 \); so, being \( \pi \mathcal{L} - (-K_{\mathbb{F}_s}) \) effective, the general position assumption with respect to \( \pi \mathcal{L} \) \([Li]\) implies the same condition with respect to \(-K_{\mathbb{F}_s} \). Therefore \(-K_{S} \) is nef by \((2.4)\) and then the assertion follows from the theorem.

The above corollary is no longer true for \( g(\mathcal{L}) = 3 \). Actually the list of rational surfaces with a very ample line bundle \( \mathcal{L} \) with \( g(\mathcal{L}) = 3 \) is the following \((\text{see } [Li, \text{pp. 161-169}])\):

a) \( S = B_s(\mathbb{P}^2), s \leq 10, \mathcal{L} = \pi^*O_{\mathbb{P}^2}(4) - sp \)
b) \( S = B_{7+4}(\mathbb{P}^2), \ s = 1, 2, \ L = \pi^*O_{\mathbb{P}^2}(6) - 7p^2 - sp \)

c) \( S = B_4(F_e), \ e \leq 3, \ s \leq 9, \ L = \pi^*[2\sigma + (e + 4)f] - sp. \)

In case a) \(-K_S\) is nef unless \( s = 10, \) in which case however \((S, L)\) is an adjoint surface as we have seen in section 1. In case b) \(-K_S\) is nef. As to case c) the situation is more involved. First of all note that, for \( e = 3, \) the line bundle \( L = L - K_S\) is not even ample, since \( L \cdot \pi^{-1}(\sigma) = 0. \) Let \( e \leq 2; \) if \( s \leq 8, \) then \(-K_S\) is nef by the above lemma and the general position argument as in the proof of (2.5). On the other hand when \( s = 9, \) then \(-K_S\) is certainly not nef, as \( K_S^2 = -1; \) nevertheless \( L \) is a very ample line bundle of genus 6 [Li, p. 169, case 13]. So, recalling the theorem, we get the following.

**Corollary 2.6.** Let \( S \) be a rational surface polarized by a very ample line bundle \( L \) with \( g(L) = 3. \) Then \((S, L)\) is in \( \text{Im} A \) if and only if it is one of the pairs in a) or b) or c) with \( e \neq 3. \) In the case c) when \( e = 3, \) \((S, L)\) is not even in \( \text{Im} RA. \)

§3. Surfaces of degree \( \leq 9 \)

In this section we consider polarized surfaces \((S, H) \in S\) with \( d' = H^2 \leq 9. \)

**Theorem 3.1.** Let \((S, H) \in S\) with \( d' = H^2 \leq 9. \) We have that \((S, H) \in \text{Im} RA\) if and only if \((S, H) \in \text{Im} A\) if and only if \((S, H)\) is one of the following pairs:

1) pairs with \( d' = H^2 \leq 5 \) except the quintic surfaces in \( \mathbb{P}^3 \) and \((F_3, [\sigma + 4f]).\)

2) pairs with \( d' = 6 \) and \( h^0(H) = 5, \)

(for the precise list of surfaces in 1) and 2) see section 1).

3) pairs with \( g' = g(H) = 0, 1, 2 \) except the following scrolls

(see also 1.2 and 1.3)

- \( g' = 0, \ S = F_e, \ H \approx [\sigma + (e + 1)f] \) and \( e = 3, \ldots, 7 \) or \( H \approx [\sigma + (e + 3)f] \) and \( e = 4, 5 \)

- \( g' = 1, \ S = P(E), \ H \approx [\sigma + (3 + e)f] \) and \( e = 1, 2, 3 \) or \( H \approx [\sigma + 5f] \) and \( e = 1 \)

- \( g' = 2, \ S = P(E), \ H \approx [\sigma + 4f] \) and \( e = -1 \) or possibly \( H \approx [\sigma + 3f] \) and \( e = -2. \)

4) the pairs in corollary 2.6.

5) \( S \) is a minimal surface with \( \kappa(S) = 0. \)

6) \( S \) is a geometrically ruled surface over a curve of genus 1, with \( e = -1 \) and \( H \approx [2\sigma + f] \) or \( \approx [3\sigma]. \)

7) \((S, H) = (B_{10}(\mathbb{P}^2), \ H = \pi^*O_{\mathbb{P}^2}(6) - 6p^2 - 4p),\)

\((S, H) = (B_{10}(\mathbb{P}^2), \ \pi^*O_{\mathbb{P}^2}(7) - 10p^2),\)

\((S, H) = (B_8(\mathbb{P}^2), \ \pi^*O_{\mathbb{P}^2}(9) - 8p^3),\)

\((S, H) = (B_9(F_0), \ \pi^*[3\sigma + 3f] - 9p).\)

8) The following pairs are possibly in \( \text{Im} RA \) or in \( \text{Im} A: \)
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\[(S, H) = (B_{10}(F_0), \pi^*[3\sigma + 3f] - 10p),\]
\[(S, H) = (B_{11}(F_e) e = 0, 1, 2, 3, \pi^*[2\sigma + (e + 5)f] - 11p),\]
\[(S, H) = (B_{10}(F_e) e = 0, 1, 2, \pi^*[4\sigma + (2e + 5)f] - 7p^2 - 3p).\]

**Proof.** First observe that 1) follows from the remark 2.2; 2) follows from proposition 1.7; 3) comes from the classification of surfaces of sectional genus \(\leq 2\) combined with examples b1.2 and 1.3 and corollary 2.5; 4) and 5) follow from corollaries 2.6 and 2.3.

We can therefore assume from now on the following: \(n \geq 5, d' \geq 6, g' \geq 3,\) and even \(g' \geq 4\) if \(q = 0\) and finally that \(S\) is not a minimal surface with \(\kappa(S) = 0.\)

3.1.1. Assume that \((S, H) \in \text{Im} \mathcal{R} \mathcal{A}:\) then, by 0.4.1, i), \(L = H - K_S\) is ample. Let now \(C \in |H|\) be a smooth element, then, since \(h^1(H) = 0,\) by the exact sequence

\[
0 \longrightarrow O_S \longrightarrow H \longrightarrow H_C \longrightarrow 0,
\]
we have that \(h^0(H|_C) = h^0(H) + q - 1 = n + q = 1.\) Therefore \(C \subset P^{n-1}\) is the projection of a linearly normal curve in \(P^{n-1+q} = P^{n+1}.\)

If \(H \cdot K_S > 0,\) for the ampleness of \(L\) we have that

\[d' < K_S \cdot H + H^2 = 2g' - 2.\]

This, together with the Castelnuovo bound 0.5, gives

\[
(d' + 2)/2 < \left[\frac{(d' - 2)}{(n + q - 2)}\right] \cdot \left(d' - n - q + 1 - \left[\frac{d' - n - q}{n + q - 2}\right]\right) \left(\frac{n + q - 2}{2}\right)
\]

which is impossible for \(d' \leq 8.\)

If \(d' = 9\) then all above implies that \(n = 5, q = 0\) and \(g' = 6\) or 7. Using the equalities in 0.4.2 we have that \(g = 5\) or 4, and that \(p_g = 1\) or 2 respectively. In particular \(S\) is not ruled.

We now look at the classification of non ruled polarized pairs with this numerical invariants ([Li] p. 174) and we have that in the first case \((S, H)\) is the projection in \(P^5\) of the K3 surface of degree 10 of \(P^6\) from a point of itself; in particular \(c_1^2 = -1\) and therefore \(d = 6.\) Looking again at the classification we see that the pairs \((S, L)\) with \(S\) as above and \(L\) very ample with \(g = 5\) and \(d \leq 6\) do not exist, that is \((S, H)\) is not in \(\text{Im} \mathcal{R} \mathcal{A}\) (see 0.4.1,ii)).

If \(g' = 7\) and \(p_g = 2\) then, by the classification we see that \(S\) is elliptic. But there are no very ample line bundles of genus 4 on these surfaces and therefore they are not in \(\text{Im} \mathcal{R} \mathcal{A}.\)

Now let \(H \cdot K_S < 0.\) Then, according to the assumption that \(S\) is not a minimal surface with \(\kappa(S) = 0,\) we know that \(S\) is ruled.
Using again lemma 0.5, and the assumption at the beginning, we see that 
\( d' \leq 7 \) is impossible.

If \( d' = 8 \) then either i) \( g = 5 \), ii) \( g' = 4 \) and \( (n, q) = (5, 0) \), or iii) \( g' = 3 \) and 
\( (n, q) = (5, 1) \).

In case i) equality holds in the Castelnuovo bound; so \( S \) is a Castelnuovo surface in the sense of Harris [Ha]; in particular \( p_g(S) = 1 \), contradicting the ruledness of \( S \).

In case iii), looking at the classification of surfaces ([Li]), we see that \( S \) is a geometrically ruled surface over a curve of genus 1 with \( e = -1 \) and \( H = 2\sigma + f \); it is easy to check in this case, e.g. using [Re, theorem 1] that \( L = H - K_S \) is very ample. This gives one of the two surfaces in 6).

Finally, in case ii), we use, now very strongly, the classification results in [Li].

More precisely we use first the table at p. 161 and we find that 
\[ S = B_{10}(\mathbb{P}^2), \quad H = \pi^*O_{\mathbb{P}^2}(6) - 6p^2 - 4p \quad \text{(case 1 at page 161)}. \]

Note that \((S, H) \in S\) provided the points blown-up are in general position and 
\((S, H)\) is the adjoint pair to the pair in case 7 of the table in [Li]. This gives the first pair listed in 7).

Using then the table at page 169 we find that either 
\[ S = B_{12}(F_e), \quad e \leq 4, \quad H = \pi^*[2\sigma + (e + 5)f] - 12p \quad \text{(case 5 with } t_0 = 12), \]

or 
\[ S = B_{10}(F_0), \quad H = \pi^*[3\sigma + 3f] - 10p \quad \text{(case 6 with } t_0 = 10). \]

The very ampleness of \( H \) in the first case is doubtful. Anyway in the same table one can see that \( L = H - K_S - tp \), for any \( p \), is not very ample, that is \((S, H)\) is not in \( \text{Im} \mathcal{R} \mathcal{A} \).

In the second case \((S, H) \in S\) provided the 10 points are in general position. 
In this case \( H - K_S - tp = L - tp = \pi^*[5\sigma + 5f] - 10p^2 - tp \) turns out to be as in case 14, with \( t = t_0 = 0 \) or 1, in table at p. 169. This gives the first case in 8).

Finally let \( d' = 9 \). Then in view of the Castelnuovo bound and the equalities in 0.4.2 only the following cases can occur:

\((g', q, n, g) = (5, 0, 5, 6), (4, 1, 5, 7), (4, 0, 6, 7), (3, 2, 5, 8), (3, 1, 6, 8) \).

By using as above the classification in [Li] we can see that the two last cases do not actually occur, while the second one gives rise to the remaining pair in 6).

Moreover the first and the third give the pairs \((S, H)\) listed in the following table. 

The values of \( t \) given in the last column are the number of possible blown-ups of points in \( S \) to obtain a pair \((\tilde{S}, \tilde{L})\) such that \((S, H) = \mathcal{R} \mathcal{A}(\tilde{S}, \tilde{L})\), or \((S, L)\) the reduction of \((\tilde{S}, \tilde{L})\).

<table>
<thead>
<tr>
<th>( g' )</th>
<th>( S )</th>
<th>( H )</th>
<th>( L )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( 5 )</td>
<td>( B_{10}(\mathbb{P}^2) )</td>
<td>( \pi^*O_{\mathbb{P}^2}(7) - 10p^2 )</td>
<td>( \pi^*O_{\mathbb{P}^2}(10) - 10p^3 )</td>
</tr>
<tr>
<td>( 5 )</td>
<td>( B_{10}(F_e) )</td>
<td>( \pi^*[4\sigma + (2e + 5)f] )</td>
<td>( \pi^*[6\sigma + (3e + 7)f] )</td>
</tr>
<tr>
<td>( e = 0, 1, 2 )</td>
<td>( -(7p^3 + 3p) )</td>
<td>( -(7p^3 + 3p^2) )</td>
<td></td>
</tr>
</tbody>
</table>
Projective surfaces arising from an adjunction process

4 $B_3(\mathbb{P}^2)$ \hspace{1cm} $\pi^*O_{\mathbb{P}^2}(9) - 8p^3$ \hspace{1cm} $\pi^*O_{\mathbb{P}^2}(12) - 8p^4$ \hspace{1cm} $t = 0, 1, 2, (3, \ldots, 7)$

4 $B_{11}(F_0)$ \hspace{1cm} $\pi^*[2\sigma + (e + 5)f]$ \hspace{1cm} $\pi^*[4\sigma + (2e + 7)f]$ \hspace{1cm} $t = (0, \ldots, 3)$

4 $B_9(F_0)$ \hspace{1cm} $\pi^*[3\sigma + 3f] - 9p$ \hspace{1cm} $\pi^*[5\sigma + 5f] - 9p^2$ \hspace{1cm} $t = 0, 1, 2, (3, 4, 5).$

The line bundles $L$ are surely very ample for those values of $t$ not in a parenthesis, which gives the last three pairs in 7), while for the others this is not known. In particular we notice that in the second and fourth cases the very ampleness of $L$ is not known for all the allowable values of $t$. They give rise to the last two cases in 8). This concludes the proof.

Remark 3.2. For the geometrically ruled surfaces $S$ over a curve of genus 2 with $e = -2$ and for $H \approx [\sigma + 3f]$ (the doubtful case in 3) of the theorem 3.1) we know that $L = H - KS$ is not very ample, but we do not know whether $(S, L)$ can be the reduction of a pair in $S$. It would be interesting to answer this question in connection with the problem quoted in 0.4.

References


