Irreducible tensor power representations of the free 2×2 matrix bialgebra

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§1.

Our base field k is assumed to have characteristic zero. Let $F = T(M_2(k)^*)$ be the free bialgebra on the dual coalgebra of the 2×2 matrix algebra. Namely, if $\{e_{ij}\}$ is the matrix unit basis of $M_2(k)$ and $\{x_{ij}\}$ is its dual basis, then F is the free algebra on $\{x_{ij}\}$, and the comultiplication Δ and the counit ϵ are given as follows.

$$\Delta(x_{ij}) = \sum_k x_{ik} \otimes x_{kj}$$

$$\epsilon(x_{ij}) = \delta_{ij}.$$

See Sweedler [3, p. 60]. If M, N are F -modules, we have the F -module $M \otimes N$ through the map Δ and the natural $F \otimes F$ -module structure.

Let $\alpha_1, \alpha_2, \beta, \gamma, \delta_1, \delta_2 \in k - \{0\}$. Consider the linear map $\varphi : M_2(k)^* \rightarrow M_2(k)$ defined by

$$\begin{aligned} x_{11} &\mapsto \begin{pmatrix} \alpha_1 & 0 \\ 0 & \alpha_2 \end{pmatrix} & x_{12} &\mapsto \begin{pmatrix} 0 & \beta \\ 0 & 0 \end{pmatrix} \\ x_{21} &\mapsto \begin{pmatrix} 0 & 0 \\ \gamma & 0 \end{pmatrix} & x_{22} &\mapsto \begin{pmatrix} \delta_1 & 0 \\ 0 & \delta_2 \end{pmatrix}. \end{aligned}$$

This makes $V = k^2$ an F -module. Our results are the following.

THEOREM 1.1. *The F -modules $V^{\otimes n}$ are simple for all $n \geq 0$.*

THEOREM 1.2. *The F -module $\bigoplus_{n \geq 0} V^{\otimes n}$ is faithful.*

Therefore we have the dense embedding $T(M_2(k)^*) \rightarrow \prod_n \text{End}(V^{\otimes n})$. In other words, if $\pi : M_2(k)^* \times M_2(k)^* \rightarrow k$ denotes the pairing corresponding to φ , the unique bialgebra pairing $T(M_2(k)^*) \times T(M_2(k)^*) \rightarrow k$ extending π is nondegenerate.

We prove Theorem 1.1 in Section 2 and Theorem 1.2 in Section 3. The theorems are true even if $\alpha_2 = \delta_1 = 0$ and $\text{char } k$ is arbitrary, which we show at the end.

§2. Proof of Theorem 1.1

We prove Theorem 1.1. Besides elementary linear algebra we use a known theorem in combinatorics about a poset of partitions (Theorem 2.1), whose proof itself requires the representation theory of the Lie algebra $\mathfrak{sl}_2$. This confines us to the characteristic zero case. Actually, when $\alpha_1/\alpha_2, \delta_1/\delta_2$ are not roots of 1, we have a shorter proof of Theorem 1.1 for any field k without the help of the combinatorial result. But we give here only the proof for the case where $\text{char } k = 0$ and α_i, δ_i are arbitrary.

Throughout this section we fix a positive integer n . Let $\mathcal{P}(n)$ be the set of maps $p : \{1, \dots, n\} \rightarrow \{1, 2\}$. Such p is determined by the subset $p^{-1}(1)$ of $\{1, \dots, n\}$, whose members we arrange in the increasing order: $p^{-1}(1) = \{i_1 < \dots < i_c\}$. We set

$$h(p) = i_1 + \dots + i_c - \frac{1}{2}c(n+1).$$

When $n = c + d$ with $c, d \in \mathbf{N}$ and $l \in \frac{1}{2}\mathbf{Z}$, we set

$$\mathcal{P}(c, d) = \{p \in \mathcal{P}(n) \mid \#p^{-1}(1) = c\}$$

$$\mathcal{P}(c, d)_l = \{p \in \mathcal{P}(c, d) \mid h(p) = l\}.$$

We define a partial order $\leq$ on $\mathcal{P}(n)$ as follows. For $p, q \in \mathcal{P}(n)$ with $p^{-1}(1) = \{i_1 < \dots < i_c\}$, $q^{-1}(1) = \{j_1 < \dots < j_d\}$, we write $p \leq q$ if $c = d$ and $i_1 \leq j_1, \dots, i_c \leq j_c$. The poset $\mathcal{P}(n)$ is then the disjoint union of the subposets $\mathcal{P}(c, d)$ with $c + d = n$. Furthermore we write $p \dashv q$ if $p < q$ and no r satisfies $p < r < q$. It is easy to see that for $p, q \in \mathcal{P}(n)$ with $p^{-1}(1) = \{i_1 < \dots < i_c\}$, $q^{-1}(1) = \{j_1 < \dots < j_c\}$ we have $p \dashv q$ if and only if there is $b \in [1, c]$ such that $i_a = j_a$ for $a \neq b$ and $i_b + 1 = j_b$. It follows that h is a height function of the poset $\mathcal{P}(n)$, that is, $h(p) + 1 = h(q)$ whenever $p \dashv q$.

Let $\mathbf{Q}[\mathcal{P}(n)]$ be the $\mathbf{Q}$ -vector space on the basis $\mathcal{P}(n)$. Define a $\mathbf{Q}$ -linear map $\Phi : \mathbf{Q}[\mathcal{P}(n)] \rightarrow \mathbf{Q}[\mathcal{P}(n)]$ by

$$\Phi(q) = \sum_{p \dashv q} p, \quad q \in \mathcal{P}(n).$$

Our proof of Theorem 1.1 depends on the following.

THEOREM 2.1. *Let $n = c + d$ with $c, d \in \mathbf{N}$ and $l \in \frac{1}{2}\mathbf{N}$. Then Φ^{2l} maps $\mathbf{Q}[\mathcal{P}(c, d)_l]$ onto $\mathbf{Q}[\mathcal{P}(c, d)_{-l}]$ isomorphically.*

This is a part of the hard Lefschetz theorem for a Grassmann variety. The poset $\mathcal{P}(c, d)$ is isomorphic to the poset of Young diagrams contained in the $c \times d$ rectangular diagram ordered by inclusion. For the details and the proof see Stanley [2], Griffiths and Harris [1].

Apart from this we need some elementary properties of the poset $\mathcal{P}(n)$. The following are easy to see.

(2.2) For $p, q \in \mathcal{P}(c, d)$ we have $p \leq q$ if and only if $\#p^{-1}(1) \cap [1, k] \geq \#q^{-1}(1) \cap [1, k]$ for $k = 0, \dots, n$.

(2.3) For $p, q \in \mathcal{P}(n)$ we have $p \dashv q$ if and only if there is $m \in [1, n-1]$ such that

$$p(k) - q(k) = \begin{cases} 0 & \text{if } k \neq m, m+1 \\ -1 & \text{if } k = m \\ 1 & \text{if } k = m+1. \end{cases}$$

Let $\leq_l$ be the lexicographic order on $\mathcal{P}(n)$. Namely, for $p, q \in \mathcal{P}(n)$ we write $p <_l q$ if $p(1) = q(1), \dots, p(k-1) = q(k-1)$, $p(k) < q(k)$ for some $k \in [1, n]$. Using (2.2), we see easily

(2.4) If $p \leq q$, then $p \leq_l q$.

Let $n = c + d$ with $c > 0$. Define a map $\tau : \mathcal{P}(c, d) \rightarrow \mathcal{P}(c-1, d+1)$ as follows. For $p \in \mathcal{P}(c, d)$, let $(q(1), \dots, q(n))$ be the sequence obtained by changing the last 1 of the sequence $(p(1), \dots, p(n))$ into 2. Then we set $\tau(p) = q$.

(2.5) Let $p, q \in \mathcal{P}(c, d)$ with $c > 0$. If $p <_l q$ and $h(p) = h(q)$, then $\tau(p) <_l \tau(q)$.

PROOF. We always have $q <_l \tau(q)$. Suppose that $p <_l q$ and $\tau(p) \geq_l \tau(q)$. Then $\tau(p) \geq_l \tau(q) >_l q >_l p$. Set $k = \max p^{-1}(1)$. Then p and $\tau(p)$ coincide on $[1, k-1]$, and so p and q do also. Since $\#p^{-1}(1) = \#q^{-1}(1)$, the letter 1 appears exactly once in $q(k), \dots, q(n)$. Thus $p \leq q$. As $p \neq q$, we must have $h(p) < h(q)$, finishing the proof.

Each poset $(\mathcal{P}(c, d), \leq)$ has the smallest element p with $p^{-1}(1) = \{1, 2, \dots, c\}$ and $h(p) = -\frac{1}{2}cd$. We set

$$\begin{aligned} l_0 &= \min\{-\frac{1}{2}cd \mid c + d = n\} \\ &= \begin{cases} -\frac{1}{2}\left(\frac{n}{2}\right)^2 & \text{if } n \text{ is even} \\ -\frac{1}{2}\frac{n+1}{2}\frac{n-1}{2} & \text{if } n \text{ is odd} \end{cases} \end{aligned}$$

so that $l_0 = \min\{h(p) \mid p \in \mathcal{P}(n)\}$.

(2.6) Let $p \in \mathcal{P}(c, d)$. If $c \geq n/2$ and $h(p) > l_0$, then $h(\tau(p)) < h(p)$.

PROOF. Let $p^{-1}(1) = \{i_1 < \cdots < i_c\}$. Then $\tau(p)^{-1}(1) = \{i_1 < \cdots < i_{c-1}\}$, so $h(p) - h(\tau(p)) = i_c - (n+1)/2$. If $c \geq n/2$ and $h(p) \leq h(\tau(p))$, then $n/2 \leq c \leq i_c \leq (n+1)/2$, hence $c = i_c = n/2$ or $(n+1)/2$ according as n is even or odd. Thus $p^{-1}(1) = \{1, \dots, c\}$ and $h(p) = l_0$. This proves (2.6).

Define an involutive map $p \mapsto \bar{p}$ on $\mathcal{P}(n)$ by $\bar{p}(k) = 3 - p(n+1-k)$ for $1 \leq k \leq n$. It is easily seen that $p \leq q$ if and only if $\bar{p} \leq \bar{q}$ and that $h(p) = h(\bar{p})$. A total order $\leq_r$ on $\mathcal{P}(n)$ is defined so that $p \leq_r q$ and $\bar{p} \leq_l \bar{q}$ are equivalent. When $n = c + d$ with $d > 0$, we define a map $\rho : \mathcal{P}(c, d) \rightarrow \mathcal{P}(c+1, d-1)$ by $\rho(\bar{p}) = \tau(p)^-$ for $p \in \mathcal{P}(d, c)$.

Let $V = k^2$ with the standard basis v_1, v_2 . We set $U = V^{\otimes n}$ and $v_p = v_{p(1)} \otimes \cdots \otimes v_{p(n)} \in U$ for $p \in \mathcal{P}(n)$. With respect to the basis $\{v_p \mid p \in \mathcal{P}(n)\}$ of U , the (p, q) -entries of the matrix of a map $f \in \text{End}(U)$ are denoted by f_{pq} for $p, q \in \mathcal{P}(n)$. When $n = c + d$ and $l \in \frac{1}{2}\mathbf{Z}$, we define subspaces of U as follows.

$$U^{c,d} = \langle v_p \mid p \in \mathcal{P}(c, d) \rangle$$

$$U_l = \langle v_p \mid h(p) = l \rangle$$

$$U_l^{c,d} = U^{c,d} \cap U_l.$$

Let $A_{ij} \in \text{End}(V)$ be

$$\begin{aligned} A_{11} &= \begin{pmatrix} \alpha_1 & 0 \\ 0 & \alpha_2 \end{pmatrix} & A_{12} &= \begin{pmatrix} 0 & \beta \\ 0 & 0 \end{pmatrix} \\ A_{21} &= \begin{pmatrix} 0 & 0 \\ \gamma & 0 \end{pmatrix} & A_{22} &= \begin{pmatrix} \delta_1 & 0 \\ 0 & \delta_2 \end{pmatrix} \end{aligned}$$

where $\alpha_1, \alpha_2, \beta, \gamma, \delta_1, \delta_2 \in k - \{0\}$. Define $B_{ij} \in \text{End}(U)$ by

$$B_{ij} = \sum_s A_{s(0)s(1)} \otimes \cdots \otimes A_{s(n-1)s(n)}$$

where the summation is taken over the maps $s : \{0, 1, \dots, n\} \rightarrow \{1, 2\}$ such that $s(0) = i$, $s(n) = j$. If $F = T(M_2(k)^*) \rightarrow \text{End}(V)$ is the representation taking x_{ij} to A_{ij} , then the n^{th} tensor power representation $F \rightarrow \text{End}(U)$ takes x_{ij} to B_{ij} for $i, j = 1, 2$. Theorem 1.1 asserts that no nontrivial subspace of U is stable under all B_{ij} .

Let $p, q \in \mathcal{P}(n)$ and $i, j \in \{1, 2\}$. Let $\{k \mid p(k) \neq q(k)\} = \{k_1 < \cdots < k_e\}$. Consider the following condition.

$$(*, i, j, p, q) \quad p(k_1) = i, \quad q(k_e) = j, \quad p(k_a) \neq p(k_{a+1}) \text{ for } a = 1, \dots, e-1.$$

If $(*, i, j, p, q)$ holds, we have $m \equiv i - j \pmod{2}$ and $\#p^{-1}(1) - \#q^{-1}(1) = j - i$, and we define a map $s_{i,j,p,q} : \{0, 1, \dots, n\} \rightarrow \{1, 2\}$ as

$$s_{i,j,p,q}(k) = \begin{cases} i & \text{if } 0 \leq k < k_1 \\ q(k_a) & \text{if } k_a \leq k < k_{a+1}, \quad a = 1, \dots, e-1 \\ j & \text{if } k_e \leq k \leq n. \end{cases}$$

(2.7) $(B_{ij})_{pq} \neq 0 \Leftrightarrow (*, i, j, p, q)$. In this case

$$(B_{ij})_{pq} = \prod_{k=1}^n (A_{s(k-1)s(k)})_{p(k)q(k)}$$

with $s = s_{i,j,p,q}$.

PROOF. We have

$$(B_{ij})_{pq} = \sum_s \prod_{k=1}^n (A_{s(k-1)s(k)})_{p(k)q(k)}$$

where s runs over the maps $\{0, 1, \dots, n\} \rightarrow \{1, 2\}$ such that $s(0) = i$, $s(n) = j$. If the term for s is nonzero, we must have

$$\begin{aligned} p(k) \neq q(k) &\Rightarrow s(k-1) = p(k), \quad s(k) = q(k) \\ p(k) = q(k) &\Rightarrow s(k-1) = s(k) \end{aligned}$$

for $k = 1, \dots, n$. Then $(*, i, j, p, q)$ holds and $s = s_{i,j,p,q}$.

(2.8) If $(B_{11})_{pq} \neq 0$, then $p \leq q$. If $p \in \mathcal{P}(c, d)$, then $(B_{11})_{pp} = \alpha_1^c \alpha_2^d$.

PROOF. If $(*, 1, 1, p, q)$ holds, then

$$\#p^{-1}(1) \cap [1, k] - \#q^{-1}(1) \cap [1, k] = 0 \text{ or } 1$$

for $k \in [0, n]$. Hence $p \leq q$. The second assertion is clear.

(2.9) Let $p, q \in \mathcal{P}(n)$ with $h(p) + 1 = h(q)$. Then $(B_{11})_{pq} \neq 0$ if and only if $p < q$. In this case we have $(B_{11})_{pq} = \beta \gamma \alpha_1^{c-1} \alpha_2^{d-1}$, where $p, q \in \mathcal{P}(c, d)$.

PROOF. Assume $p < q$. Then $p \dashv q$. By (2.3) we have $\{k \mid p(k) \neq q(k)\} = \{m, m+1\}$ for some m and $p(m) = q(m+1) = 1$. Hence $(*, 1, 1, p, q)$ holds and $s_{1,1,p,q}$ takes value 2 only at m . So $(B_{11})_{pq} = \beta \gamma \alpha_1^{c-1} \alpha_2^{d-1}$ with $c = \#p^{-1}(1)$, $d = n - c$.

Similarly we have

(2.10) If $(B_{22})_{pq} \neq 0$, then $p \geq q$. If $p \in \mathcal{P}(c, d)$, then $(B_{22})_{pp} = \delta_1^c \delta_2^d$.

(2.11) Let $p, q \in \mathcal{P}(n)$ with $h(p) = h(q) + 1$. Then $(B_{22})_{pq} \neq 0$ if and only if $p > q$. In this case we have $(B_{22})_{pq} = \beta \gamma \delta_1^{c-1} \delta_2^{d-1}$, where $p, q \in \mathcal{P}(c, d)$.

(2.12) Suppose $p, p' \in \mathcal{P}(c, d)$ with $c + d = n$ and $h(p) = h(p')$.

- (i) $c > 0$, $(B_{12})_{p\tau(p')} \neq 0 \Rightarrow p \leq_l p'$.
- (ii) $c > 0$, $(B_{21})_{\tau(p)p'} \neq 0 \Rightarrow p \geq_l p'$.
- (iii) $d > 0$, $(B_{12})_{\rho(p)p'} \neq 0 \Rightarrow p \geq_r p'$.
- (iv) $d > 0$, $(B_{21})_{p\rho(p')} \neq 0 \Rightarrow p \leq_r p'$.
- (v) $c > 0 \Rightarrow (B_{12})_{p\tau(p)} \neq 0$, $(B_{21})_{\tau(p)p} \neq 0$.
- (vi) $d > 0 \Rightarrow (B_{12})_{\rho(p)p} \neq 0$, $(B_{21})_{p\rho(p)} \neq 0$.

PROOF. First observe that $(*, 1, 2, p, q)$ implies $(*, 1, 1, \tau(p), q)$ for any $p, q \in \mathcal{P}(n)$. Then

$$\begin{aligned} (B_{12})_{p\tau(p')} \neq 0 &\Rightarrow (*, 1, 2, p, \tau(p')) \Rightarrow (*, 1, 1, \tau(p), \tau(p')) \\ &\Rightarrow \tau(p) \leq \tau(p') \Rightarrow \tau(p) \leq_l \tau(p'). \end{aligned}$$

But if $p >_l p'$, then $\tau(p) >_l \tau(p')$ by (2.5). Thus (i) is true. Since

$$(B_{12})_{pq} \neq 0 \Leftrightarrow (B_{21})_{qp} \neq 0 \Leftrightarrow (B_{21})_{\bar{p}\bar{q}} \neq 0,$$

(ii), (iii), (iv) follow from (i). (v), (vi) are clear by (2.7).

We need two elementary lemmas in linear algebra.

LEMMA 2.13. Let $X = (x_{ij})$ be an $m \times m$ matrix such that if $x_{ij} \neq 0$, then $i \leq j$ and $x_{ii} = x_{jj}$. Then the diagonal matrix $\text{diag}(x_{11}, \dots, x_{mm})$ is a polynomial of X .

PROOF. Let $e_1, \dots, e_m$ be the standard basis of k^m . Let $\lambda_1, \dots, \lambda_d$ be the distinct diagonal entries of X . The subspaces $V_s := \langle e_i \mid x_{ii} = \lambda_s \rangle$ are stable under X for $s = 1, \dots, d$ and $k^m = \bigoplus_s V_s$. There is $m_s \in \mathbb{N}$ such that $(X - \lambda_s)^{m_s} = 0$ on V_s for each s . Take a polynomial $f(T)$ such that $f(T) \equiv \lambda_s \pmod{(T - \lambda_s)^{m_s}}$ for all s . Then $f(X) = \text{diag}(x_{11}, \dots, x_{mm})$.

LEMMA 2.14. Let $e \in \frac{1}{2}\mathbb{N}$ and let U be a vector space with decomposition $U = \bigoplus_l U_l$, where the sum is taken over the set $\{l \in \frac{1}{2}\mathbb{Z} \mid |l| \leq e\}$. Let $f, g \in \text{End}(U)$ satisfy the following conditions.

(a) For each m

$$f(U_m) \subset \bigoplus_{m-l \in \mathbb{N}_+} U_l, \quad g(U_m) \subset \bigoplus_{l-m \in \mathbb{N}_+} U_l.$$

(b) For each $l \in \frac{1}{2}\mathbb{N}$ such that $l \leq e$, the compositions

$$U_l \xrightarrow{\text{incl}} U \xrightarrow{f^{2l}} U \xrightarrow{\text{proj}} U_{-l}, \quad U_{-l} \xrightarrow{\text{incl}} U \xrightarrow{g^{2l}} U \xrightarrow{\text{proj}} U_l$$

are isomorphic.

If a subspace Z of U is stable under f and g , then $Z = \bigoplus_l (Z \cap U_l)$.

PROOF. We use the induction on e . The case $e = 0$ is clear. Assume $e > 0$. Then $f^{2e}g^{2e}$ (resp. $g^{2e}f^{2e}$) maps U_{-e} (resp. U_e) isomorphically onto itself and is zero on the other summands U_l . By the Cayley-Hamilton theorem, the linear span of positive powers of $f^{2e}g^{2e}$ (resp. $g^{2e}f^{2e}$) contains the projection onto U_{-e} (resp. U_e). Hence $Z = (Z \cap U_{-e}) \oplus (Z \cap U') \oplus (Z \cap U_e)$, where $U' = \bigoplus_{|l| < e} U_l$. Define $f', g' \in \text{End}(U')$ as

$$f' : U' \xrightarrow{\text{incl}} U \xrightarrow{f} U \xrightarrow{\text{proj}} U' \quad g' : U' \xrightarrow{\text{incl}} U \xrightarrow{g} U \xrightarrow{\text{proj}} U'.$$

Then $U', e - \frac{1}{2}, f', g', Z \cap U'$ satisfy the same conditions as U, e, f, g, Z . By the induction hypothesis we have $Z \cap U' = \bigoplus_{|l| < e} (Z \cap U_l)$. Hence $Z = \bigoplus_{|l| \leq e} (Z \cap U_l)$.

For any $f \in \text{End}(U)$ and $l, m \in \frac{1}{2}\mathbf{Z}$, the composite

$$U_m \xrightarrow{\text{incl}} U \xrightarrow{f} U \xrightarrow{\text{proj}} U_l$$

is written by f_{lm} . Let $B_{11}^+, B_{22}^+ \in \text{End}(U)$ be the maps whose matrices are obtained from those of B_{11}, B_{22} by changing their diagonal entries into zero respectively.

(2.15) (i) B_{11}, B_{22} satisfy the assumption of Lemma 2.13 when they are represented as matrices by means of the basis $\{v_p \mid p \in \mathcal{P}(n)\}$ and the total ordering $\leq_l$ on $\mathcal{P}(n)$.

(ii) B_{11}^+, B_{22}^+ satisfy the conditions (a), (b) of Lemma 2.14 with respect to the decomposition $U = \bigoplus_{|l| \leq l_0} U_l$.

PROOF. We prove these only for B_{11} . (i) If $(B_{11})_{pq} \neq 0$ for $p, q \in \mathcal{P}(n)$, then $p \leq q$. Hence $p \leq_l q$ and $\#p^{-1}(1) = \#q^{-1}(1)$, which implies $(B_{11})_{pp} = (B_{11})_{qq}$.

(ii) If $(B_{11})_{lm} \neq 0$ for $l, m \in \frac{1}{2}\mathbf{Z}$, then there are $p, q \in \mathcal{P}(n)$ such that $(B_{11})_{pq} \neq 0$, $h(p) = l$, $h(q) = m$. As $p \leq q$, we have $m - l \in \mathbf{N}$. Furthermore $(B_{11})_{ll}$ are diagonal matrices. Therefore if $(B_{11}^+)_{lm} \neq 0$, then $m - l \in \mathbf{N}_+$. Thus B_{11}^+ satisfies the condition for f in (a) of Lemma 2.14. Let $B_{11}^{(1)} \in \text{End}(U)$ be such that

$$(B_{11}^{(1)})_{lm} = \begin{cases} (B_{11})_{lm} & \text{if } l = m - 1 \\ 0 & \text{otherwise} \end{cases}$$

for $l, m \in \frac{1}{2}\mathbf{Z}$. Then

$$((B_{11}^+)^{2l})_{-l,l} = ((B_{11}^{(1)})^{2l})_{-l,l}$$

for $l \in \frac{1}{2}\mathbf{N}$. To prove that B_{11}^+ satisfies the condition for f in (b) of Lemma 2.14, it is enough to show that $(B_{11}^{(1)})^{2l}$ maps $U_l^{c,d}$ isomorphically onto $U_{-l}^{c,d}$ for each $l \in \frac{1}{2}\mathbf{N}$ and $c, d \in \mathbf{N}$ such that $c + d = n$. We may assume $0 < l \leq \frac{1}{2}cd$. By (2.9) we have

$$B_{11}^{(1)}(v_q) = \beta\gamma\alpha_1^{c-1}\alpha_2^{d-1} \sum_{p \perp q} v_p$$

for any $q \in \mathcal{P}(c, d)$. Then, by Theorem 2.1 $(B_{11}^{(1)})^{2l}$ restricted to $U^{c,d}$ has the desired property.

(2.16) Let $c, d \in \mathbf{N}$, $c + d = n$, $l \in \frac{1}{2}\mathbf{Z}$, $l > l_0$.

(i) If $c \geq n/2$, then

$$\bigoplus_{l' < l} U_{l'}^{c-1, d+1} \xrightarrow{\text{incl}} U^{c-1, d+1} \xrightarrow{B_{12}} U^{c, d} \xrightarrow{\text{proj}} U_l^{c, d}$$

is surjective and

$$U_l^{c, d} \xrightarrow{\text{incl}} U^{c, d} \xrightarrow{B_{21}} U^{c-1, d+1} \xrightarrow{\text{proj}} \bigoplus_{l' < l} U_{l'}^{c-1, d+1}$$

is injective.

(ii) If $d \geq n/2$, then

$$\bigoplus_{l' < l} U_{l'}^{c+1, d-1} \xrightarrow{\text{incl}} U^{c+1, d-1} \xrightarrow{B_{21}} U^{c, d} \xrightarrow{\text{proj}} U_l^{c, d}$$

is surjective and

$$U_l^{c, d} \xrightarrow{\text{incl}} U^{c, d} \xrightarrow{B_{12}} U^{c+1, d-1} \xrightarrow{\text{proj}} \bigoplus_{l' < l} U_{l'}^{c+1, d-1}$$

is injective.

PROOF. For all $p \in \mathcal{P}(c, d)_l$ we have by (2.6) that $h(\tau(p)) < l$ if $c \geq n/2$, and similarly that $h(\rho(p)) < l$ if $d \geq n/2$. Therefore the matrices of the above four maps contain respectively the square matrices of maximal size

$$((B_{12})_{p\tau(p')}), \quad ((B_{21})_{\tau(p)p'}), \quad ((B_{21})_{p\rho(p)}), \quad ((B_{12})_{\rho(p)p'}),$$

where p, p' range over $\mathcal{P}(c, d)_l$. By (2.12) these are triangular with respect to the ordering $\leq_l$ or $\leq_r$ on $\mathcal{P}(c, d)_l$ and have nonzero diagonal entries. Hence they are nonsingular. This proves (2.16).

(2.17) If $l > l_0$, then

$$(\bigoplus_{l' < l} U_{l'}) \oplus (\bigoplus_{l' < l} U_{l'}) \xrightarrow{\text{incl}} U \oplus U \xrightarrow{(B_{12}, B_{21})} U \xrightarrow{\text{proj}} U_l$$

is surjective and

$$U_l \xrightarrow{\text{incl}} U \xrightarrow{(B_{12}, B_{21})} U \oplus U \xrightarrow{\text{proj}} (\bigoplus_{l' < l} U_{l'}) \oplus (\bigoplus_{l' < l} U_{l'})$$

is injective.

PROOF. The first (resp. second) assertion follows by combining the first (resp. second) assertions of (2.16) (i) and (ii) for all pairs (c, d) with $c + d = n$.

PROOF OF THEOREM 1.1. Let Z be a subspace of U which is stable under $B_{11}, B_{12}, B_{21}, B_{22}$. By (2.15) (i) and Lemma 2.13 B_{11}^+, B_{22}^+ lie in the subalgebras $k[B_{11}], k[B_{22}]$ of $\text{End}(U)$ respectively. So Z is stable under B_{11}^+, B_{22}^+ . Then, by (2.15) (ii) and Lemma 2.14 we have $Z = \bigoplus_l Z_l$ with $Z_l \subset U_l$. Clearly $(B_{12})_{lm}, (B_{21})_{lm}$ send Z_m into Z_l for $l, m \in \frac{1}{2}\mathbf{Z}$. If n is even, then $\dim U_{l_0} = 1$. If n is odd, then $\dim U_{l_0} = 2$ and

$$(B_{12})_{l_0 l_0} = \begin{pmatrix} 0 & * \\ 0 & 0 \end{pmatrix}, \quad (B_{21})_{l_0 l_0} = \begin{pmatrix} 0 & 0 \\ * & 0 \end{pmatrix}$$

with $*$ nonzero. In any case U_{l_0} is a simple $((B_{12})_{l_0 l_0}, (B_{21})_{l_0 l_0})$ -module. Thus $Z_{l_0} = U_{l_0}$ or 0. In the first (resp. second) case we get inductively $Z_l = U_l$ (resp. $Z_l = 0$) for all l by the first (resp. second) half of (2.17). Thus $Z = U$ or 0, finishing the proof.

§3. Proof of Theorem 1.2

Let $E = V = k^2$ and identify $\text{End}(E) = \text{End}(V) = M_2(k)$. Let $x_{ij} \in \text{End}(E)^*$, $y_{ij} \in \text{End}(V)^*$ be the dual matrix units. Let $\varphi : \text{End}(E)^* \rightarrow \text{End}(V)$ be the map $x_{ij} \mapsto A_{ij}$ of Section 2 and $\varphi : T(\text{End}(E)^*) \rightarrow \text{End}(V)$ the algebra map extending φ . The tensor power representations of $\tilde{\varphi}$ make up the algebra map $\Phi : T(\text{End}(E)^*) \rightarrow \prod_{n \geq 0} \text{End}(V^{\otimes n})$. The transpose $\psi : \text{End}(V)^* \rightarrow \text{End}(E)$ of φ induces an algebra map $\Psi : T(\text{End}(V)^*) \rightarrow \prod_{n \geq 0} \text{End}(E^{\otimes n})$ similarly. One sees easily that the components

$$\varphi_{nm} : \text{End}(E)^{* \otimes m} \rightarrow \text{End}(V^{\otimes n})$$

$$\psi_{mn} : \text{End}(V)^{* \otimes n} \rightarrow \text{End}(E^{\otimes m})$$

of Φ and Ψ are dual to each other for $m, n \in \mathbf{N}$. We have therefore the following equivalences.

Φ is injective.

$\Leftrightarrow$ For any $m \in \mathbf{N}$ there exists $n \in \mathbf{N}$ such that the composite

$$\begin{array}{ccc} T(\text{End}(E)^*) & \xrightarrow{\Phi} & \prod_{k \geq 0} \text{End}(V^{\otimes k}) \\ \text{incl} \uparrow & & \downarrow \text{proj} \\ \bigoplus_{k=0}^m \text{End}(E)^{* \otimes k} & & \prod_{k=0}^n \text{End}(V^{\otimes k}) \end{array}$$

is injective.

$\Leftrightarrow$ For any $m \in \mathbf{N}$ there exists $n \in \mathbf{N}$ such that the composite

$$\begin{array}{ccc} T(\text{End}(V)^*) & \xrightarrow{\Psi} & \prod_{k \geq 0} \text{End}(E^{\otimes k}) \\ \text{incl} \uparrow & & \downarrow \text{proj} \\ \bigoplus_{k=0}^n \text{End}(V)^{* \otimes k} & & \prod_{k=0}^m \text{End}(E^{\otimes k}) \end{array}$$

is surjective.

$\Leftrightarrow$ For any $m \in \mathbf{N}$ the map

$$T(\text{End}(V)^*) \xrightarrow{\Psi} \prod_{k \geq 0} \text{End}(E^{\otimes k}) \xrightarrow{\text{proj}} \prod_{k=0}^m \text{End}(E^{\otimes k})$$

is surjective.

$\Leftrightarrow$ The $T(\text{End}(V)^*)$ -modules $E^{\otimes k}$ for all $k \in \mathbf{N}$ are simple and mutually nonisomorphic.

Now the matrices $\psi(y_{ij})$ have forms similar to $\varphi(x_{ij}) = A_{ij}$. We can therefore apply Theorem 1.1 to ψ and conclude that $E^{\otimes k}$ are simple $T(\text{End}(V)^*)$ -modules. Thus Φ is injective and Theorem 1.2 is proved.

Finally we show that Theorems 1.1, 1.2 are also true if $\alpha_2 = \delta_1 = 0$. Here $\text{char } k$ is arbitrary. Let $E = V = k^d$ with $d > 1$ and let $\varphi : \text{End}(E)^* \rightarrow \text{End}(V)$ be the map $x_{ij} \mapsto \alpha_{ij} e_{ij}$ with $\alpha_{ij} \in k - \{0\}$. With the above notation, the map $\varphi_{nn} : \text{End}(E)^{* \otimes n} \rightarrow \text{End}(V)^{\otimes n}$ turns out to be $x_{i_1 j_1} \otimes \cdots \otimes x_{i_n j_n} \mapsto (\text{scalar}) e_{i_1 j_1} \otimes \cdots \otimes e_{i_n j_n}$, where

$$\text{scalar} = \alpha_{i_1 i_2} \alpha_{i_2 i_3}^2 \cdots \alpha_{i_n j_1}^n \cdots \alpha_{j_{n-2} j_{n-1}}^2 \alpha_{j_{n-1} j_n} \neq 0.$$

So φ_{nn} is bijective and the $T(\text{End}(E)^*)$ -module $V^{\otimes n}$ is simple for any n . The faithfulness of $\bigoplus_n V^{\otimes n}$ is proved in the same way as above.

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