On the exponential decay of the correlation functions for KMO-Langevin equations

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§1. Introduction

We consider the first continuous KMO-Langevin equation

$$(1.1) \quad \dot{\mathbf{X}} = -\beta_1 \mathbf{X} - \lim_{\varepsilon \downarrow 0} \gamma_{1,\varepsilon} * \dot{\mathbf{X}} + \alpha_1 \dot{\mathbf{B}}$$

and the second continuous KMO-Langevin equation

$$(1.2) \quad \dot{\mathbf{X}} = -\beta_2 \mathbf{X} - \lim_{\varepsilon \downarrow 0} \gamma_{2,\varepsilon} * \dot{\mathbf{X}} + \alpha_2 \mathbf{I}.$$

Here, for $j = 1, 2$, α_j and β_j are positive constants and for $t \in \mathbf{R}$ and $\varepsilon > 0$,

$$(1.3) \quad \gamma_{j,\varepsilon}(t) = \chi_{(0,\infty)}(t) \int_{\varepsilon}^{\infty} e^{-t\lambda} \rho_j(d\lambda)$$

with a Borel measure ρ_j on $[0, \infty)$ satisfying

$$(1.4) \quad \rho_j(\{0\}) = 0, \quad \int_0^{\infty} (\lambda + 1)^{-1} \rho_j(d\lambda) < \infty.$$

$\dot{\mathbf{B}}$ is a Gaussian white noise and $\mathbf{I}$ is a stationary Gaussian random tempered distribution named the *Kubo noise* associated with the solution of (1.2) (see [3]).

The equation (1.1) (resp. (1.2)) has a unique solution $X_1 = (X_1(t); t \in \mathbf{R})$ (resp. $\mathbf{X}_2 = (X_2(t); t \in \mathbf{R})$) which is a real stationary Gaussian process with mean zero and covariance function R_1 (resp. R_2) of the form

$$(1.5) \quad R_j(t) = \int_0^{\infty} e^{-|t|\lambda} \sigma_j(d\lambda) \quad (t \in \mathbf{R}).$$

Here σ_j is a non-zero bounded Borel measure on $[0, \infty)$ satisfying

$$(1.6) \quad \sigma_1(\{0\}) = 0, \quad \int_0^{\infty} (\lambda + \lambda^{-1}) \sigma_1(d\lambda) < \infty,$$

$$(1.7) \quad \sigma_2(\{0\}) = 0, \quad \int_0^{\infty} \lambda^{-1} \sigma_2(d\lambda) < \infty.$$

In fact, σ_j is given by $\sigma_j = L_j^{-1}((\alpha_j, \beta_j, \rho_j))$, where L_1 and L_2 are defined by (3.2) and (2.3) respectively. The property (1.5) is called the *reflection positivity*. Conversely, it is known that any real stationary Gaussian process $\mathbf{X}$ with the reflection positivity becomes a solution of both (1.1) and (1.2) for some adequate triples $(\alpha_1, \beta_1, \gamma_1)$ and $(\alpha_2, \beta_2, \gamma_2)$, respectively. For the details, see [3].

We put for $j = 1, 2$ and $t \in \mathbf{R}$,

$$(1.8) \quad \gamma_j(t) = \chi_{(0, \infty)}(t) \int_0^\infty e^{-t\lambda} \rho_j(d\lambda).$$

In [7] and [1], the long time tail behavior of R_1 (resp. R_2) was characterized by γ_1 (resp. γ_2) as follows: when $t \rightarrow \infty$,

$$(1.9) \quad \gamma_1(t) \sim t^{-p} L(t) \iff R_1(t) \sim \frac{\alpha_1^2 p}{\beta_1^3} t^{-(1+p)} L(t),$$

$$(1.10) \quad \gamma_2(t) \sim t^{-p} L(t) \iff R_2(t) \sim \frac{\sqrt{2\pi} \alpha_2 p}{\beta_2^2} t^{-(1+p)} L(t),$$

where $0 < p < \infty$ and L is a slowly varying function at infinity.

In this paper, we show that the exponential decay of R_j can be also characterized by the decay of γ_j . We put

$$(1.11) \quad \lambda_0(\sigma_j) = \inf\{\lambda; \lambda \in \text{supp } \sigma_j\}, \quad \lambda_0(\rho_j) = \inf\{\lambda; \lambda \in \text{supp } \rho_j\}.$$

Then we see that

$$(1.12) \quad \lim_{t \rightarrow \infty} \frac{\log R_j(t)}{t} = -\lambda_0(\sigma_j), \quad \lim_{t \rightarrow \infty} \frac{\log \gamma_j(t)}{t} = -\lambda_0(\rho_j)$$

(Proposition 3.1). Using (1.12), we show the following theorem in §3.

THEOREM 1.1. *Let $j = 1$ or 2 . Then the following (i) and (ii) are equivalent:*

- (i)
$$\lim_{t \rightarrow \infty} \frac{\log R_j(t)}{t} < 0.$$
- (ii)
$$\lim_{t \rightarrow \infty} \frac{\log \gamma_j(t)}{t} < 0.$$

Theorem 1.1 shows that the correlation function R_j decays exponentially if and only if γ_j decays exponentially.

We show similar facts also for the discrete KMO-Langevin equations.

§2. Relation between the supports of two measures

First we recall the notation of [3]:

$$(2.1) \quad \Sigma_2 = \left\{ \sigma; \sigma \text{ is a Borel measure on } [0, \infty) \text{ such that } \sigma(\{0\}) = 0, \right. \\ \left. 0 < \sigma([0, \infty)) < \infty \text{ and } \int_0^\infty \lambda^{-1} \sigma(d\lambda) < \infty \right\},$$

$$(2.2) \quad \mathcal{L}_1 = \left\{ (\alpha, \beta, \rho); \alpha > 0, \beta > 0 \text{ and } \rho \text{ is a Borel measure on } [0, \infty) \right. \\ \left. \text{such that } \rho(\{0\}) = 0 \text{ and } \int_0^\infty \frac{1}{1+t} \rho(dt) < \infty \right\}.$$

By Theorem 8.5 in [3], the relation

$$(2.3) \quad \frac{1}{2\pi} \int_0^\infty \frac{1}{\lambda - i\zeta} \sigma(d\lambda) = \frac{\alpha}{\sqrt{2\pi}} \left(\beta - i\zeta - i\zeta \int_0^\infty \frac{1}{\lambda - i\zeta} \rho(d\lambda) \right)^{-1} \quad (\zeta \in \mathbb{C}^+)$$

defines a bijection $\sigma \rightarrow L_2(\sigma) = (\alpha, \beta, \rho)$ from Σ_2 onto $\mathcal{L}_1$.

In this section, we show a relation between the supports of two measures σ and ρ in (2.3). Our principal tool is the approximation method by discrete measures. We define

$$(2.4) \quad \mathcal{L}_d = \left\{ (\alpha, \beta, \rho) \in \mathcal{L}_1; \rho = \sum_{n=1}^N \rho_n \delta_{q_n} \text{ for some } N \in \mathbb{N}, \text{ where} \right. \\ \left. \rho_n > 0 \ (n = 1, \dots, N) \text{ and } 0 < q_1 < q_2 < \dots < q_N < \infty \right\},$$

$$(2.5) \quad \Sigma_d = \left\{ \sigma \in \Sigma_2; \sigma = \sum_{n=0}^N \sigma_n \delta_{p_n} \text{ for some } N \in \mathbb{N}, \text{ where} \right. \\ \left. \sigma_n > 0 \ (n = 0, \dots, N) \text{ and } 0 < p_0 < p_1 < \dots < p_N < \infty \right\}.$$

PROPOSITION 2.1. (i) $L_2(\Sigma_d) = \mathcal{L}_d$.

(ii) Let $\sigma \in \Sigma_d$ and $(\alpha, \beta, \rho) = L_2(\sigma)$. Then we have for some $N \in \mathbb{N}$,

$$(2.6) \quad \sigma = \sum_{n=0}^N \sigma_n \delta_{p_n}, \quad \sigma_n > 0 \ (n = 0, \dots, N),$$

$$(2.7) \quad \rho = \sum_{n=1}^N \rho_n \delta_{q_n}, \quad \rho_n > 0 \ (n = 1, \dots, N)$$

with

$$(2.8) \quad 0 < p_n < q_{n+1} < p_{n+1} \quad (n = 0, \dots, N-1).$$

PROOF. (i) First let $\sigma = \sum_{n=0}^N \sigma_n \delta_{p_n} \in \Sigma_d$. We put

$$(2.9) \quad A(z) = \int_{[0, \infty)} \frac{1}{\lambda - z} \sigma(d\lambda) \quad (\operatorname{Re} z < 0).$$

Then

$$(2.10) \quad A(z) = \sum_{n=0}^N \frac{\sigma_n}{p_n - z} = f(z) \Big/ \prod_{n=0}^N (p_n - z),$$

where

$$f(z) = \sum_{n=0}^N \sigma_n \prod_{k \neq n} (p_k - z) = (-1)^N \sigma([0, \infty)) z^N + \dots.$$

Since $f(p_n) = \sigma_n \prod_{k \neq n} (p_k - p_n)$, we have $\operatorname{sgn}(f(p_n)) = (-1)^n$ ($n = 0, \dots, N$). Therefore, there exist N positive numbers q_n ($n = 1, \dots, N$) such that both (2.8) and the following hold:

$$f(z) = \sigma([0, \infty)) \prod_{n=1}^N (q_n - z).$$

We put $\alpha = \sigma([0, \infty))/\sqrt{2\pi}$, $\beta = \prod_{n=0}^N p_n / \prod_{n=1}^N q_n$ and

$$(2.11) \quad \frac{\prod_{n=0}^N (p_n - z)}{\prod_{n=1}^N (q_n - z)} - (\beta - z) = \frac{h(z)}{\prod_{n=1}^N (q_n - z)},$$

where

$$h(z) = \prod_{n=0}^N (p_n - z) - (\beta - z) \prod_{n=1}^N (q_n - z).$$

Since $h(0) = 0$, $k(z) = h(z)/z$ is a polynomial of degree $\leq N-1$ and it holds that

$$(2.12) \quad \frac{k(z)}{\prod_{n=1}^N (q_n - z)} = - \sum_{n=1}^N \frac{\rho_n}{q_n - z},$$

where

$$\rho_n = - \frac{k(q_n)}{\prod_{k \neq n} (q_k - q_n)} = - \frac{\prod_{k=0}^N (p_k - q_n)}{q_n \prod_{k \neq n} (q_k - q_n)} > 0, \quad (n = 1, \dots, N).$$

Thus if we put $\rho = \sum_{n=1}^N \rho_n \delta_{q_n}$, σ and (α, β, ρ) satisfy (2.3), so that $L_2(\sigma) = (\alpha, \beta, \rho)$ follows.

(ii) Conversely, let $(\alpha, \beta, \rho) \in \mathcal{L}_d$. Then ρ is of the form $\rho = \sum_{n=1}^N \rho_n \delta_{q_n}$. We put

$$(2.13) \quad B(z) = \alpha \sqrt{2\pi} \left(\beta - z - z \int_0^\infty \frac{1}{\lambda - z} \rho(d\lambda) \right)^{-1}.$$

Then we have

$$B(z) = \alpha \sqrt{2\pi} \prod_{n=1}^N (q_n - z) / g(z),$$

where

$$g(z) = (\beta - z) \prod_{n=1}^N (q_n - z) - z \sum_{n=1}^N \rho_n \prod_{k \neq n} (q_k - z) = (-1)^{N+1} z^{N+1} + \dots.$$

Since $g(q_n) = -q_n \rho_n \prod_{k \neq n} (q_k - q_n)$, it holds that $\text{sgn}(g(q_n)) = (-1)^n$ ($n = 1, \dots, N$). We also see that $\text{sgn}(g(0)) = 1$. Therefore, there exist $N+1$ positive numbers p_n ($n = 0, \dots, N$) such that both (2.8) and the following hold:

$$g(z) = \prod_{n=0}^N (p_n - z).$$

Then we have

$$(2.14) \quad B(z) = \frac{\alpha \sqrt{2\pi} \prod_{n=1}^N (q_n - z)}{\prod_{n=0}^N (p_n - z)} = \sum_{n=0}^N \frac{\sigma_n}{p_n - z},$$

where

$$\sigma_n = \alpha \sqrt{2\pi} \prod_{k=1}^N (q_k - p_n) / \prod_{k \neq n} (p_k - p_n) > 0 \quad (n = 0, \dots, N).$$

Thus if we put $\sigma = \sum_{n=0}^N \sigma_n \delta_{p_n}$, then (2.3) and therefore $\sigma = L_2^{-1}((\alpha, \beta, \rho))$ hold. This completes the proof of the proposition. $\blacksquare$

Let I be an interval of $\mathbf{R}$. Let σ be a bounded Borel measure on I . We denote by $\lambda_0(\sigma)$ and $\lambda_\infty(\sigma)$ the infimum and the supremum of $\text{supp } \sigma$, respectively:

$$(2.15) \quad \lambda_0(\sigma) = \inf\{\lambda \in I; \lambda \in \text{supp } \sigma\}, \quad \lambda_\infty(\sigma) = \sup\{\lambda \in I; \lambda \in \text{supp } \sigma\}.$$

THEOREM 2.2. *Let $\sigma \in \Sigma_2$ and $(\alpha, \beta, \rho) = L_2(\sigma)$. Then*

$$(2.16) \quad \lambda_0(\sigma) \leq \lambda_0(\rho) \leq \lambda_\infty(\rho) \leq \lambda_\infty(\sigma).$$

PROOF. The proof of the third inequality is almost the same as the first inequality, so that we prove only the first one. We may assume $\lambda_0(\sigma) > 0$. We choose $\sigma^{(n)} \in \Sigma_d$ ($n = 1, 2, \dots$) so that $\text{supp } \sigma^{(n)} \subset [\lambda_0(\sigma), \infty)$ and $\text{w-lim}_{n \rightarrow \infty} \sigma^{(n)} = \sigma$ on $[\lambda_0(\sigma), \infty]$. Here we regard $\sigma^{(n)}$ and σ as bounded Borel measures on $[\lambda_0(\sigma), \infty]$ by $\sigma^{(n)}(\infty) = \sigma(\infty) = 0$. If we put $L_2(\sigma^{(n)}) = (\alpha^{(n)}, \beta^{(n)}, \rho^{(n)})$, then it holds that

$$(2.17) \quad \int_{[\lambda_0(\sigma), \infty)} \frac{1}{\lambda + 1} \rho^{(n)}(d\lambda) = \frac{\alpha^{(n)} \sqrt{2\pi}}{\int_{[\lambda_0(\sigma), \infty)} (\lambda + 1)^{-1} \sigma^{(n)}(d\lambda)} - \beta^{(n)} - 1.$$

In view of Theorem 8.2 in [3], we have

$$\lim_{n \rightarrow \infty} \alpha^{(n)} = \frac{\sigma([\lambda_0(\sigma), \infty))}{\sqrt{2\pi}} = \alpha, \quad \lim_{n \rightarrow \infty} \beta^{(n)} = \frac{\sigma([\lambda_0(\sigma), \infty))}{\int_{[\lambda_0(\sigma), \infty)} \lambda^{-1} \sigma(d\lambda)} = \beta.$$

Furthermore, we have

$$\lim_{n \rightarrow \infty} \int_{[\lambda_0(\sigma), \infty)} \frac{1}{\lambda + 1} \sigma^{(n)}(d\lambda) = \int_{[\lambda_0(\sigma), \infty)} \frac{1}{\lambda + 1} \sigma(d\lambda).$$

Then if we put $\tilde{\rho}^{(n)} = (\lambda + 1)^{-1} \rho^{(n)}$, it holds that $\sup_n \tilde{\rho}^{(n)}([\lambda_0(\sigma), \infty]) < \infty$. Here we put $\tilde{\rho}^{(n)}(\{\infty\}) = 0$. Therefore there exist a subsequence n_k ($k = 1, 2, \dots$) and a bounded Borel measure $\tilde{\rho}$ on $[\lambda_0(\sigma), \infty]$ such that

$$\text{w-} \lim_{k \rightarrow \infty} \tilde{\rho}^{(n_k)} = \tilde{\rho} \quad \text{on } [\lambda_0(\sigma), \infty].$$

We put $\rho' = (\lambda + 1)\tilde{\rho}$. Then using this subsequence, we obtain for any $\zeta \in \mathbf{C}^+$,

$$\begin{aligned} & \frac{1}{2\pi} \int_{[\lambda_0(\sigma), \infty)} \frac{1}{\lambda - i\zeta} \sigma(d\lambda) \\ &= \frac{\alpha}{\sqrt{2\pi c}} \left(\frac{\beta}{c} - i\zeta - i\zeta \int_{[\lambda_0(\sigma), \infty)} \frac{1}{\lambda - i\zeta} \frac{\rho'}{c}(d\lambda) \right)^{-1}, \end{aligned}$$

where $c = \tilde{\rho}(\{\infty\}) + 1$. Then it follows from the uniqueness that $c = 1$ and $\rho = \rho'$, so that $\lambda_0(\sigma) \leq \lambda_0(\rho)$. This completes the proof of the theorem. $\blacksquare$

THEOREM 2.3. Let $\sigma \in \Sigma_2$ and $L_2(\sigma) = (\alpha, \beta, \rho)$.

- (i) If $\lambda_0(\rho) > 0$, then for some $\sigma_0 \geq 0$, $p_0 \in (0, \lambda_0(\rho))$, and $\mu_1 \in \Sigma_2$ with $\text{supp } \mu_1 \subset [\lambda_0(\rho), \infty)$, it holds that $\sigma = \sigma_0 \delta_{p_0} + \mu_1$.
- (ii) If $\lambda_\infty(\rho) < \infty$, then for some $\sigma_\infty \geq 0$, $p_\infty \in (\lambda_\infty(\rho), \infty)$, and $\mu_2 \in \Sigma_2$ with $\text{supp } \mu_2 \subset [0, \lambda_\infty(\rho)]$ it holds that $\sigma = \mu_2 + \sigma_\infty \delta_{p_\infty}$.

PROOF. The proof of (ii) is almost the same as (i), so that we prove only (i). We put $\tilde{\rho} = (\lambda + 1)^{-1}\rho$. We regard ρ as a bounded Borel measure on $[\lambda_0(\rho), \infty]$ by $\tilde{\rho}(\{\infty\}) = 0$. We choose $\tilde{\rho}^{(n)}$ ($n = 1, 2, \dots$) so that $(\alpha, \beta, \tilde{\rho}^{(n)}) \in \mathcal{L}_d$, $\text{supp } \tilde{\rho}^{(n)} \subset [\lambda_0(\rho), \infty)$ and $\text{w-lim}_{n \rightarrow \infty} \tilde{\rho}^{(n)} = \tilde{\rho}$ on $[\lambda_0(\rho), \infty]$. By considering $\tilde{\rho}^{(n)} + n^{-1}\delta_{\lambda_0(\rho)}$ instead of $\tilde{\rho}^{(n)}$ if necessary, we may assume that $\lambda_0(\rho) \in \text{supp}(\tilde{\rho}^{(n)})$ ($n = 1, 2, \dots$). We put $\rho^{(n)} = (\lambda + 1)\tilde{\rho}^{(n)}$. Then $\rho^{(n)}$ is of the form

$$\rho^{(n)} = \sum_{k=1}^{N^{(n)}} \rho_k^{(n)} \delta_{q_k^{(n)}}$$

with

$$\lambda_0(\rho) = q_1^{(n)} < q_2^{(n)} < \dots < q_{N^{(n)}}^{(n)}, \quad \rho_k^{(n)} > 0 \quad (k = 1, 2, \dots, N^{(n)}).$$

We put $\sigma^{(n)} = L_2^{-1}((\alpha, \beta, \rho^{(n)}))$. By Proposition 2.1, we see that

$$\sigma^{(n)} = \sigma_0^{(n)} \delta_{p_0^{(n)}} + \mu_1^{(n)},$$

where $\sigma_0^{(n)} > 0$, $0 < p_0^{(n)} < \lambda_0(\rho)$ and $\mu_1^{(n)} \in \Sigma_d$ with $\text{supp } \mu_1^{(n)} \subset [\lambda_0(\rho), \infty)$. It follows from Theorem 8.2 in [3] that $\sigma^{(n)}([0, \infty)) = \sqrt{2\pi}\alpha$, and so there exist a subsequence n_k ($k = 1, 2, \dots$), constants $\sigma_0 \geq 0$, $p_0 \in [0, \lambda_0(\rho)]$ and a bounded Borel measure $\mu'_1 \in \Sigma_2$ with $\text{supp } \mu'_1 \subset [\lambda_0(\rho), \infty]$ such that

$$\lim_{k \rightarrow \infty} \sigma_0^{(n_k)} = \sigma_0, \quad \lim_{k \rightarrow \infty} p_0^{(n_k)} = p_0, \quad \text{w-}\lim_{k \rightarrow \infty} \mu_1^{(n_k)} = \mu'_1 \quad \text{on } [\lambda_0(\rho), \infty].$$

We define $\mu_1 \in \Sigma_2$ by $\mu_1(d\lambda) = \mu'_1(d\lambda \cap [\lambda_0(\rho), \infty))$. Then letting $k \rightarrow \infty$, we obtain

$$\begin{aligned} & \frac{\sigma_0}{2\pi(p_0 - i\zeta)} + \frac{1}{2\pi} \int_{[\lambda_0(\rho), \infty)} \frac{1}{\lambda - i\zeta} \mu_1(d\lambda) \\ &= \frac{\alpha}{\sqrt{2\pi}} \left(\beta - i\zeta - i\zeta \int_{[\lambda_0(\rho), \infty)} \frac{\lambda + 1}{\lambda - i\zeta} \tilde{\rho}(d\lambda) \right)^{-1} \quad (\zeta \in \mathbf{C}^+). \end{aligned}$$

If $p_0 = 0$, we obtain, by substituting iy into ζ in (2.3),

$$\frac{\sigma_0}{y} \leq \frac{\alpha\sqrt{2\pi}}{\beta} \quad (y > 0),$$

which shows $\sigma_0 = 0$. Thus we see that $\sigma = \sigma_0 \delta_{p_0} + \mu_1$ and this completes the proof. $\blacksquare$

§3. Characterization of the exponential decay (1)

In this section, we consider the continuous KMO-Langevin equations and prove Theorem 1.1.

First we recall additional notations in [3]. Let

$$(3.1) \quad \Sigma_1 = \left\{ \sigma; \sigma \text{ is a Borel measure on } [0, \infty) \text{ such that } \sigma(\{0\}) = 0, \right. \\ \left. 0 < \sigma([0, \infty)) < \infty \text{ and } \int_0^\infty (\lambda + \lambda^{-1}) \sigma(d\lambda) < \infty \right\}.$$

Then by Theorems 2.1, 3.1, Lemma 2.6 and (2.19) in [3], the relation

$$(3.2) \quad \sigma(d\lambda) = \frac{1}{2\pi} \left(\int_0^\infty \frac{1}{\lambda + \lambda'} \nu(d\lambda') \right) \nu(d\lambda), \quad L_2(\nu) = (\alpha, \beta, \rho)$$

defines a bijection $\sigma \rightarrow L_1(\sigma) = (\alpha, \beta, \rho)$ from Σ_1 onto $\mathcal{L}_1$.

THEOREM 3.1. *Let $j = 1$ or 2 . Let $\sigma_j \in \Sigma_j$ and $L_j(\sigma_j) = (\alpha_j, \beta_j, \rho_j)$. Then*

- (i) $\lambda_0(\sigma_j) > 0$ if and only if $\lambda_0(\rho_j) > 0$.
- (ii) $\lambda_\infty(\sigma_j) < \infty$ if and only if $\lambda_\infty(\rho_j) < \infty$.

PROOF. The case of $j = 2$ follows immediately from Theorems 2.2 and 2.3. Next we consider the case of $j = 1$. We put $\nu = L_2^{-1}((\alpha_1, \beta_1, \rho_1))$. Then it follows from the case of $j = 2$ that

$$\lambda_0(\nu) > 0 \iff \lambda_0(\rho_1) > 0 \quad \text{and} \quad \lambda_\infty(\nu) < \infty \iff \lambda_\infty(\rho_1) < \infty.$$

On the other hand, by (3.2), we see that $\text{supp } \sigma_1 = \text{supp } \nu$. Thus the case of $j = 1$ follows and this completes the proof. $\blacksquare$

PROPOSITION 3.2. *Let σ be a bounded Borel measure on $[0, \infty)$. We put for $t \in \mathbf{R}$, $R(t) = \int_0^\infty e^{-t\lambda} \sigma(d\lambda)$. Then we have*

$$(3.3) \quad \lim_{t \rightarrow \infty} \frac{\log R(t)}{t} = -\lambda_0(\sigma).$$

PROOF. For any $\varepsilon > 0$, take positive numbers a, b such that $\lambda_0(\sigma) \leq a < b < \lambda_0(\sigma) + \varepsilon$ and $\sigma([a, b]) > 0$. Since

$$e^{-t(\lambda_0(\sigma) + \varepsilon)} \sigma([a, b]) \leq R(t) = \int_{\lambda_0(\sigma)}^\infty e^{-t\lambda} \sigma(d\lambda) \leq e^{-t\lambda_0(\sigma)} \sigma([\lambda_0(\sigma), \infty)),$$

we have

$$-(\lambda_0(\sigma) + \varepsilon) \leq \liminf_{t \rightarrow \infty} \frac{\log R(t)}{t} \leq \limsup_{t \rightarrow \infty} \frac{\log R(t)}{t} \leq -\lambda_0(\sigma).$$

Since ε is arbitrary, we obtain (3.1). $\blacksquare$

PROOF OF THEOREM 1.1. In view of Proposition 3.2, it is enough to show that $\lambda_0(\sigma_j) > 0 \iff \lambda_0(\rho_j) > 0$ but this is a straight consequence of Theorem 3.1. Thus the proof of the theorem is complete. $\blacksquare$

§4. Characterization of the exponential decay (2)

In this section, we show a result which is similar to Theorem 1.1 for the discrete KMO-Langevin equations.

We define

$$(4.1) \quad \Sigma_1^d = \left\{ \sigma; \sigma \text{ is a bounded Borel measure on } [-1, 1] \text{ such that} \right.$$

$$\left. \sigma(\{-1, 1\}) = 0 \text{ and } \int_{-1}^1 \left(\frac{1}{1+t} + \frac{1}{1-t} \right) \sigma(dt) < \infty \right\},$$

$$(4.2) \quad \mathcal{L}_1^d = \{(\alpha, \beta, \rho); \alpha > 0, \beta > 0 \text{ and } \rho \text{ is a bounded}$$

$$\text{Borel measure on } [-1, 1] \text{ such that } \rho(\{-1, 1\}) = 0\}.$$

By Theorem 3.1 in [6], the relation

$$(4.3) \quad \frac{1}{2\pi} \int_{-1}^1 \frac{1}{1-tz} \sigma(dt) \\ = \frac{\alpha}{\sqrt{2\pi}} \left(\beta(1+z) + 1-z + (1-z^2) \int_{-1}^1 \frac{1}{1-tz} \rho(dt) \right)^{-1} \quad (|z| < 1)$$

defines a bijection $\sigma \rightarrow L_2(\sigma) = (\alpha, \beta, \rho)$ from Σ_1^d onto $\mathcal{L}_1^d$. Furthermore, by Theorems 4.1, 4.2 and 4.4 in [5], we see that the relation

$$(4.4) \quad \sigma(dt) = \frac{1}{2\pi} \left(\int_{-1}^1 \frac{1}{1-ts} \nu(ds) \right) \nu(dt), \quad L_2(\nu) = (\alpha, \beta, \rho)$$

defines another bijection $\sigma \rightarrow L_1(\sigma) = (\alpha, \beta, \rho)$ from Σ_1^d onto $\mathcal{L}_1^d$.

Now we consider the first discrete KMO-Langevin equation

$$(4.5) \quad X(n) - X(n-1) \\ = -\beta_1(X(n) + X(n-1)) - (\gamma_1 * X)(n) + \alpha_1 \xi(n) \quad \text{a.s. } (n \in \mathbf{Z})$$

and the second discrete KMO-Langevin equation

$$(4.6) \quad X(n) - X(n-1) \\ = -\beta_2(X(n) + X(n-1)) - (\gamma_2 * X)(n) + \alpha_2 I(n) \quad \text{a.s. } (n \in \mathbf{Z}).$$

Here for $j = 1, 2$,

$$(4.7) \quad \gamma_j(n) = \begin{cases} 0 & (n = -1, -2, \dots) \\ \int_{-1}^1 t^n \rho_j(dt) & (n = 0, 1) \\ \int_{-1}^1 t^{n-2}(t^2 - 1)\rho_j(dt) & (n = 2, 3, \dots) \end{cases}$$

with $(\alpha_j, \beta_j, \rho_j) \in \mathcal{L}_1^d$. $\Xi = (\xi(n); n \in \mathbf{Z})$ in (4.5) is a normalized Gaussian white noise or a sequence of independent Gaussian random variables with mean 0 and variance 1. $\mathbf{I} = (I(n); n \in \mathbf{Z})$ in (4.6) is a real stationary Gaussian process named the *Kubo noise* associated with the solution $\mathbf{X} = (X(n); n \in \mathbf{Z})$ of (4.6) (see [6]).

The equation (4.5) (resp. (4.6)) has a unique real stationary solution $\mathbf{X}_1 = (X_1(n); n \in \mathbf{Z})$ (resp. $\mathbf{X}_2 = (X_2(n); n \in \mathbf{Z})$) with mean 0 and covariance R_1 (resp. R_2) of the form

$$(4.8) \quad R_j(n) = \int_{-1}^1 t^{|n|} \sigma_j(dt) \quad (n \in \mathbf{Z}),$$

where $\sigma_j = L_j^{-1}((\alpha_j, \beta_j, \rho_j)) \in \Sigma_1^d$ ($j = 1, 2$). The property (4.8) is called the *reflection positivity*.

In this section, we use the following notation: a real sequence $(a(n))_{n=0}^\infty$ is said to be of *exponential decay* if there exist two constants $c > 0$ and $p \in (0, 1)$ such that

$$|a(n)| \leq cp^n \quad (n = 0, 1, \dots).$$

In the same way as Proposition 2.1, we obtain

PROPOSITION 4.1. *Let σ be a bounded Borel measure on $[-1, 1]$ such that $\sigma(\{-1, 1\}) = 0$. We put $R(n) = \int_{-1}^1 t^n \sigma(dt)$ ($n = 0, 1, \dots$). Then $(R(n))_{n=0}^\infty$ is of exponential decay if and only if*

$$(4.9) \quad -1 < \lambda_0(\sigma) \quad \text{and} \quad \lambda_\infty(\sigma) < 1.$$

The following theorem is a discrete analogue to Theorem 1.1.

THEOREM 4.2. *Let $j = 1$ or 2 . Then the following (i) and (ii) is equivalent:*

- (i) $(\gamma_j(n))_{n=0}^\infty$ is of exponential decay.
- (ii) $(R_j(n))_{n=0}^\infty$ is of exponential decay.

PROOF. Let $\sigma_j \in \Sigma_1^d$ and $(\alpha_j, \beta_j, \rho_j) = L_j(\sigma_j)$. In view of Proposition 4.1, it is enough to show that

$$\lambda_0(\sigma_j) > -1, \quad \lambda_\infty(\sigma_j) < 1 \iff \lambda_0(\rho_j) > -1, \quad \lambda_\infty(\rho_j) < 1.$$

When $j = 1$, we use a homeomorphism φ from $(-1, 1]$ to $[0, \infty)$:

$$\varphi(t) = \frac{1-t}{1+t}$$

to get a bounded Borel measure σ_c on $[0, \infty)$ by $\sigma_c = \varphi(\sigma)$. We also define a new triple $(\alpha_c, \beta_c, \rho_c)$ by

$$\alpha_c = \frac{\alpha_1}{\sqrt{2}}, \quad \beta_c = \beta_1 \quad \text{and} \quad \rho_c = \varphi\left(\frac{1}{1+\cdot}\rho_1\right).$$

By the proofs of Theorems 4.1 and 5.1 in [5], we see that $\sigma_c = L_1(\sigma_c, \beta_c, \rho_c)$. Therefore by Theorem 3.1, it holds that

$$(4.10) \quad \lambda_0(\sigma_c) > 0, \quad \lambda_\infty(\sigma_c) < \infty$$

is equivalent to

$$(4.11) \quad \lambda_0(\rho_c) > 0, \quad \lambda_\infty(\rho_c) < \infty.$$

On the other hand, (4.10) and (4.11) are equivalent to

$$\lambda_0(\sigma_j) > -1, \quad \lambda_\infty(\sigma_j) < 1$$

and

$$\lambda_0(\rho_j) > -1, \quad \lambda_\infty(\rho_j) < 1,$$

respectively. Thus the case of $j = 1$ is proved.

Next we consider the case of $j = 2$. We put $\nu = L_1^{-1}((\alpha_2, \beta_2, \rho_2))$. Then it follows from the case of $j = 1$ that

$$\lambda_0(\nu) > -1, \quad \lambda_\infty(\nu) < 1 \iff \lambda_0(\rho_2) > -1, \quad \lambda_\infty(\rho_2) < 1.$$

On the other hand, by (4.4), we see that $\text{supp}(\sigma_2) = \text{supp}(\nu)$. Thus the case of $j = 2$ is proved and this completes the proof of the theorem. ■

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