Fourier coefficients of Klingen's Eisenstein series for principal congruence subgroups of the symplectic group

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Let us explain the motivation. Let us consider the following problem for positive definite integral matrices S, T of degree m, n respectively: when does the equation $S[X] := {}^tX S X = T$ have a solution in $M_{m,n}(\mathbf{Z})$? Suppose $n = 1$. The condition $m \geq 2n + 3 = 5$ yields that the solvability of $S[X] = T$ over $\mathbf{Z}_p$ for every prime p implies the solvability over $\mathbf{Z}$ if T is sufficiently large. The similar result for $m = 2n + 2 = 4$ is also valid with some minor restriction on T . These can be proved arithmetically, and we have the similar result for $m = 2n + 1 = 3$ by an arithmetic proof with the generalized Riemann hypothesis. Let us look at the above from the view of modular forms. Put $r(S, T) := \#\{X \in M_{m,n}(\mathbf{Z}) \mid S[X] = T\}$, and $\theta_n(z, S) := \sum_g \exp(\pi i \operatorname{tr} S[g]z)$ where g runs over $M_{m,n}(\mathbf{Z})$ and z is in Siegel's upper half-space $H_n := \{z \in M_n(\mathbf{C}) \mid z = {}^t\bar{z}, \Im z > 0\}$. Then $\theta_n(z, S)$ is obviously equal to $\sum_{P \geq 0} r(S, P) \exp(\pi i \operatorname{tr} Pz)$, which is a modular form. When $n = 1$, we decompose it as a sum of Eisenstein series and a cusp form. Thus $r(S, T)$ is a sum of Fourier coefficients of Eisenstein series and a cusp form. If we can show that Fourier coefficients of Eisenstein series are larger than those of a cusp form, we get an asymptotic formula of $r(S, T)$, in particular $r(S, T) > 0$. In this way, we can get an asymptotic formula of $r(S, T)$ if $m \geq 2n + 1 = 3$, but we put minor assumptions on T when $m = 2n + 1 = 3$ and $m = 2n + 2 = 4$. Apart from Fourier coefficients of Eisenstein series, estimates of Fourier coefficients of cusp forms which we need is the standard estimate by Hecke when $m \geq 2n + 3 = 5$, Kloosterman or Eichler's improvement when $m = 2n + 2 = 4$ and a variation of Linnik-Selberg's conjecture on Kloosterman sums when $m = 2n + 1 = 3$. When $n \geq 2$, $\theta_n(z, S)$ is not a sum of Eisenstein series and a cusp form in general. Other kind of modular forms appear. When m is comparatively larger than n , they are given by Klingen's Eisenstein series. So, if we get a good estimate of their Fourier coefficients, then we have also an asymptotic formula of $r(S, T)$ and as a by-product $r(S, T) > 0$. This strategy works when $m \geq 4n + 4$. (We do not refer to other strategies here.) It does not seem that pushing this way on is wrong. When the weight is sufficiently large, we have

already a complicated expression for the formula of Fourier coefficients of Klingen's Eisenstein series with level > 1 . From it we can get necessary estimates to have an asymptotic formula of $r(S, T)$ when $m \geq 4n + 4$. But even the well-definedness of Klingen's Eisenstein series of low weight does not seem to have been established at present, and Fourier coefficients do not converge when we consider them for the low weight formally. So, our aim here is to present them in a simpler form so as to discuss the well-definedness.

Through the paper, r, m, q denote positive integers, put $n := m + r$ and suppose $q \geq 3$.

For a commutative ring R , $M_{k,h}(R)$ is the set of $k \times h$ matrices with entries in R and we put $M_k(R) := M_{k,k}(R)$ and $\Lambda_k(R) := \{\lambda = {}^t\lambda \in M_k(R)\}$. For a matrix $T \in M_k(R)$, we write $T^{(k)}$ when we need to clarify the size. For simplicity, we put $M_{k,h} := M_{k,h}(\mathbf{Z})$, $M_k := M_k(\mathbf{Z})$ and $\Lambda_k := \Lambda_k(\mathbf{Z})$ if there is no confusion.

We put $\Lambda'_n := \{\lambda \in \Lambda_n(\mathbf{Q}) \mid 2\lambda \in \Lambda_n, \text{ all diagonals of } \lambda \in \mathbf{Z}\}$.

For a subset $G \subset M_n(\mathbf{Z})$, $G(q)$ denotes $\{g \in G \mid g \equiv 1_n \pmod{q}\}$.

For matrices A, B , $A[B]$ denotes tBAB if it is defined.

Γ_n, U_n stand for $Sp(n, \mathbf{Z}), SL_n(\mathbf{Z})$ respectively and we put

$$\Gamma_n^\infty := \left\{ \begin{pmatrix} a^{(n)} & b \\ 0 & d \end{pmatrix} \in \Gamma_n \right\}.$$

A matrix M is often written as

$$M = \begin{pmatrix} M_1 & M_2 \\ M_3 & M_4 \end{pmatrix}$$

with specification of the size of one of the M_i as in $M = \begin{pmatrix} M_1^{(r)} & M_2 \\ M_3 & M_4 \end{pmatrix}$. Thus for $M \in \Gamma_n$, the usual "c"-part of M is written as $M_3^{(n)}$. If there is no confusion, M_i is such a submatrix.

We denote Siegel's upper half space $\{z \in M_n(\mathbf{C}) \mid z = {}^tz, \Im z > 0\}$ by H_n .

$e(w)$ denotes $\exp(2\pi iw)$ for a complex number w .

For a natural numbers k, h , $k|h$ means that k divides h , and $k \nmid h$ means that k does not divide h . $k|h^\infty$ means that k divides h^t for a sufficiently large integer t .

§1. Preparations

For $\sigma := \begin{pmatrix} a^{(m)} & b \\ c & d \end{pmatrix} \in \Gamma_m$ and $v \in U_n$ we put

$$\sigma \downarrow := \left(\begin{pmatrix} 1_r & \\ & a \end{pmatrix} \begin{pmatrix} 0 & b \\ 1_r & d \end{pmatrix} \right) \in \Gamma_n, N(v, \sigma) := \begin{pmatrix} v & \\ & {}_t v^{-1} \end{pmatrix} \sigma \downarrow \in \Gamma_n.$$

$\Delta_{n,r}$ denotes the subgroup of Γ_n consisting of $\sigma \in \Gamma_n$ such that the left lower $(m, n+r)$ submatrix $\sigma_3^{(m, n+r)} := (0, 1_m)\sigma(1_{n+r}, 0)$ is zero. If so, decomposing σ as

$\sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ at present, $\sigma \in \Delta_{n,r}$ yields $a_2 = c_2 = 0, c_3 = d_3 = 0, c_4 = 0$, where we decompose an $n \times n$ matrix M as $M = \begin{pmatrix} M_1^{(r)} & M_2 \\ M_3 & M_4 \end{pmatrix}$.

$P_{n,r}$ denotes $\left\{ \begin{pmatrix} v_1^{(r)} & v_2 \\ 0 & v_4 \end{pmatrix} \in U_n \right\}$.

LEMMA 1. Let $u = \begin{pmatrix} u_1^{(r)} & u_2 \\ u_3 & u_4 \end{pmatrix} \in GL_n(\mathbf{C})$ and put $v := {}^t u^{-1} = \begin{pmatrix} v_1^{(r)} & v_2 \\ v_3 & v_4 \end{pmatrix}$. Then we have $\det u_4 = \det u \det v_1$, and if $\det u_4 \neq 0$, then we have

$$\begin{aligned} v_1 &= {}^t(u_1 - u_2 u_4^{-1} u_3)^{-1}, \\ v_2 &= -{}^t(u_1 - u_2 u_4^{-1} u_3)^{-1} {}^t u_3 {}^t u_4^{-1}, \\ v_3 &= -{}^t u_4^{-1} {}^t u_2 {}^t(u_1 - u_2 u_4^{-1} u_3)^{-1}, \\ v_4 &= {}^t u_4^{-1} {}^t u_2 {}^t(u_1 - u_2 u_4^{-1} u_3)^{-1} {}^t u_3 {}^t u_4^{-1} + {}^t u_4^{-1}. \end{aligned}$$

PROOF. Suppose $\det u_4 \neq 0$. Then $\begin{pmatrix} 1 & -u_2 u_4^{-1} \\ 0 & 1 \end{pmatrix} u = \begin{pmatrix} u_1 - u_2 u_4^{-1} u_3 & 0 \\ u_3 & u_4 \end{pmatrix}$ and then $\det u = \det(u_1 - u_2 u_4^{-1} u_3) \det u_4 \neq 0$. Taking the transpose and the inverse, we have

$$\begin{pmatrix} 1 & 0 \\ {}^t u_4^{-1} {}^t u_2 & 1 \end{pmatrix} v = \begin{pmatrix} {}^t(u_1 - u_2 u_4^{-1} u_3)^{-1} & -{}^t(u_1 - u_2 u_4^{-1} u_3)^{-1} {}^t u_3 {}^t u_4^{-1} \\ 0 & {}^t u_4^{-1} \end{pmatrix}.$$

Thus v has a desired form, and $\det v_1 = \det(u_1 - u_2 u_4^{-1} u_3)^{-1}$ implies $\det u = \det v_1^{-1} \det u_4$. If, conversely $\det v_1 \neq 0$, then $\begin{pmatrix} 1 & 0 \\ -v_3 v_1^{-1} & 1 \end{pmatrix} v = \begin{pmatrix} v_1 & v_2 \\ 0 & v_4 - v_3 v_1^{-1} v_2 \end{pmatrix}$ and so $\begin{pmatrix} 1 & {}^t v_1^{-1} {}^t v_3 \\ 0 & 1 \end{pmatrix} u = \begin{pmatrix} {}^t v_1^{-1} & 0 \\ -{}^t(v_4 - v_3 v_1^{-1} v_2)^{-1} {}^t v_2 {}^t v_1^{-1} & {}^t(v_4 - v_3 v_1^{-1} v_2)^{-1} \end{pmatrix}$. Thus we have

$$\begin{aligned} u_1 &= {}^t v_1^{-1} + {}^t v_1^{-1} {}^t v_3 {}^t(v_4 - v_3 v_1^{-1} v_2)^{-1} {}^t v_2 {}^t v_1^{-1}, \\ u_2 &= -{}^t v_1^{-1} {}^t v_3 {}^t(v_4 - v_3 v_1^{-1} v_2)^{-1}, \\ u_3 &= -{}^t(v_4 - v_3 v_1^{-1} v_2)^{-1} {}^t v_2 {}^t v_1^{-1}, \\ u_4 &= {}^t(v_4 - v_3 v_1^{-1} v_2)^{-1} \end{aligned}$$

and $\det v_1 \neq 0$ implies $\det u_4 = \det {}^t(v_4 - v_3 v_1^{-1} v_2)^{-1} \neq 0$. $\blacksquare$

LEMMA 2. Let $K = \begin{pmatrix} A^{(n)} & B \\ C & D \end{pmatrix} \in \Gamma_n$ and suppose $C = \begin{pmatrix} C_1^{(r)} & C_2 \\ 0 & C_4 \end{pmatrix}$. Then the coset $\Delta_{n,r} K$ contains $N(v, \sigma)$ for $v \in U_n, \sigma \in \Gamma_m$, where v, σ can be taken from any fixed representatives of ${}^t P_{n,r} \backslash U_n, \Gamma_m^\infty \backslash \Gamma_m$, and we have $|\det C_4| = |\det v_1^{(r)} \times \det \sigma_3^{(m)}|$.

PROOF. Putting $D := \begin{pmatrix} D_1^{(r)} & D_2 \\ D_3 & D_4 \end{pmatrix}$, $K \in \Gamma_n$ implies $C^t D \in \Lambda_n$ and so $C_4 {}^t D_4 \in \Lambda_m$. By the elementary divisor theorem, we can write $(C_4, D_4) = \tilde{u}_4(c, d)$ for

$\tilde{u}_4, c, d \in M_m$ such that $\det \tilde{u}_4 \neq 0$ and (c, d) is a primitive matrix. $C_4 {}^t D_4 \in \Lambda_m$ implies $c {}^t d \in \Lambda_m$ and hence there are $a, b \in M_m$ such that $\sigma := \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_m$. By modifying the decomposition for (C_4, D_4) and a, b , we may assume that σ is one of the fixed representatives of $\Gamma_m^\infty \backslash \Gamma_m$. The last m rows of $K\sigma^{-1} \downarrow$ is $(0, C_4, D_3, D_4) \begin{pmatrix} \begin{pmatrix} 1_r & t_d \\ 0 & -t_c \end{pmatrix} \begin{pmatrix} 0 & -t_b \\ 1_r & t_a \end{pmatrix} \end{pmatrix} = (0, 0, D_3, \tilde{u}_4)$. Hence $(D_3, \tilde{u}_4)$ is a primitive matrix and there is an extension $\tilde{u} = \begin{pmatrix} * & * \\ D_3 & \tilde{u}_4 \end{pmatrix} \in U_n$. Put $\tilde{u} = wu$ for $w \in P_{n,r}$ and u which is one of the fixed representatives of $P_{n,r} \backslash U_n$. Then the last m rows of $K\sigma^{-1} \downarrow \begin{pmatrix} t_u & \\ & u^{-1} \end{pmatrix} = (0, 0, 0, 1_m) \begin{pmatrix} t_w^{-1} & \\ & w \end{pmatrix} = (0, 0, 0, *)$ and hence $K\sigma^{-1} \downarrow \begin{pmatrix} t_u & \\ & u^{-1} \end{pmatrix} \in \Delta_{n,r}$. Putting $v := t_u^{-1}$, we have $\Delta_{n,r} K \ni \begin{pmatrix} v & \\ & t_{v^{-1}} \end{pmatrix} \sigma \downarrow = N(v, \sigma)$. $\tilde{u} = wu$ yields $\det \tilde{u}_4 = \pm \det u_4^{(m)}$. Since $\det u_4 = \det v_1$ by the previous lemma, we have $|\det C_4| = |\det v_1| |\det c|$. ■

LEMMA 3. *We have the following disjoint decomposition:*

$$\Delta_n^r := \{K \in \Gamma_n \mid \text{rank}(0, 1_m) K_3^{(n)} = m\} = \sqcup_{\sigma, v, w} \Delta_{n,r} N(v, \sigma) \begin{pmatrix} w^{-1} & \\ & t_w \end{pmatrix},$$

where σ, v, w run over

$$\{\sigma \in \Gamma_m^\infty \backslash \Gamma_m \mid \det \sigma_3^{(m)} \neq 0\}, \{v \in {}^t P_{n,r} \backslash U_n \mid \det v_1^{(r)} \neq 0\}, U_n / P_{n,r},$$

respectively. Here σ, v, w are taken in any fixed representatives.

PROOF. Suppose that $K \in \Delta_{n,r} N(v, \sigma)$ for $v \in U_n, \sigma \in \Gamma_m$. Then $K_3^{(n)}$ is $Cv \begin{pmatrix} 1_r & \\ & \sigma_1 \end{pmatrix} + D {}^t v^{-1} \begin{pmatrix} 0 & \\ & \sigma_3 \end{pmatrix}$ for $C = \begin{pmatrix} C_1^{(r)} & \\ & 0 \end{pmatrix}, D = \begin{pmatrix} D_1^{(r)} & D_2 \\ 0 & D_4 \end{pmatrix}$. Writing ${}^t v^{-1} = \begin{pmatrix} u_1^{(r)} & u_2 \\ u_3 & u_4 \end{pmatrix}$, K_3 is equal to $\begin{pmatrix} * & * \\ 0 & D_4 u_4 \sigma_3 \end{pmatrix}$.

If $K \in \Delta_n^r$, then there is $w \in U_n$ such that $K_3 w = \begin{pmatrix} C_1^{(r)} & * \\ 0 & C_4^{(m)} \end{pmatrix}$ and w is one of the given representatives. By Lemma 2, there exist $v \in U_n, \sigma \in \Gamma_m$ such that $K \begin{pmatrix} w & \\ & t_{w^{-1}} \end{pmatrix} \in \Delta_{n,r} N(v, \sigma)$. $\text{rank } C_4 = m$ implies $|\det C_4| = |\det v_1| |\det \sigma_3^{(m)}| \neq 0$.

Conversely suppose that $K \in \Delta_{n,r} N(v, \sigma) \begin{pmatrix} w^{-1} & \\ & t_w \end{pmatrix}$ for $v, w \in U_n, \sigma \in \Gamma_m$ with $\det v_1^{(r)} \neq 0, \det \sigma_3^{(m)} \neq 0$. As above K_3 is equal to $\begin{pmatrix} * & * \\ 0 & D_4 u_4 \sigma_3 \end{pmatrix} w^{-1}$ for $D_4 \in GL_m(\mathbf{Z})$. Since $\det(D_4 u_4 \sigma_3) = \det(D_4 v_1 \sigma_3) \neq 0$, $\text{rank}(0, 1_m) K_3$ is equal to m . Thus $K \in \Delta_n^r$. It remains to show the disjointness. Let σ, v, w and $\tilde{\sigma}, \tilde{v}, \tilde{w}$ be one of the given representatives and suppose $\det \sigma_3^{(m)} \tilde{\sigma}_3^{(m)} \det v_1 \tilde{v}_1 \neq 0$ and

$$(1) \quad \Delta_{n,r} N(v, \sigma) \begin{pmatrix} w^{-1} & \\ & t_w \end{pmatrix} = \Delta_{n,r} N(\tilde{v}, \tilde{\sigma}) \begin{pmatrix} \tilde{w}^{-1} & \\ & t_{\tilde{w}} \end{pmatrix}.$$

Since

$$\begin{aligned} N(v, \sigma) &= \begin{pmatrix} v & \\ & {}^t v^{-1} \end{pmatrix} \sigma \downarrow \\ &= \begin{pmatrix} v \begin{pmatrix} 1_r & \\ & \sigma_1 \end{pmatrix} & v \begin{pmatrix} 0 & \\ & \sigma_2 \end{pmatrix} \\ u \begin{pmatrix} 0 & \\ & \sigma_3 \end{pmatrix} & u \begin{pmatrix} 1_r & \\ & \sigma_4 \end{pmatrix} \end{pmatrix} \quad \text{for } u := {}^t v^{-1} \\ &= \begin{pmatrix} * & * & * & * \\ 0 & u_4 \sigma_3 & u_3 & u_4 \sigma_4 \end{pmatrix} \quad \text{for } u = \begin{pmatrix} u_1 & u_2 \\ u_3 & u_4^{(m)} \end{pmatrix}, \end{aligned}$$

(1) is equivalent to

$$(0, u_4 \sigma_3, u_3, u_4 \sigma_4) = g(0, \tilde{u}_4 \tilde{\sigma}_3, \tilde{u}_3, \tilde{u}_4 \tilde{\sigma}_4) \begin{pmatrix} \tilde{w}^{-1} w & \\ & {}^t(\tilde{w}^{-1} w) \end{pmatrix}$$

for some $g \in GL_m(\mathbf{Z})$, where we put $\tilde{u} := {}^t \tilde{v}^{-1}$. Hence $(0, u_4 \sigma_3) = (0, g \tilde{u}_4 \tilde{\sigma}_3) \tilde{w}^{-1} \cdot w$ and $\tilde{w}^{-1} w \in P_{n,r}$ since $\det g \tilde{u}_4 \tilde{\sigma}_3 \neq 0$. From the assumption $w = \tilde{w}$ follows. Thus $u_4(\sigma_3, \sigma_4) = g \tilde{u}_4(\tilde{\sigma}_3, \tilde{\sigma}_4)$ and hence $\sigma \tilde{\sigma}^{-1} \in \Gamma_m^\infty$ and so $\sigma = \tilde{\sigma}$, which implies $u_4(\sigma_3, \sigma_4) = g \tilde{u}_4(\tilde{\sigma}_3, \tilde{\sigma}_4) = g \tilde{u}_4(\sigma_3, \sigma_4)$. Hence $u_4 = g \tilde{u}_4$ and $(u_3, u_4) = g(\tilde{u}_3, \tilde{u}_4)$. This yields $u \in P_{n,r} \tilde{u}$ and so $v \in {}^t P_{n,r} \tilde{v}$. This completes the proof. ■

LEMMA 4. For $v \in U_n, \sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_m, \lambda = \begin{pmatrix} \lambda_1^{(r)} & \lambda_2 \\ \lambda_3 & \lambda_4 \end{pmatrix} \in \Lambda_n$, we have

$$N(v, \sigma) \begin{pmatrix} 1_n & \lambda \\ 0 & 1_n \end{pmatrix} = \begin{pmatrix} 1_n & x \\ 0 & 1_n \end{pmatrix} N \left(v \begin{pmatrix} 1_r & -\lambda_2 {}^t c \\ 0 & 1_m \end{pmatrix}, \sigma \begin{pmatrix} 1_m & \lambda_4 \\ 0 & 1_m \end{pmatrix} \right)$$

where

$$x = v \begin{pmatrix} \lambda_1 - \lambda_2 {}^t a c \lambda_3 & \lambda_2 {}^t a \\ a \lambda_3 & 0 \end{pmatrix} {}^t v.$$

PROOF. Putting $v' := v \begin{pmatrix} 1_r & -\lambda_2 {}^t c \\ 0 & 1_m \end{pmatrix}$, we have

$$\begin{aligned} (1) \quad N(v, \sigma) \begin{pmatrix} 1_n & \lambda \\ 0 & 1_n \end{pmatrix} N(v', \sigma \begin{pmatrix} 1_m & \lambda_4 \\ 0 & 1_m \end{pmatrix})^{-1} \\ = \begin{pmatrix} v & {}^t v^{-1} \end{pmatrix} \sigma \downarrow \begin{pmatrix} 1_n & \begin{pmatrix} \lambda_1 & \lambda_2 \\ \lambda_3 & 0 \end{pmatrix} \\ 0 & 1_n \end{pmatrix} \sigma^{-1} \downarrow \begin{pmatrix} v'^{-1} & \\ & {}^t v' \end{pmatrix}. \end{aligned}$$

Here $\sigma \downarrow \begin{pmatrix} 1_n & \begin{pmatrix} \lambda_1 & \lambda_2 \\ \lambda_3 & 0 \end{pmatrix} \\ 0 & 1_n \end{pmatrix} \sigma^{-1} \downarrow$ is equal to

$$\begin{aligned} & \begin{pmatrix} \begin{pmatrix} 1_r & \\ & a \end{pmatrix} \begin{pmatrix} 0 & \\ & b \end{pmatrix} \\ \begin{pmatrix} 0 & \\ & c \end{pmatrix} \begin{pmatrix} 1_r & \\ & d \end{pmatrix} \end{pmatrix} \begin{pmatrix} 1_n & \begin{pmatrix} \lambda_1 & \lambda_2 \\ \lambda_3 & 0 \end{pmatrix} \\ 0 & 1_n \end{pmatrix} \begin{pmatrix} \begin{pmatrix} 1_r & {}^t d \\ 0 & -{}^t c \end{pmatrix} \begin{pmatrix} 0 & -{}^t b \\ 1_r & {}^t a \end{pmatrix} \end{pmatrix} \\ & = \begin{pmatrix} \begin{pmatrix} 1_r & -\lambda_2 {}^t c \\ 0 & 1_m \end{pmatrix} \begin{pmatrix} \lambda_1 & \lambda_2 {}^t a \\ a \lambda_3 & 0 \end{pmatrix} \\ 0 & \begin{pmatrix} 1_r & 0 \\ c \lambda_3 & 1_m \end{pmatrix} \end{pmatrix}. \end{aligned}$$

Thus (1) is equal to

$$\begin{aligned} \begin{pmatrix} v & \\ & {}^t v^{-1} \end{pmatrix} \begin{pmatrix} \begin{pmatrix} 1_r & -\lambda_2 {}^t c \\ 0 & 1_m \end{pmatrix} & \begin{pmatrix} \lambda_1 & \lambda_2 {}^t a \\ a\lambda_3 & 0 \end{pmatrix} \\ 0 & \begin{pmatrix} 1_r & 0 \\ c\lambda_3 & 1_m \end{pmatrix} \end{pmatrix} \begin{pmatrix} \begin{pmatrix} 1_r & \lambda_2 {}^t c \\ 0 & 1_m \end{pmatrix} & \\ & \begin{pmatrix} 1_r & 0 \\ -c {}^t \lambda_2 & 1_m \end{pmatrix} \end{pmatrix} \begin{pmatrix} v^{-1} & \\ & {}^t v \end{pmatrix} \\ = \begin{pmatrix} v & \\ & {}^t v^{-1} \end{pmatrix} \begin{pmatrix} 1_n & y \\ 0 & 1_n \end{pmatrix} \begin{pmatrix} v^{-1} & \\ & {}^t v \end{pmatrix}, \end{aligned}$$

where $y = \begin{pmatrix} \lambda_1 - \lambda_2 {}^t a c {}^t \lambda_2 & \lambda_2 {}^t a \\ a\lambda_3 & 0 \end{pmatrix} = v^{-1} x {}^t v^{-1}$, and hence (1) is equal to $\begin{pmatrix} 1_n & x \\ 0 & 1_n \end{pmatrix}$. Thus we have completed the proof. $\blacksquare$

LEMMA 5. For $M \in \Gamma_n$, we have

$$\begin{aligned} \{K \in \Gamma_n \mid \text{rank}(0, 1_m)K_3^{(n)} = m, K \equiv M \bmod q\} \\ = \sqcup_{w, \lambda, \sigma, v, \eta} \Delta_{n,r}(q) \eta N(v, \sigma) \begin{pmatrix} 1_n & \lambda \\ 0 & 1_n \end{pmatrix} \begin{pmatrix} w^{-1} & \\ & {}^t w \end{pmatrix}, \end{aligned}$$

where w, λ, σ, v and η run correspondingly as follows:

$$\begin{aligned} w &\in U_n / P_{n,r}, \\ \{\lambda &\in q\Lambda_n \mid \lambda_1^{(r)} \neq 0\}, \\ \{\sigma &\in \Gamma_m^\infty \setminus \Gamma_m / \begin{pmatrix} 1_m & q\Lambda_m \\ 0 & 1_m \end{pmatrix} \mid \det \sigma_3^{(m)} \neq 0\}, \\ \{v &\in {}^t P_{n,r} \setminus U_n / \begin{pmatrix} 1_r & qM_{r,m} {}^t \sigma_3 \\ 0 & 1_m \end{pmatrix} \mid \det v_1^{(r)} \neq 0\}, \\ \text{and } \{\eta &\in \Delta_{n,r}(q) \setminus \Delta_{n,r} \mid \eta N(v, \sigma) \begin{pmatrix} w^{-1} & \\ & {}^t w \end{pmatrix} \equiv M \bmod q\}. \end{aligned}$$

PROOF. Let $\sigma \in \Gamma_m$ with $\det \sigma_3^{(m)} \neq 0$ and $v \in U_n$ with $\det v_1^{(r)} \neq 0$.

We show first that $\Gamma_m^\infty \sigma = \Gamma_m^\infty \sigma \begin{pmatrix} 1_m & \lambda_4 \\ 0 & 1_m \end{pmatrix}$ for $\lambda_4 \in \Lambda_m$ implies $\lambda_4 = 0$, and ${}^t P_{n,r} v = {}^t P_{n,r} v \begin{pmatrix} 1_r & \lambda_2 \\ 0 & 1_m \end{pmatrix}$ for $\lambda_2 \in M_{r,m}$ implies $\lambda_2 = 0$. If $\sigma \begin{pmatrix} 1_r & \lambda_4 \\ 0 & 1_m \end{pmatrix} \sigma^{-1} \in \Gamma_m^\infty$, then $\sigma \begin{pmatrix} 1_r & \lambda_4 \\ 0 & 1_m \end{pmatrix} \sigma^{-1} = \begin{pmatrix} * & * \\ -\sigma_3 \lambda_4 {}^t \sigma_3 & * \end{pmatrix} \in \Gamma_m^\infty$ implies $\lambda_4 = 0$ by the assumption $\det \sigma_3 \neq 0$. If $v \begin{pmatrix} 1_r & \lambda_2 \\ 0 & 1_m \end{pmatrix} v^{-1} \in {}^t P_{n,r}$, then $v \begin{pmatrix} 1_r & \lambda_2 \\ 0 & 1_m \end{pmatrix} = \begin{pmatrix} x_1^{(r)} & 0 \\ x_3 & x_4 \end{pmatrix} v$ for some $\begin{pmatrix} x_1 & 0 \\ x_3 & x_4 \end{pmatrix} \in {}^t P_{n,r}$ and it implies $(v_1, v_1 \lambda_2 + v_2) = (x_1 v_1, x_1 v_2)$. The assumption $\det v_1 \neq 0$ yields $x_1 = 1_r$ and hence $v_1 \lambda_2 + v_2 = x_1 v_2 = v_2$ implies $\lambda_2 = 0$.

Now the above facts and Lemma 3 imply

$$\begin{aligned} \{K \in \Gamma_n \mid \text{rank}(0, 1_m)K_3^{(n)} = m\} \\ = \sqcup \Delta_{n,r} N \left(v \begin{pmatrix} 1_r & -\lambda_2 {}^t \sigma_3 \\ 0 & 1_m \end{pmatrix}, \sigma \begin{pmatrix} 1_m & \lambda_4 \\ 0 & 1_m \end{pmatrix} \right) \begin{pmatrix} w^{-1} & \\ & {}^t w \end{pmatrix}, \end{aligned}$$

where $w, \lambda_2, \lambda_4, \sigma, v$ run as follows:

$$\begin{aligned} w &\in U_n/P_{n,r}, \quad \lambda_2 \in qM_{r,m}, \quad \lambda_4 \in q\Lambda_m, \\ \sigma &\in \Gamma_m^\infty \backslash \Gamma_m / \begin{pmatrix} 1_m & q\Lambda_m \\ 0 & 1_m \end{pmatrix} \quad \text{with } \det \sigma_3^{(m)} \neq 0, \\ v &\in {}^tP_{n,r} \backslash U_n / \begin{pmatrix} 1_r & qM_{r,m} {}^t\sigma_3 \\ 0 & 1_m \end{pmatrix} \quad \text{with } \det v_1^{(r)} \neq 0. \end{aligned}$$

From Lemma 4 follows

$$\Delta_{n,r} N(v, \sigma) \begin{pmatrix} 1_n & \lambda \\ 0 & 1_n \end{pmatrix} = \Delta_{n,r} N \left(v \begin{pmatrix} 1_r & -\lambda_2^{(r,m)} {}^t\sigma_3^{(m)} \\ 0 & 1_m \end{pmatrix}, \sigma \begin{pmatrix} 1_m & \lambda_4 \\ 0 & 1_m \end{pmatrix} \right).$$

Decomposing $\Delta_{n,r} = \sqcup_\eta \Delta_{n,r}(q)\eta$ ($\eta \in \Delta_{n,r}(q) \backslash \Delta_{n,r}$), the assertion is clear. ■

LEMMA 6. *We have a disjoint decomposition*

$$\{K \in \Gamma_n \mid \det K_3^{(n)} \neq 0\} = \sqcup_{\sigma, v, M} \Gamma_n^\infty M \uparrow N(v, \sigma),$$

where σ, v, M run over the following sets, respectively:

$$\begin{aligned} &\{\sigma \in \Gamma_m^\infty \backslash \Gamma_m \mid \det \sigma_3^{(m)} > 0\}, \\ &\{v \in {}^tP_{n,r} \backslash U_n \mid \det v_1^{(r)} > 0\}, \\ &\{M \in \Gamma_r^\infty \backslash \Gamma_r \mid \det M_3^{(r)} > 0\}. \end{aligned}$$

Here $M \uparrow$ denotes $\begin{pmatrix} \begin{pmatrix} M_1 & \\ & 1_m \end{pmatrix} & \begin{pmatrix} M_2 & \\ & 0 \end{pmatrix} \\ \begin{pmatrix} M_3 & \\ & 0 \end{pmatrix} & \begin{pmatrix} M_4 & \\ & 1_m \end{pmatrix} \end{pmatrix} \in \Delta_{n,r}$. Putting $M \uparrow N(v, \sigma) = \begin{pmatrix} * & * \\ \gamma^{(n)} & \delta \end{pmatrix}$, we have $\det \gamma = \det M_3 \det \sigma_3 \det v_1^2$ and

$$\gamma^{-1} \delta = {}^t(\gamma^{-1} \delta) = \begin{pmatrix} M_3^{-1} M_4 [{}^t v_1^{-1}] + \sigma_1 \sigma_3^{-1} [{}^t(v_1^{-1} v_2)] & * \\ -\sigma_3^{-1} {}^t(v_1^{-1} v_2) & \sigma_3^{-1} \sigma_4 \end{pmatrix}.$$

PROOF. For $M \in \Gamma_r, v \in U_n, \sigma \in \Gamma_m$, we have

$$\begin{aligned} &M \uparrow N(v, \sigma) \\ &= \begin{pmatrix} \begin{pmatrix} M_1 & \\ & 1_m \end{pmatrix} & \begin{pmatrix} M_2 & \\ & 0 \end{pmatrix} \\ \begin{pmatrix} M_3 & \\ & 0 \end{pmatrix} & \begin{pmatrix} M_4 & \\ & 1_m \end{pmatrix} \end{pmatrix} \begin{pmatrix} v & \\ & u \end{pmatrix} \begin{pmatrix} \begin{pmatrix} 1_r & \\ & \sigma_1 \end{pmatrix} & \begin{pmatrix} 0 & \\ & \sigma_2 \end{pmatrix} \\ \begin{pmatrix} 0 & \\ & \sigma_3 \end{pmatrix} & \begin{pmatrix} 1_r & \\ & \sigma_4 \end{pmatrix} \end{pmatrix} \quad (u := {}^t v^{-1}). \end{aligned}$$

Denoting it by $\begin{pmatrix} * & * \\ \gamma & \delta \end{pmatrix}$,

$$\begin{aligned} \gamma &= \begin{pmatrix} M_3 & \\ & 0 \end{pmatrix} v \begin{pmatrix} 1 & \\ & \sigma_1 \end{pmatrix} + \begin{pmatrix} M_4 & \\ & 1 \end{pmatrix} u \begin{pmatrix} 0 & \\ & \sigma_3 \end{pmatrix} = \begin{pmatrix} M_3 v_1 & M_3 v_2 \sigma_1 + M_4 u_2 \sigma_3 \\ 0 & u_4 \sigma_3 \end{pmatrix}, \\ \delta &= \begin{pmatrix} M_3 & \\ & 0 \end{pmatrix} v \begin{pmatrix} 0 & \\ & \sigma_2 \end{pmatrix} + \begin{pmatrix} M_4 & \\ & 1 \end{pmatrix} u \begin{pmatrix} 1 & \\ & \sigma_4 \end{pmatrix} = \begin{pmatrix} M_4 u_1 & M_3 v_2 \sigma_2 + M_4 u_2 \sigma_4 \\ u_3 & u_4 \sigma_4 \end{pmatrix}. \end{aligned}$$

Thus $\det \gamma = \det M_3 \det \sigma_3 \det v_1^2$ since $\det v_1 = \det u_4$ by Lemma 1.

$$\gamma^{-1}\delta = {}^t(\gamma^{-1}\delta) = \begin{pmatrix} M_3^{-1}M_4[{}^tv_1^{-1}] + \sigma_1\sigma_3^{-1}[{}^t(v_1^{-1}v_2)] & * \\ -\sigma_3^{-1}{}^t(v_1^{-1}v_2) & \sigma_3^{-1}\sigma_4 \end{pmatrix}$$

is easy to see, using $v_1^{-1}v_2 = -{}^tu_3{}^tu_4^{-1}$, $u_1 - u_2u_4^{-1}u_3 = {}^tv_1^{-1}$ by Lemma 1.

Suppose that $\widetilde{M}_3^{(n)}$ for $\widetilde{M} \in \Gamma_n$ is regular. Then we can take $w \in U_n$ so that $\begin{pmatrix} {}^tw^{-1} \\ w \end{pmatrix} \widetilde{M} = \begin{pmatrix} (C_1^{(r)} C_2)^* \\ 0 C_4 \end{pmatrix}^*$. From Lemma 2 follows that $\begin{pmatrix} {}^tw^{-1} \\ w \end{pmatrix} \widetilde{M}$ is equal to $KN(v, \sigma)$ for $K \in \Delta_{n,r}$, $v \in {}^tP_{n,r} \setminus U_n$, $\sigma \in \Gamma_m^\infty \setminus \Gamma_m$ with $\det C_4 = \pm \det v_1^{(r)} \det \sigma_3^{(m)} \neq 0$. Writing $K = \begin{pmatrix} K_1^{(r)} & 0 & K_2^{(r)} & * \\ * & * & * & * \\ K_3^{(r)} & 0 & K_4^{(r)} & * \\ 0 & 0 & 0 & * \end{pmatrix}$, we have $K \begin{pmatrix} K_1 & K_2 \\ K_3 & K_4 \end{pmatrix}^{-1} \uparrow \in \Gamma_n^\infty$. Hence $\Gamma_n^\infty \widetilde{M} = \Gamma_n^\infty \begin{pmatrix} K_1 & K_2 \\ K_3 & K_4 \end{pmatrix} \uparrow N(v, \sigma)$ and $|\det \widetilde{M}_3| = |\det K_3 \det \sigma_3 \det v_1^2|$ from the above. We have to prove the disjointness of the union. Let $\sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $v = \begin{pmatrix} v_1 & v_2 \\ v_3 & v_4 \end{pmatrix}$, $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ and $\widetilde{\sigma} = \begin{pmatrix} \widetilde{a} & \widetilde{b} \\ \widetilde{c} & \widetilde{d} \end{pmatrix}$, $\widetilde{v} = \begin{pmatrix} \widetilde{v}_1 & \widetilde{v}_2 \\ \widetilde{v}_3 & \widetilde{v}_4 \end{pmatrix}$, $\widetilde{M} = \begin{pmatrix} \widetilde{A} & \widetilde{B} \\ \widetilde{C} & \widetilde{D} \end{pmatrix}$ be representatives in the assertion of Lemma, and suppose $\Gamma_n^\infty M \uparrow N(v, \sigma) = \Gamma_n^\infty \widetilde{M} \uparrow N(\widetilde{v}, \widetilde{\sigma})$ with $\det C \widetilde{C} \det v_1 \widetilde{v}_1 \det c \widetilde{c} \neq 0$. Then for some $U \in GL_n(\mathbf{Z})$ we have $\begin{pmatrix} * & * \\ 0 & U \end{pmatrix} M \uparrow N(v, \sigma) = \widetilde{M} \uparrow N(\widetilde{v}, \widetilde{\sigma})$. Comparing the C -parts of the both sides, we have

$$U \begin{pmatrix} Cv_1 & Cv_2a + Du_2c \\ 0 & u_4c \end{pmatrix} = \begin{pmatrix} \widetilde{C}\widetilde{v}_1 & \widetilde{C}\widetilde{v}_2\widetilde{a} + \widetilde{D}\widetilde{u}_2\widetilde{c} \\ 0 & \widetilde{u}_4\widetilde{c} \end{pmatrix}.$$

Hence $U \in P_{n,r}$, and $M \uparrow, \widetilde{M} \uparrow \in \Delta_{n,r}$ implies $\Delta_{n,r}N(v, \sigma) = \Delta_{n,r}N(\widetilde{v}, \widetilde{\sigma})$. By Lemma 3, we have $v = \widetilde{v}$, $\sigma = \widetilde{\sigma}$. It implies $\Gamma_n^\infty M \uparrow = \Gamma_n^\infty \widetilde{M} \uparrow$ and $(\widetilde{M}M^{-1}) \uparrow \in \Gamma_n^\infty$. Thus we have $\widetilde{M}M^{-1} \in \Gamma_r^\infty$ and $\widetilde{M} = M$. ■

LEMMA 7. $\begin{pmatrix} 1_n & \Lambda_n \\ 0 & 1_n \end{pmatrix}$ acts faithfully on $\Gamma_m^\infty \setminus \{K \in \Gamma_n \mid \det K_3^{(n)} \neq 0\}$ from the right, and as a complete set of representatives of orbits we can take $M \uparrow N(v, \sigma)$, where σ, v, M run, respectively, over

$$\begin{aligned} & \{\sigma \in \Gamma_m^\infty \setminus \Gamma_m / \begin{pmatrix} 1_m & \Lambda_m \\ 0 & 1_m \end{pmatrix} \mid \det \sigma_3^{(m)} > 0\}, \\ & \left\{v \in {}^tP_{n,r} \setminus U_n / \begin{pmatrix} 1_r & M_{r,m} {}^t\sigma_3 \\ 0 & 1_m \end{pmatrix} \mid \det v_1^{(r)} > 0\right\}, \\ & \left\{M \in \Gamma_r^\infty \setminus \Gamma_r / \begin{pmatrix} 1_r & \Lambda_r [{}^tv_1] \\ 0 & 1_r \end{pmatrix} \mid \det M_3^{(r)} > 0\right\}. \end{aligned}$$

PROOF. Let $\sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_m$, $v = \begin{pmatrix} v_1 & v_2 \\ v_3 & v_4 \end{pmatrix} \in U_n$, $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_r$ and $\lambda = \begin{pmatrix} \lambda_1^{(r)} & \lambda_2 \\ \lambda_3 & \lambda_4 \end{pmatrix} \in \Lambda_n$. By Lemma 4, we have

$$M \uparrow N(v, \sigma) \begin{pmatrix} 1_n & \lambda \\ 0 & 1_n \end{pmatrix} = M \uparrow \begin{pmatrix} 1_n & x \\ 0 & 1_n \end{pmatrix} N\left(v \begin{pmatrix} 1_r & -\lambda_2 {}^tc \\ 0 & 1_m \end{pmatrix}, \sigma \begin{pmatrix} 1_m & \lambda_4 \\ 0 & 1_m \end{pmatrix}\right)$$

where $x = v \begin{pmatrix} \lambda_1 - \lambda_2 {}^t ac \lambda_3 & \lambda_2 {}^t a \\ a \lambda_3 & 0 \end{pmatrix} {}^t v$. Putting $x = \begin{pmatrix} x_1^{(r)} & x_2 \\ x_3 & x_4 \end{pmatrix}$, we have

$$\begin{aligned} M \uparrow \begin{pmatrix} 1_n & x \\ 0 & 1_n \end{pmatrix} &= \begin{pmatrix} \begin{pmatrix} A & \\ & 1_m \end{pmatrix} & \begin{pmatrix} B & \\ & 0 \end{pmatrix} \\ \begin{pmatrix} C & \\ & 0 \end{pmatrix} & \begin{pmatrix} D & \\ & 1_m \end{pmatrix} \end{pmatrix} \begin{pmatrix} 1_n & \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix} \\ 0 & 1_n \end{pmatrix} \\ &= \begin{pmatrix} \begin{pmatrix} A & \\ & 1_m \end{pmatrix} & \begin{pmatrix} Ax_1 + B & Ax_2 \\ x_3 & x_4 \end{pmatrix} \\ \begin{pmatrix} C & \\ & 0 \end{pmatrix} & \begin{pmatrix} Cx_1 + D & Cx_2 \\ 0 & 1_m \end{pmatrix} \end{pmatrix} \\ &= \begin{pmatrix} \begin{pmatrix} 1_r & 0 \\ -x_3 {}^t C & 1_m \end{pmatrix} & \begin{pmatrix} 0 & Ax_2 \\ x_3 {}^t A & x_4 \end{pmatrix} \\ 0 & \begin{pmatrix} 1_r & Cx_2 \\ 0 & 1_m \end{pmatrix} \end{pmatrix} \left(\begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} 1_r & x_1 \\ 0 & 1_r \end{pmatrix} \right) \uparrow. \end{aligned}$$

Thus we have

$$\begin{aligned} &\Gamma_n^\infty M \uparrow N(v, \sigma) \begin{pmatrix} 1_n & \lambda \\ 0 & 1_n \end{pmatrix} \\ (1) \quad &= \Gamma_n^\infty (M \begin{pmatrix} 1_r & x_1 \\ 0 & 1_r \end{pmatrix}) \uparrow N \left(v \begin{pmatrix} 1_r & -\lambda_2 {}^t c \\ 0 & 1_m \end{pmatrix}, \sigma \begin{pmatrix} 1_m & \lambda_4 \\ 0 & 1_m \end{pmatrix} \right) \end{aligned}$$

where $x_1 := \begin{pmatrix} \lambda_1 - \lambda_2 {}^t ac \lambda_3 & \lambda_2 {}^t a \\ a \lambda_3 & 0 \end{pmatrix} \begin{bmatrix} {}^t v_1 \\ {}^t v_2 \end{bmatrix} = \lambda_1 [{}^t v_1] - {}^t ac [\lambda_3 {}^t v_1] + v_2 a \lambda_3 {}^t v_1 + v_1 \lambda_2 {}^t a {}^t v_2$. By virtue of the previous lemma, our assertion is obvious except for the faithfulness of the action and the completeness of representatives. Let us show these facts. First we remark that, for $k = r$ or m , $\begin{pmatrix} 1_k & \Lambda_k \\ 0 & 1_k \end{pmatrix}$ acts faithfully from the right on $\Gamma_k^\infty \setminus \{\sigma \in \Gamma_k \mid \det \sigma_3^{(k)} > 0\}$ and $\begin{pmatrix} 1_r & M_{r,m} \\ 0 & 1_m \end{pmatrix}$ acts also faithfully on ${}^t P_{n,r} \setminus \{v \in U_n \mid \det v_1^{(r)} > 0\}$ from the right, as was shown at the beginning of the proof of Lemma 5. Let both σ, v, M and $\tilde{\sigma}, \tilde{v}, \tilde{M}$ be as in the assertion of Lemma and assume

$$\Gamma_n^\infty M \uparrow N(v, \sigma) \begin{pmatrix} 1_n & \lambda \\ 0 & 1_n \end{pmatrix} = \Gamma_n^\infty \tilde{M} \uparrow N(\tilde{v}, \tilde{\sigma})$$

for $\lambda = \begin{pmatrix} \lambda_1^{(r)} & \lambda_2 \\ \lambda_3 & \lambda_4 \end{pmatrix} \in \Lambda_n$. The previous lemma and (1) yield $\Gamma_m^\infty \tilde{\sigma} = \Gamma_m^\infty \sigma \begin{pmatrix} 1_m & \lambda_4 \\ 0 & 1_m \end{pmatrix}$. Since $\tilde{\sigma}, \sigma$ are representatives of $\Gamma_m^\infty \setminus \Gamma_m / \begin{pmatrix} 1_m & \Lambda_m \\ 0 & 1_m \end{pmatrix}$, we have $\tilde{\sigma} = \sigma$ and then $\lambda_4 = 0$ by the above remark. Next ${}^t P_{n,r} v \begin{pmatrix} 1_r & -\lambda_2 {}^t c \\ 0 & 1_m \end{pmatrix} = {}^t P_{n,r} \tilde{v}$ holds. By the similar reason we have $v = \tilde{v}$ and $\lambda_2 = 0$. Then x_1 is equal to $\lambda_1 [{}^t v_1]$. Lastly $\Gamma_r^\infty M \begin{pmatrix} 1_r & x_1 \\ 0 & 1_r \end{pmatrix} = \Gamma_r^\infty \tilde{M}$ implies $M = \tilde{M}$ and $x_1 = 0$, i.e. $\lambda_1 = 0$. Thus we have $M = \tilde{M}, v = \tilde{v}, \sigma = \tilde{\sigma}$ and $\lambda = 0$, which complete the proof. ■

Next we generalize the Möbius transformation. Let us define the Möbius function for finite abelian groups. Let N be a finite abelian group. If $N = \{0\}$, then we put $\mu(N) = 0$. If N contains an element whose order is not square-free, then $\mu(N) = 0$. Otherwise $N \cong \oplus_{i=1}^k \oplus^{r_i} \mathbf{Z}/p_i \mathbf{Z}$ where p_i is a prime number and $p_i \neq p_j$

if $i \neq j$. Then we put $\mu(N) = \prod_{i=1}^k (-1)^{r_i} p_i^{r_i(r_i-1)/2}$. For a natural number h , $\mu(\mathbf{Z}/h\mathbf{Z})$ is equal to the authentic Möbius function $\mu(h)$.

LEMMA 8. *Let M be a finite abelian group and S the set of subgroups of M . Then we have*

$$\sum_{N \in S} \mu(N) = \sum_{N \in S} \mu(M/N) = \begin{cases} 1 & \text{if } M = 0, \\ 0 & \text{otherwise,} \end{cases}$$

and for functions ϕ, ψ on S

$$\psi(K) = \sum_{H \in S, H \supset K} \phi(H) \quad \text{for every } K \in S$$

if and only if

$$\phi(H) = \sum_{K \in S, K \supset H} \mu(K/H) \psi(K) \quad \text{for every } H \in S.$$

This follows from [D].

Now we introduce exponential sums k and K . For $T \in \Lambda_n(\mathbf{Q})$, $c, d \in M_m$ with $\det c \neq 0$, $\tilde{\sigma} \in \Gamma_m$, $v_1 \in M_r(\mathbf{Q})$, $\tilde{v}_2 \in M_{r,m}(\mathbf{Q})$, we define k, K as follows:

Suppose that the following condition (‡) on $T, c, d, \tilde{\sigma}, v_1, \tilde{v}_2$ is satisfied:

(‡): $T \in \Lambda'_n$, $v_1 \in M_r$ with $\det v_1 \neq 0$, $T_1^{(r)}[v_1^{-1}] \in \Lambda'_r$, $\tilde{v}_2 \in M_{r,m}$ and there exist $a, b \in M_m$ such that $\Gamma_m \ni \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \tilde{\sigma} \pmod{q}$.

Then we put

$$\begin{aligned} k(T, (c, d), v_1; \tilde{\sigma}, \tilde{v}_2) \\ := \sum_{\substack{v_2 \in M_{r,m} \pmod{qv_1 M_{r,m}{}^t c} \\ v_2 \equiv \tilde{v}_2 \pmod{q}, (v_1, v_2): \text{primitive}}} e \left(\text{tr } T \begin{pmatrix} ac^{-1} [{}^t(v_1^{-1} v_2)] & -v_1^{-1} v_2 {}^t c^{-1} \\ -c^{-1} {}^t(v_1^{-1} v_2) & c^{-1} d \end{pmatrix} / q \right), \end{aligned}$$

$$\begin{aligned} K(T, (c, d), v_1; \tilde{\sigma}, \tilde{v}_2) \\ := \sum_{\substack{v_2 \in M_{r,m} \pmod{qv_1 M_{r,m}{}^t c} \\ v_2 \equiv \tilde{v}_2 \pmod{q}}} e \left(\text{tr } T \begin{pmatrix} ac^{-1} [{}^t(v_1^{-1} v_2)] & -v_1^{-1} v_2 {}^t c^{-1} \\ -c^{-1} {}^t(v_1^{-1} v_2) & c^{-1} d \end{pmatrix} / q \right). \end{aligned}$$

If the condition (‡) is not satisfied, then we put $k = K = 0$.

LEMMA 9. *Two exponential sums k and K are well-defined.*

PROOF. Suppose that the condition $(\mathfrak{h})$ is satisfied. If $\Gamma_m \ni \begin{pmatrix} a' & b' \\ c & d \end{pmatrix} \equiv \sigma \pmod{q}$, then $\begin{pmatrix} a' & b' \\ c & d \end{pmatrix} = \begin{pmatrix} 1_m & \lambda \\ 0 & 1_m \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ for $\lambda \in q\Lambda_m$. Putting $x = v_1^{-1}v_2$ ($v_2 \in M_{r,m}$), we want to show

$$\begin{aligned} & \operatorname{tr} T \begin{pmatrix} (a + \lambda c)c^{-1}[{}^t(x + qy{}^t c)] & -(x + qy{}^t c){}^t c^{-1} \\ -c^{-1}{}^t(x + qy{}^t c) & c^{-1}d \end{pmatrix} \\ & \equiv \operatorname{tr} T \begin{pmatrix} ac^{-1}[{}^t x] & -x{}^t c^{-1} \\ -c^{-1}{}^t x & c^{-1}d \end{pmatrix} \pmod{q} \end{aligned}$$

for $y \in M_{r,m}$, which means that the sums k, K are independent of the choice of $v_2 \pmod{qv_1 M_{r,m}{}^t c}$ and the extension $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_m$. So the above congruence implies that k and K are well-defined. The left hand side minus the right hand side is equal to

$$\begin{aligned} & \operatorname{tr} T_1((a + \lambda c)c^{-1}[{}^t(x + qy{}^t c)] - ac^{-1}[{}^t x]) \pmod{q} \\ & \equiv \operatorname{tr} T_1(qxa{}^t y + qy{}^t a{}^t x + q^2 y{}^t ca{}^t y + \lambda[{}^t(x + qy{}^t c)]) \pmod{q} \\ & \equiv q \operatorname{tr} T_1[v_1^{-1}](v_2 a{}^t y{}^t v_1 + v_1 y{}^t a{}^t v_2 + q{}^t ca[{}^t y{}^t v_1]) \\ & \quad + \operatorname{tr} T_1[v_1^{-1}] \cdot \lambda[{}^t(v_2 + qv_1 y{}^t c)] \pmod{q} \\ & \equiv 0 \pmod{q} \end{aligned}$$

since $T_1[v_1^{-1}] \in \Lambda'_r$ and $\lambda \in q\Lambda_m$. $\blacksquare$

It is easy to see that k, K are invariant by replacing $v_1, \tilde{v}_2$ by $uv_1, u\tilde{v}_2$ for $u \in GL_r(\mathbf{Z})$.

LEMMA 10. For $T, c, d, v_1, \tilde{\sigma}, \tilde{v}_2$ satisfying the condition $(\mathfrak{h})$, we have

$$\begin{aligned} & K(T, (c, d), v_1; \tilde{\sigma}, \tilde{v}_2) \\ & = \sum_{g \in M_r/GL_r(\mathbf{Z})} \sum_{\substack{\tilde{v}_2 \in M_{r,m} \pmod{qM_{r,m}} \\ g\tilde{v}_2 \equiv \tilde{v}_2 \pmod{q}}} k(T, (c, d), g^{-1}v_1; \tilde{\sigma}, \tilde{V}_2), \\ & k(T, (c, d), v_1; \tilde{\sigma}, \tilde{v}_2) \\ & = \sum_{g \in M_r/GL_r(\mathbf{Z})} \mu(\mathbf{Z}^r/\mathbf{Z}^r g) \sum_{\substack{\tilde{v}_2 \in M_{r,m} \pmod{qM_{r,m}} \\ g\tilde{v}_2 \equiv \tilde{v}_2 \pmod{q}}} K(T, (c, d), g^{-1}v_1; \tilde{\sigma}, \tilde{V}_2). \end{aligned}$$

Here the sums on g are finite.

PROOF. $K(T, (c, d), v_1; \tilde{\sigma}, \tilde{v}_2)$ is equal to

$$\begin{aligned}
& \sum_{g \in M_r / GL_r(\mathbf{Z})} \sum_{\substack{v_2 \in M_{r,m} \bmod q v_1 M_{r,m} t c \\ v_2 \equiv \tilde{v}_2 \bmod q, (g^{-1} v_1, g^{-1} v_2): \text{primitive}}} e \left(\text{tr} T \begin{pmatrix} ac^{-1} [{}^t(v_1^{-1} v_2)] & * \\ -c^{-1} {}^t(v_1^{-1} v_2) & c^{-1} d \end{pmatrix} / q \right) \\
&= \sum_{g \in M_r / GL_r(\mathbf{Z})} \sum_{\substack{V_2 \in M_{r,m} \bmod q g^{-1} v_1 M_{r,m} t c \\ g V_2 \equiv \tilde{v}_2 \bmod q, (g^{-1} v_1, V_2): \text{primitive}}} e \left(\text{tr} T \begin{pmatrix} ac^{-1} [{}^t((g^{-1} v_1)^{-1} V_2)] & * \\ -c^{-1} {}^t((g^{-1} v_1)^{-1} V_2) & c^{-1} d \end{pmatrix} / q \right) \\
&= \sum_{g \in M_r / GL_r(\mathbf{Z})} \sum_{\substack{\tilde{V}_2 \in M_{r,m} \bmod q M_{r,m} \\ g \tilde{V}_2 \equiv \tilde{v}_2 \bmod q}} k(T, (c, d), g^{-1} v_1; \tilde{\sigma}, \tilde{V}_2).
\end{aligned}$$

Let us identify $GL_r(\mathbf{Q})/GL_r(\mathbf{Z})$ with lattices on $\mathbf{Q}^r$ by $g \in GL_r(\mathbf{Q}) \Leftrightarrow \mathbf{Z}^r g^{-1}$. Denote $\mathbf{Z}^r$ by F . Then for $g \in GL_r(\mathbf{Q})$, $g \in M_r$ is equivalent to $F \subset Fg^{-1}$. For a lattice $L = Fg^{-1} \supset F$ ($g \in M_r$), we put $\phi(L) := \sum_{\substack{\tilde{v}_2 \in M_{r,m} \bmod q M_{r,m} \\ g \tilde{V}_2 \equiv \tilde{v}_2 \bmod q}} k(T, (c, d), g^{-1} v_1; \tilde{\sigma}, \tilde{V}_2)$. For another choice g' for $L = F(g')^{-1}$, $g' = gu$ for $u \in GL_r(\mathbf{Z})$, we have $\sum_{\substack{\tilde{v}_2 \in M_{r,m} \bmod q M_{r,m} \\ g' \tilde{V}_2 \equiv \tilde{v}_2 \bmod q}} k(T, (c, d), (g')^{-1} v_1; \tilde{\sigma}, \tilde{V}_2) = \sum_{\substack{\tilde{v}_2 \in M_{r,m} \bmod q M_{r,m} \\ g \tilde{V}_2 \equiv \tilde{v}_2 \bmod q}} k(T, (c, d), u^{-1} \cdot g^{-1} v_1; \tilde{\sigma}, u^{-1} \tilde{V}_2) = \sum_{\substack{\tilde{v}_2 \in M_{r,m} \bmod q M_{r,m} \\ g \tilde{V}_2 \equiv \tilde{v}_2 \bmod q}} k(T, (c, d), g^{-1} v_1; \tilde{\sigma}, \tilde{V}_2)$, by the remark just before Lemma 10. Thus $\phi(L)$ is well-defined for $L \supset F$ and $K(T, (c, d), v_1; \tilde{\sigma}, \tilde{v}_2) = \sum_{L \supset F} \phi(L)$ by the equation above. Note that $[L : F] = |\det g| > |\det v_1|$ implies $\phi(L) = 0$ since $g^{-1} v_1 \notin M_r$. Hence there is a lattice $M \supset F$ such that $L \supset F$ and $\phi(L) \neq 0$ imply $L \subset M$. For a lattice $J = Fg_1^{-1} \supset F$, we have

$$\begin{aligned}
\psi(J) &:= \sum_{L \supset J} \phi(L) \\
&= \sum_{\substack{g \in M_r / GL_r(\mathbf{Z}) \\ g_1^{-1} g \in M_r}} \sum_{\substack{\tilde{v}_2 \in M_{r,m} \bmod q M_{r,m} \\ g \tilde{V}_2 \equiv \tilde{v}_2 \bmod q}} k(T, (c, d), g^{-1} v_1; \tilde{\sigma}, \tilde{V}_2) \\
&= \sum_{G \in M_r / GL_r(\mathbf{Z})} \sum_{\substack{\tilde{v}_2 \in M_{r,m} \bmod q M_{r,m} \\ g_1 G \tilde{V}_2 \equiv \tilde{v}_2 \bmod q}} k(T, (c, d), G^{-1} g_1^{-1} v_1; \tilde{\sigma}, \tilde{V}_2) \\
&= \sum_{G \in M_r / GL_r(\mathbf{Z})} \sum_{\substack{x \in M_{r,m} \bmod q M_{r,m} \\ g_1 x \equiv \tilde{v}_2 \bmod q}} \sum_{\substack{\tilde{v}_2 \in M_{r,m} \bmod q M_{r,m} \\ G \tilde{V}_2 \equiv x \bmod q}} k(T, (c, d), G^{-1} g_1^{-1} v_1; \tilde{\sigma}, \tilde{V}_2) \\
&= \sum_{\substack{x \in M_{r,m} \bmod q M_{r,m} \\ g_1 x \equiv \tilde{v}_2 \bmod q}} K(T, (c, d), g_1^{-1} v_1; \tilde{\sigma}, x).
\end{aligned}$$

Applying Lemma 8 to $M/F, \phi, \psi$, we have $\phi(F) = \sum_{L \supset F} \mu(L/F) \psi(L)$, which is the second desired equation. ■

LEMMA 11. For $T \in \Lambda'_n, \tilde{\sigma} \in \Gamma_m, v_1 \in M_r, \tilde{v}_2 \in M_{r,m}, \Gamma_m \ni \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \tilde{\sigma} \pmod{q}$ with $\det v_1 \neq 0, \det c \neq 0$, and $u, v \in GL_m(\mathbf{Z})$, we have

$$(1) \quad K(T, (c, d), v_1; \tilde{\sigma}, \tilde{v}_2) = |\det v_1|^m K\left(T \left[\begin{pmatrix} v_1 & \\ & 1_m \end{pmatrix}^{-1} \right], (c, d), 1_r; \tilde{\sigma}, \tilde{v}_2\right),$$

$$(2) \quad K\left(T, (ucv, ud^t v^{-1}), 1_r; \begin{pmatrix} {}^t u^{-1} & \\ & u \end{pmatrix} \tilde{\sigma} \begin{pmatrix} v & \\ & {}^t v^{-1} \end{pmatrix}, \tilde{v}_2\right) \\ = K(T \left[\begin{pmatrix} 1_r & \\ & v^{-1} \end{pmatrix} \right], (c, d), 1_r; \tilde{\sigma}, \tilde{v}_2 {}^t u^{-1}),$$

$$(3) \quad K(T \left[\begin{pmatrix} u & \\ & 1_m \end{pmatrix} \right], (c, d), 1_r; \tilde{\sigma}, \tilde{v}_2) \\ = K(T, (c, d), 1_r; \tilde{\sigma}, u \tilde{v}_2) \quad \text{for } u \in GL_r(\mathbf{Z}_p).$$

PROOF. Writing $v_2 = x + qy^t c$, we can see that v_2 runs over $M_{r,m} \pmod{qv_1}$ with $v_2 \equiv \tilde{v}_2 \pmod{q}$ if and only if x runs over $M_{r,m} \pmod{qM_{r,m}}$ with $x \equiv \tilde{v}_2 \pmod{q}$ and y runs over $M_{r,m} \pmod{v_1 M_{r,m}}$. Then $ac^{-1}[{}^t(v_1^{-1}v_2)] = ac^{-1}[{}^t(v_1^{-1}x)] + qv_1^{-1}(y^t a^t x + xa^t y)^t v_1^{-1} + q^2 {}^t ca[{}^t y^t v_1^{-1}]$ and $c^{-1} {}^t(v_1^{-1}v_2) = c^{-1} {}^t(v_1^{-1}x) + q^t y^t v_1^{-1}$ imply $K(T, (c, d), v_1; \tilde{\sigma}, \tilde{v}_2) = \sum_x e\left(\text{tr } T \begin{pmatrix} ac^{-1}[{}^t(v_1^{-1}x)] & -v_1^{-1}x^t c^{-1} \\ -c^{-1} {}^t(v_1^{-1}x) & c^{-1}d \end{pmatrix} / q\right) \times \sum_y e\left(\text{tr } T \begin{pmatrix} (y^t a^t x + xa^t y + q^t {}^t ca[{}^t y]) [{}^t v_1^{-1}] & -v_1^{-1}y \\ -{}^t y^t v_1^{-1} & 0 \end{pmatrix}\right)$. If $T_1[v_1^{-1}] \notin \Lambda'_r$, then both sides of (1) are equal to 0 by definition, and so we may assume $T_1[v_1^{-1}] \in \Lambda'_r$. Therefore the partial sum on y turns out to be equal to $\sum_y e(-2 \text{tr } T_2 {}^t y^t v_1^{-1}) = |\det v_1|^m$ if $2 {}^t v_1^{-1} T_2 \in M_{r,m}$ and = 0 otherwise. Thus we have $K(T, (c, d), v_1; \tilde{\sigma}, \tilde{v}_2) = |\det v_1|^m K\left(T \left[\begin{pmatrix} v_1 & \\ & 1_m \end{pmatrix}^{-1} \right], (c, d), 1_r; \tilde{\sigma}, \tilde{v}_2\right)$ if $T \left[\begin{pmatrix} v_1 & \\ & 1_m \end{pmatrix}^{-1} \right] \in \Lambda'_n$ and = 0 otherwise. Since the right hand side of (1) is equal to 0 if $T \left[\begin{pmatrix} v_1 & \\ & 1_m \end{pmatrix}^{-1} \right] \notin \Lambda'_n$, we have proved (1). Noting $\begin{pmatrix} {}^t u^{-1} & \\ & u \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} v & \\ & {}^t v^{-1} \end{pmatrix} \equiv \begin{pmatrix} {}^t u^{-1} & \\ & u \end{pmatrix} \tilde{\sigma} \begin{pmatrix} v & \\ & {}^t v^{-1} \end{pmatrix} \pmod{q}$, the left hand side of (2) is equal to

$$\sum_{\substack{v_2 \in M_{r,m} \pmod{qM_{r,m}} \\ v_2 \equiv \tilde{v}_2 \pmod{q}}} e\left(\text{tr } T \begin{pmatrix} {}^t u^{-1} ac^{-1} u^{-1} [{}^t v_2] & {}^*_{v^{-1} c^{-1} d {}^t v^{-1}} \\ -v^{-1} c^{-1} u^{-1} {}^t v_2 & \end{pmatrix} / q\right) \\ = \sum_{\substack{V_2 \in M_{r,m} \pmod{qM_{r,m}} \\ V_2 \equiv \tilde{v}_2 {}^t u^{-1} \pmod{q}}} e\left(\text{tr } T \begin{pmatrix} ac^{-1} [{}^t V_2] & {}^*_{-v^{-1} c^{-1} {}^t V_2} \\ -v^{-1} c^{-1} {}^t V_2 & c^{-1} d [{}^t v^{-1}] \end{pmatrix} / q\right) \\ = \sum e\left(\text{tr } T \left[\begin{pmatrix} 1_r & \\ & v \end{pmatrix}^{-1} \right] \begin{pmatrix} ac^{-1} [{}^t V_2] & {}^*_{-c^{-1} {}^t V_2} \\ -c^{-1} {}^t V_2 & c^{-1} d \end{pmatrix} / q\right) \\ = K\left(T \left[\begin{pmatrix} 1_r & \\ & v \end{pmatrix}^{-1} \right], (c, d), 1_r; \tilde{\sigma}, \tilde{v}_2 {}^t u^{-1}\right).$$

(3) is easy. ■

§2. Fourier coefficients of Klingen's Eisenstein series

From now on, we fix a natural number k . Let f be a cusp form of degree r , weight k and level q . For $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_r$, $N \in \Gamma_n$, we put

$$\begin{aligned} j(M, z) &:= \det(cz + d) \quad \text{for } z \in H_r, \\ (f|M)(z) &:= j(M, z)^{-k} f(M(z)) \quad \text{for } z \in H_r, \\ (f\|_s N)(z) &:= j(N, z)^{-k} f(N(z)_1) \left(\frac{\det \Im N(z)}{\det \Im N(z)_1} \right)^s \quad \text{for } z \in H_n, \end{aligned}$$

where $N(z)_1 = N(z) \begin{bmatrix} 1_r \\ 0 \end{bmatrix}$. By definition, $f(z)$ is holomorphic on H_r and satisfies

$$f|M = f \quad \text{for } M \in \Gamma_r(q) \quad (:= \{\sigma \in \Gamma_r \mid \sigma \equiv 1_{2r} \pmod{q}\}).$$

Note that we consider principal subgroups. It is more convenient to treat cusps impartially. We often use $\|$ instead of $\|_s$ for simplicity.

LEMMA 13. For $N = \begin{pmatrix} a_1^{(r)} & 0 & b_1^{(r)} & * \\ * & * & * & * \\ c_1^{(r)} & 0 & d_1^{(r)} & * \\ 0 & 0 & 0 & d_4^{(m)} \end{pmatrix} \in \Delta_{n,r}$ and $M \in \Gamma_n$, we have

$$f\|NM = \det d_4^{-k} \left(f \left| \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} \right. \right) \|M.$$

PROOF. The proof is straightforward.

$$\begin{aligned} j(NM, z) &= j(N, M(z))j(M, z) = \det \left(\begin{pmatrix} c_1 & 0 \\ 0 & 0 \end{pmatrix} M(z) + \begin{pmatrix} d_1 & * \\ 0 & d_4 \end{pmatrix} \right) j(M, z) \\ &= \det d_4 \det(c_1 M(z)_1 + d_1) j(M, z). \end{aligned}$$

$(NM)(z)_1 = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} (M(z)_1)$ is easy. $\Im NM(z) = {}^t \left(\begin{pmatrix} c_1 & 0 \\ 0 & 0 \end{pmatrix} M(\bar{z}) + \begin{pmatrix} d_1 & * \\ 0 & d_4 \end{pmatrix} \right)^{-1}$
 $\Im M(z) \left(\begin{pmatrix} c_1 & 0 \\ 0 & 0 \end{pmatrix} M(z) + \begin{pmatrix} d_1 & * \\ 0 & d_4 \end{pmatrix} \right)^{-1}$ implies $\det \Im NM(z) = \det \Im M(z) |\det(c_1 M(z)_1 + d_1)|^{-2}$ and $\det \Im NM(z)_1 = \det \Im M(z)_1 |\det(c_1 M(z)_1 + d_1)|^{-2}$. Thus we have

$$\begin{aligned} f\|NM(z) &= \det d_4^{-k} \det(c_1 M(z)_1 + d_1)^{-k} j(M, z)^{-k} \\ &\quad \times f \left(\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} (M(z)_1) \right) \left(\frac{\det \Im M(z)}{\det \Im M(z)_1} \right)^s \\ &= \det d_4^{-k} j(M, z)^{-k} \left(f \left| \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} \right. \right) (M(z)_1) \left(\frac{\det \Im M(z)}{\det \Im M(z)_1} \right)^s, \end{aligned}$$

which completes the proof. $\blacksquare$

LEMMA 14. For $M \in \Gamma_n, \lambda \in \Lambda_n$ and $w \in U_n$, we have

$$(f \| M \begin{pmatrix} 1_n & \lambda \\ 0 & 1_n \end{pmatrix} \begin{pmatrix} w^{-1} & \\ & {}_t w \end{pmatrix})(z) = (f \| M)(z[{}^t w^{-1}] + \lambda).$$

PROOF. This follows immediately from $j \left(M \begin{pmatrix} 1_n & \lambda \\ 0 & 1_n \end{pmatrix} \begin{pmatrix} w^{-1} & \\ & {}_t w \end{pmatrix}, z \right) = j(M, z[{}^t w^{-1}] + \lambda)$. $\blacksquare$

LEMMA 15. For $v \in U_n$ with $\det v_1^{(r)} \neq 0$ and $\sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_m$ with $\det c \neq 0$, we have

$$\begin{aligned} & (f \| N(v, \sigma))(z) \\ &= j(\sigma, z_4)^{-k} f(z_1[{}^t v_1] - (z_4 + c^{-1}d)^{-1}[z_3 {}^t v_1 - c^{-1} {}^t v_2] + ac^{-1}[{}^t v_2]) \\ & \quad \times |\det c \det v_1|^{-2s} \det \left(y^{-1} \begin{bmatrix} x_2 - v_1^{-1} v_2 {}^t c^{-1} \\ x_4 + c^{-1}d \end{bmatrix} + y_4 \right)^{-s}, \end{aligned}$$

where $z = \begin{pmatrix} z_1^{(r)} & z_2 \\ z_3 & z_4 \end{pmatrix}$ and $x_i = \Re z_i, y_i = \Im z_i, y = \Im z$.

PROOF. Since $N(v, \sigma) = \begin{pmatrix} v & \\ & {}_t v^{-1} \end{pmatrix} \sigma \downarrow$, we have

$$j(N(v, \sigma), z) = \det \left({}^t v^{-1} \begin{pmatrix} 0 & \\ & c \end{pmatrix} z + {}^t v^{-1} \begin{pmatrix} 1 & \\ & d \end{pmatrix} \right) = j(\sigma, z_4),$$

and

$$\begin{aligned} N(v, \sigma)(z) &= v \left(\begin{pmatrix} 1 & \\ & a \end{pmatrix} z + \begin{pmatrix} 0 & \\ & b \end{pmatrix} \right) \left(\begin{pmatrix} 0 & \\ & c \end{pmatrix} z + \begin{pmatrix} 1 & \\ & d \end{pmatrix} \right)^{-1} {}^t v \\ &= v \begin{pmatrix} z_1 - (z_4 + c^{-1}d)^{-1}[z_3] & z_2(z_4 + c^{-1}d)^{-1}c^{-1} \\ {}^t c^{-1}(z_4 + c^{-1}d)^{-1}z_3 & (az_4 + b)(cz_4 + d)^{-1} \end{pmatrix} {}^t v. \end{aligned}$$

Hence $N(v, \sigma)(z)_1 := N(v, \sigma)(z) \begin{bmatrix} 1_r \\ 0 \end{bmatrix} = (z_1 - (z_4 + c^{-1}d)^{-1}[z_3])[{}^t v_1] + v_1 z_2 (z_4 + c^{-1}d)^{-1}c^{-1} {}^t v_2 + v_2 {}^t c^{-1} (z_4 + c^{-1}d)^{-1} z_3 {}^t v_1 + (az_4 + b)(cz_4 + d)^{-1}[{}^t v_2] = z_1[{}^t v_1] - (z_4 + c^{-1}d)^{-1}[z_3 {}^t v_1 - c^{-1} {}^t v_2] + ac^{-1}[{}^t v_2]$, because of $(az_4 + b)(cz_4 + d)^{-1} = ac^{-1} - (z_4 + c^{-1}d)^{-1}[c^{-1}]$. It remains to verify

$$\frac{\det \Im N(v, \sigma)(z)}{\det \Im N(v, \sigma)(z)_1} = \det c^{-2} \det v_1^{-2} \det \left(y^{-1} \begin{bmatrix} x_2 - v_1^{-1} v_2 {}^t c^{-1} \\ x_4 + c^{-1}d \end{bmatrix} + y_4 \right)^{-1}.$$

For complex matrices A, B , we denote ${}^t \bar{B} A B$ by $A\{B\}$ if it is defined. First let us see for $z = x + iy \in H_n$

$$(1) \quad \det y\{z^{-1}\} / \det y\{z^{-1}\}_1 = \det(y^{-1} \begin{bmatrix} x_2 \\ x_4 \end{bmatrix} + y_4)^{-1}.$$

Putting $Y := y\{z^{-1}\}$, we have $Y = (xy^{-1}x + y)^{-1}$. Decomposing Y as $Y = \begin{pmatrix} Y_1^{(r)} & \\ & Y_4 - Y_1^{-1}[Y_2] \end{pmatrix} \begin{bmatrix} \begin{pmatrix} 1_r & Y_1^{-1}Y_2 \\ 0 & 1_m \end{pmatrix} \end{bmatrix}$, we have $\det Y / \det Y_1 = \det(Y_4 - Y_1^{-1}[Y_2])$ and $xy^{-1}x + y = Y^{-1} = \begin{pmatrix} * & \\ * & (Y_4 - Y_1^{-1}[Y_2])^{-1} \end{pmatrix}$. From these follows (1). Next let us show

$$\det \Im j(z) / \det \Im j(z)_1 = \det \Im z / \det \Im z_1$$

for $j := \begin{pmatrix} 0 & -1_r \\ 1_r & 0 \end{pmatrix} \uparrow \in \Gamma_n$. This follows from $\det \Im j(z) / \det \Im j(z)_1 = \det y \left\{ \left(\begin{pmatrix} 1_r & \\ & 0 \end{pmatrix} z + \begin{pmatrix} 0 & \\ & 1_m \end{pmatrix} \right)^{-1} \right\} / \det \Im(-z_1^{-1}) = \det y / \det y_1$. We apply this to $N(v, \sigma)z$ instead of z . To do so, we calculate $\Im jN(v, \sigma)(z)$. Since

$$\begin{aligned} jN(v, \sigma) &= \begin{pmatrix} \begin{pmatrix} 0 & \\ & 1_m \end{pmatrix} \begin{pmatrix} -1_r & 0 \\ 0 & 1_m \end{pmatrix} \\ \begin{pmatrix} 1_r & 0 \\ 0 & 1_m \end{pmatrix} \end{pmatrix} \begin{pmatrix} v & \\ & u \end{pmatrix} \begin{pmatrix} \begin{pmatrix} 1_r & \\ & a \end{pmatrix} \begin{pmatrix} 0 & b \\ 1_r & d \end{pmatrix} \end{pmatrix} \quad (u := {}^t v^{-1}) \\ &= \begin{pmatrix} 0 & -u_2 c & -u_1 & -u_2 d \\ v_3 & v_4 a & 0 & v_4 b \\ v_1 & v_2 a & 0 & v_2 b \\ 0 & u_4 c & u_3 & u_4 d \end{pmatrix}, \end{aligned}$$

$$\begin{aligned} \Im jN(v, \sigma)(z) &= y \left\{ \left(\begin{pmatrix} v_1 & v_2 a \\ 0 & u_4 c \end{pmatrix} z + \begin{pmatrix} 0 & v_2 b \\ u_3 & u_4 d \end{pmatrix} \right)^{-1} \right\} \\ &= y \left\{ \left(z + \begin{pmatrix} ac^{-1} [{}^t(v_1^{-1} v_2)] & * \\ -c^{-1} {}^t(v_1^{-1} v_2) & c^{-1} d \end{pmatrix} \right)^{-1} \right\} \left[\begin{pmatrix} v_1 & v_2 a \\ 0 & u_4 c \end{pmatrix}^{-1} \right], \end{aligned}$$

because of $u_4^{-1} u_3 = -{}^t(v_1^{-1} v_2)$. Thus we have

$$\begin{aligned} &\det \Im N(v, \sigma)(z) / \det \Im N(v, \sigma)(z)_1 \\ &= \det \Im jN(v, \sigma)(z) / \det \Im jN(v, \sigma)(z)_1 \\ &= \frac{\det y \left\{ \left(z + \begin{pmatrix} ac^{-1} [{}^t(v_1^{-1} v_2)] & * \\ -c^{-1} {}^t(v_1^{-1} v_2) & c^{-1} d \end{pmatrix} \right)^{-1} \right\} \det v_1^{-2} \det(u_4 c)^{-2}}{\det y \left\{ \left(z + \begin{pmatrix} ac^{-1} [{}^t(v_1^{-1} v_2)] & * \\ -c^{-1} {}^t(v_1^{-1} v_2) & c^{-1} d \end{pmatrix} \right)^{-1} \right\}_1 \det v_1^{-2}} \\ &= \det \left(y^{-1} \begin{bmatrix} x_2 - v_1^{-1} v_2 {}^t c^{-1} \\ x_4 + c^{-1} d \end{bmatrix} + y_4 \right)^{-1} \det c^{-2} \det v_1^{-2} \quad \text{by (1)}, \end{aligned}$$

which is what we want. ■

LEMMA 16. *Matrices with index 1 are upper left $r \times r$ submatrices here. Let $f(z) = \sum_{0 < t \in \Lambda'_r} b(t) e(\text{tr } tz/q)$ be a cusp form of degree r , weight k and level q , and $\sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_m$ with $\det c \neq 0$ and $v \in U_n$ with $\det v_1 \neq 0$. Then we have, for $T \in \Lambda'_n$*

$$\begin{aligned}
& \int_{x \in \Lambda_n(\mathbf{R})/q\Lambda_n} \sum_{\substack{\lambda \in q\Lambda_n \\ \lambda_1=0}} (f\|_s N(v, \sigma) \begin{pmatrix} 1_n & \lambda \\ 0 & 1_n \end{pmatrix})(z) e(-\operatorname{tr} Tx/q) dx \\
&= q^{r(r+1)/2} H(q^{-1}T, y; s) |\det c^{-k}| |\det c \det v_1|^{-2s} b(T_1[v_1^{-1}]) \\
&\quad \times e\left(\operatorname{tr} T \begin{pmatrix} ac^{-1} [{}^t(v_1^{-1}v_2)] & * \\ -c^{-1} {}^t(v_1^{-1}v_2) & c^{-1}d \end{pmatrix} / q\right),
\end{aligned}$$

where $x := \Re z, y := \Im z$ and we assume $b(T_1[v_1^{-1}]) = 0$ unless $0 < T_1[v_1^{-1}] \in \Lambda'_r$, and

$$\begin{aligned}
H(T, y; s) &:= e(i \operatorname{tr} T_1 y_1) \int_{\substack{\sigma \in \Lambda_n(\mathbf{R}) \\ \sigma_1=0}} \det(\sigma_4 + iy_4)^{-k} \det(y^{-1} \begin{bmatrix} \sigma_2 \\ \sigma_4 \end{bmatrix} + y_4)^{-s} \\
&\quad \times e\left(-\operatorname{tr} T_1 \cdot z_4^{-1}[z_3] - \operatorname{tr} \begin{pmatrix} 0 & \sigma_2 \\ \sigma_3 & \sigma_4 \end{pmatrix} T\right) d\sigma_2 d\sigma_4,
\end{aligned}$$

where we put $z_j = \sigma_j + iy_j$ and $d\sigma_2 = \prod_{i,j} da_{ij}$ for $\sigma_2 = (a_{ij})$ and $d\sigma_4 = \prod_{i \leq j} db_{ij}$ for $\sigma_4 = (b_{ij})$.

PROOF. Since $(f\|N(v, \sigma) \begin{pmatrix} 1_n & \lambda \\ 0 & 1_n \end{pmatrix})(z) = (f\|N(v, \sigma))(z + \lambda)$ by Lemma 14, the integral in question is equal to

$$\begin{aligned}
& \int_{x \in \Lambda_n(\mathbf{R})/q\Lambda_n} \sum_{\substack{\lambda \in q\Lambda_n \\ \lambda_1=0}} (f\|N(v, \sigma))(z + \lambda) e(-\operatorname{tr} Tx/q) dx \\
&= \int_{\substack{x_1 \in \Lambda_r(\mathbf{R})/q\Lambda_r, x_2 \in M_{r,m}(\mathbf{R}) \\ x_4 \in \Lambda_m(\mathbf{R})}} (f\|N(v, \sigma))(x + iy) e(-\operatorname{tr} Tx/q) dx \\
&= \int_{\substack{x_1 \in \Lambda_r(\mathbf{R})/q\Lambda_r, x_2 \in M_{r,m}(\mathbf{R}) \\ x_4 \in \Lambda_m(\mathbf{R})}} \det(cz_4 + d)^{-k} |\det c \det v_1|^{-2s} \det\left(y^{-1} \begin{bmatrix} x_2 - v_1^{-1} v_2 {}^t c^{-1} \\ x_4 + c^{-1} d \end{bmatrix} + y_4\right)^{-s} \\
&\quad \times \sum_{0 < t \in \Lambda'_r} b(t) e(\operatorname{tr} t(z_1[{}^t v_1] - (z_4 + c^{-1}d)^{-1}[z_3 {}^t v_1 - c^{-1} {}^t v_2] + ac^{-1}[{}^t v_2])/q) \\
&\quad \times e(-\operatorname{tr} Tx/q) dx, \\
&= |\det c^{-k}| |\det c \det v_1|^{-2s} \sum_{0 < t \in \Lambda'_r} b(t) e(i \operatorname{tr} t[v_1] y_1/q + \operatorname{tr} t \cdot ac^{-1}[{}^t v_2]/q) \\
&\quad \times \int_{x_1 \in \Lambda_r(\mathbf{R})/q\Lambda_r} e(\operatorname{tr}(t[v_1] - T_1)x_1/q) dx_1
\end{aligned}$$

$$\begin{aligned}
& \times \int_{\substack{x_2 \in M_{r,m}(\mathbf{R}) \\ x_4 \in \Lambda_m(\mathbf{R})}} \det(z_4 + c^{-1}d)^{-k} \det \left(y^{-1} \begin{bmatrix} x_2 - v_1^{-1}v_2 {}^t c^{-1} \\ x_4 + c^{-1}d \end{bmatrix} + y_4 \right)^{-s} \\
& \times e \left(\operatorname{tr} t(- (z_4 + c^{-1}d)^{-1} [z_3 {}^t v_1 - c^{-1} {}^t v_2]) / q - \operatorname{tr} \begin{pmatrix} 0 & T_2 \\ T_3 & T_4 \end{pmatrix} \begin{pmatrix} 0 & x_2 \\ {}^t x_2 & x_4 \end{pmatrix} / q \right) dx_2 dx_4 \\
& = q^{r(r+1)/2} \det c^{-k} |\det c \det v_1|^{-2s} b(T_1[v_1^{-1}]) e(\operatorname{tr} T_1 \cdot ac^{-1} [{}^t(v_1^{-1}v_2)] / q) \\
& \times e \left(\operatorname{tr} \begin{pmatrix} 0 & T_2 \\ T_3 & T_4 \end{pmatrix} \begin{pmatrix} 0 & -v_1^{-1}v_2 {}^t c^{-1} \\ -c^{-1} {}^t(v_1^{-1}v_2) & c^{-1}d \end{pmatrix} / q \right) H(q^{-1}T, y; s). \quad \blacksquare
\end{aligned}$$

REMARK. Although we did not refer to them explicitly, the convergence and the commutativity of sums and integrals are justified if $\Re s$ is sufficiently large. $H(T, y; s)$ is absolutely convergent when $\Re s > (n - k)/2$, and $H(T, y; s) = O((\det T_1^{(r)})^{-m/2} (\det y_4^{(m)})^{(n+1)/2-k-s} \exp(-2\pi \operatorname{tr} T_1(y_1 - y_4^{-1} [{}^t y_2])))$ if $\Re s > (n - k)/2$ and $\Re s \geq 0$. For $w = \begin{pmatrix} w_1^{(r)} & w_2 \\ 0 & w_4 \end{pmatrix} \in GL_n(\mathbf{R})$,

$$H(T[w], y[{}^t w^{-1}]; s) = |\det w_1|^{-m} |\det w_4|^{2k-n-1+2s} H(T, y; s).$$

In particular, $H(T, T^{-1}; s) = \det T^{-(n+1)/2+k+s} \det T_1^{(r+1)/2-k-s} H(1_n, 1_n; s)$. It is not difficult to see

$$\begin{aligned}
H(T, y; 0) &= 2^{-(n-1)m/2} i^{-mk} (2\pi)^{m(k-r/2)} \pi^{-m(m-1)/4} \prod_{j=0}^{m-1} \Gamma(k - (r+j)/2)^{-1} \\
& \times \exp(-2\pi \operatorname{tr} Ty) \begin{cases} \det T_1^{(r+1)/2-k} \det T^{k-(n+1)/2} & \text{if } T > 0, \\ 0 & \text{otherwise} \end{cases}
\end{aligned}$$

when $k > n$.

LEMMA 17. Let $f(z)$ be the cusp form in the previous lemma, and for $\sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_m$ with $\det c \neq 0$ and for $\tilde{v}_1 \in M_r, \tilde{v}_2 \in M_{r,m}$, we have

$$\begin{aligned}
& q^{-r(r+1)/2} \int_{x \in \Lambda_n(\mathbf{R})/q\Lambda_n} \sum_{\lambda, v_1, v_2: (C)} (f \| N(v, \sigma) \begin{pmatrix} 1_n & \lambda \\ 0 & 1_n \end{pmatrix})(z) e(-\operatorname{tr} Tx/q) dx \\
& = [GL_r(\mathbf{Z}) : U_r(q)]^{-1} H(q^{-1}T, y; s) \det c^{-k} |\det c|^{-2s} \\
& \times \sum_{v_1 \in U_r(q) \setminus (M_r \cap GL_r(\mathbf{Q}))} |\det v_1|^{m-2s} \\
& \times \sum_{\substack{g \in U_r(q) \setminus (M_r \cap GL_r(\mathbf{Q})) \\ gv_1 \equiv \tilde{v}_1 \pmod{q}}} \mu(\mathbf{Z}^r / \mathbf{Z}^r g) |\det g|^{-2s} b(T_1[v_1^{-1}g^{-1}]) \\
& \times \sum_{\substack{\tilde{v}_2 \in M_{r,m} \pmod{q} M_{r,m} \\ g\tilde{v}_2 \equiv \tilde{v}_2 \pmod{q}}} K \left(T \left[\begin{pmatrix} v_1^{-1} & \\ & 1_m \end{pmatrix} \right], (c, d), 1_r; \tilde{\sigma}, \tilde{V}_2 \right),
\end{aligned}$$

where the condition (C) on λ, v_1, v_2 is

(C): λ runs over $q\Lambda_n$ with $\lambda_1^{(r)} = 0$, v_1 runs over $U_r(q) \backslash M_r$ with $\det v_1 \neq 0$ and $v_1 \equiv \tilde{v}_1 \pmod{q}$, and v_2 runs over $M_{r,m} \pmod{qv_1 M_{r,m} {}^t c}$ with (v_1, v_2) being primitive and $v_2 \equiv \tilde{v}_2 \pmod{q}$, and for v_1, v_2 above $v = \begin{pmatrix} v_1 & v_2 \\ * & * \end{pmatrix}$ is any extension to an element in U_n .

PROOF. Denoting the left hand side of the equation in Lemma by I , the previous lemma yields

$$I = H(q^{-1}T, y; s) \det c^{-k} |\det c|^{-2s} \\ \times \sum_{v_1} |\det v_1|^{-2s} b(T_1[v_1^{-1}]) k(T, (c, d), v_1; \tilde{\sigma}, \tilde{v}_2),$$

where v_1 runs over the set in (C). By Lemma 10, the partial sum on v_1 is equal to

$$\sum_{v_1} |\det v_1|^{-2s} b(T_1[v_1^{-1}]) \sum_{g \in M_r / GL_r(\mathbf{Z})} \mu(\mathbf{Z}^r / \mathbf{Z}^r g) \\ \times \sum_{\substack{\tilde{v}_2 \in M_{r,m} \pmod{qM_{r,m}} \\ g\tilde{v}_2 \equiv \tilde{v}_2 \pmod{q}}} K(T, (c, d), g^{-1}v_1; \tilde{\sigma}, \tilde{V}_2) \\ = [GL_r(\mathbf{Z}) : U_r(q)]^{-1} \sum_{v_1} |\det v_1|^{-2s} b(T_1[v_1^{-1}]) \sum_{g \in M_r / U_r(q)} \mu(\mathbf{Z}^r / \mathbf{Z}^r g) \\ \times \sum_{\substack{\tilde{V}_2 \in M_{r,m} \pmod{qM_{r,m}} \\ g\tilde{V}_2 \equiv \tilde{v}_2 \pmod{q}, g^{-1}v_1 \in M_r}} |\det g^{-1}v_1|^m K\left(T \left[\begin{pmatrix} v_1^{-1}g & \\ & 1_m \end{pmatrix} \right], (c, d), 1_r; \tilde{\sigma}, \tilde{V}_2\right) \\ = [GL_r(\mathbf{Z}) : U_r(q)]^{-1} \sum_{v_1} |\det v_1|^{m-2s} b(T_1[v_1^{-1}]) \sum_{\substack{g \in M_r / U_r(q) \\ g^{-1}v_1 \in M_r}} \mu(\mathbf{Z}^r / \mathbf{Z}^r g) \\ \times |\det g|^{-m} \sum_{\substack{\tilde{V}_2 \in M_{r,m} \pmod{qM_{r,m}} \\ g\tilde{V}_2 \equiv \tilde{v}_2 \pmod{q}}} K\left(T \left[\begin{pmatrix} v_1^{-1}g & \\ & 1_m \end{pmatrix} \right], (c, d), 1_r; \tilde{\sigma}, \tilde{V}_2\right),$$

where v_1 runs over the set in (C). It is easy to see that $\begin{pmatrix} g \\ v_1 \end{pmatrix}$ runs over

$$U_r(q) \backslash S_1 / \begin{pmatrix} U_r(q) & \\ & 1_r \end{pmatrix}$$

for $S_1 := \left\{ \begin{pmatrix} g \\ v \end{pmatrix} \mid g, v \in M_r \cap GL_r(\mathbf{Q}), g^{-1}v \in M_r, v \equiv \tilde{v}_1 \pmod{q} \right\}$.

Let us consider the mapping $\phi : \begin{pmatrix} g \\ v \end{pmatrix} \mapsto \begin{pmatrix} g \\ {}^t(g^{-1}v) \end{pmatrix}$ from this set to

$$(\mathfrak{h}) : \left(\begin{pmatrix} U_r(q) & \\ & 1_r \end{pmatrix} \backslash S_2 / \{ \begin{pmatrix} w & \\ & {}^t w^{-1} \end{pmatrix} \mid w \in U_r(q) \} \right. \\ \left. \text{for } S_2 := \left\{ \begin{pmatrix} G \\ V \end{pmatrix} \mid G, V \in M_r \cap GL_r(\mathbf{Q}), G^t V \equiv \tilde{v}_1 \bmod q \right\} \right).$$

For $u, v \in U_r(q)$, $\phi \begin{pmatrix} ugw \\ uv \end{pmatrix} = \begin{pmatrix} u & \\ & 1_r \end{pmatrix} \begin{pmatrix} g & \\ & {}^t v^{-1} {}^t g^{-1} \end{pmatrix} \begin{pmatrix} w & \\ & {}^t w^{-1} \end{pmatrix}$ yields that the mapping ϕ is well-defined. If $\phi \begin{pmatrix} g_1 \\ v_1 \end{pmatrix} = \begin{pmatrix} u & \\ & 1 \end{pmatrix} \phi \begin{pmatrix} g_2 \\ v_2 \end{pmatrix} \begin{pmatrix} w & \\ & {}^t w^{-1} \end{pmatrix}$ for $u, w \in U_r(q)$, then $g_1 = u g_2 w$ and ${}^t v_1 {}^t g_1^{-1} = {}^t v_2 {}^t g_2^{-1} {}^t w^{-1}$, which imply $\begin{pmatrix} g_1 \\ v_1 \end{pmatrix} = u \begin{pmatrix} g_2 \\ v_2 \end{pmatrix} \begin{pmatrix} w & \\ & 1 \end{pmatrix}$. Thus ϕ is injective. $\phi \begin{pmatrix} G \\ G^t V \end{pmatrix} = \begin{pmatrix} G \\ V \end{pmatrix}$ means that ϕ is surjective and hence ϕ is bijective. In the set $(\mathfrak{h})$, we can choose first $V \in (M_r \cap GL_r(\mathbf{Q})) / U_r(q)$ and then $G \in U_r(q) \backslash (M_r \cap GL_r(\mathbf{Q}))$. Thus the above sum turns out to be equal to

$$[GL_r(\mathbf{Z}) : U_r(q)]^{-1} \sum_{V \in (M_r \cap GL_r(\mathbf{Q})) / U_r(q)} \sum_{\substack{G \in U_r(q) \backslash (M_r \cap GL_r(\mathbf{Q})) \\ G^t V \equiv \tilde{v}_1 \bmod q}} |\det GV|^{m-2s} \\ \times b(T_1[{}^t V^{-1} G^{-1}]) \mu(\mathbf{Z}^r / \mathbf{Z}^r G) |\det G|^{-m} \\ \times \sum_{\substack{\tilde{V}_2 \in M_{r,m} \bmod q M_{r,m} \\ G \tilde{V}_2 \equiv \tilde{v}_2 \bmod q}} K \left(T \left[\begin{pmatrix} {}^t V^{-1} & \\ & 1_m \end{pmatrix} \right], (c, d), 1_r; \tilde{\sigma}, \tilde{V}_2 \right).$$

Replacing V by ${}^t v_1$ and G by g , we get the assertion. $\blacksquare$

LEMMA 18. For the cusp form $f(z)$ in Lemma 16 and $M \in \Gamma_n$, we put

$$\mathcal{F}(T, y; f, M) := q^{-r(r+1)/2} \int_{x \in \Lambda_n(\mathbf{R})/q\Lambda_n} \sum_K (f \| K)(z) e(-\operatorname{tr} T x / q) dx,$$

where $y := \Im z$, $x := \Re z$ and σ runs over

$$\Delta_{n,r}(q) \backslash \{ K \in \Gamma_n \mid \operatorname{rank}(0, 1_m) K_3^{(n)} = m, \sigma \equiv M \bmod q \}.$$

For $\eta := \begin{pmatrix} a_1^{(r)} & 0 & b_1^{(r)} & * \\ * & * & * & * \\ c_1^{(r)} & 0 & d_1^{(r)} & * \\ 0 & 0 & 0 & d_4 \end{pmatrix} \in \Delta_{n,r}$, we put

$$\det d_4^{-k} \left(f \left| \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} \right. \right) (\tau) := \sum_{t > 0} b(t; \eta) e(\operatorname{tr} t \tau / q) \quad (\tau \in H_r).$$

Then we have

$$\begin{aligned}
& \mathcal{F}(T, y; f, M) \\
&= [\Gamma_m^\infty : \Gamma_m^\infty(q)]^{-1} [P_{n,r} : P_{n,r}(q)]^{-2} [GL_r(\mathbf{Z}) : U_r(q)]^{-1} \\
&\quad \times \sum_w H(q^{-1}T[w], y[{}^t w^{-1}]; s) \\
&\quad \times \sum_{\tilde{v}, \eta, \tilde{\sigma}, v_1} |\det v_1|^{m-2s} \sum_g \mu(\mathbf{Z}^r / \mathbf{Z}^r g) |\det g|^{-2s} b(T[w]_1 [v_1^{-1} g^{-1}]; \eta) \\
&\quad \times \sum_{\tilde{V}_2} \sum_{(c,d)} \det c^{-k} |\det c|^{-2s} K(T[w({}^{v_1} {}_1 m)^{-1}], (c, d), 1_r; \tilde{\sigma}, \tilde{V}_2),
\end{aligned}$$

where the ranges of sums are

$$\begin{aligned}
& w \in U_n / P_{n,r}(q), \\
& \tilde{v} = \begin{pmatrix} \tilde{v}_1^{(r)} & \tilde{v}_2 \\ * & * \end{pmatrix} \in U_n / U_n(q), \\
& \eta \in \Delta_{n,r}(q) \setminus \Delta_{n,r}, \\
& \tilde{\sigma} = \begin{pmatrix} \tilde{a} & \tilde{b} \\ \tilde{c} & \tilde{d} \end{pmatrix} \in \Gamma_m / \Gamma_m(q) \quad \text{with} \quad \eta N(\tilde{v}, \tilde{\sigma}) \begin{pmatrix} w^{-1} & \\ & t_w \end{pmatrix} \equiv M \pmod{q}, \\
& v_1 \in U_r(q) \setminus (M_r \cap GL_r(\mathbf{Q})), \\
& g \in U_r(q) \setminus (M_r \cap GL_r(\mathbf{Q})) \quad \text{with} \quad gv_1 \equiv \tilde{v}_1 \pmod{q}, \\
& \tilde{V}_2 \in M_{r,m} \pmod{qM_{r,m}} \quad \text{with} \quad g\tilde{V}_2 \equiv \tilde{v}_2 \pmod{q},
\end{aligned}$$

and lastly (c, d) runs over

$$U_m(q) \setminus \{(c, d) \mid c, d \in M_m, \det c \neq 0\} / \begin{pmatrix} 1_m & q\Lambda_m \\ 0 & 1_m \end{pmatrix}.$$

PROOF. First, let us see that the number of inverse images of the canonical mapping from $\Gamma_m^\infty(q) \setminus \Gamma_m / \begin{pmatrix} 1 & q\Lambda_m \\ 0 & 1 \end{pmatrix}$ to $\Gamma_m^\infty \setminus \Gamma_m / \begin{pmatrix} 1 & q\Lambda_m \\ 0 & 1 \end{pmatrix}$ is $[\Gamma_m^\infty : \Gamma_m^\infty(q)]$. For $\sigma, \eta \in \Gamma_m$, $\Gamma_m^\infty \sigma \begin{pmatrix} 1 & q\Lambda_m \\ 0 & 1 \end{pmatrix} = \Gamma_m^\infty \eta \begin{pmatrix} 1 & q\Lambda_m \\ 0 & 1 \end{pmatrix}$ implies $\Gamma_m^\infty(q) \sigma \begin{pmatrix} 1 & q\Lambda_m \\ 0 & 1 \end{pmatrix} = \xi \Gamma_m^\infty(q) \eta \begin{pmatrix} 1 & q\Lambda_m \\ 0 & 1 \end{pmatrix}$ for $\xi \in \Gamma_m^\infty$, and then $\Gamma_m^\infty(q) \sigma \begin{pmatrix} 1 & q\Lambda_m \\ 0 & 1 \end{pmatrix} = \Gamma_m^\infty(q) \eta \begin{pmatrix} 1 & q\Lambda_m \\ 0 & 1 \end{pmatrix}$ holds if and only if $\xi \in \Gamma_m^\infty(q)$, considering both sides modulo q . Thus the number in question is $[\Gamma_m^\infty : \Gamma_m^\infty(q)]$. Next, let us see that the number of inverse images of the canonical mapping from ${}^t P_{n,r}(q) \setminus U_n / \begin{pmatrix} 1 & qM_{r,m} {}^t c \\ 0 & 1 \end{pmatrix}$ to ${}^t P_{n,r} \setminus U_n / \begin{pmatrix} 1 & qM_{r,m} {}^t c \\ 0 & 1 \end{pmatrix}$ for $c \in M_m$ with $\det c \neq 0$ is $[P_{n,r} : P_{n,r}(q)]$. For $u, v \in U_n$, ${}^t P_{n,r} u \begin{pmatrix} 1 & qM_{r,m} {}^t c \\ 0 & 1 \end{pmatrix} =$

${}^tP_{n,r}v \begin{pmatrix} 1 & qM_{r,m} & {}^tc \\ 0 & 1 & \end{pmatrix}$ implies $u = wv \begin{pmatrix} 1 & qy & {}^tc \\ 0 & 1 & \end{pmatrix}$ for $w \in {}^tP_{n,r}$, $y \in M_{r,m}$, and then it is easy to see that ${}^tP_{n,r}(q)u \begin{pmatrix} 1 & qM_{r,m} & {}^tc \\ 0 & 1 & \end{pmatrix} = {}^tP_{n,r}(q)v \begin{pmatrix} 1 & qM_{r,m} & {}^tc \\ 0 & 1 & \end{pmatrix}$ if and only if $w \equiv 1_n \pmod{q}$. Thus the number of inverse images is $[P_{n,r} : P_{n,r}(q)]$. Now applying Lemma 5, we have

$$\begin{aligned} & \mathcal{F}(T, y; f, M) \\ &= q^{-r(r+1)/2} [\Gamma_m^\infty : \Gamma_m^\infty(q)]^{-1} [P_{n,r} : P_{n,r}(q)]^{-2} \\ & \quad \times \sum_{w, \lambda, \sigma, v, \eta} \int_{x \in \Lambda_n(\mathbf{R})/q\Lambda_n} (f \| \eta N(v, \sigma) \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix} \begin{pmatrix} w^{-1} & \\ & {}_w \end{pmatrix})(z) e(-\operatorname{tr} Tx/q) dx, \end{aligned}$$

where sums on $w, \lambda, \sigma, v, \eta$ are taken as follows: $w \in U_n/P_{n,r}(q)$, $\lambda \in q\Lambda_n$ with $\lambda_1^{(r)} = 0$, $\sigma \in \Gamma_m^\infty(q) \setminus \Gamma_m / \begin{pmatrix} 1_m & q\Lambda_m \\ 0 & 1_m \end{pmatrix}$ with $\det \sigma_3^{(m)} \neq 0$, $v \in {}^tP_{n,r}(q) \setminus U_n / \begin{pmatrix} 1_r & qM_{r,m} & {}^t\sigma_3 \\ 0 & 1_m & \end{pmatrix}$ with $\det v_1^{(r)} \neq 0$ and $\eta \in \Delta_{n,r}(q) \setminus \Delta_{n,r}$ with $\eta N(v, \sigma) \begin{pmatrix} w^{-1} & \\ & {}_w \end{pmatrix} \equiv M \pmod{q}$.

Hence the sum on $w, \lambda, \sigma, v, \eta$ in the above can be replaced by the sum on $w, \tilde{v}, \tilde{\eta}, \tilde{\sigma}, \sigma, v, \lambda$ where $w, \tilde{v}, \tilde{\eta}, \tilde{\sigma}$ are as in the assertion of Lemma and $\sigma \in \Gamma_m^\infty(q) \setminus \Gamma_m / \begin{pmatrix} 1 & q\Lambda_m \\ 0 & 1 \end{pmatrix}$ with $\det \sigma_3^{(m)} \neq 0$, $\sigma \equiv \tilde{\sigma} \pmod{q}$ and v runs over ${}^tP_{n,r}(q) \setminus U_n / \begin{pmatrix} 1 & qM_{r,m} & {}^t\sigma_3 \\ 0 & 1 & \end{pmatrix}$ with $\det v_1^{(r)} \neq 0$, $v \equiv \tilde{v} \pmod{q}$ and λ runs over $q\Lambda_n$ with $\lambda_1^{(r)} = 0$. And the above integral is equal to

$$\int_{x \in \Lambda_n(\mathbf{R})/q\Lambda_n} \left(f \| \eta N(v, \sigma) \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix} \right) (x + iy[{}^tw^{-1}]) e(-\operatorname{tr} T[w]x/q) dx.$$

Let us define the mapping $v \mapsto (1_r, 0)v$ from $\left\{ v = \begin{pmatrix} v_1^{(r)} & v_2 \\ * & * \end{pmatrix} \in {}^tP_{n,r}(q) \setminus U_n / \begin{pmatrix} 1 & qM_{r,m} & {}^t\sigma_3 \\ 0 & 1 & \end{pmatrix} \mid \det v_1 \neq 0, v \equiv \tilde{v} \pmod{q} \right\}$ to $U_r(q) \setminus \{(v_1, v_2) \mid v_1 \in M_r, v_1 \equiv \tilde{v}_1^{(r)} \pmod{q}, \det v_1 \neq 0, v_2 \in M_{r,m} \pmod{qv_1 M_{r,m} {}^t\sigma_3}, v_2 \equiv \tilde{v}_2 \pmod{q}, (v_1, v_2) : \text{primitive}\}$ and verify that this is bijective. The well-definedness is clear.

Suppose that for $v, v' \in U_n$, $(v'_1, v'_2) = u(v_1, v_2 + qv_1 x {}^t\sigma_3)$ for $u \in U_r(q)$ and $x \in M_r$.

Then there exist $y \in M_{m,r}$ and $z \in M_r$ such that $v' = \begin{pmatrix} u & 0 \\ y & z \end{pmatrix} v \begin{pmatrix} 1_r & qx {}^t\sigma_3 \\ 0 & 1_m \end{pmatrix}$. $v' \equiv v \equiv \tilde{v} \pmod{q}$ yields then $\begin{pmatrix} u & 0 \\ y & z \end{pmatrix} \equiv 1_n \pmod{q}$ and hence it is in ${}^tP_{n,r}(q)$ and so v and v' are in the same double coset. So the mapping is injective. Conversely suppose that v_1, v_2 are given. Since (v_1, v_2) is primitive, they can be extended to $v = \begin{pmatrix} v_1 & v_2 \\ * & * \end{pmatrix} \in U_n$, and then $v\tilde{v}^{-1} \equiv \begin{pmatrix} 1_r & 0 \\ \gamma & \delta \end{pmatrix} \pmod{q}$ with $\det \delta \equiv 1 \pmod{q}$ yields that there exists $w = \begin{pmatrix} 1_r & 0 \\ * & * \end{pmatrix} \in {}^tP_{n,r}(q)$ such that $v\tilde{v}^{-1} \equiv w \pmod{q}$. Now (v_1, v_2) is the image of $w^{-1}v$. Thus the mapping is surjective. Note that the conditions on v_1, v_2

are ones given in (C) in Lemma 17, putting $c = \sigma_3$. From Lemma 17 follows now that

$$\begin{aligned} & \mathcal{F}(T, y; f, M) \\ &= [\Gamma_m^\infty : \Gamma_m^\infty(q)]^{-1} [P_{n,r} : P_{n,r}(q)]^{-2} [GL_r(\mathbf{Z}) : U_r(q)]^{-1} \sum_w H(q^{-1}T[w], y[tw^{-1}]; s) \\ & \times \sum_{\tilde{v}, \eta, \tilde{\sigma}, \sigma} (\det \sigma_3^{(m)})^{-k} |\det \sigma_3|^{-2s} \sum_{v_1} |\det v_1|^{m-2s} \\ & \times \sum_g \mu(\mathbf{Z}^r / \mathbf{Z}^r g) |\det g|^{-2s} b(T[w]_1 [v_1^{-1} g^{-1}]; \eta) \\ & \times \sum_{\tilde{V}_2} K(T[w(\begin{smallmatrix} v_1 & \\ & 1_m \end{smallmatrix})^{-1}], (\sigma_3, \sigma_4), 1_r; \tilde{\sigma}, \tilde{V}_2), \end{aligned}$$

where $w, \tilde{v}, \eta, \tilde{\sigma}, v_1, g, \tilde{V}_2$ run over the sets given in the assertion of Lemma and σ runs over the set as above, and $T[w]_1 = T[w] \begin{bmatrix} 1_r \\ 0 \end{bmatrix}$ by definition. Finally let us see that $\left\{ \sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_m^\infty(q) \backslash \Gamma_m / \begin{pmatrix} 1 & q\Lambda_m \\ 0 & 1 \end{pmatrix} \mid \det c \neq 0, \sigma \equiv \tilde{\sigma} \bmod q \right\}$ and $U_m(q) \backslash \{(c, d) \mid \begin{pmatrix} * & * \\ c & d \end{pmatrix} \in \Gamma_m, \det c \neq 0, (c, d) \equiv (\tilde{c}, \tilde{d}) \bmod q\} / \begin{pmatrix} 1 & q\Lambda_m \\ 0 & 1_m \end{pmatrix}$ correspond bijectively by $\sigma \mapsto (c, d)$, where we put $\tilde{\sigma} = \begin{pmatrix} * & * \\ \tilde{c} & \tilde{d} \end{pmatrix}$. The well-definedness is obvious. If $(c_1, d_1) = u(c_2, d_2) \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix}$ for $u \in U_m(q), \lambda \in q\Lambda_m$ and $\sigma_i = \begin{pmatrix} * & * \\ c_i & d_i \end{pmatrix} \in \Gamma_m$, then $\sigma_1 = \begin{pmatrix} v & w \\ 0 & u \end{pmatrix} \sigma_2 \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix}$ for $v \in U_m, w \in M_m$, and $\sigma_1 \equiv \sigma_2 \equiv \tilde{\sigma} \bmod q$ implies $\begin{pmatrix} v & w \\ 0 & u \end{pmatrix} \in \Gamma_m^\infty(q)$. Thus the mapping is injective. Conversely, suppose that (c, d) are given in accordance with our conditions. Then there exist $\sigma = \begin{pmatrix} * & * \\ c & d \end{pmatrix} \in \Gamma_m$. $\sigma \tilde{\sigma}^{-1} \equiv \begin{pmatrix} * & * \\ 0 & 1_m \end{pmatrix} \bmod q$ yields that $\sigma \tilde{\sigma}^{-1} \equiv \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix} \bmod q$ for some $\lambda \in \Lambda_m$. Thus $\begin{pmatrix} 1 & -\lambda \\ 0 & 1 \end{pmatrix} \sigma = \begin{pmatrix} * & * \\ c & d \end{pmatrix} \equiv \tilde{\sigma} \bmod q$ and it means that the mapping is bijective. Thus rewriting the condition on σ in terms of c, d , we have completed the proof. ■

§3. Singular series

We study the singular series on c, d in the formula for $\mathcal{F}(T, y; f, M)$ in Lemma 18. The first aim is to write it as a sum of infinite products. For $T \in \Lambda'_n, \tilde{\sigma} \in \Gamma_m, \tilde{v}_2 \in M_{r,m}$, a positive integer α , and $c, d \in M_m$ with $\det c \neq 0$, we put

$$K_\alpha(T, (c, d); \tilde{\sigma}, \tilde{v}_2) := \sum_{\substack{v_2 \in M_{r,m} \bmod \alpha M_{r,m} \\ v_2 \equiv \tilde{v}_2 \bmod \alpha}} e \left(\operatorname{tr} T \begin{pmatrix} ac^{-1} \begin{bmatrix} t v_2 \\ -c^{-1} t v_2 \end{bmatrix} & -v_2 {}^t c^{-1} \\ & c^{-1} d \end{pmatrix} / \alpha \right),$$

if there exist $a, b \in M_m$ such that $\Gamma_m \ni \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \tilde{\sigma} \bmod \alpha$, and otherwise put $K_\alpha(T, (c, d); \tilde{\sigma}, \tilde{v}_2) := 0$. The sum K_α is independent of the choice of a , since another

a is written as $a + \lambda c$ for $\lambda \in \alpha \Lambda_m$. It is clear that

$$K_q(T, (c, d); \tilde{\sigma}, \tilde{v}_2) = K(T, (c, d), 1_r; \tilde{\sigma}, \tilde{v}_2).$$

When $\alpha = 1$, K_α is independent of $\tilde{\sigma}$ and $\tilde{v}_2$ and so we put

$$K(T, (c, d)) := K_1(T, (c, d); \tilde{\sigma}, \tilde{v}_2).$$

So, $K(T, (c, d)) = 0$ unless $\begin{pmatrix} * & * \\ c & d \end{pmatrix} \in \Gamma_m$.

LEMMA 19. Let α_i, β_i ($i = 1, 2$) be natural numbers such that $(\beta_1, \beta_2) = 1$ and $\alpha_i | \beta_i$ ($i = 1, 2$), and we decompose $c \in M_m$ with $\det c \neq 0$ as $c = uc_1c_2v$ where $u \in GL_m(\mathbf{Z})$, $v \in U_m$, and c_1, c_2 are diagonal matrices with $|\det c_i| | \beta_i$ ($i = 1, 2$). Put $D_i := \{d_i \in M_m \bmod \alpha_i uc_i v \Lambda_m \mid \begin{pmatrix} * & * \\ uc_i v & d_i \end{pmatrix} \in \Gamma_m\}$ ($i = 1, 2$) and $D := \{d \in M_m \bmod \alpha_1 \alpha_2 c \Lambda_m \mid \begin{pmatrix} * & * \\ c & d \end{pmatrix} \in \Gamma_m\}$. We define a mapping ϕ from $D_1 \times D_2$ to D by $\phi(d_1, d_2) := \alpha_2 uc_2 u^{-1} d_1 + \alpha_1 uc_1 u^{-1} d_2$, taking representatives d_1, d_2 such that $d_1 \equiv 0 \bmod \beta_2$ and $d_2 \equiv 0 \bmod \beta_1$. Then ϕ is bijective, and if $\begin{pmatrix} a_i & b_i \\ c_i v & u^{-1} d_i \end{pmatrix} \in \Gamma_m$, then there is an element $\sigma = \begin{pmatrix} * & * \\ c & \phi(d_1, d_2) \end{pmatrix} \in \Gamma_m$ such that

$$\sigma \equiv \begin{cases} \begin{pmatrix} {}^t(uc_2)^{-1} & \\ & uc_2 \end{pmatrix} \begin{pmatrix} \alpha_2^{-1} a_1 & b_1 \\ c_1 v & \alpha_2 u^{-1} d_1 \end{pmatrix} \bmod \beta_1, \\ \begin{pmatrix} {}^t(uc_1)^{-1} & \\ & uc_1 \end{pmatrix} \begin{pmatrix} \alpha_1^{-1} a_2 & b_2 \\ c_2 v & \alpha_1 u^{-1} d_2 \end{pmatrix} \bmod \beta_2. \end{cases}$$

PROOF. If $\begin{pmatrix} * & * \\ uc_1 v & d_1 \end{pmatrix} \in \Gamma_m$, then we can take $\lambda \in \Lambda_m$ such that $\alpha_1 uc_1 v \lambda + d_1 \equiv 0 \bmod \beta_2$ since $(\det(\alpha_1 uc_1 v), \beta_2) = 1$ and $\alpha_1 uc_1 v {}^t d_1 \in \Lambda_m$. Then $\begin{pmatrix} * & * \\ uc_1 v & d_1 \end{pmatrix} \cdot \begin{pmatrix} 1 & \alpha_1 \lambda \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} * & * \\ uc_1 v & \alpha_1 uc_1 v \lambda + d_1 \end{pmatrix} \in \Lambda_m$. Hence we can choose representatives d_1 of D_1 such that $d_1 \equiv 0 \bmod \beta_2$. Similarly we can choose d_2 with $d_2 \equiv 0 \bmod \beta_1$ as representatives of D_2 . Let us see that ϕ is well-defined. We have only to verify $\begin{pmatrix} * & * \\ c & \phi(d_1, d_2) \end{pmatrix} \in \Gamma_m$, which is equivalent to that $c {}^t \phi(d_1, d_2) \in \Lambda_m$ and $(c, \phi(d_1, d_2))$ is primitive. It is easy to see $c {}^t \phi(d_1, d_2) = \alpha_2 uc_2 u^{-1} (uc_1 v) {}^t d_1 {}^t u^{-1} {}^t c_2 {}^t u + \alpha_1 uc_1 u^{-1} (uc_2 v) {}^t d_2 {}^t u^{-1} {}^t c_1 {}^t u \in \Lambda_m$. Next let us see that $(c, \phi(d_1, d_2))$ is primitive. To see this, we have only to see its validity over $\mathbf{Z}/p\mathbf{Z}$ for every prime p . It is clear if p does not divide $\beta_1 \beta_2$, since $\det c \not\equiv 0 \bmod p$. If $p | \beta_2$, then $(c, \phi(d_1, d_2)) \equiv (uc_1 c_2 v, \alpha_1 uc_1 u^{-1} d_2) \bmod p \equiv uc_1 u^{-1} (uc_2 v, d_2) \begin{pmatrix} 1_m & \\ & \alpha_1 1_m \end{pmatrix} \bmod p$ is primitive. If $p | \beta_1$, then $(c, \phi(d_1, d_2)) \equiv (uc_1 c_2 v, \alpha_2 uc_2 u^{-1} d_1) \equiv uc_2 u^{-1} (uc_1 v, d_1) \begin{pmatrix} 1_m & \\ & \alpha_2 1_m \end{pmatrix} \bmod p$ is also primitive. Thus ϕ is well-defined. Let us see ϕ is injective. Suppose that $d_i, d'_i \in D_i$ ($i = 1, 2$) satisfy $\phi(d'_1, d'_2) = \phi(d_1, d_2) + \alpha_1 \alpha_2 c \lambda$, $\lambda \in \Lambda_m$. Then $\lambda = \alpha_1^{-1} v^{-1} c_1^{-1} u^{-1} (d'_1 - d_1) + \alpha_2^{-1} v^{-1} c_2^{-1} u^{-1} (d'_2 - d_2) \in \Lambda_m$, and prime divisors of denominators of entries

of $\alpha_i^{-1}v^{-1}c_i^{-1}u^{-1}(d'_i - d_i)$ divide β_i ($i = 1, 2$). Hence $\alpha_i^{-1}v^{-1}c_i^{-1}u^{-1}(d'_i - d_i) \in \Lambda_m$ ($i = 1, 2$) follows and it means $d'_i = d_i$ in D_i for $i = 1, 2$. Thus ϕ is injective. Let us verify that ϕ is surjective. For this purpose, we choose integral diagonal matrices f_1, f_2 such that $f_i c_i$ ($i = 1, 2$) are scalar matrices and $\alpha_1 c_1 f_1 + \alpha_2 c_2 f_2 = 1_m$ and $f_i \equiv 0 \pmod{\beta_i}$ ($i = 1, 2$). We put $d_1 = u f_2 u^{-1} d, d_2 = u f_1 u^{-1} d$ for $d \in D$. Let us see $d_i \in D_i$ ($i = 1, 2$). $uc_1 v^t d_1 = (f_2 c_2)^{-1} (u f_2 u^{-1}) c^t d^t (u f_2 u^{-1})$, $uc_2 v^t d_2 = (f_1 c_1)^{-1} (u f_1 u^{-1}) c^t d^t (u f_1 u^{-1})$ are symmetric. If p does not divide β_1 , then $(uc_1 v, d_1)$ is primitive over $\mathbf{Z}/p\mathbf{Z}$, and if $p|\beta_1$, then $(uc_1 v, d_1) = (u f_2 u^{-1})(c, d) \begin{pmatrix} (c_2 f_2)^{-1} & \\ & 1_m \end{pmatrix}$ is primitive over $\mathbf{Z}/p\mathbf{Z}$. Thus $(uc_1 v, d_1)$ and similarly $(uc_2 v, d_2)$ are primitive and we have proved $d_i \in D_i$. $d_1 \equiv 0 \pmod{\beta_2}$ and $d_2 \equiv 0 \pmod{\beta_1}$ follow from the assumption, and $\phi(d_1, d_2) = d$ is clear. Therefore we have proved the surjectivity and so the bijectivity of ϕ . Next, assume that $\begin{pmatrix} a_i & b_i \\ c_i v & u^{-1} d_i \end{pmatrix} \in \Gamma_m$ with $d_1 \equiv 0 \pmod{\beta_2}, d_2 \equiv 0 \pmod{\beta_1}$. Then $(c, \phi(d_1, d_2)) \equiv (uc_1 c_2 v, \alpha_2 uc_2 u^{-1} d_1) \pmod{\beta_1} \equiv (0, 1_m) \begin{pmatrix} {}^t(uc_2)^{-1} & \\ & uc_2 \end{pmatrix} \begin{pmatrix} \alpha_2^{-1} a_1 & b_1 \\ c_1 v & \alpha_2 u^{-1} d_1 \end{pmatrix} \pmod{\beta_1}$, where $\begin{pmatrix} \alpha_2^{-1} a_1 & b_1 \\ c_1 v & \alpha_2 u^{-1} d_1 \end{pmatrix} \in Sp(m, \mathbf{Z}_p)$ for $p|\beta_1$. We have also $(c, \phi(d_1, d_2)) \equiv (uc_1 c_2 v, \alpha_1 uc_1 u^{-1} d_2) \pmod{\beta_2} \equiv (0, 1_m) \begin{pmatrix} {}^t(uc_1)^{-1} & \\ & uc_1 \end{pmatrix} \begin{pmatrix} \alpha_1^{-1} a_2 & b_2 \\ c_2 v & \alpha_1 u^{-1} d_2 \end{pmatrix} \pmod{\beta_2}$. Thus we have for $\sigma' = \begin{pmatrix} * & * \\ c & \phi(d_1, d_2) \end{pmatrix} \in \Gamma_m$

$$\sigma' \equiv \begin{cases} \begin{pmatrix} 1 & \lambda_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} {}^t(uc_2)^{-1} & \\ & uc_2 \end{pmatrix} \begin{pmatrix} \alpha_2^{-1} a_1 & b_1 \\ c_1 v & \alpha_2 u^{-1} d_1 \end{pmatrix} \pmod{\beta_1}, \\ \begin{pmatrix} 1 & \lambda_2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} {}^t(uc_1)^{-1} & \\ & uc_1 \end{pmatrix} \begin{pmatrix} \alpha_1^{-1} a_2 & b_2 \\ c_2 v & \alpha_1 u^{-1} d_2 \end{pmatrix} \pmod{\beta_2}. \end{cases}$$

for $\lambda_i \in \Lambda_m$. Taking $\lambda \in \Lambda_m$ such that $\lambda \equiv \lambda_i \pmod{\beta_i}$, $\sigma = \begin{pmatrix} 1 & -\lambda \\ 0 & 1 \end{pmatrix} \sigma'$ is what we want. ■

LEMMA 20. Let α_i, β_i be natural numbers such that $\alpha_1 \alpha_2 = q$, $\alpha_i | \beta_i$ and $(\beta_1, \beta_2) = 1$. Let $c \in M_m$ satisfy that $\det c \neq 0$ and $\det c | (\beta_1 \beta_2)^\infty$. Write $c = uc_1 c_2 v$, where $u \in GL_m(\mathbf{Z}), v \in U_m$ and c_i is diagonal such that $\det c_i | \beta_i^\infty$ ($i = 1, 2$). If $\begin{pmatrix} a_i & b_i \\ c_i v & u^{-1} d_i \end{pmatrix} \in \Gamma_m$ ($i = 1, 2$) with $d_1 \equiv 0 \pmod{(\beta_2 \det c_2)}, d_2 \equiv 0 \pmod{(\beta_1 \det c_1)}$, then we have

$$\begin{aligned} & K_q(T, (uc_1 c_2 v, \alpha_2 uc_2 u^{-1} d_1 + \alpha_1 uc_1 u^{-1} d_2); \tilde{\sigma}, \tilde{v}_2) \\ &= K_{\alpha_2} \left(T, (c_2 v, u^{-1} d_2); \left[\alpha_1 \ddagger \begin{pmatrix} {}^t(uc_1) & \\ & (uc_1)^{-1} \end{pmatrix} \sigma \right]_{\alpha_2}, \alpha_1^{-1} \tilde{v}_2 {}^t u^{-1} c_1^{-1} \right) \\ & \quad \times K_{\alpha_1} \left(T, (c_1 v, u^{-1} d_1), \left[\alpha_2 \ddagger \begin{pmatrix} {}^t(uc_2) & \\ & (uc_2)^{-1} \end{pmatrix} \sigma \right]_{\alpha_1}, \alpha_2^{-1} \tilde{v}_2 {}^t u^{-1} c_2^{-1} \right), \end{aligned}$$

where $[\alpha \ddagger \begin{pmatrix} a & b \\ c & d \end{pmatrix}]_\beta$ stands for an element of Γ_m which is congruent to $\begin{pmatrix} \alpha a & b \\ c & \alpha^{-1} d \end{pmatrix}$ modulo β .

PROOF. By virtue of Lemma 19, there exist $\sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_m$ with $d := \alpha_2 u c_2 u^{-1} d_1 + \alpha_1 u c_1 u^{-1} d_2$ such that

$$(1) \quad \sigma \equiv \begin{cases} \begin{pmatrix} {}^t(uc_2)^{-1} & \\ & uc_2 \end{pmatrix} \begin{pmatrix} \alpha_2^{-1} a_1 & b_1 \\ c_1 v & \alpha_2 u^{-1} d_1 \end{pmatrix} \bmod \beta_1 \det c_1, \\ \begin{pmatrix} {}^t(uc_1)^{-1} & \\ & uc_1 \end{pmatrix} \begin{pmatrix} \alpha_1^{-1} a_2 & b_2 \\ c_2 v & \alpha_1 u^{-1} d_2 \end{pmatrix} \bmod \beta_2 \det c_2. \end{cases}$$

If there is an element $\begin{pmatrix} * & * \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix} \sigma \in \Gamma_m$ such that $\begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix} \sigma \equiv \tilde{\sigma} \bmod q$, then $\begin{pmatrix} 1 & -\lambda \\ 0 & 1 \end{pmatrix} \tilde{\sigma} \equiv \begin{pmatrix} {}^t(uc_2)^{-1} & \\ & uc_2 \end{pmatrix} \cdot \begin{pmatrix} \alpha_2^{-1} a_1 & b_1 \\ c_1 v & \alpha_2 u^{-1} d_1 \end{pmatrix} \bmod \alpha_1$ and hence

$$(2) \quad \begin{aligned} & \left[\alpha_2 \dagger \begin{pmatrix} {}^t(uc_2) & \\ & (uc_2)^{-1} \end{pmatrix} \tilde{\sigma} \right]_{\alpha_1} \\ & \equiv \begin{pmatrix} 1 & \alpha_2 \lambda [uc_2] \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a_1 & b_1 \\ c_1 v & \alpha_2 u^{-1} d_1 \end{pmatrix} \bmod \alpha_1 \equiv \begin{pmatrix} * & * \\ c_1 v & u^{-1} d_1 \end{pmatrix} \bmod \alpha_1 \end{aligned}$$

Similarly we have

$$(3) \quad \begin{aligned} & \left[\alpha_1 \dagger \begin{pmatrix} {}^t(uc_1) & \\ & (uc_1)^{-1} \end{pmatrix} \tilde{\sigma} \right]_{\alpha_2} \\ & \equiv \begin{pmatrix} 1 & \alpha_1 \lambda [uc_1] \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a_2 & b_2 \\ c_2 v & \alpha_1 u^{-1} d_2 \end{pmatrix} \bmod \alpha_2 \equiv \begin{pmatrix} * & * \\ c_2 v & u^{-1} d_2 \end{pmatrix} \bmod \alpha_2. \end{aligned}$$

If conversely $\left[\alpha_2 \dagger \begin{pmatrix} {}^t(uc_2) & \\ & (uc_2)^{-1} \end{pmatrix} \tilde{\sigma} \right]_{\alpha_1} \equiv \begin{pmatrix} a'_1 & b'_1 \\ c_1 v & u^{-1} d_1 \end{pmatrix} \bmod \alpha_1$, $\left[\alpha_1 \dagger \begin{pmatrix} {}^t(uc_1) & \\ & (uc_1)^{-1} \end{pmatrix} \tilde{\sigma} \right]_{\alpha_2} \equiv \begin{pmatrix} a'_2 & b'_2 \\ c_2 v & u^{-1} d_2 \end{pmatrix} \bmod \alpha_2$ for $\begin{pmatrix} a'_i & b'_i \\ c_i v & u^{-1} d_i \end{pmatrix} \in \Gamma_m$, then taking $\lambda_i \in \Lambda_m$ such that $\begin{pmatrix} a'_i & b'_i \\ c_i v & u^{-1} d_i \end{pmatrix} = \begin{pmatrix} 1 & \lambda_i \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a_i & b_i \\ c_i v & u^{-1} d_i \end{pmatrix}$, we choose $\lambda \in \Lambda_m$ such that $\alpha_2 \lambda [uc_2] \equiv \lambda_1 \bmod \alpha_1$, $\alpha_1 \lambda [uc_1] \equiv \lambda_2 \bmod \alpha_2$. Then we have $\begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix} \sigma \equiv \tilde{\sigma} \bmod q$. Thus (c, d) has an extension to Γ_m which is congruent to $\tilde{\sigma}$ modulo q if and only if $(c_1 v, u^{-1} d_1)$ (resp. $(c_2 v, u^{-1} d_2)$) has an extension to Γ_m which is congruent to $\left[\alpha_2 \dagger \begin{pmatrix} {}^t(uc_2) & \\ & (uc_2)^{-1} \end{pmatrix} \tilde{\sigma} \right]_{\alpha_1}$ modulo α_1 (resp. $\left[\alpha_1 \dagger \begin{pmatrix} {}^t(uc_1) & \\ & (uc_1)^{-1} \end{pmatrix} \tilde{\sigma} \right]_{\alpha_2}$ modulo α_2). Hence $K_q = K_{\alpha_1} = K_{\alpha_2} = 0$ if there are no such extensions.

Now we may suppose that (c, d) has an extension to Γ_m which is congruent to $\tilde{\sigma} \bmod q$. Let it be $\begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix} \sigma$ for $\lambda \in \Lambda_m$ and $\sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ satisfies (1). Now we have $K_q := K_q(T, (uc_1 c_2 v, \alpha_2 u c_2 u^{-1} d_1 + \alpha_1 u c_1 u^{-1} d_2); \tilde{\sigma}, \tilde{v}_2) = \sum_{\substack{V_2 \in M_{r,m} \bmod \alpha_1 \alpha_2 M_{r,m} {}^t c \\ V_2 \equiv \tilde{v}_2 \bmod \alpha_1 \alpha_2}} e \left(\text{tr } T \left(\begin{pmatrix} (a+\lambda c)(uc_1 c_2 v)^{-1} [{}^t V_2] \\ -(uc_1 c_2 v)^{-1} {}^t V_2 \end{pmatrix} \begin{pmatrix} * \\ \alpha_2 v^{-1} c_1^{-1} u^{-1} d_1 + \alpha_1 v^{-1} c_2^{-1} u^{-1} d_2 \end{pmatrix} \right) / (\alpha_1 \alpha_2) \right)$. Now we take integers f_1, f_2 such that $\alpha_1 f_1 (\det c_1)^2 + \alpha_2 f_2 (\det c_2)^2 = 1$.

Let us see that $V_2 \in M_{r,m} \bmod \alpha_1 \alpha_2 M_{r,m} {}^t c$ with $V_2 \equiv \tilde{v}_2 \bmod \alpha_1 \alpha_2$ can be written as

$$V_2 = (\alpha_1 x c_1 + \alpha_2 y c_2) {}^t u,$$

where $x \in M_{r,m} \bmod \alpha_2 M_{r,m} c_2$ with $\alpha_1 x c_1 \equiv \tilde{v}_2 {}^t u^{-1} \bmod \alpha_2$, and $y \in M_{r,m} \bmod \alpha_1 M_{r,m} c_1$ with $\alpha_2 y c_2 \equiv \tilde{v}_2 {}^t u^{-1} \bmod \alpha_1$. Suppose that V_2 is given with $V_2 \equiv \tilde{v}_2 \bmod \alpha_1 \alpha_2$. Putting $x = f_1 \det c_1^2 V_2 {}^t u^{-1} c_1^{-1}$, $y = f_2 \det c_2^2 V_2 {}^t u^{-1} c_2^{-1}$, we have $V_2 {}^t u^{-1} = \alpha_1 x c_1 + \alpha_2 y c_2$, and $V_2 \equiv \tilde{v}_2 \bmod \alpha_1 \alpha_2$ if and only if $\alpha_2 y c_2 \equiv \tilde{v}_2 {}^t u^{-1} \bmod \alpha_1$ and $\alpha_1 x c_1 \equiv \tilde{v}_2 {}^t u^{-1} \bmod \alpha_2$. $V_2 \in \alpha_1 \alpha_2 M_{r,m} {}^t c$ if and only if $V_2 {}^t u^{-1} \in \alpha_1 \alpha_2 M_{r,m} c_1 c_2$, which implies $\alpha_1 x c_1 \in \alpha_2 M_{r,m} c_2$ and $\alpha_2 y c_2 \in \alpha_1 M_{r,m} c_1$. ($\alpha_1 \times \det c_1, \alpha_2 \det c_2$) = 1 yields then $x \in \alpha_2 M_{r,m} c_2$ and $y \in \alpha_1 M_{r,m} c_1$. Conversely they imply $V_2 = (\alpha_1 x c_1 + \alpha_2 y c_2) {}^t u \in \alpha_1 \alpha_2 M_{r,m} {}^t c$. Thus the validity of the decomposition has been proved.

For the above decomposition $V_2 = (\alpha_1 x c_1 + \alpha_2 y c_2) {}^t u$, we have $(a + \lambda c) (uc_1 c_2 v)^{-1} [{}^t V_2] = av^{-1} c_2^{-1} c_1^{-1} u^{-1} [u(\alpha_1 c_1 {}^t x + \alpha_2 c_2 {}^t y)] + \lambda [{}^t V_2] = \alpha_1^2 x c_1 {}^t u a v^{-1} c_2^{-1} {}^t x + \alpha_1 \alpha_2 x c_1 {}^t u a v^{-1} c_1^{-1} {}^t y + \alpha_1 \alpha_2 y c_1^{-1} {}^t v^{-1} {}^t a u c_1 {}^t x + \alpha_2^2 y c_2 {}^t u a v^{-1} c_1^{-1} {}^t y + \lambda [{}^t V_2] \equiv \alpha_1 (a_2 (c_2 v)^{-1}) [{}^t x] + \alpha_2 (a_1 (c_1 v)^{-1}) [{}^t y] + \lambda [{}^t V_2] \bmod q$ by (1), $\alpha_1 \alpha_2 = q, \alpha_i | \beta_i$. $(uc_1 c_2 v)^{-1} {}^t V_2 = \alpha_1 v^{-1} c_2^{-1} {}^t x + \alpha_2 v^{-1} c_1^{-1} {}^t y$ is clear. Thus we have

$$\begin{aligned} & K_q(T, (uc_1 c_2 v, \alpha_2 uc_2 u^{-1} d_1 + \alpha_1 uc_1 u^{-1} d_2); \tilde{\sigma}, \tilde{v}_2) \\ &= \sum_x e \left(\text{tr } T \left(\begin{pmatrix} (a_2 + \alpha_1 \lambda [uc_1] c_2 v) (c_2 v)^{-1} [{}^t x] \\ -(c_2 v)^{-1} {}^t x \end{pmatrix} \begin{pmatrix} * \\ (c_2 v)^{-1} u^{-1} d_2 \end{pmatrix} \right) / \alpha_2 \right) \\ & \quad \times \sum_y e \left(\text{tr } T \left(\begin{pmatrix} (a_1 + \alpha_2 \lambda [uc_2] c_1 v) (c_1 v)^{-1} [{}^t y] \\ -(c_1 v)^{-1} {}^t y \end{pmatrix} \begin{pmatrix} * \\ (c_1 v)^{-1} u^{-1} d_1 \end{pmatrix} \right) / \alpha_1 \right), \end{aligned}$$

where x runs over $M_{r,m} \bmod \alpha_2 M_{r,m} c_2$ with $\alpha_1 x c_1 \equiv \tilde{v}_2 {}^t u^{-1} \bmod \alpha_2$, y runs over $M_{r,m} \bmod \alpha_1 M_{r,m} c_1$ with $\alpha_2 y c_2 \equiv \tilde{v}_2 {}^t u^{-1} \bmod \alpha_1$, and this is equal to

$$\begin{aligned} & K_{\alpha_2} \left(T, (c_2 v, u^{-1} d_2); \begin{pmatrix} 1 & \alpha_1 \lambda [uc_1] \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a_2 & b_2 \\ c_2 v & u^{-1} d_2 \end{pmatrix}, \alpha_1^{-1} \tilde{v}_2 {}^t u^{-1} c_1^{-1} \right) \\ & \times K_{\alpha_1} \left(T, (c_1 v, u^{-1} d_1); \begin{pmatrix} 1 & \alpha_2 \lambda [uc_2] \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a_1 & b_1 \\ c_1 v & u^{-1} d_1 \end{pmatrix}, \alpha_2^{-1} \tilde{v}_2 {}^t u^{-1} c_2^{-1} \right), \end{aligned}$$

which completes the proof by (2) and (3). $\blacksquare$

LEMMA 21. Let α_1, α_2 be relatively prime natural numbers. We decompose $c \in M_m$ with $\det c \neq 0$ as $c = uc_1 c_2 v_1 v_2$ where $u \in GL_m(\mathbf{Z})$, $v_1 \in c_2^{-1} U_m c_2 \cap U_m$, $v_2 \in U_m(\alpha_1)$, $(\alpha_1 \det c_1, \alpha_2 \det c_2) = 1$. Then

$$\begin{aligned} & \sum_{d \bmod \alpha_1 \alpha_2 c \Lambda_m} K_{\alpha_1 \alpha_2}(T, (c, d); \tilde{\sigma}, \tilde{v}_2) \\ &= \sum_{d_1 \bmod \alpha_1 c_1 v_1 \Lambda_m} K_{\alpha_1} \left(T, (c_1 v_1, d_1); \left[\alpha_2 \dagger \begin{pmatrix} {}^t (uc_2) \\ (uc_2)^{-1} \end{pmatrix} \tilde{\sigma} \right]_{\alpha_1}, \alpha_2^{-1} \tilde{v}_2 {}^t (uc_2)^{-1} \right) \\ & \times \sum_{d_2 \bmod \alpha_2 c_2 v_2 \Lambda_m} K_{\alpha_2} \left(T, (c_2 v_2, d_2); \left[\alpha_1 \dagger \begin{pmatrix} {}^t (uc_1 x) \\ (uc_1 x)^{-1} \end{pmatrix} \tilde{\sigma} \right]_{\alpha_2}, \alpha_1^{-1} \tilde{v}_2 {}^t (uc_1 x)^{-1} \right) \end{aligned}$$

where $x := c_2 v_1 c_2^{-1} \in U_m$.

PROOF. Put $\beta_i = \alpha_i \det c_i$ ($i = 1, 2$); then we have $(\beta_1, \beta_2) = 1$. Then by Lemmas 19, 20 we have, putting $v := v_1 v_2$

$$\begin{aligned} & \sum_{d \bmod \alpha_1 \alpha_2 c \Lambda_m} K_{\alpha_1 \alpha_2}(T, (c, d); \tilde{\sigma}, \tilde{v}_2) \\ &= \sum_{d_1 \bmod \alpha_1 u c_1 v \Lambda_m} K_{\alpha_1} \left(T, (c_1 v, u^{-1} d_1); \left[\alpha_2 \dagger \left(\begin{smallmatrix} {}^t(uc_2) \\ (uc_2)^{-1} \end{smallmatrix} \right) \tilde{\sigma} \right]_{\alpha_1}, \alpha_2^{-1} \tilde{v}_2 {}^t(uc_2)^{-1} \right) \\ & \times \sum_{d_2 \bmod \alpha_2 u c_2 v \Lambda_m} K_{\alpha_2} \left(T, (c_2 v, u^{-1} d_2); \left[\alpha_1 \dagger \left(\begin{smallmatrix} {}^t(uc_1) \\ (uc_1)^{-1} \end{smallmatrix} \right) \tilde{\sigma} \right]_{\alpha_2}, \alpha_1^{-1} \tilde{v}_2 {}^t(uc_1)^{-1} \right). \end{aligned}$$

Put $u^{-1} d_1 = D_1 {}^t v_2^{-1}$. Then D_1 runs over $M_m \bmod \alpha_1 c_1 v_1 \Lambda_m$ and $v_2 \equiv 1 \bmod \alpha_1$ yields that the sum on d_1 above is equal to the sum on d_1 in the assertion of Lemma by (2) in Lemma 11. $x := c_2 v_1 c_2^{-1} \in U_m$ implies $c_2 v = x c_2 v_2$, and then the condition $d_2 \in M_m \bmod \alpha_2 u c_2 v \Lambda_m$ is equivalent to $u^{-1} d_2 \in M_m \bmod \alpha_2 x c_2 v_2 \Lambda_m$. Thus putting $u^{-1} d_2 = x D_2$, we get the sum on d_2 above to be equal to $\sum_{D_2 \in M_m \bmod \alpha_2 c_2 v_2 \Lambda_m} K_{\alpha_2} \left(T, x(c_2 v_2, D_2); \left[\alpha_1 \dagger \left(\begin{smallmatrix} {}^t(uc_1) \\ (uc_1)^{-1} \end{smallmatrix} \right) \tilde{\sigma} \right]_{\alpha_2}, \alpha_1^{-1} \tilde{v}_2 {}^t(uc_1)^{-1} \right)$. Applying (2) in Lemma 11 for $u = x, \hat{u} = \hat{v} = 0, v = 1_m$ to this sum, we see that it is the same as the sum on d_2 in the equality to be proved. ■

LEMMA 22. If $c_i \in M_m$ ($i = 1, 2$) satisfies $c_1 c_2 = c_2 c_1, \det(c_1 c_2) \neq 0$, and $(\det c_1, \det c_2) = 1$. Then we have

- (i) $(c_1 c_2)^{-1} U_m c_1 c_2 \cap U_m = c_1^{-1} U_m c_1 \cap c_2^{-1} U_m c_2 \cap U_m$,
- (ii) a complete set of representatives of $((c_1 c_2)^{-1} U_m (c_1 c_2) \cap U_m) \setminus (c_2^{-1} U_m c_2 \cap U_m)$ becomes a complete set of representatives of $(c_1^{-1} U_m c_1 \cap U_m) \setminus U_m$,
- (iii) a complete set of representatives of $GL_m(\mathbf{Z}) \setminus GL_m(\mathbf{Z})_{c_1 c_2} GL_m(\mathbf{Z})$ is given by $c_1 c_2 v_1 v_2$, where

$$\begin{aligned} v_1 &\in ((c_1 c_2)^{-1} U_m c_1 c_2 \cap U_m) \setminus (c_2^{-1} U_m c_2 \cap U_m) \\ v_2 &\in (c_2^{-1} U_m c_2 \cap U_m) \setminus U_m \quad \text{with } v_2 \equiv 1_m \bmod \alpha, \end{aligned}$$

where α is any given natural number relatively prime to $\det c_2$.

PROOF. (i) If $g = (c_1 c_2)^{-1} g' (c_1 c_2)$ for $g, g' \in U_m$, then $c_1 g c_1^{-1} = c_2^{-1} g' c_2$ follows, and denominators of entries of both sides are relatively prime. Hence $c_1 g c_1^{-1} = c_2^{-1} g' c_2 \in M_m$. Then $g \in c_1^{-1} M_m c_1$, and similarly we get $g \in c_2^{-1} M_m c_2$ using $g = (c_2 c_1)^{-1} g' (c_2 c_1)$. So the left hand side of (i) is contained in the right hand side. If $g = c_i^{-1} g_i c_i$ for $g_i \in U_m$ ($i = 1, 2$), then we have $(c_1 c_2) g (c_1 c_2)^{-1} = c_1 g_2 c_1^{-1} = c_2 g_1 c_2^{-1}$. $c_1 g_2 c_1^{-1} \in (\det c_1)^{-1} M_m$ and $c_2 g_1 c_2^{-1} \in (\det c_2)^{-1} M_m$ yield $(c_1 c_2) g (c_1 c_2)^{-1} \in M_m$. Thus the right hand side is in the left hand side.

(ii) Suppose that $g_i \in c_2^{-1}U_m c_2 \cap U_m$ ($i = 1, 2$) satisfies $(c_1^{-1}U_m c_1 \cap U_m)g_1 = (c_1^{-1}U_m c_1 \cap U_m)g_2$. Then $g_1 g_2^{-1} \in c_1^{-1}U_m c_1 \cap U_m \cap c_2^{-1}U_m c_2 = (c_1 c_2)^{-1}U_m c_1 c_2 \cap U_m$ by (i). It remains to show that we can choose elements of $c_2^{-1}U_m c_2 \cap U_m$ as representatives of $(c_1^{-1}U_m c_1 \cap U_m) \backslash U_m$. Take any element $g \in U_m$ and choose an element $v \in U_m$ by $v \equiv 1_m \pmod{\det c_1}$ and $v \equiv g^{-1} \pmod{\det c_2}$. Putting $u := vg$, we have $c_2 u c_2^{-1} = c_2 v g c_2^{-1} \in M_m$ because of $v \equiv g^{-1} \pmod{\det c_2}$. These imply $u \in c_2^{-1}U_m c_2 \cap U_m$ and $v \in c_1^{-1}U_m c_1 \cap U_m$, which are what we want.

(iii) is clear except for condition $v_2 \equiv 1_m \pmod{\alpha}$. It is enough to show that for a given $g \in U_m$, there exists $u \in U_m(\alpha)$ such that $u \in (c_2^{-1}U_m c_2 \cap U_m)g$. Take $v \in U_m$ such that $v \equiv 1_m \pmod{\det c_2}$, $v \equiv g^{-1} \pmod{\alpha}$. Then v satisfies $v \in c_2^{-1}U_m c_2 \cap U_m$ and $u := vg \equiv 1_m \pmod{\alpha}$. Therefore u is the required one. ■

LEMMA 23. Let C_i ($i = 1, 2$) be integral and diagonal regular matrix in M_m such that $(\det C_1, \det C_2) = 1$. For natural numbers α_1, α_2 with $(\alpha_1 \det C_1, \alpha_2 \det C_2) = 1$, we have

$$\begin{aligned} & \sum_{c \in U_m(\alpha_1 \alpha_2) \backslash U_m C_1 C_2 U_m} \sum_{d \in M_m \bmod \alpha_1 \alpha_2 c \Lambda_m} K_{\alpha_1 \alpha_2}(T, (c, d); \tilde{\sigma}, \tilde{v}_2) \\ &= \sum_{c_1} \sum_{d_1} K_{\alpha_1} \left(T, (c_1, d_1); \left[\alpha_2 \dagger \left(\widehat{\det C_2} \widehat{\det C_2}^{-1} \right) \tilde{\sigma} \right]_{\alpha_1}, \alpha_2^{-1} \tilde{v}_2 \widehat{\det C_2}^{-1} \right) \\ & \quad \times \sum_{c_2} \sum_{d_2} K_{\alpha_2} \left(T, (c_2, d_2); \left[\alpha_1 \dagger \left(\widehat{\det C_1} \widehat{\det C_1}^{-1} \right) \tilde{\sigma} \right]_{\alpha_2}, \alpha_1^{-1} \tilde{v}_2 \widehat{\det C_1}^{-1} \right) \end{aligned}$$

where c_i, d_i run over $U_m(\alpha_i) \backslash U_m C_i U_m$, $M_m \bmod \alpha_i c_i \Lambda_m$ respectively, and $\widehat{\det C_i}$ stands for $\text{diag}(1_{m-1}, \det C_i)$.

PROOF. In the assertion of Lemma 21, putting $u = u_1 u_2$ where $u_i \equiv 1_m \pmod{\alpha_i}$, $u_i \in U_m$, we have

$$\begin{aligned} (1) \quad & \sum_{d \bmod \alpha_1 \alpha_2 u c_1 c_2 v_1 v_2 \Lambda_m} K_{\alpha_1 \alpha_2}(T, (u_1 u_2 c_1 c_2 v_1 v_2, d); \tilde{\sigma}, \tilde{v}_2) \\ &= \sum_{d_1} K_{\alpha_1} \left(T, (c_1 v_1, d_1); \left[\alpha_2 \dagger \left({}^t(u_2 c_2) (u_2 c_2)^{-1} \right) \tilde{\sigma} \right]_{\alpha_1}, \alpha_2^{-1} \tilde{v}_2 {}^t(u_2 c_2)^{-1} \right) \\ & \quad \times \sum_{d_2} K_{\alpha_2} \left(T, (c_2 v_2, d_2); \left[\alpha_1 \dagger \left({}^t(u_1 c_1 x) (u_1 c_1 x)^{-1} \right) \tilde{\sigma} \right]_{\alpha_2}, \alpha_1^{-1} \tilde{v}_2 {}^t(u_1 c_1 x)^{-1} \right) \end{aligned}$$

where $x = c_2 v_1 c_2^{-1} \in U_m$ and d_i runs over $M_m \bmod \alpha_i c_i v_i \Lambda_m$. Note that if u_1 runs over $U_m(\alpha_1 \alpha_2) \backslash U_m(\alpha_1)$, then $u_1 c_1 x$ runs over $\{y \in M_m \bmod \alpha_2 M_m \mid \det y \equiv \det c_1 \pmod{\alpha_2}\}$, and if u_2 runs over $U_m(\alpha_1 \alpha_2) \backslash U_m(\alpha_2)$, then $u_2 c_2$ runs over $\{z \in M_m \bmod \alpha_1 M_m \mid \det z \equiv \det c_2 \pmod{\alpha_1}\}$. Hence, if $u = u_1 u_2$ runs over $U_m(\alpha_1 \alpha_2) \backslash$

U_m as above, then u_2c_2 and u_1c_1x in (1) runs over $\{z \in M_m \bmod \alpha_1 M_m \mid \det z \equiv \det c_2 \bmod \alpha_1\}$ and $\{y \in M_m \bmod \alpha_2 M_m \mid \det y \equiv \det c_1 \bmod \alpha_2\}$ respectively. Now adding (1) on $u = u_1u_2 \in U_m(\alpha_1\alpha_2) \setminus U_m$, we have

$$\begin{aligned} & \sum_{u \in U_m(\alpha_1\alpha_2) \setminus U_m} \sum_{d \bmod \alpha_1\alpha_2 u c_1 c_2 v_1 v_2 \Lambda_m} K_{\alpha_1\alpha_2}(T, (u_1u_2c_1c_2v_1v_2, d); \tilde{\sigma}, \tilde{v}_2) \\ &= \sum_{\tilde{u}_1 \in U_m(\alpha_1) \setminus U_m} \sum_{d_1} K_{\alpha_1}\left(T, (\tilde{u}_1c_1v_1, \tilde{u}_1d_1); \right. \\ & \quad \left. \left[\alpha_2 \ddagger \left(\begin{smallmatrix} {}^t \widehat{\det c_2} \\ \widehat{\det c_2}^{-1} \end{smallmatrix} \right) \tilde{\sigma} \right]_{\alpha_1}, \alpha_2^{-1} \tilde{v}_2 \begin{smallmatrix} {}^t \widehat{\det c_2}^{-1} \end{smallmatrix} \right) \\ & \times \sum_{\tilde{u}_2 \in U_m(\alpha_2) \setminus U_m} \sum_{d_2} K_{\alpha_2}\left(T, (\tilde{u}_2c_2v_2, \tilde{u}_2d_2); \right. \\ & \quad \left. \left[\alpha_1 \ddagger \left(\begin{smallmatrix} {}^t \widehat{\det c_1} \\ \widehat{\det c_1}^{-1} \end{smallmatrix} \right) \tilde{\sigma} \right]_{\alpha_2}, \alpha_1^{-1} \tilde{v}_2 \begin{smallmatrix} {}^t \widehat{\det c_1}^{-1} \end{smallmatrix} \right), \end{aligned}$$

using (2) in Lemma 11. Adding them on $v_1 \in ((c_1c_2)^{-1}U_m c_1c_2 \cap U_m) \setminus (c_2^{-1}U_m c_2 \cap U_m)$ and $v_2 \in (c_2^{-1}U_m c_2 \cap U_m) \setminus U_m(\alpha_1)$ and replacing c_i by C_i , we get the formula in Lemma. ■

LEMMA 24. For $\tilde{\sigma} = \begin{pmatrix} * & * \\ \tilde{c} & \tilde{d} \end{pmatrix} \in \Gamma_m$, $\tilde{V}_2 \in M_{r,m}$ and $T \in \Lambda'_n$, we put

$$S(T; \tilde{\sigma}, \tilde{V}_2) := \sum_{c,d} \det c^{-k} |\det c|^{-2s} K(T, (c, d), 1_r; \tilde{\sigma}, \tilde{V}_2)$$

where (c, d) runs over $U_m(q) \setminus \{(c, d) \mid c, d \in M_m, \det c \neq 0\} / \begin{pmatrix} 1_m & q\Lambda_m \\ 0 & 1_m \end{pmatrix}$.

Then $S(T; \tilde{\sigma}, \tilde{V}_2)$ is equal to

$$\begin{aligned} & 2\varphi(q)^{-1} \sum_{\chi} \left\{ \sum_{c_1} \chi(\det c_1) (\det c_1)^{-k-2s} \sum_{d_1} K(T, (c_1, d_1)) \right\} \\ & \times \left\{ \sum_a \bar{\chi}(a) \sum_{c_2} (\det c_2)^{-k-2s} \sum_{d_2} K_q(T, (c_2, d_2); \begin{pmatrix} {}^t \hat{a} \\ \hat{a}^{-1} \end{pmatrix} \tilde{\sigma}, \tilde{V}_2 \hat{a}^{-1}) \right\}, \end{aligned}$$

where φ is Euler's function, χ runs over the set of Dirichlet characters defined modulo q with $\chi(-1) = (-1)^k$, and c_i, d_i run over

$$\begin{aligned} c_1 &\in U_m \setminus \{c_1 \in M_m \mid \det c_1 > 0, (\det c_1, q) = 1\}, \quad d_1 \in M_m \bmod c_1 \Lambda_m, \\ c_2 &\in U_m(q) \setminus \{c_2 \in M_m \mid 0 < \det c_2 | q^\infty\}, \quad d_2 \in M_m \bmod qc_2 \Lambda_m, \end{aligned}$$

a runs over $(\mathbf{Z}/q\mathbf{Z})^\times$, and $\hat{a}$ stands for $\text{diag}(1_{m-1}, a)$.

PROOF. By definition, $K(T, (c, d), 1_r; \tilde{\sigma}, \tilde{v}_2) = K_q(T, (c, d); \tilde{\sigma}, \tilde{v}_2)$, $K(T, (c, d)) = K_1(T, (c, d); \tilde{\sigma}, \tilde{v}_2)$. Putting $\kappa = \text{diag}(1_{m-1}, -1)$, we have

$$\begin{aligned} & S(T; \tilde{\sigma}, \tilde{V}_2) \\ &= \sum_{\substack{c \in U_m(q) \setminus M_m \\ \det c > 0}} (\det c)^{-k-2s} \sum_d K_q(T, (c, d); \tilde{\sigma}, \tilde{V}_2) \\ &+ \sum_{\substack{c \in U_m(q) \setminus M_m \\ \det c > 0}} (-1)^k (\det c)^{-k-2s} \sum_d K_q(T, (\kappa c, \kappa d); \tilde{\sigma}, \tilde{V}_2) \\ &= \sum_{\substack{c \in U_m(q) \setminus M_m \\ \det c > 0}} (\det c)^{-k-2s} \sum_d K_q(T, (c, d); \tilde{\sigma}, \tilde{V}_2) \\ &+ (-1)^k \sum_{\substack{c \in U_m(q) \setminus M_m \\ \det c > 0}} (\det c)^{-k-2s} \sum_d K_q\left(T, (c, d); \begin{pmatrix} {}^t \kappa & \\ & \kappa^{-1} \end{pmatrix} \tilde{\sigma}, \tilde{V}_2 {}^t \kappa^{-1}\right) \end{aligned}$$

by Lemma 11. Thus putting $S'(T; \tilde{\sigma}, \tilde{V}_2) := \sum_{c,d} (\det c)^{-k-2s} K_q(T, (c, d); \tilde{\sigma}, \tilde{V}_2)$, where $c \in U_m(q) \setminus \{c \in M_m \mid \det c > 0\}$, $d \in M_m \bmod qc\Lambda_m$, we have $S(T; \tilde{\sigma}, \tilde{V}_2) = S'(T; \tilde{\sigma}, \tilde{V}_2) + (-1)^k S'\left(T; \begin{pmatrix} {}^t \kappa & \\ & \kappa^{-1} \end{pmatrix} \tilde{\sigma}, \tilde{V}_2 {}^t \kappa^{-1}\right)$. Applying Lemma 23 to $\alpha_1 = 1$, $\alpha_2 = q$, we have

$$\begin{aligned} & S'(T; \tilde{\sigma}, \tilde{V}_2) \\ &= \sum_{c_1} \sum_{c_2} (\det(c_1 c_2))^{-k-2s} \sum_{d_1} K(T, (c_1, d_1)) \\ &\quad \times \sum_{d_2} K_q\left(T, (c_2, d_2); \left(\widehat{\det c_1} \widehat{\det c_1}^{-1}\right) \tilde{\sigma}, \tilde{V}_2 \widehat{\det c_1}^{-1}\right) \end{aligned}$$

where c_i, d_i runs over the same sets as in Lemma. This is equal to

$$\begin{aligned} & \sum_{c_1} \sum_{c_2} (\det(c_1 c_2))^{-k-2s} \sum_{d_1} K(T, (c_1, d_1)) \\ & \times \varphi(q)^{-1} \sum_a \sum_{\chi} \bar{\chi}(a) \chi(\det c_1) \sum_{d_2} K_q\left(T, (c_2, d_2); \left(\hat{a} \hat{a}^{-1}\right) \tilde{\sigma}, \tilde{V}_2 \hat{a}^{-1}\right) \end{aligned}$$

where a runs over $(\mathbf{Z}/q\mathbf{Z})^\times$ and χ runs over the set of Dirichlet characters defined modulo q , and this is equal to

$$\begin{aligned} & \varphi(q)^{-1} \sum_{\chi} \left\{ \sum_{c_1} \chi(\det c_1) (\det c_1)^{-k-2s} \sum_{d_1} K(T, (c_1, d_1)) \right\} \\ & \times \left\{ \sum_a \bar{\chi}(a) \sum_{c_2} (\det c_2)^{-k-2s} \sum_{d_2} K_q\left(T, (c_2, d_2); \left(\hat{a} \hat{a}^{-1}\right) \tilde{\sigma}, \tilde{V}_2 \hat{a}^{-1}\right) \right\}. \end{aligned}$$

Thus we have $S(T; \tilde{\sigma}, \tilde{V}_2) = S'(T; \tilde{\sigma}, \tilde{V}_2) + (-1)^k S'(T; \left(\begin{smallmatrix} t\kappa \\ \kappa^{-1} \end{smallmatrix} \right) \tilde{\sigma}, \tilde{V}_2 {}^t\kappa^{-1}) = S'(T; \tilde{\sigma}, \tilde{V}_2) + \chi(-1)(-1)^k S'(T; \tilde{\sigma}, \tilde{V}_2)$, which completes the proof. $\blacksquare$

LEMMA 25. For $T \in \Lambda'_n$ and a Dirichlet character χ defined modulo q , we have

$$\begin{aligned} & \sum_{c_1 \in U_m \setminus \left\{ c_1 \in M_m \mid \begin{array}{l} \det c_1 > 0, \\ (\det c_1, q) = 1 \end{array} \right\}} \chi(\det c_1) (\det c_1)^{-k-2s} \sum_{d_1 \in M_m \bmod c_1 \Lambda_m} K(T, (c_1, d_1)) \\ &= \prod_{p \nmid q} \left\{ \sum_{c \in U_m \setminus \{c \in M_m \mid 0 < \det c / p^\infty\}} (\chi(p) p^{-k-2s})^{\text{ord}_p(\det c)} \sum_{d \in M_m \bmod c \Lambda_m} K(T, (c, d)) \right\}. \end{aligned}$$

PROOF. Applying Lemma 23 for $\alpha_1 = \alpha_2 = 1$, we get the assertion. $\blacksquare$

LEMMA 26. Let $q = \prod_i q_i$ be a decomposition such that $(q_i, q_j) = 1$ for $i \neq j$ and q_i is a power of a prime number.

We decompose a Dirichlet character χ as $\chi = \prod \chi_i$ where χ_i is a Dirichlet character defined modulo q_i . Then we have

$$\begin{aligned} & \sum_{a \bmod q} \bar{\chi}(a) \sum_{c_2} (\det c_2)^{-k-2s} \sum_{d_2 \in M_m \bmod qc_2 \Lambda_m} K_q \left(T, (c_2, d_2); \left(\begin{smallmatrix} t\hat{a} \\ \hat{a}^{-1} \end{smallmatrix} \right) \tilde{\sigma}; \tilde{v}_2 \hat{a}^{-1} \right) \\ &= \prod_i \left\{ \sum_{a_i \bmod q_i} \bar{\chi}_i(a_i) \sum_{c_i} \left\{ \prod_{j \neq i} \chi_j(\det c_j) \right\} (\det c_i)^{-k-2s} \right. \\ & \quad \times \left. \sum_{d_i \in M_m \bmod q_i c_i \Lambda_m} K_{q_i} \left(T, (c_i, d_i); \left[qq_i^{-1} \dagger \left(\begin{smallmatrix} t\hat{a}_i \\ \hat{a}_i^{-1} \end{smallmatrix} \right) \tilde{\sigma} \right]_{q_i}, (qq_i^{-1})^{-1} \tilde{v}_2 \hat{a}_i^{-1} \right) \right\}, \end{aligned}$$

where c_2 in the first line runs over $U_m(q) \setminus \{c_2 \in M_m \mid 0 < \det c_2 / q^\infty\}$ and c_i runs over $U_m(q_i) \setminus \{c_i \in M_m \mid 0 < \det c_i / q_i^\infty\}$.

PROOF. We use induction on the number of prime factors of q . When q is a power of a prime, the assertion is obvious. Let α_1, α_2 be relatively prime natural numbers and $\chi = \chi_1 \chi_2$ a Dirichlet character with χ_i being defined modulo α_i . Let ξ be another Dirichlet character defined modulo $\alpha_1 \alpha_2$ and decomposing $a \in (\mathbf{Z} / \alpha_1 \alpha_2 \mathbf{Z})^\times$ as $a = a_1 a_2$, $a_i \equiv 1 \pmod{\alpha_i}$, we have, by Lemma 23

$$\begin{aligned} & \sum_{a \bmod \alpha_1 \alpha_2} \bar{\chi}(a) \sum_{c \in U_m(\alpha_1 \alpha_2) \setminus \left\{ c \in M_m \mid \begin{array}{l} \det c > 0 \\ \det c \mid (\alpha_1 \alpha_2)^\infty \end{array} \right\}} \xi(\det c) (\det c)^{-k-2s} \\ & \times \sum_{d \in M_m \bmod \alpha_1 \alpha_2 c \Lambda_m} K_{\alpha_1 \alpha_2} \left(T, (c, d); \left(\begin{smallmatrix} t\hat{a} \\ \hat{a}^{-1} \end{smallmatrix} \right) \tilde{\sigma}, \tilde{v}_2 \hat{a}^{-1} \right) \end{aligned}$$

$$\begin{aligned}
&= \sum_{a_1, a_2} \bar{\chi}_1(a_2) \bar{\chi}_2(a_1) \\
&\quad \times \sum_{c_i \in U_m(\alpha_i) \setminus \left\{ c_i \in M_m \mid \frac{\det c_i > 0}{\det c_i | \alpha_i^\infty} \right\}} \xi(\det c_1)(\det c_1)^{-k-2s} \xi(\det c_2)(\det c_2)^{-k-2s} \\
&\quad \times \sum_{d_1 \in M_m \bmod \alpha_1 c_1 \Lambda_m} K_{\alpha_1} \left(T, (c_1, d_1); \left[\alpha_2 \ddagger \left(\begin{smallmatrix} {}^t \widehat{\det c_2} & \\ & \widehat{\det c_2}^{-1} \end{smallmatrix} \right) \begin{pmatrix} {}^t \hat{a}_2 & \\ & \hat{a}_2^{-1} \end{pmatrix} \tilde{\sigma} \right]_{\alpha_1}, \right. \\
&\quad \left. \alpha_2^{-1} \tilde{v}_2 \hat{a}_2^{-1} \widehat{\det c_2}^{-1} \right) \\
&\quad \times \sum_{d_2 \in M_m \bmod \alpha_2 c_2 \Lambda_m} K_{\alpha_2} \left(T, (c_2, d_2); \left[\alpha_1 \ddagger \left(\begin{smallmatrix} {}^t \widehat{\det c_1} & \\ & \widehat{\det c_1}^{-1} \end{smallmatrix} \right) \begin{pmatrix} {}^t \hat{a}_1 & \\ & \hat{a}_1^{-1} \end{pmatrix} \tilde{\sigma} \right]_{\alpha_2}, \right. \\
&\quad \left. \alpha_1^{-1} \tilde{v}_2 \hat{a}_1^{-1} \widehat{\det c_1}^{-1} \right) \\
&= \left\{ \sum_{a_2 \bmod \alpha_1} \bar{\chi}_1(a_2) \sum_{c_1 \in U_m(\alpha_1) \setminus \left\{ c_1 \in M_m \mid \frac{\det c_1 > 0}{\det c_1 | \alpha_1^\infty} \right\}} \xi \chi_2(\det c_1)(\det c_1)^{-k-2s} \right. \\
&\quad \times \sum_{d_1 \in M_m \bmod \alpha_1 c_1 \Lambda_m} K_{\alpha_1} \left(T, (c_1, d_1); \left[\alpha_2 \ddagger \begin{pmatrix} {}^t \hat{a}_2 & \\ & \hat{a}_2^{-1} \end{pmatrix} \tilde{\sigma} \right]_{\alpha_1}, \alpha_2^{-1} \tilde{v}_2 \hat{a}_2^{-1} \right) \\
&\quad \times \left\{ \sum_{a_1 \bmod \alpha_2} \bar{\chi}_2(a_1) \sum_{c_2 \in U_m(\alpha_2) \setminus \left\{ c_2 \in M_m \mid \frac{\det c_2 > 0}{\det c_2 | \alpha_2^\infty} \right\}} \xi \chi_1(\det c_2)(\det c_2)^{-k-2s} \right. \\
&\quad \times \sum_{d_2 \in M_m \bmod \alpha_2 c_2 \Lambda_m} K_{\alpha_2} \left(T, (c_2, d_2); \left[\alpha_1 \ddagger \begin{pmatrix} {}^t \hat{a}_1 & \\ & \hat{a}_1^{-1} \end{pmatrix} \tilde{\sigma} \right]_{\alpha_2}, \alpha_1^{-1} \tilde{v}_2 \hat{a}_1^{-1} \right) \Big\}.
\end{aligned}$$

When $q = q_1 q_2$, putting $\alpha_i = q_i$ and $\xi = 1$ and exchanging a_1 and a_2 , we get the assertion. Using the above equation inductively, we complete the proof. ■

From Lemmas 24, 25, 26 follows

LEMMA 27. Let $q = \prod_i q_i$ be a decomposition such that $(q_i, q_j) = 1$ for $i \neq j$ and q_i is a power of a prime number. For $\tilde{\sigma} = \begin{pmatrix} * & * \\ \tilde{c} & \tilde{d} \end{pmatrix} \in \Gamma_m$, $\tilde{v}_2 \in M_{r,m}$ and $T \in \Lambda'_n$, $S(T; \tilde{\sigma}, \tilde{v}_2)$ in Lemma 24 is equal to

$$\begin{aligned}
&2\varphi(q)^{-1} \sum_{\substack{\chi \bmod q \\ \chi(-1)=(-1)^k}} \prod_{p \nmid q} \left\{ \sum_{c \in U_m \setminus \{c \in M_m \mid 0 < \det c | p^\infty\}} (\chi(p)p^{-k-2s})^{\text{ord}_p \det c} \right. \\
&\quad \times \sum_{d \bmod c \Lambda_m} K(T, (c, d)) \Big\} \prod_i \left\{ \sum_{a_i \bmod q_i} \bar{\chi}_i(a_i) \right.
\end{aligned}$$

$$\begin{aligned} & \times \sum_{c_i \in U_m(q_i) \setminus \{c_i \in M_m \mid \frac{\det c_i > 0}{\det c_i | q_i^\infty}\}} \left\{ \prod_{j \neq i} \chi_j(\det c_i) \right\} (\det c_i)^{-k-2s} \\ & \times \sum_{d_i \bmod q_i c_i \Lambda_m} K_{q_i} \left(T, (c_i, d_i); \left[qq_i^{-1} \dagger \left(\begin{smallmatrix} {}^t \hat{a}_i & \\ & \hat{a}_i^{-1} \end{smallmatrix} \right) \tilde{\sigma} \right]_{q_i}, (qq_i^{-1})^{-1} \tilde{v}_2 \hat{a}_i^{-1} \right) \Big\}, \end{aligned}$$

where $\chi = \prod_i \chi_i$ and χ_i is a Dirichlet character defined modulo q_i .

Let us study factors in the above infinite product for $p \nmid q$. To evaluate the power series

$$\sum_{c \in U_n \setminus \left\{ c \in M_n \mid \frac{\det c > 0}{\det c | p^\infty} \right\}} x^{\text{ord}_p \det c} \sum_{d \in M_n \bmod c \Lambda_n} K(T, (c, d))$$

for $T \in \Lambda'_n$ with $\det T_1^{(r)} \neq 0$, we consider

$$\beta_p(T, x) := \sum_{c, d} x^{\text{ord}_p \det c} e(\text{tr } T c^{-1} d)$$

for $T \in \Lambda'_n$, where c, d run over $GL_n(\mathbf{Z}) \setminus \{(c, d) \mid \begin{pmatrix} * & * \\ c & d \end{pmatrix} \in \Gamma_n, \det c | p^\infty\} / \begin{pmatrix} 1_n & \Lambda_n \\ 0 & 1_n \end{pmatrix}$. If $T \notin \Lambda'_n$, then we put $\beta_p(T, x) = 0$. It is known [K2] that if $\det T \neq 0$, then

$$\begin{aligned} & \beta_p(T, p^{-k}) \\ & = \lim_{t \rightarrow \infty} (p^t)^{(n+1)n/2-2nk} \sharp \{X \in M_{2k,n}(\mathbf{Z}_p/p^t \mathbf{Z}_p) \mid H_k[X] \equiv T \bmod p^t \Lambda'_n(\mathbf{Z}_p)\}, \end{aligned}$$

where $H_k := \frac{1}{2} \begin{pmatrix} 0 & 1_k \\ 1_k & 0 \end{pmatrix}$ and $\Lambda'_n(\mathbf{Z}_p) := \Lambda'_n \otimes_{\mathbf{Z}} \mathbf{Z}_p$ if $2k \geq n$, and

$$\beta_p(T, x) = \sum_G (p^{n+1} x^2)^{\text{ord}_p \det G} \gamma_p(T[G^{-1}], x),$$

where G runs over $GL_n(\mathbf{Z}_p) \setminus (M_n(\mathbf{Z}_p) \cap GL_n(\mathbf{Q}_p))$ and $\gamma_p(T, x) := 0$ if $T \notin \Lambda'_n(\mathbf{Z}_p)$, otherwise we put

$$\gamma_p(T, x) := (1-x)(1+\epsilon p^{n-d/2}x) \prod_{1 \leq i < n-d/2} (1-p^{2i}x^2),$$

where d, ϵ are defined as follows:

Let V be a quadratic space over $\mathbf{Z}/p\mathbf{Z}$ with basis $\{v_i\}_{i=1}^n$ such that the quadratic form q is given by

$$q\left(\sum x_i v_i\right) := T \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \bmod p,$$

and the associated bilinear form b is defined by $b(x, y) := q(x+y) - q(x) - q(y)$. Then $n-d$ is the dimension of the radical $\text{Rad}(V) := \{v \in V \mid q(v) = 0, b(v, V) = 0\}$. If d is odd, then we put $\epsilon = 0$. Suppose that d is even and put $V = V_0 \perp \text{Rad}(V)$. If V_0 is hyperbolic, that is $V_0 = 0$ or V_0 is isometric to the quadratic space corresponding to H_k for some k , then we put $\epsilon = 1$, otherwise $\epsilon = -1$.

LEMMA 28. For $T \in \Lambda'_n$ with $\det T_1^{(r)} \neq 0$, we have

$$\begin{aligned} \beta_p(T, x) = & \sum_{\substack{\sigma \in \Gamma_m^\infty \setminus \Gamma_m / \left(\begin{smallmatrix} 1 & \Lambda_m \\ 0 & 1 \end{smallmatrix} \right) \\ 0 < \det \sigma_3^{(m)} | p^\infty}} x^{\text{ord}_p \det \sigma_3} \sum_{\substack{V_1 \in U_r \setminus M_r \\ 0 < \det V_1 | p^\infty}} (\det V_1)^{n+1} x^{2 \text{ord}_p \det V_1} \\ & \times \gamma_p(T_1[V_1^{-1}], x) K(T \left(\begin{smallmatrix} V_1 & \\ & 1_m \end{smallmatrix} \right)^{-1}, (\sigma_3, \sigma_4)). \end{aligned}$$

PROOF. From Lemma 6,7 follows that a complete set of representatives of $\Gamma_n^\infty \setminus \{K \in \Gamma_n \mid \det K_3^{(n)} | p^\infty\} / \left(\begin{smallmatrix} 1 & \Lambda_n \\ 0 & 1 \end{smallmatrix} \right)$ is given by $M \uparrow N(v, \sigma)$, where $\sigma \in \Gamma_m^\infty \setminus \Gamma_m / \left(\begin{smallmatrix} 1 & \Lambda_m \\ 0 & 1 \end{smallmatrix} \right)$ with $0 < \det \sigma_3^{(m)} | p^\infty$, $v \in {}^t P_{n,r} \setminus U_n / \left(\begin{smallmatrix} 1_r & M_{r,m} {}^t \sigma_3 \\ & 1_m \end{smallmatrix} \right)$ with $0 < \det v_1^{(r)} | p^\infty$, $M \in \Gamma_r^\infty \setminus \Gamma_r / \left(\begin{smallmatrix} 1_r & \Lambda_r [{}^t v_1] \\ & 1_r \end{smallmatrix} \right)$ with $0 < \det M_3^{(r)} | p^\infty$. Lemma 6 implies

$$\begin{aligned} \beta_p(T, x) &= \sum_{\sigma, v, M} x^{\text{ord}_p (\det M_3 \det \sigma_3 \det v_1^2)} e \left(\text{tr } T \left(\begin{smallmatrix} M_3^{-1} M_4 [{}^t v_1^{-1}] + \sigma_1 \sigma_3^{-1} [{}^t (v_1^{-1} v_2)] & * \\ -\sigma_3^{-1} {}^t (v_1^{-1} v_2) & \sigma_3^{-1} \sigma_4 \end{smallmatrix} \right) \right) \end{aligned}$$

where σ, v, M run over above sets, and this is equal to

$$\sum_{\sigma, v} (\det v_1)^{r+1} \beta_p(T_1[v_1^{-1}], x) x^{\text{ord}_p (\det \sigma_3 \det v_1^2)} e \left(\text{tr } T \left(\begin{smallmatrix} \sigma_1 \sigma_3^{-1} [{}^t (v_1^{-1} v_2)] & * \\ -\sigma_3^{-1} {}^t (v_1^{-1} v_2) & \sigma_3^{-1} \sigma_4 \end{smallmatrix} \right) \right)$$

since $[\Lambda_r : \Lambda_r [{}^t v_1]] = \det v_1^{r+1}$

$$\begin{aligned} &= \sum_{\sigma, v} \det v_1^{r+1} \left\{ \sum_g (p^{r+1} x^2)^{\text{ord}_p \det g} \gamma_p(T_1[v_1^{-1} g^{-1}], x) \right\} x^{\text{ord}_p (\det \sigma_3 \det v_1^2)} \\ &\quad \times e \left(\text{tr } T \left(\begin{smallmatrix} \sigma_1 \sigma_3^{-1} [{}^t (v_1^{-1} v_2)] & * \\ -\sigma_3^{-1} {}^t (v_1^{-1} v_2) & \sigma_3^{-1} \sigma_4 \end{smallmatrix} \right) \right) \end{aligned}$$

where g runs over $U_r \setminus \{g \in M_r \mid 0 < \det g | p^\infty\}$. The mapping $v \mapsto (v_1^{(r)}, v_2)$ gives a bijection from $\{v \in {}^t P_{n,r} \setminus U_n / \left(\begin{smallmatrix} 1_r & M_{r,m} {}^t \sigma_3 \\ & 1_m \end{smallmatrix} \right) \mid 0 < \det v_1 | p^\infty\}$ to $\tilde{V}_{pr} := U_r \setminus \{(v_1, v_2) \in M_r \times M_{r,m} \mid 0 < \det v_1 | p^\infty, (v_1, v_2) : \text{primitive}, v_2 \bmod v_1 M_{r,m} {}^t \sigma_3\}$ and the mapping $g, (v_1, v_2) \mapsto (gv_1, gv_2)$ gives also a bijection from $\sqcup_{(v_1, v_2) \in \tilde{V}_{pr}} U_r \setminus \{g \in M_r \mid 0 < \det g | p^\infty\} \times (v_1, v_2)$ to $\tilde{V} := U_r \setminus \{(V_1, V_2) \in M_r \times M_{r,m} \mid 0 <$

$\det V_1 | p^\infty, V_2 \bmod V_1 M_{r,m} {}^t \sigma_3 \}$. The first one is easy. To see the second bijectivity, suppose that $(v_1, v_2), (v'_1, v'_2) \in \tilde{V}_{pr}, g, g' \in M_r$ with $0 < \det g, \det g'$ and $\det gg' | p^\infty$ are given and $(g'v'_1, g'v'_2) = u(gv_1, gv_2) \begin{pmatrix} 1_r & x \\ 0 & 1_m \end{pmatrix}$ for $u \in U_r, x \in M_{r,m} {}^t \sigma_3$. Then the primitivity of $(v_1, v_2), (v'_1, v'_2)$ implies $(ug)^{-1}g' \in GL_r(\mathbf{Z})$, and so $(ug)^{-1}g' \in U_r$ which means $g' \in U_r g$, and it yields that (v_1, v_2) and (v'_1, v'_2) are the same in $\tilde{V}_{pr}$. Thus the injectivity has been proved. Elementary divisor theorem yields the surjectivity. Putting $(V_1, V_2) := (gv_1, gv_2)$ and continuing the calculation, we have

$$\begin{aligned}
 \beta_p(T, x) &= \sum_{\sigma} \sum_{(V_1, V_2) \in \tilde{V}} (\det V_1)^{r+1} x^{\text{ord}_p(\det \sigma_3) + 2 \text{ord}_p \det V_1} \\
 &\quad \times \gamma_p(T_1[V_1^{-1}], x) e \left(\text{tr} T \begin{pmatrix} \sigma_1 \sigma_3^{-1} [{}^t(V_1^{-1} V_2)] & * \\ -\sigma_3^{-1} {}^t(V_1^{-1} V_2) & \sigma_3^{-1} \sigma_4 \end{pmatrix} \right) \\
 &= \sum_{\sigma} \sum_{V_1 \in U_r \setminus \{V_1 \in M_r \mid 0 < \det V_1 \mid p^\infty\}} (\det V_1)^{r+1} x^{\text{ord}_p(\det \sigma_3) + 2 \text{ord}_p \det V_1} \\
 (1) \quad &\quad \times \gamma_p(T_1[V_1^{-1}], x) \sum_{V_2 \in M_{r,m} \bmod V_1 M_{r,m} {}^t \sigma_3} e \left(\text{tr} T \begin{pmatrix} \sigma_1 \sigma_3^{-1} [{}^t(V_1^{-1} V_2)] & * \\ -\sigma_3^{-1} {}^t(V_1^{-1} V_2) & \sigma_3^{-1} \sigma_4 \end{pmatrix} \right).
 \end{aligned}$$

The partial sum on V_2 is, by writing $V_2 = x + y {}^t \sigma_3$ ($x \in M_{r,m} \bmod M_{r,m} {}^t \sigma_3, y \in M_{r,m} \bmod V_1 M_{r,m}$),

$$\begin{aligned}
 &\sum_{x, y} e \left(\text{tr} T \begin{pmatrix} \sigma_1 \sigma_3^{-1} [{}^t(V_1^{-1}(x + y {}^t \sigma_3))] & * \\ -\sigma_3^{-1} {}^t(V_1^{-1}(x + y {}^t \sigma_3)) & \sigma_3^{-1} \sigma_4 \end{pmatrix} \right) \\
 &= \sum_{x, y} e \left(\text{tr} T \left[\begin{pmatrix} V_1 & \\ & 1_m \end{pmatrix}^{-1} \right] \begin{pmatrix} \sigma_1 \sigma_3^{-1} [{}^t x + \sigma_3 {}^t y] & * \\ -\sigma_3^{-1} ({}^t x + \sigma_3 {}^t y) & \sigma_3^{-1} \sigma_4 \end{pmatrix} \right) \\
 &= \sum_x e \left(\text{tr} T \left[\begin{pmatrix} V_1 & \\ & 1_m \end{pmatrix}^{-1} \right] \begin{pmatrix} \sigma_1 \sigma_3^{-1} [{}^t x] & * \\ -\sigma_3^{-1} {}^t x & \sigma_3^{-1} \sigma_4 \end{pmatrix} \right) \\
 &\quad \times \sum_y e \left(\text{tr} T \left[\begin{pmatrix} V_1 & \\ & 1_m \end{pmatrix}^{-1} \right] \begin{pmatrix} x \sigma_1 {}^t y + y {}^t \sigma_1 {}^t x + {}^t \sigma_3 \sigma_1 [y] & * \\ -{}^t y & 0 \end{pmatrix} \right)
 \end{aligned}$$

here by virtue of the existence of $\gamma_p(T_1[V_1^{-1}], x)$ at (1), we may assume $T_1[V_1^{-1}] \in \Lambda'_r$ and so the above expression is equal to

$$\sum_x e \left(\text{tr} T \left[\begin{pmatrix} V_1 & \\ & 1_m \end{pmatrix}^{-1} \right] \begin{pmatrix} \sigma_1 \sigma_3^{-1} [{}^t x] & * \\ -\sigma_3^{-1} {}^t x & \sigma_3^{-1} \sigma_4 \end{pmatrix} \right) \times \begin{cases} \det V_1^m & \text{if } 2 {}^t V_1^{-1} T_2 \in M_{r,m}, \\ 0 & \text{otherwise.} \end{cases}$$

The conditions $T_1[V_1^{-1}] \in \Lambda'_r$ and $2 {}^t V_1^{-1} T_2 \in M_{r,m}$ are equivalent to $T \left[\begin{pmatrix} V_1 & \\ & 1_m \end{pmatrix}^{-1} \right] \in \Lambda'_n$ and then the sum on x becomes $K \left(T \left[\begin{pmatrix} V_1 & \\ & 1_m \end{pmatrix}^{-1} \right], (\sigma_3, \sigma_4) \right)$. Thus, we get the desired result. $\blacksquare$

LEMMA 29. For $T \in \Lambda'_n$ with $\det T_1^{(r)} \neq 0$, we have

$$\begin{aligned} & \sum_{c \in U_m \setminus \{c \in M_m \mid 0 < \det c \mid p^\infty\}} x^{\text{ord}_p \det c} \sum_{d \in M_m \bmod c\Lambda_m} K(T, (c, d)) \\ &= \gamma_p(T_1, x)^{-1} \sum_{\substack{V_1 \in U_r \setminus M_r \\ 0 < \det V_1 \mid p^\infty}} \mu(\mathbf{Z}^r / \mathbf{Z}^r V_1) \det V_1^{n+1} x^{2 \text{ord}_p \det V_1} \beta_p \left(T \left[\begin{pmatrix} V_1 & \\ & 1_m \end{pmatrix}^{-1} \right], x \right). \end{aligned}$$

PROOF. Consider a vector space $V = \mathbf{Q}[v_1, \dots, v_n]$ and introduce a bilinear form B on it by $(B(v_i, v_j)) = T$. Put $L_0 := \mathbf{Z}[v_1, \dots, v_r]$, $L_1 := \mathbf{Z}[v_{r+1}, \dots, v_n]$ and $L := L_0 \oplus L_1$. Now we choose a lattice $\tilde{L} \supset L_0$ so that $[\tilde{L} : L_0] \mid p^\infty$ and if a lattice M on $\mathbf{Q}L_0$ satisfies $M \supset L_0$, $[M : L_0] \mid p^\infty$ and $B(x, x) \in \mathbf{Z}$ for every $x \in M \oplus L_1$, then $M \subset \tilde{L}$. Such a lattice $\tilde{L}$ exists since $\det T_1 \neq 0$. For a lattice $N = \mathbf{Z}[u_1, \dots, u_r]$ on $\mathbf{Q}L_0$, we put

$$\gamma_p(N, x) := \gamma_p(((B(u_i, u_j))_{1 \leq i, j \leq r}, x),$$

$$K(N \oplus L_1, (c, d)) := K(((B(u_i, u_j))_{1 \leq i, j \leq n}, (c, d))$$

where $u_i := v_i$ if $i > r$. If $(B(u_i, u_j)) \notin \Lambda'_r$ or Λ'_n , then $\gamma_p(N, x)$ or $K(N \oplus L_1, (c, d)) = 0$ by definitions. Since $\gamma_p(N, x)$ is independent of a choice of bases of N , $\gamma_p(N, x)$ is well-defined. Suppose $(u_1, \dots, u_r)w$ ($w \in GL_r(\mathbf{Z})$) be another basis of N . Then the corresponding matrix is $(B(u_i, u_j)) \left[\begin{pmatrix} w & \\ & 1_m \end{pmatrix} \right]$ and then (3) in Lemma 11 yields $K(((B(u_i, u_j)) \left[\begin{pmatrix} w & \\ & 1_m \end{pmatrix} \right], (c, d)) = K(((B(u_i, u_j)), (c, d))$. Thus $K(N \oplus L_1, (c, d))$ is well-defined. For a lattice N on $\mathbf{Q}L_0$, we put

$$\phi(N) := \sum_{\substack{\sigma \in \Gamma_m^\infty \setminus \Gamma_m / \begin{pmatrix} 1 & \Lambda_m \\ 0 & 1 \end{pmatrix} \\ 0 < \det \sigma_3^{(m)} \mid p^\infty}} x^{\text{ord}_p \det \sigma_3} \gamma_p(N, x) K(N \oplus L_1, (\sigma_3, \sigma_4)).$$

Then the previous lemma yields for $\tilde{L} \supset M \supset L_0$

$$\beta_p(M \oplus L_1, x) := \beta_p(\tilde{T}, x) = \sum_{\tilde{L} \supset N \supset M} [N : M]^{n+1} x^{2 \text{ord}_p [N : M]} \phi(N),$$

where $\tilde{T} = (B(y_i, y_j))$ for a basis $\{y_i\}$ of $M \oplus L_1$. It means

$$[M : L_0]^{n+1} x^{2 \text{ord}_p [M : L_0]} \beta_p(M \oplus L_1, x) = \sum_{\tilde{L} \supset N \supset M} [N : L_0]^{n+1} x^{2 \text{ord}_p [N : L_0]} \phi(N).$$

Applying Lemma 8 to $\tilde{L}/L_0$, we have

$$\phi(L_0) = \sum_{\tilde{L} \supset N \supset L_0} \mu(N/L_0) [N : L_0]^{n+1} x^{2 \text{ord}_p [N : L_0]} \beta_p(N \oplus L_1, x),$$

which is our assertion. $\blacksquare$

Lemmas 27 and 29 imply

LEMMA 30. For $T \in \Lambda'_n$ with $\det T_1^{(r)} \neq 0, \tilde{\sigma} \in \Gamma_m, \tilde{v}_2 \in M_{r,m}, S(T; \tilde{\sigma}, \tilde{v}_2)$ defined in Lemma 24 is equal to

$$\begin{aligned} & 2\varphi(q)^{-1} \sum_{\substack{\chi \bmod q \\ \chi(-1)=(-1)^k}} \left\{ \prod_{p \nmid q} \gamma_p(T_1, \chi(p)p^{-k-2s})^{-1} \right\} \left\{ \sum_{\substack{V_1 \in U_r \setminus M_r \\ \det V_1 > 0, (\det V_1, q)=1}} \mu(\mathbf{Z}^r / \mathbf{Z}^r V_1) \right. \\ & \times \chi(\det V_1)^2 (\det V_1)^{n+1-2k-4s} \prod_{p \nmid q} \beta_p \left(T \left[\begin{pmatrix} V_1 & \\ & 1_m \end{pmatrix}^{-1} \right], \chi(p)p^{-k-2s} \right) \Big\} \\ & \times \prod_i \left\{ \sum_{a_i \bmod q_i} \bar{\chi}_i(a_i) \sum_{c_i} \left\{ \prod_{j \neq i} \chi_j(\det c_i) \right\} (\det c_i)^{-k-2s} \right. \\ & \times \sum_{d_i} K_{q_i} \left(T, (c_i, d_i); \left[qq_i^{-1} \dagger \begin{pmatrix} {}^t \hat{a}_i & \\ & \hat{a}_i^{-1} \end{pmatrix} \tilde{\sigma} \right]_{q_i}, (qq_i^{-1})^{-1} \tilde{v}_2 \hat{a}_i^{-1} \right) \Big\}, \end{aligned}$$

where c_i runs over $U_m(q_i) \setminus \{c_i \in M_m \mid 0 < \det c_i \mid q_i^\infty\}$ and d_i runs over $M_m \bmod q_i c_i \Lambda_m$.

REMARK. The final sum on c_i, d_i seems to be a finite sum if $\det T \neq 0$. It is true if $m = r = 1, n = 2$. I do not know whether it is true or not for $n > 2$, nor the meaning of the sums.

Now we have from Lemmas 18, 30

THEOREM 1 Let $f(z)$ be a cusp form of degree r , weight k and level q , and let $M \in \Gamma_n$.

$$\text{For } \eta = \begin{pmatrix} a_1^{(r)} & 0 & b_1^{(r)} & * \\ * & * & * & * \\ c_1^{(r)} & 0 & d_1^{(r)} & * \\ 0 & 0 & 0 & d_4^{(m)} \end{pmatrix} \in \Delta_{n,r}, \text{ we put}$$

$$(\det d_4)^k f \left| \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} \right. (\tau) = \sum_{t>0} b(t; \eta) e(\text{tr } t\tau/q) \quad (\tau \in H_r).$$

Then putting for $T \in \Lambda'_n$

$$\mathcal{F}(T, y; f, M) := q^{-r(\tau+1)/2} \int_{x \in \Lambda_n(\mathbf{R})/q\Lambda_n} \sum_K (f \| K)(z) e(-\text{tr } Tx/q) dx,$$

where K runs over

$$\Delta_{n,r} \setminus \{K \in \Gamma_n \mid \text{rank}(0, 1_m) K_3^{(n)} = m, \sigma \equiv M \bmod q\},$$

we have

$$\begin{aligned}
& 2^{-1} \varphi(q) \mathcal{F}(T, y; f, M) \\
&= [\Gamma_m^\infty : \Gamma_m^\infty(q)]^{-1} [P_{n,r} : P_{n,r}(q)]^{-2} [GL_r(\mathbf{Z}) : U_r(q)]^{-1} \\
&\times \sum_{w \in U_n/P_{n,r}(q)} H(q^{-1}T[w], y[{}^t w^{-1}]; s) \\
&\times \sum_{\tilde{v} = \begin{pmatrix} \tilde{v}_1^{(r)} & \tilde{v}_2 \\ * & * \end{pmatrix} \in U_n/U_n(q)} \sum_{\eta \in \Delta_{n,r}(q) \setminus \Delta_{n,r}} \sum_{\tilde{\sigma} = \begin{pmatrix} \tilde{a} & \tilde{b} \\ \tilde{c} & \tilde{d} \end{pmatrix} \in \Gamma_m/\Gamma_m(q)} \sum_{v_1 \in U_r(q) \setminus (M_r \cap GL_r(\mathbf{Q}))} \\
&\quad \eta N(\tilde{v}, \tilde{\sigma}) \begin{pmatrix} w^{-1} & \\ & t_w \end{pmatrix} \equiv M \pmod{q} \\
&\times |\det v_1|^{m-2s} \left\{ \sum_{\substack{g \in U_r(q) \setminus (M_r \cap GL_r(\mathbf{Q})) \\ gv_1 \equiv \tilde{v}_1 \pmod{q}}} \mu(\mathbf{Z}^r/\mathbf{Z}^r g) |\det g|^{-2s} b(T[w]_1^{(r)} [v_1^{-1} g^{-1}]; \eta) \right\} \\
&\times \sum_{\substack{\chi \pmod{q} \\ \chi(-1) = (-1)^k}} \left\{ \prod_{p \nmid q} \gamma_p(T[w]_1 [v_1^{-1}], \chi(p) p^{-k-2s})^{-1} \right\} \\
&\times \left\{ \sum_{\substack{V_1 \in U_r \setminus M_r \\ \det V_1 > 0, (\det V_1, q) = 1}} \mu(\mathbf{Z}^r/\mathbf{Z}^r V_1) \chi(\det V_1)^2 \right. \\
&\times (\det V_1)^{n+1-2k-4s} \prod_{p \nmid q} \beta_p \left(T \left[w \begin{pmatrix} V_1 v_1 & \\ & 1_m \end{pmatrix}^{-1} \right], \chi(p) p^{-k-2s} \right) \Big\} \\
&\times \sum_{\substack{\tilde{V}_2 \in M_{r,m} \pmod{q} \\ g \tilde{V}_2 \equiv \tilde{v}_2 \pmod{q}}} \prod_i \left\{ \sum_{a_i \pmod{q_i}} \bar{\chi}_i(a_i) \right. \\
&\times \sum_{c_i \in U_m(q_i) \setminus \{c_i \in M_m \mid 0 < \det c_i \mid q_i^\infty\}} \left\{ \prod_{j \neq i} \chi_j(\det c_i) \right\} (\det c_i)^{-k-2s} \\
&\times \sum_{d_i \in M_m \pmod{q_i} c_i \Lambda_m} K_{q_i} \left(T \left[w \begin{pmatrix} v_1 & \\ & 1_m \end{pmatrix}^{-1} \right], (c_i, d_i); \right. \\
&\quad \left. \left[qq_i^{-1} \dagger \begin{pmatrix} {}^t \hat{a}_i & \\ & \hat{a}_i^{-1} \end{pmatrix} \tilde{\sigma} \right]_{q_i}, (qq_i^{-1})^{-1} \tilde{V}_2 \hat{a}_i^{-1} \right) \Big\}.
\end{aligned}$$

Here $q = \prod_i q_i$ is a decomposition such that $(q_i, q_j) = 1$ for $i \neq j$ and q_i is a power of a prime number, χ_i is a Dirichlet character defined modulo q_i such that $\chi = \prod \chi_i$, K_{q_i} is defined at the beginning of this section, $\hat{a} := \text{diag}(1_{m-1}, a)$ and $[\alpha \dagger \sigma]_\beta$ stands for an element in Γ_m which is congruent modulo β to $\begin{pmatrix} \alpha a & b \\ c & \alpha^{-1} d \end{pmatrix}$ for α where $\sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

REMARK. Since $T[w]_1^{(r)}$ is the Bigl upper $r \times r$ submatrix of $T[w]$ by

definition, $b(T[w]_1[v_1^{-1}g^{-1}]; \eta) = 0$ unless $0 < T[w]_1[v_1^{-1}g^{-1}] \in \Lambda'_r$. Also $\beta_p\left(T\left[w\begin{pmatrix} V_1 v_1 & \\ & 1_m \end{pmatrix}^{-1}\right], \chi(p)p^{-k-2s}\right) = 0$ unless $T\left[w\begin{pmatrix} V_1 v_1 & \\ & 1_m \end{pmatrix}^{-1}\right] \in \Lambda'_n$. Therefore the sum on g, v_1 and V_1 is a finite sum, and $\mathcal{F}(T, y; f, M) = 0$ if $\text{rank } T < r$.

THEOREM 2. Let f be the cusp form in Theorem 1, and put

$$E_{n,r}^k(f, s; q, M)(z) := \sum_K (f \|_s K)(z) \quad \text{for } z \in H_n$$

where K runs over $\Delta_{n,r}(q) \setminus \{K \in \Gamma_n \mid K \equiv M \pmod{q}\}$. The Fourier coefficient at $0 < T \in \Lambda'_n$ of $E_{n,r}^k(f, s; q, M)$ is given by $\mathcal{F}(T, y; f, M)$ in Theorem 1.

PROOF. A complete set of representatives of $\Delta_{n,r} \setminus \Gamma_n$ is given by

$$\sqcup_{\mu=r}^n L_n^{r,\mu},$$

where $L_n^{r,\mu} := \left\{ \sigma \uparrow \begin{pmatrix} t u & \\ & u^{-1} \end{pmatrix} \in \Gamma_n \mid \sigma \in \Delta_{\mu,r} \setminus \Gamma_\mu, \text{rank}(0, 1_{\mu-r}) \sigma_3^{(\mu)} = \mu - r, u \in U_n/P_{n,\mu} \right\}$. The proof proceeds as follows:

If, for $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_n, \text{rank}(0, 1_m)C = \mu - r$ ($r \leq \mu \leq n$), then there exist $g \in GL_m(\mathbf{Z})$ and $v \in GL_n(\mathbf{Z})$ such that $g((0, 1_m)C)^t v^{-1} = \begin{pmatrix} * & * \\ 0 & 1_{n-\mu} \end{pmatrix}$. Then taking $w \in GL_n(\mathbf{Z})$ such that $g((0, 1_m)D)vw = \begin{pmatrix} * & * \\ 0 & 1_{n-\mu} \end{pmatrix}$, we have $\begin{pmatrix} 1_r & & \\ & {}^t g^{-1} & \\ & & 1_r \end{pmatrix}_g M \begin{pmatrix} t(vv)^{-1} & \\ & vw \end{pmatrix} = \begin{pmatrix} C_1^{(\mu,n)} & * \\ 0 & 1_{n-\mu} \end{pmatrix} \begin{pmatrix} * & * \\ 0 & 1_{n-\mu} \end{pmatrix} \in \Delta_{n,\mu} \subset \Delta_{n,r} \times \Gamma_\mu \uparrow$. Thus $M \in \Delta_{n,r} \Gamma_\mu \uparrow \times \left\{ \begin{pmatrix} t u & \\ & u^{-1} \end{pmatrix} \mid u \in GL_n(\mathbf{Z}) \right\}$. By the assumption, $\text{rank}(0, 1_m) \begin{pmatrix} 1_r & \\ & g^{-1} \end{pmatrix} \begin{pmatrix} C_1 \\ 0 \end{pmatrix} = \mu - r$ holds and hence $\text{rank}(0, 1_{\mu-r})C_1 = \mu - r$. Thus we have shown that representatives of $\Delta_{n,r} \setminus \Gamma_n$ are in $\left\{ \sigma \uparrow \begin{pmatrix} t u & \\ & u^{-1} \end{pmatrix} \mid \sigma \in \Gamma_\mu \text{ with } \text{rank}(0, 1_{\mu-r}) \sigma_3^{(\mu)} = \mu - r, u \in GL_n(\mathbf{Z}) \right\}$. The completeness is easy.

For $\sigma \in \Gamma_\mu$, it is easy to see

$$(f \| \sigma \uparrow)(z) = j(\sigma, z^*)^{-k} f(\sigma(z^*)_1) \left(\frac{\det \Im \sigma(z^*)}{\det \Im \sigma(z^*)_1} \right)^s \left(\frac{\det y}{\det y^*} \right)^s,$$

where $z^* = z \begin{bmatrix} 1_\mu \\ 0 \end{bmatrix}$ and the suffix 1 means $r \times r$ upper left submatrix. Thus $E_{n,r}^k(f, s; q, M)$ turns out to be equal to

$$\sum_{\mu=r}^n \sum_{u \in U_n/P_{n,\mu}} \sum_{\sigma \in \Delta_{\mu,r}(q) \setminus \Gamma_\mu, \sigma \uparrow \begin{pmatrix} t u & \\ & u^{-1} \end{pmatrix} \equiv M \pmod{q}, \text{rank}(0, 1_{\mu-r}) \sigma_3^{(\mu)} = \mu - r} j(\sigma, z[u]^*)^{-k} \\ \times f(\sigma(z[u]^*)_1) \left(\frac{\det \Im \sigma(z[u]^*)}{\det \Im \sigma(z[u]^*)_1} \right)^s \left(\frac{\det y}{\det y[u]^*} \right)^s,$$

where $w^* = w_1^{(\mu)}$. Hence for a regular half-integral matrix $T \in \Lambda'_n$, the integration $\int_{x \in \Lambda_n(\mathbf{R})/q\Lambda_n} (f \parallel \sigma^\dagger)(z) e(-\operatorname{tr} Tx/q) dx$ vanishes for $\sigma \in \Gamma_\mu$ with $\mu < n$ and then by termwise integration Theorem 1 shows

$$\int_{x \in \Lambda_n(\mathbf{R})/q\Lambda_n} E_{n,r}^k(f, s; q, M) e(-\operatorname{tr} Tx/q) dx = q^{r(r+1)/2} \mathcal{F}(T, y; f, M). \quad \blacksquare$$

REMARK 1. In Theorems 1, 2, $\Re s$ should be sufficiently large to justify the calculation. If $k > n + r + 1$, then $\Re s \geq 0$ is enough and hence we can put $s = 0$ in Theorems 1, 2.

REMARK 2. To get Γ_0 -type Klingen's Eisenstein series, we have only to add $E_{n,r}^k(f, s; q, M)$ on $M \in N\Gamma_0$ for some $N \in \Gamma_n$. When we consider Γ_0 -type, Euler factors turn out to be simpler for primes dividing level. When $N = 1_{2n}$, there is a result of Böcherer-Schulze-Pillot [BS].

REMARK 3. The author does not know when $E_{n,r}^k(f, 0; q, M)$ is a well-defined modular form ($k \leq n + r + 1$) and whether they span the space of modular forms of degree n , weight k , level q with cusp forms and Eisenstein series or not. For the application to quadratic forms we need an estimate on $\mathcal{F}(T, T^{-1}; f, M)$ at $s = 0$ for $0 < T \in \Lambda'_n$.

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