On stable homotopy of the complex projective space

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§0. Introduction

This paper is a continuation of [16] and also regarded as a complex version of [19]. Throughout the paper FP^n stands for the n -dimensional F -projective space, where $F = \mathbf{R}$ (real), $\mathbf{C}$ (complex) or $\mathbf{H}$ (quaternionic). We set $FP = FP^\infty$ and denote by $Q_n(F)$ the quasi-projective space [3].

The first purpose of the present note is to determine the group structure of the k -th stable homotopy group $\pi_k^S(\mathbf{C}P^n)$ for $11 \leq k \leq 18$. The 2-primary torsion of $\pi_k^S(\mathbf{C}P)$ for $9 \leq k \leq 19$ was obtained by Mosher [13], up to group extension.

We shall use the composition methods. We shall often discuss problems from the unstable point of view. To determine group extensions, we investigate Toda brackets in the homotopy group $\pi_{m+k}(\Sigma^k X)$, where $X = FP^n$ or $Q_n(\mathbf{H})$ and $\Sigma^k X$ is the k -fold reduced suspension of X . It is essential to choose a Toda bracket having least indeterminacy. The James splitting theorem (7.10 of [3]) is also essentially used. It enables us to obtain some relations in $\pi_*^S(X)$ from relations in $\pi_*(Y)$, where Y is the classical group SU_n or Sp_n .

The second purpose of this paper is to determine the image of the S^1 -transfer: $B_k = \text{Im} \{g_{n*} : \pi_{k-1}^S(\mathbf{C}P^{n-1}) \rightarrow \pi_k^S(S^0), k \leq 2n-1\}$ [16] for $12 \leq k \leq 19$.

We denote by $M(X) = \{X, X\}$ the stable self-homotopy group of X and by $\varepsilon(X)$ the group of stable homotopy classes of self-homotopy equivalences of X . The third purpose of the present note is to determine the group structures of $M(\mathbf{C}P^n)$ and $\varepsilon(\mathbf{C}P^n)$ for $n \leq 7$. We shall use McGibbon's method [6]. We denote by $D(n, \mathbf{C})$ the subgroup of $\mathbf{Z}^n$ corresponding to the torsion-free subgroup of $M(\mathbf{C}P^n)$ [6].

We shall freely use the information about $\pi_{n+k}^S(S^n)$ [25] and do the group structure of $\pi_k^S(\mathbf{C}P^n)$ for $k \leq 10$ [16].

Our result is stated as follows.

THEOREM 1. i) $\pi_{11}^S(\mathbf{C}P) \cong \mathbf{Z}_4$, $\pi_{12}^S(\mathbf{C}P) \cong \mathbf{Z} \oplus \mathbf{Z}_3$, $\pi_{13}^S(\mathbf{C}P) \cong \mathbf{Z}_{24} \oplus \mathbf{Z}_6$ and $\pi_{14}^S(\mathbf{C}P) \cong \mathbf{Z} \oplus \mathbf{Z}_2$.

ii) $\pi_{15}^S(\mathbf{C}P) \cong \mathbf{Z}_8 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_3$ and $\pi_{16}^S(\mathbf{C}P) \cong \mathbf{Z} \oplus (\mathbf{Z}_2)^3$.

iii) $\pi_{17}^S(\mathbf{C}P) \cong \mathbf{Z}_{32} \oplus \mathbf{Z}_8 \oplus \mathbf{Z}_{15}$ and $\pi_{18}^S(\mathbf{C}P) \cong \mathbf{Z} \oplus \mathbf{Z}_4$.

THEOREM 2. $B_{12} = 0$, $B_{13} = \pi_{13}^S(S^0)$, $B_{14} = \pi_{14}^S(S^0)$, $B_{15} = \mathbf{Z}_{240}\{2\rho\} \oplus \mathbf{Z}_2\{\eta\kappa\}$, $B_{16} = \mathbf{Z}_2\{\omega^*\}$, $B_{17} = \mathbf{Z}_2\{\eta\omega^*\} \oplus \mathbf{Z}_2\{\eta^2\rho\} \oplus \mathbf{Z}_2\{\nu\kappa\}$, $B_{18} = \mathbf{Z}_8\{\nu^*\}$ and $B_{19} = \pi_{19}^S(S^0)$, where $\omega^* = \eta^*$ or $\eta^* + \eta\rho$.

THEOREM 3. i) $M(\mathbf{CP}^n) \cong D(n, \mathbf{C})$ for $2 \leq n \leq 6$ and $M(\mathbf{CP}^7) \cong D(7, \mathbf{C}) \oplus \mathbf{Z}_2$.

ii) $\varepsilon(\mathbf{CP}^n) \cong \mathbf{Z}_2 \times \mathbf{Z}_2$ for $2 \leq n \leq 6$ and $\varepsilon(\mathbf{CP}^7) \cong \mathbf{Z}_2 \times \mathbf{Z}_2 \times \mathbf{Z}_2$.

iii) $\varepsilon(\mathbf{CP}^n)$ contains a subgroup $H \cong \mathbf{Z}_2 \times \mathbf{Z}_2$ as a direct summand for $n \geq 2$.

Theorem 1 overlaps with Theorem 2.2 of [13] and Knapp's result about the 2-component of $\pi_{13}^S(\mathbf{CP})$ [4]. Theorem 3 overlaps with the results of [21] and [26].

This paper consists of 6 sections. In §1 we shall prepare fundamental facts. §2-§4 are devoted to proving Theorem 1. In §5 we shall prove Theorem 2. §6 is devoted to proving Theorem 3.

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§1. Some fundamental lemmas

We set $\tilde{Q}_n(F) = Q_n(F)/Q_1(F) = \Sigma^{d-1}FP^{n-1}$ for $F = \mathbf{R}$ or $\mathbf{C}$ and $\tilde{Q}_n(\mathbf{H}) = Q_n(\mathbf{H})$, where $d = \dim_{\mathbf{R}} F$. Let $i_n(F) : FP^n \rightarrow FP^{n+1}$ be the inclusion. Let $p_n(F) : FP^n \rightarrow FP^n/FP^{n-1} = S^{dn}$ and $\tilde{p}_n(F) : \tilde{Q}_n(F) \rightarrow \tilde{Q}_n(F)/\tilde{Q}_{n-1}(F) = S^{dn-1}$ be the collapsing maps. Let $t_n : \tilde{Q}_n(\mathbf{H}) \rightarrow \tilde{Q}_{2n}(\mathbf{C})$ be the reduced Toda-Kozima's map [17]. Then $\tilde{p}_n(F) = \Sigma^{d-1}p_{n-1}(F)$ for $F = \mathbf{R}$ or $\mathbf{C}$ and $\tilde{p}_{2n}(\mathbf{C})t_n = \tilde{p}_n(\mathbf{H})$. Let $\gamma_n(F) : S^{d(n+1)-1} \rightarrow FP^n$ and $\omega_n(F) : S^{d(n+1)-2} \rightarrow Q_n(F)$ be the projections and let $\tilde{\omega}_n(F) : S^{d(n+1)-2} \rightarrow \tilde{Q}_n(F)$ be $\omega_n(F)$ followed by the projection $Q_n(F) \rightarrow \tilde{Q}_n(F)$. Then $\tilde{\omega}_n(F) = \Sigma^{d-1}\gamma_{n-1}(F)$ for $F = \mathbf{R}$ or $\mathbf{C}$ and $\tilde{\omega}_n(\mathbf{H}) = \omega_n(\mathbf{H})$. We recall that $\Sigma\gamma_{2n}(\mathbf{C}) = (-1)^{n+1}\Sigma i_{2n-1}(\mathbf{C})t_n\omega_n(\mathbf{H})$ [17].

Let ι_n be the identity class of S^n . Let $\eta_n \in \pi_{n+1}(S^n)$ for $n \geq 2$, $\nu_n \in \pi_{n+3}(S^n)$ for $n \geq 4$ and $\sigma_n \in \pi_{n+7}(S^n)$ for $n \geq 8$ be suspensions of the Hopf maps. Then the following relations are well known [3]: $p_n(\mathbf{C})\gamma_n(\mathbf{C}) = n\eta_{2n}$ and $p_n(\mathbf{H})\gamma_n(\mathbf{H}) = n\nu_{4n}$ for $n \geq 1$; $\omega_1(\mathbf{H}) = \nu'$ and $\tilde{P}_n(\mathbf{H})\omega_n(\mathbf{H}) = (n+1)\nu_{4n-1}$ for $n \geq 2$, where ν' is a generator of $\pi_6(S^3) \cong \mathbf{Z}_{12}$. Set $\eta = \Sigma^\infty\eta_2$, $\nu = \Sigma^\infty\nu_4$ and $\sigma = \Sigma^\infty\sigma_8$. By abuse of notation, we often use the same letter to denote generators of a group and its 2-component. Let $h_n : \mathbf{CP}^{2n+1} \rightarrow \mathbf{HP}^n$ be the standard fibration induced from $\gamma_n(\mathbf{H})$. We set $q_n = p_n(\mathbf{H})h_n$. Then, from the definition, we have the following

$$(1) \quad q_n \mid \mathbf{CP}^{2n} = p_{2n}(\mathbf{C}) \quad \text{and} \quad q_n\gamma_{2n+1}(\mathbf{C}) = n\nu_{4n}.$$

We set $O_n(F) = O_n$, U_n or Sp_n (resp. $\tilde{O}_n(F) = SO_n$, SU_n or Sp_n) for $F = \mathbf{R}$, $\mathbf{C}$ or $\mathbf{H}$, respectively. We define an S^{d-1} -transfer map $g_n(F) : \Sigma^{dn}\tilde{Q}_n(F) \rightarrow S^{dn}$ as the adjoint of the composition of the canonical maps; $\tilde{Q}_n(F) \xrightarrow{j_n} \tilde{O}_n(F) \hookrightarrow \Omega^{dn}S^{dn}$,

where $j_n = j_n(F)$ is the reflection map [3] and $\Omega^{dn}S^{dn}$ stands for a space of all self-maps of S^{dn} . From the definition, we have

$$(2) \quad g_n(\mathbf{H}) = g_{2n}(\mathbf{C}) \circ \Sigma^{4n}t_n.$$

We know that $\Sigma(g_n(\mathbf{C}) \circ \Sigma^{2n+1}\gamma_{n-1}(\mathbf{C})) = g_{2n+1}(\mathbf{R}) \circ \Sigma^{2n+1}\gamma_{2n}(\mathbf{R})$ and $\Sigma^2(g_n(\mathbf{H}) \circ \Sigma^{4n}\omega_n(\mathbf{H})) = g_{2n+1}(\mathbf{C}) \circ \Sigma^{4n+3}\gamma_{2n}(\mathbf{C})$. By (4.4) of [18], $g_n(\mathbf{R}) \circ \Sigma^n\gamma_{n-1}(\mathbf{R}) = \pm[\iota_n, \iota_n]$. So we have

$$(3) \quad \Sigma^{d-1}(g_n(F) \circ \Sigma^{dn}\tilde{\omega}_n(F)) = \pm[\iota_k, \iota_k] \quad \text{for } k = d(n+1) - 1.$$

Hereafter we often omit the letter F , unless any confusion occurs. Especially we omit the letter $\mathbf{C}$ in almost all cases and ω_n means $\omega_n(\mathbf{H})$. We also use i, i' and i'' (resp. p, p' and p'') for the inclusion (resp. collapsing) maps. By abuse of notation, we often use the same letter for a mapping and its suspension.

Let $\zeta_n \in \pi_{2n+1}(\Sigma P^n)$ be Toda's map satisfying $p_*\zeta_n = n!\iota_{2n+1}$ ([24], [16]). First we recall the following well-known result ([24], [13]).

LEMMA 1.1. $\pi_{2n}^S(P^n) = \mathbf{Z}\{\zeta_n\} \oplus i_*\pi_{2n}^S(P^{n-1})$ and $\pi_{2n}^S(P^{n-1})$ is finite.

Let X_n be the symmetric space SU_{2n}/Sp_n . We denote by $\# \alpha$ the order of α and by (a, b) the greatest common divisor of integers a and b .

LEMMA 1.2. *In the stable range, we have the following.*

- i) $\#(t_n\omega_n) = (2n+1)!$ for $n \geq 2$;
- ii) $t_n\omega_n\eta = (n, 24)\gamma_{2n-1}\nu$;
- iii) $(24, n+2)\omega_n\nu = 0$.

PROOF. Since $\#(\Sigma^k\gamma_n) = (n+1)!$ for $k \geq 1$ [16] and $\#(j_{2n}t_n\omega_n) = (2n+1)!$ in $\pi_{4n+2}(SU_{2n})$ for $n \geq 2$ ([16], [17]), the James splitting leads us to i).

We consider an exact sequence induced from the fibration $c: Sp_n \rightarrow SU_{2n}$ for $k = 4n+3$:

$$\pi_k(Sp_n) \xrightarrow{c_*} \pi_k(SU_{2n}) \xrightarrow{p_*} \pi_k(X_n) \xrightarrow{\partial} \pi_{k-1}(Sp_n) \xrightarrow{c_*} \pi_{k-1}(SU_{2n}).$$

We know the following facts ([9], [11], [16], [17]): $j_{2n}t_n\omega_n = cj_n(\mathbf{H})\omega_n$; $\pi_{k-1}(Sp_n) = \mathbf{Z}_{a(n)(2n+1)!}\{j_n(\mathbf{H})\omega_n\}$; $\pi_{k-1}(SU_{2n}) = \mathbf{Z}_{(2n+1)!}\{j_{2n}t_n\omega_n\} \oplus \mathbf{Z}_2\{j_{2n}\Sigma\gamma_{2n-1} \circ \eta_{4n}^2\}$; $\pi_k(Sp_n) = \mathbf{Z}_2\{j_n(\mathbf{H})\omega_n\eta_{4n+2}\}$; $\pi_k(SU_{2n}) = \mathbf{Z}_{2(n,12)}\{j_{2n}\Sigma\gamma_{2n-1} \circ \nu_{4n}\}$; $\pi_k(X_n) \cong \mathbf{Z}_{a(n)(n,24)}$, where $a(n) = 1$ or 2 according as n is even or odd. So, for even n , p_* is onto and $c_*\pi_k(Sp_n) \cong 0$ or $\mathbf{Z}_2$ according as $n \equiv 0 \pmod{8}$ or not. For odd n , $c_*\pi_k(Sp_n) \cong \mathbf{Z}_2$. Therefore we have a relation $j_{2n}^*(t_n\omega_n\eta_{4n+2}) = c_*(j_n(\mathbf{H})_*(\omega_n\eta_{4n+2})) = (n, 24)j_{2n}^*(\Sigma\gamma_{2n-1} \circ \nu_{4n})$. So the James splitting leads us to ii).

By [9], we have $\pi_{4n+5}(Sp_n) \cong \mathbf{Z}_{(24,n+2)} \oplus H$, where $H \cong \mathbf{Z}_2$ or 0 according as n is even or odd. As is easily seen, the first direct summand is generated by $j_n(\mathbf{H})\omega_n\nu_{4n+2}$. So, by the James splitting, we have iii). This completes the proof.

Let $[\alpha] \in \pi_k(\tilde{O}_n(F))$ for $\alpha = \Sigma\alpha' \in \pi_k(S^{dn-1})$ be an element satisfying $\tilde{p}'_n(F)[\alpha] = \alpha$, where $\tilde{p}'_n(F) : \tilde{O}_n(F) \rightarrow S^{dn-1}$ is the projection. Let $\alpha'' \in \pi_k(\tilde{Q}_n(F))$ be a coextension of α' , i.e., an element satisfying $\tilde{p}_n(F)\alpha'' = \alpha$. Then we have

$$(4) \quad j_n(F)\alpha'' = [\alpha].$$

We recall that $\pi_5(\Sigma P^2) = \mathbf{Z}\{\zeta_2\}$, $\pi_7(\Sigma P^2) = \mathbf{Z}\{\xi\}$ and $\Sigma\xi = i\nu_4\eta_7$ [16]. Since $\eta_3\nu_4\eta_7^2 = \nu'\eta_6^3 = 0$, we can define a Toda bracket $\{i, \eta_3, \nu_4\eta_7^2\} \subset \pi_{10}(\Sigma P^2)$. The indeterminacy of this bracket is $i_*\pi_{10}(S^3) + \pi_5(\Sigma P^2) \circ \nu_5\eta_8^2 = i_*\pi_{10}(S^3) + \{\zeta_2\nu_5\eta_8^2\}$. We have $2\{i, \eta_3, \nu_4\eta_7^2\} = \{i, \eta_3, \nu_4\eta_7^2\} \circ 2\iota_{10} = -i\{\eta_3, \nu_4\eta_7^2, 2\iota_9\} \subset i_*\pi_{10}(S^3)$ and $p\{i, \eta_3, \nu_4\eta_7^2\} \ni \nu_5\eta_8^2$. So we can take a coextension $\varphi \in \pi_{10}(\Sigma P^2)$ of $\nu_4\eta_7^2$ such that $\varphi \in \{i, \eta_3, \nu_4\eta_7^2\}$ and $\#\varphi = 2$.

Let CX be a cone of X . Let $\hat{\tau} \in \pi_{k+1}(CX, X)$ be an element corresponding to $\tau \in \pi_k(X)$ by the boundary isomorphism $\partial' : \pi_{k+1}(CX, X) \rightarrow \pi_k(X)$. We denote by α' a generator of $\pi_{10}(S^3) \cong \mathbf{Z}_{15}$.

LEMMA 1.3. i) $j_{3*} : \pi_k(\Sigma P^2) \rightarrow \pi_k(SU_3)$ is a split epimorphism for $k = 10$ and 11.

$$\text{ii) } \pi_{10}(\Sigma P^2) = \mathbf{Z}_{24}\{\xi\nu_7\} \oplus \mathbf{Z}_2\{\varphi\} \oplus \mathbf{Z}_{15}\{i\alpha'\}.$$

PROOF. We consider the following exact sequence

$$\pi_{11}(\Sigma P^2) \longrightarrow \pi_{11}(SU_3) \longrightarrow \pi_{11}(SU_3, \Sigma P^2) \xrightarrow{\partial} \pi_{10}(\Sigma P^2) \longrightarrow \pi_{10}(SU_3).$$

By Theorem 2.1 of [2], we have $\pi_{11}(SU_3, \Sigma P^2) = \mathbf{Z}_{24}\{\beta\hat{\nu}_7\}$ and $\partial(\beta\hat{\nu}_7) = \xi\nu_7$, where β is a generator of $\pi_8(SU_3, \Sigma P^2) \cong \mathbf{Z}$.

We have $\pi_{10}(SU_3) = \mathbf{Z}_2\{[\nu_5\eta_8^2]\} \oplus \mathbf{Z}_{15}\{i\alpha'\}$ [10]. By (4), we have $j_{2*}\varphi = [\nu_5\eta_8^2]$. So we have i) for $k = 10$.

Since $\eta_3\nu_4^2 = \nu'\eta_6\nu_7 = 0$, we can define a coextension $\tilde{\nu}_4^2 \in \{i, \eta_3, \nu_4^2\} \subset \pi_{11}(\Sigma P^2)$ of ν_4^2 . We know that $\pi_{11}(SU_3) = \mathbf{Z}_4\{[\nu_5^2]\}$ and $2[\nu_5^2] = i\varepsilon_3$ [10]. So, by (4), we have $j_2\tilde{\nu}_4^2 = [\nu_5^2]$. We have $4\tilde{\nu}_4^2 \in \{i, \eta_3, \nu_4^2\} \circ 4\iota_{11} \subset \{i, \eta_3, 2\nu_4^2\} \circ 2\iota_{11} \subset \{i, \eta_3 \circ \nu_4 \circ 2\nu_7, \nu_7\} \circ 2\iota_{11} = \{i, 0, \nu_7\} \circ 2\iota_{11} \subset \{2i\varepsilon_3\} + \pi_8(\Sigma P^2) \circ 2\nu_8 = 0$. For $\eta_3 \circ \nu_4 \circ 2\nu_7 = \nu' \circ \eta_6 \circ 2\nu_7 = 0$ and $\pi_8(\Sigma P^2) \circ 2\nu_8 = \{\xi\eta_7, \zeta_2\nu_5\} \circ 2\nu_8 = 0$ by [16]. This leads us to the assertion i).

ii) is a direct consequence of i) and the above sequence. This completes the proof.

We recall that $\pi_{15-k}(S^{8-k}) = \mathbf{Z}_{15b(k)}\{\sigma^{(k)}\}$, where $b(k) = 2^{4-k}$ and $\sigma^{(k)} = \sigma', \sigma''$ and σ''' for $k = 1, 2$ and 3, respectively. The following lemma improves Lemma 11.5 of [16].

LEMMA 1.4. *There exists an element $\alpha \in \pi_{4n+2}(\Sigma P^{2n-2})$ for $n \geq 2$ such that $12/(12, n+1)t_n\omega_n = \Sigma\gamma_{2n-1} \circ \eta_{4n}^2 + i\alpha$ and $\#(\Sigma^\infty\alpha) = (2n+1)!(12, n+1)/12$. In particular, $\alpha \equiv \varphi + i\alpha' \pmod{4\xi\nu_7}$ for $n = 2$ and $4\Sigma^2(t_2\omega_2) = \Sigma^3\gamma_3 \circ \eta_{10}^2 + i\sigma'''$.*

PROOF. By Theorem 2.1 of [2], $p_* : \pi_{4n+2}(\Sigma P^{2n-1}, \Sigma P^{2n-2}) \rightarrow \pi_{4n+2}(S^{4n-1})$ is an isomorphism. So we have an exact sequence

$$\pi_{4n+2}(\Sigma P^{2n-2}) \xrightarrow{i_*} \pi_{4n+2}(\Sigma P^{2n-1}) \xrightarrow{p_*} \pi_{4n+2}(S^{4n-1}).$$

$p_*(t_n\omega_n) = (n+1)\nu_{4n-1}$ and $p_*(\Sigma\gamma_{2n-1} \circ \eta_{4n}^2) = \eta_{4n-1}^3 = 12\nu_{4n-1}$. This leads us to the first relation. Since $\pi_4^S(S^0) = 0, i_* : \pi_{4n+1}^S(P^{2n-2}) \rightarrow \pi_{4n+1}^S(P^{2n-1})$ is monomorphic. So, by Lemma 1.2, we have the assertion about $\#(\Sigma^\infty\alpha)$. By [16] and [17], $\#\Sigma^k\omega_2 = \#\Sigma^k t_2\omega_2 = \#\Sigma^{k+1}\gamma_4 = 120$ for $k \geq 0$ and $\pi_9^S(P^2) = \mathbf{Z}_{240}\{i\sigma\}$. So we have $\#\Sigma^k\alpha = 30$ and $\Sigma^2\alpha = i\sigma'''$. By Lemma 1.3, we can take $\alpha \equiv \varphi + i\alpha' \pmod{\xi\nu_7}$. This completes the proof.

Here we recall the group structure of $\pi_k^S(P^n)$ for $k \leq 10$.

REMARK 1 ([16]). i) $\pi_4^S(P^2) = \mathbf{Z}\{\zeta_2\}$; $\pi_5^S(P^2) = \mathbf{Z}_{12}\{i\nu\}$; $\pi_5^S(P^3) = \mathbf{Z}_2\{i\nu\}$; $\pi_6^S(P^2) = 0$; $\pi_6^S(P^3) = \mathbf{Z}\{\zeta_3\}$.

ii) $\pi_7^S(P^2) = \mathbf{Z}_{24}\{\tilde{\nu}\}$, where $\tilde{\nu} \in \langle i, \eta, \nu \rangle$; $\pi_7^S(P^3) = \mathbf{Z}_{24}\{\gamma_3\} \oplus \mathbf{Z}_2\{i\tilde{\nu} \pm \gamma_3\}$ and $2\gamma_3 = \pm i\zeta_2\nu$; $\pi_7^S(P^4) = \mathbf{Z}_2\{i\tilde{\nu}\}$; $\pi_8^S(P^2) = \mathbf{Z}_2\{i\nu^2\}$; $\pi_8^S(P^3) = \mathbf{Z}_2\{i\nu^2\} \oplus \mathbf{Z}_2\{\gamma_3\eta\}$; $\pi_8^S(P^4) = \mathbf{Z}\{\zeta_4\} \oplus \mathbf{Z}_2\{i\nu^2\}$.

iii) $\pi_9^S(P^2) = \mathbf{Z}_{240}\{i\sigma\}$; $\pi_9^S(P^3) = \mathbf{Z}_{80}\{3i\sigma\} \oplus \mathbf{Z}_{72}\{\tilde{\nu}'\}$, where $\tilde{\nu}' \in \langle i, \gamma_2, \nu \rangle$ and $3\tilde{\nu}' \equiv t_2\omega_2 \pmod{2i\sigma}$; $\pi_9^S(P^4) = \mathbf{Z}_{80}\{3i\sigma\} \oplus \mathbf{Z}_8\{15(\gamma_4 - 2i\sigma)\} \oplus \mathbf{Z}_9\{8i\tilde{\nu}'\}$; $\pi_9^S(P^5) = \mathbf{Z}_8\{i\sigma\} \oplus \mathbf{Z}_3\{8i\tilde{\nu}'\}$.

iv) $\pi_{10}^S(P^2) = \mathbf{Z}_4\{\tilde{\nu}\nu\}$ and $2\tilde{\nu}\nu = i\varepsilon$; $\pi_{10}^S(P^3) = \mathbf{Z}_4\{i\tilde{\nu}\nu\}$ and $\gamma_3\nu = \pm i\tilde{\nu}\nu$; $\pi_{10}^S(P^4) = 0$; $\pi_{10}^S(P^5) = \mathbf{Z}\{\zeta_5\}$.

We consider the standard cofibre sequence [20].

$$(5) \quad \mathbf{C}P^{2n-1} \xrightarrow{h_{n-1}} \mathbf{H}P^{n-1} \xrightarrow{f_{n-1}} Q_n \xrightarrow{t_n} \Sigma\mathbf{C}P^{2n-1} \longrightarrow \dots$$

As is easily seen, we have $\pi_8^S(Q_2) = \mathbf{Z}_2\{\tilde{\eta}'\}$ and $\pi_9^S(Q_2) = \mathbf{Z}_2\{\tilde{\eta}'\eta\} \oplus \mathbf{Z}_2\{i\nu^2\}$, where $\tilde{\eta}'$ is an element such that $\tilde{p}_{2*}\tilde{\eta}' = \eta$. By (1), we have $h_{1*}\gamma_3 = \nu$ and so, $h_{1*} : \pi_7^S(P^3) \rightarrow \pi_7^S(S^4)$ is a split epimorphism. Therefore we have $t_2\tilde{\eta}' = \gamma_3 \pm i\tilde{\nu}$ and $t_{2*}(\tilde{\eta}'\eta) = (\gamma_3 \pm i\tilde{\nu}) \circ \eta = \gamma_3\eta + i\nu^2$, because $\tilde{\nu}\eta \in \langle i, \eta, \nu \rangle \circ \eta = -i\langle \eta, \nu, \eta \rangle \ni i\nu^2 \pmod{0}$. From the relation $\tilde{p}_{2*}(\tilde{\eta}'\eta^2 - 4\omega_2) = 0$, we have $\tilde{\eta}'\eta^2 = 4\omega_2 + 8a i\sigma$ for $a = 0$ or 1 . By Lemma 1.4, we have $a = 1$. Hence we have proved the following

LEMMA 1.5. $t_2\tilde{\eta}' = \gamma_3 \pm i\tilde{\nu}$ and $\tilde{\eta}'\eta^2 = 4\omega_2 + 8i\sigma$.

LEMMA 1.6. *There exist elements $\alpha \in \pi_{12}(\Sigma P^4)$ and $\beta \in \pi_{12}(\Sigma P^3)$ such that $2\Sigma\gamma_5 = i_4\alpha$ and $6\alpha = i_3\beta$. In the stable range, we have $\#\alpha = 360$ and $\#\beta = 60$.*

PROOF. We consider a natural map between exact sequences:

$$\begin{array}{ccccccc}
 \pi_{12}(\Sigma P^4) & \xrightarrow{i_4^*} & \pi_{12}(\Sigma P^5) & \xrightarrow{p_*} & \pi_{12}(S^{11}) & & \\
 \downarrow j_{5*} & & \downarrow j_{6*} & & \parallel & & \\
 0 & \longrightarrow & \pi_{12}(SU_5) & \xrightarrow{i_*'} & \pi_{12}(SU_6) & \xrightarrow{p_*'} & \pi_{12}(S^{11}) \longrightarrow 0.
 \end{array}$$

Since $p_*(\Sigma\gamma_5) = \eta_{11}$, there exists an element $\alpha \in \pi_{12}(\Sigma P^5)$ such that $2\Sigma\gamma_5 = i_4\alpha$. Since $\gamma_4\eta^2 = 0$, $i_4^* : \pi_{11}^S(P^4) \rightarrow \pi_{11}^S(P^5)$ is monomorphic. Therefore we have $\#(\Sigma^\infty\alpha) = \#(2\Sigma^\infty\gamma_5) = 360$. Since $\pi_{12}(SU_5) \cong \mathbf{Z}_{360}$, $\pi_{12}(SU_6) = \mathbf{Z}_{720}\{j_6\Sigma\gamma_5\}$ and j_{k*} is an epimorphism for $k = 5$ and 6 [16], the above diagram shows that $j_{5*}\alpha$ is a generator of $\pi_{12}(SU_5)$.

By use of [2], $\pi_{12}(\Sigma P^4, \Sigma P^3) \cong \pi_{12}(S^9)$. So we have a commutative diagram between exact sequences:

$$\begin{array}{ccccccc}
 \pi_{12}(\Sigma P^3) & \xrightarrow{i_3^*} & \pi_{12}(\Sigma P^4) & \xrightarrow{p_*} & \pi_{12}(S^9) & \xrightarrow{\partial} & \pi_{11}(\Sigma P^3) \\
 \downarrow j_{4*} & & \downarrow j_{5*} & & \parallel & & \downarrow j_{4*} \\
 0 & \longrightarrow & \pi_{12}(SU_4) & \xrightarrow{i_*'} & \pi_{12}(SU_5) & \xrightarrow{p_*'} & \pi_{12}(S^9) \xrightarrow{\partial'} \pi_{11}(SU_4).
 \end{array}$$

We recall that $\pi_{12}(SU_4) \cong \pi_{12}(SU_3) = \mathbf{Z}_{60}\{[\sigma''']\}$ [10] and $\pi_{11}(SU_4) = \mathbf{Z}_4\{j_4\Sigma\gamma_3 \circ \nu_8\}$ [16]. So we have $\pi_{12}(SU_5) = \mathbf{Z}_{360}\{[4\nu_9]\}$ and $6[4\nu_9] = i_*[\sigma''']$. Since $\#(\Sigma^n\gamma_3 \circ \nu_{n+7}) = 4$ for $n \geq 1$ [16], we can take $\alpha \in \{i, \Sigma\gamma_3, 4\nu_8\} \subset \pi_{12}(\Sigma P^4)$ such that $p_*\alpha = 4\nu_9$ and $j_5\alpha = [4\nu_9]$ by (4). By the above diagram, there exists an element $\beta \in \pi_{12}(\Sigma P^3)$ such that $6\alpha = i_3\beta$ and $j_4\beta = i[\sigma''']$. Since $\pi_{11}^S(S^7) = 0$, $i_3^* : \pi_{11}^S(P^3) \rightarrow \pi_{11}^S(P^4)$ is monomorphic. Hence we have $\#(\Sigma^\infty\beta) = \#(6\Sigma^\infty\alpha) = 60$. This completes the proof.

§2. Determination of $\pi_k^S(CP)$ for $11 \leq k \leq 14$

We consider a cofibre sequence starting with γ_n and the exact sequence induced from it for $m = k + 2n + 1$:

$$(6) \quad S^{2n+1} \xrightarrow{\gamma_n} p^n \xrightarrow{i} p^{n+1} \xrightarrow{p} S^{2n+2} \longrightarrow \dots;$$

$$(6)_{n,m} \quad \pi_m^S(S^{2n+1}) \xrightarrow{\gamma_n^*} \pi_m^S(p^n) \xrightarrow{i_*} \pi_m^S(p^{n+1}) \xrightarrow{p_*} \pi_m^S(S^{2n+2}) \longrightarrow \dots.$$

First we deal with the exact sequence $(6)_n = (6)_{n,11}$. In $(6)_1$, $\text{Im } \gamma_{1*} = \eta\pi_8^S(S^0) = (\mathbf{Z}_2)^2\{\eta^2\sigma, \eta\bar{\nu}\}$ and $p\zeta_2\sigma = 2\sigma$. So we have $120\zeta_2\sigma \in \langle i, \eta, 2\iota \rangle \circ 120\sigma = -i\langle \eta, 2\iota, 120\sigma \rangle \ni i\mu \pmod{0}$. Therefore we have $120\zeta_2\sigma = i\mu$ and $\pi_{11}^S(P^2) = \mathbf{Z}_{240}\{\zeta_2\sigma\}$.

In $(6)_2$, $\gamma_2\nu^2 = 2i\nu^3 = 0$. So we have $\pi_{11}^S(P^3) = \mathbf{Z}_{240}\{i\zeta_2\sigma\}$. In Lemma 1.6, we can take $\beta = 4i\zeta_2\sigma$.

In $(6)_3$, $\#(\gamma_3\nu) = 4$. In Lemma 1.6, α is taken as a coextension of 4ν and $6\alpha = 4i\zeta_2\sigma$. So we have $\pi_{11}^S(P^4) = (\mathbf{Z}_{80} \oplus \mathbf{Z}_{18})\{\alpha, i\zeta_2\sigma\}$.

In $(6)_4$, $\gamma_4\eta^2 = 0$ and $p\gamma_5 = \eta$. So, by Lemma 1.6, we have $2\gamma_5 = i\alpha$, $12\gamma_5 = 4i\zeta_2\sigma$ and $\pi_{11}^S(P^5) = \mathbf{Z}_{720}\{\gamma_5\} \oplus \mathbf{Z}_4\{i\zeta_2\sigma - 3\gamma_5\}$. By $(6)_5$, we have $\pi_{11}^S(P^6) = \mathbf{Z}_4\{i\zeta_2\sigma\}$. Hence we have proved the following

- PROPOSITION 2.1. i) $\pi_{11}^S(P^2) = \mathbf{Z}_{240}\{\zeta_2\sigma\}$ and $\pi_{11}^S(P^3) = \mathbf{Z}_{240}\{i\zeta_2\sigma\}$.
 ii) $\pi_{11}^S(P^4) = (\mathbf{Z}_{80} \oplus \mathbf{Z}_{18})\{\alpha, i\zeta_2\sigma\}$, where $\alpha \in \langle i, \gamma_3, 4\nu \rangle$ and $6\alpha = 4i\zeta_2\sigma$.
 iii) $\pi_{11}^S(P^5) = \mathbf{Z}_{720}\{\gamma_5\} \oplus \mathbf{Z}_4\{i\zeta_2\sigma - 3\gamma_5\}$ and $\pi_{11}^S(P^6) = \mathbf{Z}_4\{i\zeta_2\sigma\}$, where $2\gamma_5 = i\alpha$.

We recall that $\pi_{10}^S(S^0) = \mathbf{Z}_2\{\eta\mu\} \oplus \mathbf{Z}_3\{\beta_1\}$. By $(6)_{n,12}$ for $1 \leq n \leq 3$, we have the following

PROPOSITION 2.2. $\pi_{12}^S(P^2) = \mathbf{Z}_3\{i\beta_1\}$, $\pi_{12}^S(P^3) = \mathbf{Z}_2\{t_2\omega_2\nu\} \oplus \mathbf{Z}_3\{i\beta_1\}$ and $\pi_{12}^S(P^4) \cong \pi_{12}^S(P^3)$.

By use of $(6)_{n,12}$ for $n = 4$ and 5 , we have the following

- PROPOSITION 2.3. i) $\gamma_4\nu = -it_2\omega_2\nu$ and $\pi_{12}^S(P^5) = \mathbf{Z}_2\{\gamma_5\eta\} \oplus \mathbf{Z}_3\{i\beta_1\}$.
 ii) $\pi_{12}^S(P^6) = \mathbf{Z}\{\zeta_6\} \oplus \mathbf{Z}_3\{i\beta_1\}$.

We denote by pG the p -component of a finitely generated abelian group G .

Next we use $(6)_n = (6)_{n,13}$. In $(6)_1$, $\eta^2\mu = 252\zeta$ and $\eta \circ \eta\varepsilon = \eta \circ \nu^3 = 0$. We have $p_*\tilde{\nu}\nu^2 = \nu^3$ and $\#(\tilde{\nu}\nu^2) = 2$. Let $\tilde{\eta}\varepsilon \in \langle i, \eta, \eta\varepsilon \rangle$ be a coextension of $\eta\varepsilon$. Then $2\tilde{\eta}\varepsilon \in \langle i, \eta, \eta\varepsilon \rangle \circ 2\iota = -i\langle \eta, \eta\varepsilon, 2\iota \rangle \ni -i\zeta \bmod 2i\zeta$. So, by a suitable choice of $\tilde{\eta}\varepsilon$, we have $2\tilde{\eta}\varepsilon = i\zeta$ and $\pi_{13}^S(P^2) = \mathbf{Z}_{504}\{\tilde{\eta}\varepsilon\} \oplus \mathbf{Z}_2\{\tilde{\nu}\nu^2\}$.

In $(6)_2$, $\gamma_2\pi_8^S(S^0) = 0$ and $\gamma_2\sigma = 2i\nu\sigma = 0$. Let $\tilde{\sigma}' \in \langle i, \gamma_2, \sigma \rangle$ be a coextension of σ . Then $240\tilde{\sigma}' \in -i\langle \gamma_2, \sigma, 240\iota \rangle \supset 2i\langle \pm\nu, \sigma, 240\iota \rangle \ni 2ai\zeta = 4ai\tilde{\eta}\varepsilon \bmod 240i_*\pi_{13}^S(P^2)$ for odd a .

In the 3-component, by (13.8) of [25], we have $\langle \alpha_1, \alpha_2, 3\iota \rangle \ni \alpha'_3 \bmod 3\alpha'_3 = \alpha_3$, where $\alpha'_3 = 56\zeta$. So we have $\pi_{13}^S(P^3) = \mathbf{Z}_{30240}\{\tilde{\sigma}'\} \oplus \mathbf{Z}_4\{bi\tilde{\eta}\varepsilon - 60\tilde{\sigma}'\} \oplus \mathbf{Z}_2\{i\tilde{\nu}\nu^2\}$ for odd b .

In $(6)_3$, $\gamma_3\nu^2 = i\tilde{\nu}\nu^2$ and so, $\pi_{13}^S(P^4) = \mathbf{Z}_{30240}\{\tilde{\sigma}'\} \oplus \mathbf{Z}_4\{bi\tilde{\eta}\varepsilon - 60i\tilde{\sigma}'\}$.

By $(6)_4$ and Proposition 2.3, $\gamma_4\nu = -it_2\omega_2\nu = -3i\tilde{\nu}'\nu$. By Lemma 1.2, $4\gamma_4\nu = -4it_2\omega_2\nu = 0$. Let $\tilde{\alpha}_1 \in \pi_{13}^S(P^5)$ be a coextension of $\alpha_1 = 8\nu$. Then, by (13.8) of [25] and by Lemma 1.4, we have $24\tilde{\alpha}_1 \in \langle i', \gamma_4, \alpha_1 \rangle \circ 24\iota \subset -i'\langle 8\gamma_4, \alpha_1, 3\iota \rangle \supset -i''\langle \alpha_2, \alpha_1, 3\iota \rangle \ni -i''\alpha'_3 = \pm 240i\tilde{\alpha}' \bmod 3i_*\pi_{13}^S(P^4)$. So, by a suitable choice of $\tilde{\alpha}_1$, we have ${}^3\pi_{13}^S(P^5) = \mathbf{Z}_{27}\{1120i\tilde{\sigma}'\} \oplus \mathbf{Z}_3\{\tilde{\alpha}_1\}$.

To determine the group extension in the 2-component, we examine the group structure of ${}^2\pi_{14}^S(Q_n)$ for $n = 2$ and 3 . By use of a cofibre sequence starting with

$\nu' : S^6 \rightarrow S^3$, we have an exact sequence

$$\pi_{14}^S(S^6) \xrightarrow{\nu'_*} \pi_{14}^S(S^3) \xrightarrow{i_*} \pi_{14}^S(Q_2) \xrightarrow{p_*} \pi_{14}^S(S^7) \xrightarrow{\nu'_*} \dots$$

Both ν'_* are trivial since $2\nu \circ \eta\sigma = 0$, $2\nu \circ \varepsilon = 0$ and $2\nu \circ \sigma = 0$. Let $\tilde{\sigma} \in \pi_{14}^S(Q_2)$ be a coextension of σ . Then $16\tilde{\sigma} \in \langle i, 2\nu, \sigma \rangle \circ 16\iota = -i\langle 2\nu, \sigma, 16\iota \rangle \ni -2xi\zeta \bmod 8i_*\pi_{11}^S(S^0) = 0$ for odd x . So we have the following

LEMMA 2.4. $16\tilde{\sigma} = -2xi\zeta$ for odd x and ${}^2\pi_{14}^S(Q_2) = \mathbf{Z}_{64}\{\tilde{\sigma}\} \oplus \mathbf{Z}_2\{xi\zeta + 8\tilde{\sigma}\}$.

We can take $\tilde{\sigma}' = t_2\tilde{\sigma}$.

We denote by $M^n = \Sigma^{n-2}RP^2 = S^{n-1} \cup_{2\iota_{n-1}} e^n$ be the $\mathbf{Z}_2$ -Moore space. We recall that $\nu' \in \{\eta_3, 2\iota_4, \eta_5\}_1 \bmod 2\nu'$. So we have $\nu' = \tilde{\eta}_3\tilde{\eta}_4$. Here $\tilde{\eta}_3 \in \{\eta_3, 2\iota_4, p\} \subset [M^5, S^3]$ and $\tilde{\eta}_4 \in \{i, 2\iota_4, \eta_4\} \subset \pi_6(M^5)$ are an extension of η_3 and a coextension of η_4 , respectively. We set $\tilde{\eta}_n = \Sigma^{n-3}\tilde{\eta}_3$, $\tilde{\eta}_n = \Sigma^{n-3}\tilde{\eta}_3$ for $n \geq 3$, $\tilde{\eta} = \Sigma^\infty\tilde{\eta}_3$ and $\tilde{\eta} = \Sigma^\infty\tilde{\eta}_3$, where $\tilde{\eta}_3 \in \{i, 2\iota_3, \eta_3\} \subset \pi_5(M^4)$ is a coextension of η_3 . Let $\overline{\nu}_5^2 \in \{\nu_5^2, 2\iota_{11}, p\}$ be an extension of ν_5^2 , $\overline{\nu}_n^2 = \Sigma^{n-5}\nu_5^2$ for $n \geq 5$ and $\overline{\nu}^2 = \Sigma^\infty\overline{\nu}_5^2$.

LEMMA 2.5. $\langle \eta, \nu, \tilde{\eta} \rangle \ni \overline{\nu}^2 \bmod 0$ and $\overline{\nu}^2 \circ \tilde{\eta} = \varepsilon$.

PROOF. We improve the technique of [15]. Since $\nu_7\tilde{\eta}_{10} \ni -\{\nu_7, \eta_{10}, 2\iota_{11}\}p \ni 0 \bmod 0$, we define a bracket $\{\eta_6, \nu_7, \tilde{\eta}_{10}\} \subset [M^{13}, S^6]$. By use of a cofibre sequence starting with $2\iota_{12}$, we have $\{\eta_6, \nu_7, \tilde{\eta}_{10}\} \ni \overline{\nu}_6^2 \bmod \sigma''p$. So we have $\langle \eta, \nu, \tilde{\eta} \rangle \supset \Sigma^\infty\{\eta_6, \nu_7, \tilde{\eta}_{10}\} \ni \overline{\nu}^2 \bmod \eta\{M^6, S^0\} = \eta\{\nu^2p\} = 0$. This leads us to the first relation.

Since $\pi_{13}(S^5) = \mathbf{Z}_2\{\varepsilon_3\}$ and $\langle \nu^2, 2\iota, \eta \rangle \ni \varepsilon$, we have $\{\nu_5^2, 2\iota_{11}, \eta_{11}\} \ni \varepsilon_5 \bmod 0$. So we have $\overline{\nu}_5^2 \circ \tilde{\eta}_{11} = \varepsilon_5$ and $\overline{\nu}^2 \circ \tilde{\eta} = \varepsilon$. This completes the proof.

We set $Q_k^n = Q_n/Q_{k-1}$. Since $Q_2^3 = S^7 \cup_{3\nu} e^{11}$, it is 2-equivalent to $\Sigma^3\mathbf{HP}^2$. In the stable range, we have a cellular decomposition $Q_3 = S^3 \cup_{2\nu} C(\Sigma^{-1}Q_2^3)$, where $\overline{2\nu} \in \langle 2\nu, \nu, p' \rangle$ for the collapsing map $p' : Q_2^3 \rightarrow S^{11}$. Since $\tilde{\eta}\nu \in -i\langle 2\iota, \eta, \nu \rangle \ni 0 \bmod 0$, we can define an extension $\tilde{\tilde{\eta}} \in \langle \tilde{\eta}, \nu, p' \rangle \subset \{Q_2^3, M^5\}$ of $\tilde{\eta}$. We take $\overline{2\nu} = \tilde{\tilde{\eta}} \circ \tilde{\eta}$.

By use of an exact sequence

$$0 \longrightarrow \pi_{11}^S(S^4) \xrightarrow{i_*} \pi_{11}^S(\mathbf{HP}^2) \xrightarrow{p'_*} \pi_{11}^S(S^8) \xrightarrow{\nu_*} \pi_{11}^S(S^5) \longrightarrow 0,$$

we have ${}^2\pi_{14}^S(Q_2^3) = \mathbf{Z}_4\{\widetilde{2\nu}\} \oplus \mathbf{Z}_{16}\{i\sigma\}$, where $\widetilde{2\nu} \in \langle i, \nu, 2\nu \rangle$. Since $\nu\tilde{\eta} \in -\langle \nu, \eta, 2\iota \rangle \circ p \ni 0 \bmod 0$, we can define a coextension $\tilde{\tilde{\eta}} \in \langle i, \nu, \tilde{\eta} \rangle \subset \{M^{13}, Q_2^3\}$ of $\tilde{\eta}$. We take $\widetilde{2\nu} = \tilde{\tilde{\eta}} \circ \tilde{\eta}$. We consider an exact sequence

$$\dots \longrightarrow \pi_{14}^S(S^3) \xrightarrow{i_*} \pi_{14}^S(Q_3) \xrightarrow{\tilde{p}'_*} \pi_{14}^S(Q_2^3) \xrightarrow{\overline{2\nu}_*} \pi_{14}^S(S^4) \longrightarrow \dots$$

Since $\overline{2\nu} \circ \widetilde{2\nu} \ni \langle 2\nu, \nu, 2\nu \rangle \ni 0 \bmod 0$, we define a coextension $\widetilde{2\nu}' \in \langle i, \overline{2\nu}, \widetilde{2\nu} \rangle$ of $\widetilde{2\nu}$. We have $42\widetilde{2\nu}' \in -i\langle \overline{2\nu}, \widetilde{2\nu}, 4\iota \rangle$ and $2\langle \overline{2\nu}, \widetilde{2\nu}, 4\iota \rangle \subset \langle 2\tilde{\eta} \circ \tilde{\tilde{\eta}}, \tilde{\tilde{\eta}} \circ \tilde{\eta}, 4\iota \rangle \subset \langle \eta^2p, \tilde{\tilde{\eta}} \circ$

$\widetilde{\eta} \circ \widetilde{\eta}, 4\iota\rangle \subset \langle \eta^2, \langle \eta, \nu, \widetilde{\eta} \rangle \widetilde{\eta}, 4\iota \rangle$. By Lemma 2.5, this is equal to $\langle \eta^2, \overline{\nu^2} \circ \widetilde{\eta}, 4\iota \rangle = \langle \eta^2, \varepsilon, 4\iota \rangle \supset 2\langle \eta^2, \varepsilon, 2\iota \rangle \ni 2\zeta \bmod 4\zeta$. Therefore we have $\langle \widetilde{2\nu}, \widetilde{2\nu}, 4\iota \rangle \ni \zeta \bmod 2\zeta$ and $4\widetilde{2\nu}' \equiv -i\zeta \bmod 2i\zeta$. So we can take $4\widetilde{2\nu}' = -xi\zeta$ and $8\widetilde{2\nu}' = -2xi\zeta = 16i\widetilde{\sigma}$. We set $\widetilde{2\nu}'' = \widetilde{2\nu}' - 2i\widetilde{\sigma}$.

LEMMA 2.6. i) $4\widetilde{2\nu}'' = -(xi\zeta + 8i\widetilde{\sigma})$ and ${}^2\pi_{14}^S(Q_3) = \mathbf{Z}_{64}\{i\widetilde{\sigma}\} \oplus \mathbf{Z}_8\{\widetilde{2\nu}''\}$.
 ii) $315\omega_3 = 2yi\widetilde{\sigma} + 2z\widetilde{2\nu}''$ for odd integers y and z .

PROOF. By an exact sequence

$$0 \longrightarrow \pi_{14}^S(Q_2) \xrightarrow{i_*} \pi_{14}^S(Q_3) \xrightarrow{\widetilde{p}_{3*}} \pi_{14}^S(S^{11}) \xrightarrow{\omega_{2*}} \pi_{13}^S(Q_2)$$

and by Lemma 2.4, we have i).

We know $\#\omega_3 = 2.7!$ and $\widetilde{p}_{3*}\omega_3 = 4\nu$. So, by i), we have ii). This completes the proof.

The following is easily checked.

LEMMA 2.7. $\pi_{13}^S(\mathbf{H}P^2) = \mathbf{Z}_2\{i\eta\varepsilon\} \oplus \mathbf{Z}_2\{i\mu\}$ and $\pi_{14}^S(\mathbf{H}P^2) = \mathbf{Z}_2\{i\eta\mu\} \oplus \mathbf{Z}_3\{i\beta_1\}$.

In an exact sequence induced from (5) and containing the above groups, we have $f_2(i\mu) = if_1\mu = i\eta\mu$ and $f_2(i\eta\mu) = i\eta^2\mu = 4i\zeta = 32i\widetilde{\sigma}$ since $f_1 = i\eta$. We have $h_2\gamma_5\eta^2 = \gamma_2(\mathbf{H})\eta^2 = i\eta\varepsilon$, because $\Sigma\gamma_2(\mathbf{H}) \circ \eta_{12} \in \{i, \nu_5, 2\nu_8\} \circ \eta_{12} = -i\{\nu_5, 2\nu_8, \eta_{11}\} \ni i\varepsilon_5 \bmod 0$. So we have ${}^2\pi_{13}^S(P^5) = \mathbf{Z}_{32}\{i\widetilde{\sigma}'\} \oplus \mathbf{Z}_8\{\widetilde{2\nu}'''\} \oplus \mathbf{Z}_2\{\gamma_5\eta^2\}$. Here we set $\widetilde{2\nu}''' = t_3\widetilde{2\nu}''$.

By (6)₅, ${}^2\pi_{13}^S(P^6) = \mathbf{Z}_{32}\{i\widetilde{\sigma}'\} \oplus \mathbf{Z}_8\{i'\widetilde{2\nu}'''\}$. By (1), $h_2i\widetilde{\eta}\varepsilon = ip_2\widetilde{\eta}\varepsilon = i\eta\varepsilon$ and by Lemma 1.2, $\#(t_3\omega_3) = 7!$. Therefore, by Lemma 1.4 and (6)₄, we have

$$(7) \quad \gamma_5\eta^2 = 3t_3\omega_3 + ai\widetilde{\eta}\varepsilon + 2bi\widetilde{\sigma}' \quad \text{for odd integers } a \text{ and } b.$$

By Lemma 2.6, we have $315\gamma_6 = 2yi\widetilde{\sigma}' + 2zi'\widetilde{2\nu}'''$. So, by (6)₆, we have ${}^2\pi_{13}^S(P^7) = \mathbf{Z}_8\{i'\widetilde{2\nu}'''\} \oplus \mathbf{Z}_2\{i\widetilde{\sigma}'\}$. Hence we have the following

PROPOSITION 2.8. i) $\pi_{13}^S(P^5) = \mathbf{Z}_{30240}\{i\widetilde{\sigma}'\} \oplus \mathbf{Z}_{24}\{\widetilde{2\nu}'''\} \oplus \mathbf{Z}_2\{\gamma_5\eta^2\}$.
 ii) $\pi_{13}^S(P^6) = \mathbf{Z}_{30240}\{i\widetilde{\sigma}'\} \oplus \mathbf{Z}_{24}\{i'\widetilde{2\nu}'''\}$.
 iii) $\pi_{13}^S(P^7) = \mathbf{Z}_{24}\{i'\widetilde{2\nu}'''\} \oplus \mathbf{Z}_6\{i\widetilde{\sigma}'\}$.

Finally we shall determine $\pi_{14}^S(P)$ by use of (6)_n = (6)_{n,14}. By use of (6)₁, we have $\pi_{14}^S(P^2) = \mathbf{Z}_3\{\zeta_2\beta_1\}$.

In (6)₂, $\gamma_2\pi_9^S(S^0) = 0$, $p\gamma_3\sigma = \eta\sigma$ and $p\widetilde{\nu} = \overline{\nu}$, where $\widetilde{\nu} \in \{i, \gamma_2, \overline{\nu}\}$. So we have $\pi_{14}^S(P^3) = \mathbf{Z}_2\{\widetilde{\nu}\} \oplus \mathbf{Z}_2\{\gamma_3\sigma\} \oplus \mathbf{Z}_3\{i\zeta_2\beta_1\}$.

In (6)₃, $\gamma_3\nu^2 = i\widetilde{\nu}\nu^2 \neq 0$. So we have $\pi_{14}^S(P^4) = \mathbf{Z}_2\{i\widetilde{\nu}\} \oplus \mathbf{Z}_3\{i\zeta_2\beta_1\}$. By (6)₄, we have $\pi_{14}^S(P^5) \cong \pi_{14}^S(P^4)$.

LEMMA 2.9. $\gamma_5\nu = \pm i\zeta_2\beta_1$.

PROOF. By Lemma 1.2, Proposition 2.1 and (7), we have $3\gamma_5\nu = t_3\omega_3\eta = \gamma_5\eta^3 + ai\widetilde{\eta}\varepsilon\eta = 12\gamma_5\nu = 4i\zeta_2\sigma\nu = 0$ since $\widetilde{\eta}\varepsilon\eta \in -i\langle\eta, \eta\varepsilon, \eta\rangle \subset i_*\pi_{12}^S(S^0) = 0$. We set $\gamma_5\nu = ci\zeta_2\beta_1$ for $c \in \mathbf{Z}_3$. We have $h_2\gamma_5\nu = \gamma_2(\mathbf{H})\nu \in -i\langle\nu, 2\nu, \nu\rangle = \pm i\langle\alpha_1, \alpha_1, \alpha_1\rangle \ni \pm i\beta_1 \bmod 0$. Therefore we have $\pm i\beta_1 = h_2\gamma_5\nu = ch_2i\zeta_2\beta_1 = aip_2(\mathbf{C})\zeta_2\beta_1 = 2ci\beta_1$. Hence c is a generator of $\mathbf{Z}_3$. This completes the proof.

By Lemma 2.9 and (6)_k for $k = 5$ and 6 , we have the following

PROPOSITION 2.10. $\pi_{14}^S(P^6) = \mathbf{Z}_2\{\widetilde{i\nu}\}$ and $\pi_{14}^S(P^7) = \mathbf{Z}\{\zeta_7\} \oplus \mathbf{Z}_2\{\widetilde{i\nu}\}$.

§3. Determination of $\pi_k^S(CP)$ for $k = 15$ and 16

By use of (6)_n = (6)_{n,15} for $n = 1$ and 2 , we have the following facts: $\pi_{15}^S(P^2) = \mathbf{Z}_{504}\{\widetilde{\zeta}\} \oplus \mathbf{Z}_3\{i\alpha_1\beta_1\}$; $\pi_{15}^S(P^3) = \mathbf{Z}_{504}\{i\widetilde{\zeta}\} \oplus \mathbf{Z}_2\{\widetilde{\mu}\} \oplus \mathbf{Z}_2\{\gamma_3\eta\sigma\} \oplus \mathbf{Z}_2\{\gamma_3\bar{\nu}\}$, where $\widetilde{\zeta} \in \langle i, \eta, \zeta \rangle$ and $\widetilde{\mu} \in \langle i, \gamma_2, \mu \rangle$. $\widetilde{\mu}$ is taken satisfying $\#\widetilde{\mu} = 2$ since $2\widetilde{\mu} \in -i\langle\gamma_2, \mu, 2\mu\rangle \supset \pm i\nu\langle 2\mu, \mu, 2\mu \rangle \ni i\nu\eta\mu = 0 \bmod 2i_*\pi_{15}^S(P^2)$.

In (6)₃, $\gamma_3\sigma \neq 0$ and $2\gamma_3\sigma = \pm i\zeta_2\nu\sigma = 0$. So we can define a coextension $\widetilde{2\sigma'} \in \langle i, \gamma_3, 2\sigma \rangle$ of 2σ . $120\widetilde{2\sigma'} \in -i\langle\gamma_3, 2\sigma, 120\mu\rangle$. In the 2-component, $2\langle\gamma_3, 2\sigma, 8\mu\rangle \subset \langle \pm i\zeta_2\nu, 2\sigma, 8\mu \rangle \supset \pm i\zeta_2\langle\nu, \sigma, 16\mu\rangle \ni \pm i\zeta_2\zeta = \pm 2i\widetilde{\zeta} \bmod i\zeta_2\nu\pi_8^S(S^0) + 8\pi_{15}^S(P^3) = 0$. So, by the fact that $p\langle\gamma_3, 2\sigma, 8\mu\rangle \subset \langle\eta, 2\sigma, 8\mu\rangle \ni \mu$, we take $82\widetilde{2\sigma'} = i\widetilde{\zeta} + i\widetilde{\mu}$ and $\#\widetilde{2\sigma'} = 64$. Therefore we have ${}^2\pi_{15}^S(P^4) = \mathbf{Z}_{64}\{\widetilde{2\sigma'}\} \oplus \mathbf{Z}_2\{i\widetilde{\mu}\}$.

LEMMA 3.1. i) $\langle\gamma_4, \eta, \nu\rangle \ni 0 \bmod i\zeta_2\beta_1$.

ii) $\gamma_4\nu^2 = 0$.

PROOF. By Lemma 2.9 and Proposition 2.1, we have $\pm i\zeta_2\beta_1 = 2\gamma_5\nu = i\alpha\nu$. On the other hand, $\gamma_5\nu \in \langle i, \gamma_4, \eta \rangle \circ \nu = -i\langle\gamma_4, \eta, \nu\rangle \bmod i_*\pi_{11}^S(P^4) \circ \nu \subset \{i\zeta_2\sigma\nu, i\alpha\nu\} = \{2\gamma_5\nu\}$. This leads us to i).

We have $\gamma_4\nu^2 \in \gamma_4\langle\eta, \nu, \eta\rangle = -\langle\gamma_4, \eta, \nu\rangle \circ \eta \ni 0 \bmod 0$ by i). This completes the proof.

REMARK 2. We give another proof of the fact: $\langle\gamma_4, \eta, \nu\rangle \not\ni i\widetilde{\nu}$. By (3), $g_5\langle\gamma_4, \eta, \nu\rangle = -\langle g_5, \gamma_4, \eta \rangle \circ \nu \subset \pi_{12}(S^0) \circ \nu = 0$. We take $\widetilde{\nu'} \in \langle i, 2\nu, \bar{\nu} \rangle \subset \pi_{15}^S(Q_2)$ such that $\widetilde{\nu} = t_2\widetilde{\nu'}$. Then, by (2), we have $g_5i\widetilde{\nu} = g_4t_2\widetilde{\nu'} = g_2(\mathbf{H})\widetilde{\nu'} \in \langle\nu, 2\nu, \bar{\nu}\rangle \ni \eta\kappa \bmod 0$. This leads us to the assertion.

By (6)₄ and Lemma 3.1, we have $\pi_{15}^S(P^5) \cong \pi_{15}^S(P^4)$.

To determine $2\pi_{15}^S(P^6)$, we prepare a few lemmas.

LEMMA 3.2. i) $\pi_{15}^S(Q_2) = \mathbf{Z}_2\{\widetilde{\eta'\sigma}\} \oplus \mathbf{Z}_2\{\widetilde{\nu'}\}$, $\pi_{15}^S(Q_3) \cong \pi_{15}^S(Q_2)$ and $\pi_{15}^S(Q_4) \cong \mathbf{Z} \oplus \pi_{15}^S(Q_3)$.

ii) $\pi_{16}^S(Q_2) = \mathbf{Z}_2\{\widetilde{\eta'\eta\sigma}\} \oplus \mathbf{Z}_2\{\widetilde{\eta'\bar{\nu}}\} \oplus \mathbf{Z}_2\{\widetilde{\mu'}\}$, $\pi_{16}^S(Q_3) = \mathbf{Z}_2\{i\widetilde{\eta'\eta\sigma}\} \oplus \mathbf{Z}_2\{i\widetilde{\mu'}\}$

and $\pi_{16}^S(Q_4) = \mathbf{Z}_2\{i\tilde{\eta}'\eta\sigma\} \oplus \mathbf{Z}_2\{i\tilde{\mu}'\} \oplus \mathbf{Z}_2\{\tilde{\eta}''\}$, where $\tilde{\mu}' \in \langle i, 2\nu, \mu \rangle$ and $\tilde{\eta}'' \in \langle i, \omega_3, \eta \rangle$.

PROOF. The assertion of the first two of i) and the first of ii) are obvious. By Lemma 2.6, $\omega_3\eta = 2(i\tilde{\sigma} + \widetilde{2\nu''}) \circ \eta = 0$. So we have the third of i).

Since $\tilde{p}_2\omega_2\nu^2 = \nu^3 = \eta\bar{\nu} = \tilde{p}_2\tilde{\eta}'\bar{\nu}$ for the projection $\tilde{p}_2 : Q_2 \rightarrow S^7$, we have the second of ii).

It remains to prove $\#\tilde{\eta}'' = 2$. $2\tilde{\eta}'' \in -i\langle \omega_3, \eta, 2\iota \rangle$ and $\langle \omega_3, \eta, 2\iota \rangle \subset \langle 2i\tilde{\sigma}, \eta, 2\iota \rangle + \langle 2\widetilde{2\nu''}, \eta, 2\iota \rangle \ni i\tilde{\sigma}\eta^2 + \widetilde{2\nu''} \circ \eta^2 \bmod 2\pi_{16}^S(Q_3) = 0$. Since $p\tilde{\eta}'\sigma = \eta\sigma = p\tilde{\sigma}\eta$, we have $\tilde{\eta}'\sigma = \tilde{\sigma}\eta$ and $i\tilde{\eta}'\eta\sigma = i\tilde{\sigma}\eta^2$. $\widetilde{2\nu''} \circ \eta = \widetilde{2\nu'} \circ \eta \in \langle i, \widetilde{2\nu}, \widetilde{2\nu} \rangle \circ \eta$ and $\widetilde{2\nu} \circ \eta = \tilde{\eta}\tilde{\eta}\eta \in -i\langle \nu, \bar{\eta}, \tilde{\eta}\eta \rangle \ni i\varepsilon = i\bar{\nu} + i\eta\sigma \bmod 0$, because $\eta\langle \nu, \bar{\eta}, \tilde{\eta}\eta \rangle = -\langle \eta, \nu, \bar{\eta} \rangle \circ \tilde{\eta}\eta \ni \bar{\nu}^2 \circ \tilde{\eta}\eta = \varepsilon\eta \bmod 0$ by Lemma 2.5. So we have $\widetilde{2\nu''} \circ \eta \in \langle i, \widetilde{2\nu}, i\bar{\nu} + i\eta\sigma \rangle \subset \langle i, 2\nu, \bar{\nu} + \eta\sigma \rangle \ni i\bar{\nu}' + i\tilde{\eta}'\sigma \bmod \pi_7^S(Q_2) \circ \varepsilon = \{\alpha\varepsilon\} = 0$ for $\alpha \in \langle i, 2\nu, 4\iota \rangle$, because $\alpha\varepsilon \in -i\langle 2\nu, 4\iota, \varepsilon \rangle \subset i_*\pi_{12}^S(S^0) = 0$. Hence we have $\widetilde{2\nu''} \circ \eta^2 = i\bar{\nu}'\eta + i\tilde{\eta}'\eta\sigma = i\tilde{\eta}'\bar{\nu} + i\tilde{\eta}'\eta\sigma = i\tilde{\eta}'\eta\sigma$. This leads us to the assertion $\langle \omega_3, \eta, 2\iota \rangle = 0$ and $2\tilde{\eta}'' = 0$.

Since $\#\gamma'_3 = 8!$ for $\gamma'_3 = \gamma_3(\mathbf{H})$ [17], we have the following

LEMMA 3.3. $\pi_{15}^S(\mathbf{H}P^2) = \mathbf{Z}_{128}\{\tilde{\sigma}''\}$, $\pi_{15}^S(\mathbf{H}P^3) = \mathbf{Z}_{128}\{\gamma'_3\} \oplus \mathbf{Z}_8\{i\tilde{\sigma}'' - \gamma'_3\}$ and $\pi_{15}^S(\mathbf{H}P^4) = \mathbf{Z}_8\{i\tilde{\sigma}''\}$, where $\tilde{\sigma}'' \in \langle i, \nu, \sigma \rangle$ and $16\tilde{\sigma}'' = i\zeta$.

We can set $\tilde{\mu} = t_2\tilde{\mu}'$. We set $\tilde{\eta}''' = t_4\tilde{\eta}''$.

By Lemma 2.9, we define a coextension $\tilde{\nu}'' \in \langle i, \gamma_5, \nu \rangle \subset \pi_{15}^S(P^6)$ of ν .

LEMMA 3.4. i) $\gamma_7 = i\tilde{\nu}'' + ai\widetilde{2\sigma'} + bi\tilde{\mu} + \tilde{\eta}'''$ and $\#\tilde{\nu}'' = 128$, where a and b are integers.

ii) $8\tilde{\nu}'' = 4i\widetilde{2\sigma'} + i\tilde{\mu}$.

iii) $4i\widetilde{2\sigma'} = i\tilde{\mu}$ and $i\tilde{\eta}''' \equiv i\tilde{\nu}'' \bmod i\widetilde{2\sigma'}$ in $\pi_{15}^S(P^8)$.

PROOF. By Lemma 1.5, $t_3i\tilde{\eta}'\sigma = it_2\tilde{\eta}'\sigma = i\tilde{\nu}\sigma \in -i\langle \eta, \nu, \sigma \rangle = 0$. So, by Lemmas 3.2, 3.3 and by (5), we have $f_{2*}\tilde{\sigma}'' = i\tilde{\eta}'\sigma$. By (1), $p'\gamma'_3 = p'h_4(i\tilde{\nu}'') = \nu$ for $p' = p_3(\mathbf{H})$. So, by Lemma 3.3, we have $h_{3*}(\gamma_7 - i\tilde{\nu}'') = xi\tilde{\sigma}''$ for an integer x . Therefore we have $0 = f_{3*}(xi\tilde{\sigma}'') = xif_{2*}\tilde{\sigma}'' = xi\tilde{\eta}'\sigma$ and $x = 2a$ for an integer a . Since $h_{3*}i\widetilde{2\sigma'} = 2i\tilde{\sigma}''$, we have $h_{3*}(\gamma_7 - i\tilde{\nu}'' - ai\widetilde{2\sigma'}) = 0$. So, by (5), Lemma 3.2 and by the fact that $p_7\gamma_7 = \eta = p_7\tilde{\eta}'''$, we have the first of i). By this and by (6)₆, we have $\#(i\tilde{\nu}'') = \#\tilde{\nu}'' = 128$.

By i) and by the fact that $8\tilde{\nu}'' \in -i\langle \gamma_5, \nu, 8\iota \rangle$, we have $\langle \gamma_5, \nu, 8\iota \rangle \ni 4i\widetilde{2\sigma'} \bmod i\tilde{\mu}$. $\langle \gamma_5, \nu, 8\iota \rangle \subset \langle \gamma_5, \eta^3, 2\iota \rangle \supset \langle \gamma_5\eta^2, \eta, 2\iota \rangle$. By (7) and the proof of Lemma 3.2, we have $\langle \gamma_5\eta^2, \eta, 2\iota \rangle \subset \langle t_3\omega_3, \eta, 2\iota \rangle + \langle i\tilde{\eta}\varepsilon, \eta, 2\iota \rangle + \langle 2i\tilde{\sigma'}, \eta, 2\iota \rangle \supset t_3\langle \omega_3, \eta, 2\iota \rangle + i\langle \eta\varepsilon, \eta, 2\iota \rangle + i\tilde{\sigma'}\eta^2 \ni i\tilde{\zeta}$, because $p_2\langle \tilde{\eta}\varepsilon, \eta, 2\iota \rangle \subset \langle \eta\varepsilon, \eta, 2\iota \rangle \ni \zeta \bmod 2\zeta$. So we have $\langle \gamma_5, \nu, 8\iota \rangle \ni i\tilde{\zeta} \bmod 2i\widetilde{2\sigma'}$ and $\langle \gamma_5, \nu, 8\iota \rangle \ni i\tilde{\zeta} + 8i\widetilde{2\sigma'} = i\tilde{\mu}$. Hence, by a suitable choice of $\tilde{\nu}''$, we have ii).

By i), $4i\tilde{\nu}'' = -4ai\widetilde{2\sigma'}$ in $\pi_{15}^S(P^9)$. So, by (5) and Lemma 3.2, we have $4i\tilde{\nu}'' \in$

$t_{5*}\pi_{16}^S(Q_5) = \{i\tilde{\mu}, i\tilde{\eta}'''\}$. Therefore we have $8i\tilde{\nu}'' = 0$ and $4i2\tilde{\sigma}' = i\tilde{\mu}$ by ii). Hence, by i), we have $i\tilde{\eta}''' = i\tilde{\nu}'' + (a + 4b)i2\tilde{\sigma}'$ in $\pi_{15}^S(P^9)$. This completes the proof.

By $(6)_n$ for $5 \leq n \leq 7$ and by Lemma 3.4, we have the following: ${}^2\pi_{15}^S(P^6) = \mathbf{Z}_{128}\{\tilde{\nu}''\} \oplus \mathbf{Z}_8\{i2\tilde{\sigma}' - 2\tilde{\nu}''\}$, ${}^2\pi_{15}^S(P^7) = \mathbf{Z}_{128}\{\gamma_7\} \oplus \mathbf{Z}_8\{i'(i2\tilde{\sigma}' - \tilde{\nu}'')\} \oplus \mathbf{Z}_2\{\tilde{\eta}'''\}$ and ${}^2\pi_{15}^S(P^8) = \mathbf{Z}_8\{i2\tilde{\sigma}'\} \oplus \mathbf{Z}_2\{i\tilde{\eta}'''\}$.

On the 3-component, $2\zeta = \pm\zeta_2\zeta$ and $2\gamma_3 = \pm i\zeta_2\alpha_1$. $6i2\tilde{\sigma}' \in -i'\langle 2\gamma_3, 2\sigma, 3\iota \rangle = -i'\langle \pm i\zeta_2\alpha_1, 2\alpha_2, 3\iota \rangle \ni \pm 2i\zeta_2\langle \alpha_1, \alpha_2, 3\iota \rangle \ni \pm 2i\zeta_2\alpha'_3 \pmod{3i_*\pi_{15}^S(P^3)}$. So we have ${}^3\pi_{15}^S(P^4) = \mathbf{Z}_{27}\{2\tilde{\sigma}'\}$. By $(6)_n$ for $4 \leq n \leq 6$ and by Lemma 2.9, we have ${}^3\pi_{15}^S(P^n) \cong \mathbf{Z}_{27}$ for $4 \leq n \leq 6$. By $(6)_7$ and by the fact $\#\gamma_7 = 8!$, we have ${}^3\pi_{15}^S(P^8) = \mathbf{Z}_3\{i2\tilde{\sigma}'\}$.

Summarizing the above arguments, we have the following

PROPOSITION 3.5. i) $\pi_{15}^S(P^2) = \mathbf{Z}_{504}\{\tilde{\zeta}\} \oplus \mathbf{Z}_3\{i\alpha_1\beta_1\}$ and $\pi_{15}^S(P^3) = \mathbf{Z}_{504}\{i\tilde{\zeta}\} \oplus \mathbf{Z}_2\{\tilde{\mu}\} \oplus \mathbf{Z}_2\{\gamma_3\eta\sigma\} \oplus \mathbf{Z}_2\{\gamma_3\bar{\nu}\}$.

ii) $\pi_{15}^S(P^4) = \mathbf{Z}_{8640}\{2\tilde{\sigma}'\} \oplus \mathbf{Z}_2\{i\tilde{\mu}\} \oplus \mathbf{Z}_7\{72i\tilde{\zeta}\}$ and $\pi_{15}^S(P^5) \cong \pi_{15}^S(P^4)$.

iii) $\pi_{15}^S(P^6) = \mathbf{Z}_{128}\{\tilde{\nu}''\} \oplus \mathbf{Z}_{1680}\{i2\tilde{\sigma}' - 2\tilde{\nu}''\} \oplus \mathbf{Z}_7\{72i\tilde{\zeta}\}$.

iv) $\pi_{15}^S(P^7) = \mathbf{Z}_{8!/9}\{\gamma_7\} \oplus \mathbf{Z}_{216}\{i2\tilde{\sigma}'\} \oplus \mathbf{Z}_2\{\tilde{\eta}'''\}$.

v) $\pi_{15}^S(P^8) = \mathbf{Z}_{24}\{i2\tilde{\sigma}'\} \oplus \mathbf{Z}_2\{i\tilde{\eta}'''\}$.

Next we shall examine $\pi_{16}^S(P^n)$. Hereafter we use $(6)_n = (6)_{n,16}$. By use of $(6)_1$, we have $\pi_{16}^S(P^2) = \mathbf{Z}_2\{i\sigma^2\} \oplus \mathbf{Z}_2\{i\kappa\}$.

In $(6)_2$, $\gamma_2\zeta = \pm 2i\nu\zeta = 0$ and $\gamma_2\beta_1 = \pm i\alpha_1\beta_1 \neq 0$. So we have $\pi_{16}^S(P^3) = \mathbf{Z}_2\{\gamma_3\mu\} \oplus \mathbf{Z}_2\{i\sigma^2\} \oplus \mathbf{Z}_2\{i\kappa\}$.

In $(6)_3$, $\gamma_3\eta^2\sigma \in -i\langle \gamma_2, \eta, \eta^2\sigma \rangle \supset \pm 2i\langle \nu, \eta, \eta^2\sigma \rangle \subset 2i_*\pi_{16}^S(S^2) = 0 \pmod{i_*\pi_7^S(P^2) \circ \eta^2\sigma = \{i\tilde{\nu}\eta^2\sigma\} = \{i\nu^2\eta\sigma\} = 0}$ and $\gamma_3\nu^3 = i\tilde{\nu}\nu^3 = i\tilde{\nu}\eta\bar{\nu} = i\nu^2\bar{\nu} = 0$. Therefore we have $\pi_{16}^S(P^4) = \mathbf{Z}_2\{i\sigma^2\} \oplus \mathbf{Z}_2\{i\kappa\}$.

LEMMA 3.6. $\omega_2\sigma = 0$ and $\gamma_2(\mathbf{H})\sigma = 0$.

PROOF. We note that $\omega_2 \in \langle i, 2\nu, 3\nu \rangle$ and $\gamma_2(\mathbf{H}) \in \langle i, \nu, 2\nu \rangle$. We have $\omega_2\sigma \in -i\langle 2\nu, 3\nu, \sigma \rangle \ni 0 \pmod{i\sigma^2}$. By [2], $p_* : \pi_{21}(\Sigma^4 Q_2, S^7) \rightarrow \pi_{21}(S^{11})$ for $p = \Sigma^4 \tilde{p}_2$ is an isomorphism. So we have an exact sequence

$$\pi_{21}(S^7) \xrightarrow{i_*} \pi_{21}(\Sigma^4 Q_2) \xrightarrow{p_*} \pi_{21}(S^{11}).$$

Since $p_*\Sigma^4(\omega_2\sigma_{10}) = 3\nu_{11}\sigma_{14} = 0$, $\Sigma^4(\omega_2\sigma_{10}) \in i_*\pi_{21}(S^7) = \{i\sigma'\sigma_{14}, i\kappa_7\}$. Hence we have $\omega_2\sigma = 0$ or $i\kappa$. Since $i_* : \pi_{17}^S(S^3) \rightarrow \pi_{17}^S(Q_2)$ is monomorphic, we have the first.

By an argument quite parallel to the above, we have the second. This completes the proof.

In $(6)_4$, by Lemmas 3.1 and 3.6, $\gamma_4\nu^2 = 0$ and $\gamma_4\sigma = -it_2\omega_2\sigma = 0$. Let $\tilde{\nu}^2 \in \langle i, \gamma_4, \nu^2 \rangle$ be a coextension of ν^2 . Since $\nu^2 \in \langle \eta, \nu, \eta \rangle \pmod{0}$, we can take

$\nu^2 = \bar{\eta} \circ \tilde{\eta}$. Here $\bar{\eta}$ and $\tilde{\eta}$ are an extension and a coextension of η with respect to ν , respectively. By Lemma 3.1, we have $\gamma_4 \bar{\eta} = \gamma_4 \langle \eta, \nu, p \rangle = -\langle \gamma_4, \eta, \nu \rangle p \ni 0 \bmod 0$. So there exists a coextension $\tilde{\bar{\eta}} \in \langle i, \gamma_4, \bar{\eta} \rangle$ of $\bar{\eta}$. We take $\tilde{\nu}^2 = \tilde{\bar{\eta}} \circ \tilde{\eta}$.

PROPOSITION 3.7. i) $\pi_{16}^S(P^5) = \mathbf{Z}_2\{\tilde{\nu}^2\} \oplus \mathbf{Z}_2\{i\sigma^2\} \oplus \mathbf{Z}_2\{i\kappa\}$.

ii) $\pi_{16}^S(P^6) \cong \pi_{16}^S(P^5)$ and $\pi_{16}^S(P^7) \cong \mathbf{Z}_2\{\gamma_7\eta\} \oplus \pi_{16}^S(P^6)$.

iii) $\pi_{16}^S(P^8) = \mathbf{Z}\{\zeta_8\} \oplus \mathbf{Z}_2\{i\tilde{\nu}^2\} \oplus \mathbf{Z}_2\{i\sigma^2\} \oplus \mathbf{Z}_2\{i\kappa\}$.

PROOF. Since $2\tilde{\eta} \in -i\langle \nu, \eta, 2\iota \rangle \ni 0 \bmod 0$, we have $\#\tilde{\nu}^2 = 2$. So, by (6)₄, we have i).

By (6)₅, we have the first of ii). By Lemma 1.2, $\gamma_6\nu = it_3\omega_3\nu = 0$. So, by (6)₆, we have the second of ii).

By (6)₇, we have iii). This completes the proof.

§4. Determination of $\pi_k^S(CP)$ for $k = 17$ and 18

In this section we often work in the 2-component when we deal with $\pi_{m+k}(\Sigma^k Q_n)$. We consider an exact sequence

$$0 \longrightarrow \pi_{18}^S(S^3) \xrightarrow{i_*} \pi_{18}^S(Q_2) \xrightarrow{p_*} \pi_{18}^S(S^7) \xrightarrow{(2\nu)^*} \pi_{18}^S(S^4).$$

Since $\nu'\zeta_6 = 0$, we can define $\{i, \nu', \zeta_6\}$. We take $\tilde{\zeta}_6 \in \{i, \nu', \zeta_6\}$ and set $\tilde{\zeta}_n = \Sigma^{n-6}\tilde{\zeta}_6$ for $n \geq 6$ and $\tilde{\zeta} = \Sigma^\infty\tilde{\zeta}_6$. $8\tilde{\zeta}_6 \in -i\{\nu', \zeta_6, 8\iota_{17}\}$ and this bracket is 0 or $\bar{\varepsilon}_3$. Since $\bar{\varepsilon} = \eta\kappa$ and $\langle 2\nu, \zeta, 8\iota \rangle \ni 0 \bmod 8\rho$ [7], we have $\{\nu', \zeta_6, 8\iota_{17}\} = 0$ and $8\tilde{\zeta}_6 = 0$.

In the 3-component, we have $18\tilde{\alpha}'_3 \in -i\langle 4\alpha_1, \alpha'_3, 9\iota \rangle \subset -i\langle 4\alpha_1, \alpha_3, 3\iota \rangle \ni -i\alpha_4 \bmod 3i_*\pi_{15}^S(S^0)$, where $\alpha_4 = 160\rho$. So we have the following

LEMMA 4.1. $\pi_{18}^S(Q_2) = \mathbf{Z}_{1512}\{\tilde{\zeta}\} \oplus \mathbf{Z}_{160}\{i\rho\} \oplus \mathbf{Z}_2\{i\eta\kappa\}$.

By use of the homotopy exact sequence of a pair (Sp_2, Q_2) and by [10], we have the following: $\pi_9(Q_2) \cong \pi_{10}(Sp_2, Q_2) \cong \mathbf{Z}$; $\pi_{10}(Q_2) = \mathbf{Z}_{120}\{\omega_2\} \oplus \mathbf{Z}_2\{\beta\eta_9\}$; $\pi_{11}(Q_2) = \mathbf{Z}\{[i, \beta]\} \oplus \mathbf{Z}_2\{\omega_2\eta_{10}\} \oplus \mathbf{Z}_2\{\beta\eta_9^2\}$, where β is a generator of $\pi_9(Q_2)$. By [2], $\pi_9(Q_2, S^3) = \mathbf{Z}\{[\iota_3, \alpha]\} \oplus \mathbf{Z}_2\{\alpha\hat{\eta}_6^2\}$, where α is a generator of $\pi_7(Q_2, S^3)$. Let $j : (Q_2, *) \hookrightarrow (Q_2, S^3)$ be the inclusion. Then, by the homotopy exact sequence of a pair (Q_2, S^3) , we have $\partial(\alpha\hat{\eta}_6^2) = \nu'\eta_6^2$ and $j_*\beta = [\iota_3, \alpha]$. Here $[\iota_3, \alpha]$ is the relative Whitehead product [23].

LEMMA 4.2. i) $j_2(\mathbf{H})\beta = 0$, $\tilde{p}_2\beta = 0$ and $\Sigma\beta \circ \eta_{10} = 0$.

ii) Let a be 0 or 1. $\omega_2 + a\beta\eta_9 \in \{i, \nu', 3\nu_6\}$, $\omega_2\eta_{10} + a\beta\eta_9^2 = i\varepsilon_3$, $\omega_2\eta_{10}\sigma_{11} = a\beta\eta_9^2\sigma_{11}$ and $\omega_2\varepsilon_{10} + a\beta\eta_9\varepsilon_{10} = i\bar{\varepsilon}_3$.

PROOF. i) is trivial except for the last. Let $\Sigma' : \pi_9(Q_2, S^3) \rightarrow \pi_{10}(\Sigma Q_2, S^4)$ be the relative suspension [23]. We consider an anti-commutative diagram between

exact sequences:

$$\begin{array}{ccccc}
 \pi_9(S^3) & \xrightarrow{i_*} & \pi_9(Q_2) & \xrightarrow{j_*} & \pi_9(Q_2, S^3) \\
 \downarrow \Sigma & & \downarrow \Sigma & & \downarrow \Sigma' \\
 \pi_{10}(S^4) & \xrightarrow{i'_*} & \pi_{10}(\Sigma Q_2) & \xrightarrow{j'_*} & \pi_{10}(\Sigma Q_2, S^4).
 \end{array}$$

Then $j'_*\Sigma\beta = -\Sigma'(j_*\beta) = -\Sigma'[\iota_3, \alpha] = 0$ [23]. So we have $\Sigma\beta = b\nu_4^2$ for an integer b . Hence we have $\Sigma\beta \circ \eta_{10} = 0$.

Since $\tilde{p}_2\omega_2 = 3\nu_7$, we have $\omega_2 + a\beta\eta_9 \in \{i, \nu', 3\nu_6\}$ and $\omega_2\eta_{10} + a\beta\eta_9^2 \in -i\{\nu', 3\nu_6, \eta_9\} \ni i\varepsilon_3 \bmod 0$. We have $\omega_2\eta_{10}\sigma_{11} + a\beta\eta_9^2\sigma_{11} = i\varepsilon_3\sigma_{11} = 0$ and $\omega_2\varepsilon_{10} + a\beta\eta_9\varepsilon_{10} \in -i\{\nu', 3\nu_6, \varepsilon_9\} \ni i\bar{\varepsilon}_3 \bmod 0$. This completes the proof.

LEMMA 4.3. $\{2\iota_5, \eta_5, \nu_6\} = 0$ and $\{2\iota_5, \eta_5, \zeta_6\} = 0$.

PROOF. As is well known, $i\nu_5\eta_8^2 \neq 0$ in $\pi_{10}(M^6)$. We have $i\{2\iota_5, \eta_5, \nu_6\} = -\{i, 2\iota_5, \eta_5\} \circ \nu_7 = -\{i, 2\iota_5, \eta_5\}_2 \circ \nu_7 \subset -\{i, 2\iota_5, \eta_5\nu_6\}_2 = -\{i, 2\iota_5, 0\}_2 \ni 0 \bmod i\Sigma^2\pi_8(S^3) = 0$. This leads us to the first.

By the Jacobi identity of Toda brackets, we have $\{2\iota_5, \eta_5, \zeta_6\} = \{2\iota_5, \eta_5, \{\nu_6, 8\iota_9, 2\sigma_9\}\} \equiv \{\{2\iota_5, \eta_5, \nu_6\}, 8\iota_{10}, 2\sigma_{10}\} + \{2\iota_5, \{\eta_5, \nu_6, 8\iota_9\}, 2\sigma_{10}\} \bmod 2\pi_{18}(S^5) = 0$. We have $\{\eta_5, \nu_6, 8\iota_9\} \subset \{\eta_5, \eta_6^3, 2\iota_9\} \supset \{4\nu_5, \eta_8, 2\iota_9\} \supset 2\{2\nu_5, \eta_8, 2\iota_9\} = 0 \bmod 0$. So, by the first relation, we have the second. This completes the proof.

By Lemma 4.3, we have $\tilde{\eta}_5\zeta_7 \in \{i, 2\iota_5, \eta_5\} \circ \zeta_7 = -i\{2\iota_5, \eta_5, \zeta_6\} = 0$. So we have $-\tilde{\zeta}_7 \in \{i, \Sigma\nu', \zeta_7\} = \{i, \tilde{\eta}_4\tilde{\eta}_5, \zeta_7\} \supset \{i\tilde{\eta}_4, \tilde{\eta}_5, \zeta_7\} \bmod \{i\bar{\varepsilon}_4\} + \pi_8(\Sigma Q_2) \circ \zeta_8$. We take $\tilde{\zeta}'_7 \in \{i\tilde{\eta}_4, \tilde{\eta}_5, \zeta_7\}$.

LEMMA 4.4. i) $\tilde{\zeta}'_7 \equiv -\tilde{\zeta}_7 \bmod \{i\bar{\varepsilon}_4, 12\tilde{\zeta}_7\}$.

ii) For $\tilde{\zeta}' \in \langle i\tilde{\eta}, \tilde{\eta}, \zeta \rangle$, $\tilde{\zeta}' \equiv -\tilde{\zeta} \bmod \{i\eta\kappa, 12\tilde{\zeta}\}$.

PROOF. By use of the homotopy exact sequence of a pair $(\Sigma Q_2, S^4)$, we have $\pi_8(\Sigma Q_2) = \mathbf{Z}\{\delta\} \oplus \mathbf{Z}_2\{i\nu_4\eta_7\}$, where $\delta \in \{i, \Sigma\nu', 12\nu_7\}$ is a coextension of $12\nu_7$. We have $\delta\zeta_8 \in \{i, \Sigma\nu', 12\zeta_7\} \ni -12\tilde{\zeta}_7 \bmod \{i\bar{\varepsilon}_4, 12\delta\zeta_8\}$. So we have i).

The indeterminacy of $\langle i\tilde{\eta}, \tilde{\eta}, \zeta \rangle$ is $i\tilde{\eta}\pi_{15}^S(M^2) + \pi_7^S(Q_2) \circ \zeta = \{i\eta\kappa, \delta\zeta\} = \{i\eta\kappa, 12\tilde{\zeta}\}$. This completes the proof.

Now we investigate brackets $\langle \omega_2, 8\iota, 2\sigma \rangle$ and $\langle \omega_2, \sigma, 16\iota \rangle$. By Lemma 4.3, $\tilde{\eta}_5\nu_7 \in -i\{2\iota_5, \eta_5, \nu_6\} = 0$. So, by Lemma 4.2, we have $-\Sigma\omega_2 \in \{i, \tilde{\eta}_4\tilde{\eta}_5, \nu_7\} = \{i\tilde{\eta}_4, \tilde{\eta}_5, \nu_7\} \bmod \pi_8(\Sigma Q_2) \circ \nu_8 = \{\delta\nu_8\}$. We have $\delta\nu_8 \in \{i, \Sigma\nu', 12\nu_7\} \ni 4\Sigma\omega_2 \bmod 12\pi_8(\Sigma Q_2) \circ \nu_8 = 0$. Therefore we can take $-\Sigma\omega_2 = \bar{\alpha}\tilde{\nu}_7$ for $\alpha = i\tilde{\eta}_4$, where $\bar{\alpha} \in \{i\tilde{\eta}_4, \tilde{\eta}_5, p'\} \subset [Y, \Sigma Q_2]$ and $\tilde{\nu}_7 \in \{i', \tilde{\eta}_5, \nu_7\} \subset \pi_{11}(Y)$ for $Y = M^6 \cup_{\tilde{\eta}_5} e^8$ are an extension of α and a coextension of ν_7 , respectively.

LEMMA 4.5. $\langle \omega_2, \sigma, 16\iota \rangle \ni x\tilde{\zeta} \bmod \{16i\rho, i\eta\kappa\}$ and $\langle \omega_2, 8\iota, 2\sigma \rangle \ni y\tilde{\zeta} \bmod i\eta\kappa$, where x and y are odd.

PROOF. Since $-\omega_2 = \bar{\alpha}\tilde{\nu}$ for $\alpha = i\tilde{\eta}$ and $\tilde{\nu}\sigma \in -i\langle \tilde{\eta}, \nu, \sigma \rangle \subset i_*\pi_{14}^S(M^2) = 0$, we have $\langle \omega_2, \sigma, 16\iota \rangle \supset -\bar{\alpha}\langle \tilde{\nu}, \sigma, 16\iota \rangle$ and $\bar{\alpha}\langle \tilde{\nu}, \sigma, 16\iota \rangle \subset \langle i\tilde{\eta}, \tilde{\eta}, \langle \nu, \sigma, 16\iota \rangle \rangle \subset a\langle i\tilde{\eta}, \tilde{\eta}, \zeta \rangle$ for odd a . So, by Lemma 4.4, $\theta \equiv -a\tilde{\zeta} \bmod \{12\zeta, i\eta\kappa\}$ for any element $\theta \in \bar{\alpha}\langle \tilde{\nu}, \sigma, 16\iota \rangle$. By Lemmas 4.1 and 4.2, the indeterminacy of $\langle \omega_2, \sigma, 16\iota \rangle$ is $\omega_2\pi_8^S(S^0) + 16\pi_{18}^S(Q_2) = \{i\eta\kappa, 16i\rho\}$. This leads us to the first assertion.

Similarly we have $\langle \omega_2, 8\iota, 2\sigma \rangle \supset -\bar{\alpha}\langle \tilde{\nu}, 8\iota, 2\sigma \rangle$ and $\bar{\alpha}\langle \tilde{\nu}, 8\iota, 2\sigma \rangle \subset \langle i\tilde{\eta}, \tilde{\eta}, \langle \nu, 8\iota, 2\sigma \rangle \rangle = \langle i\tilde{\eta}, \tilde{\eta}, \zeta \rangle$. The indeterminacy of $\langle \omega_2, 8\iota, 2\sigma \rangle$ is $\omega_2\pi_8^S(S^0) + \pi_{11}^S(Q_2) \circ 2\sigma = \{i\eta\kappa\}$, because $\pi_{11}^S(Q_2) \cong \pi_8^S(S^0)$. This completes the proof.

By Lemmas 1.5 and 4.5, we have $\tilde{\eta}'\eta\mu \in \tilde{\eta}'\eta\langle \eta, 2\iota, 8\sigma \rangle \subset \langle \tilde{\eta}'\eta^2, 2\iota, 8\sigma \rangle = \langle 4\omega_2 + 8i\sigma, 2\iota, 8\sigma \rangle \subset \langle 4\omega_2, 2\iota, 8\sigma \rangle + \langle 8i\sigma, 2\iota, 8\sigma \rangle \subset \langle 4\omega_2, 8\iota, 2\sigma \rangle + \langle 8i\sigma, 2\iota, 8\sigma \rangle \supset 4\langle \omega_2, 8\iota, 2\sigma \rangle + i\langle 8\sigma, 2\iota, 8\sigma \rangle \ni 4\tilde{\zeta} + 16i\rho \bmod \pi_{11}^S(Q_2) \circ 2\sigma = 0$. So we have

$$(8) \quad \tilde{\eta}'\eta\mu = 4\tilde{\zeta} + 16i\rho \text{ and } \gamma_3\eta\mu = 4\tilde{\zeta}'' + 16i\rho, \text{ where } \tilde{\zeta}'' = t_2\tilde{\zeta}.$$

Now we determine $\pi_{17}^S(P^n)$ for $2 \leq n \leq 3$. By use of $(6)_n = (6)_{n,17}$ for $n = 1$, we have $\pi_{17}^S(P^2) = \mathbf{Z}_{480}\{i\rho\} \oplus \mathbf{Z}_3\{\zeta_2\alpha_1\beta_1\}$. We consider an exact sequence induced from (5):

$$\pi_{18}^S(S^4) \xrightarrow{f_{1*}} \pi_{18}^S(Q_2) \xrightarrow{t_{2*}} \pi_{17}^S(P^3) \xrightarrow{h_{1*}} \pi_{17}^S(S^4).$$

$f_{1*}(\sigma^2) = i\eta\sigma^2 = 0$, $f_{1*}(\kappa) = i\eta\kappa$, $t_2\tilde{\zeta} = \tilde{\zeta}''$ and $h_{1*}(i\zeta_2\alpha_1\beta_1) = 2\alpha_1\beta_1$. So, by Lemma 4.1, we have the following

$$\text{PROPOSITION 4.6. } \pi_{17}^S(P^3) = \mathbf{Z}_{1512}\{\tilde{\zeta}''\} \oplus \mathbf{Z}_{160}\{i\rho\} \oplus \mathbf{Z}_3\{i\zeta_2\alpha_1\beta_1\}.$$

By use of $(6)_3$, we have an exact sequence

$$\pi_{10}^S(S^0) \xrightarrow{\gamma_{3*}} \pi_{17}^S(P^3) \xrightarrow{i_*} \pi_{17}^S(P^4) \xrightarrow{p_*} \pi_9^S(S^0) \xrightarrow{\gamma_{3*}} \dots$$

Since $\gamma_3\eta^2\sigma = \gamma_3\nu^3 = 0$, we can define coextensions $\lambda \in \langle i, \gamma_3, \eta^2\sigma \rangle$ of $\eta^2\sigma$ and $\tilde{\nu}^3 \in \langle i, \gamma_3, \nu^3 \rangle$ of ν^3 , respectively.

LEMMA 4.7. By a suitable choice of $\tilde{\nu}^3$ and λ , we have $\#\tilde{\nu}^3 = 2$, $2\lambda = \pm i\tilde{\zeta}'' \pm i\rho$ and $\#\lambda = 64$.

PROOF. $2\tilde{\nu}^3 \in -i\langle \gamma_3, \nu^3, 2\iota \rangle$. $\langle \gamma_3, \nu^3, 2\iota \rangle \supset \langle \gamma_3\nu, \nu^2, 2\iota \rangle = \langle \pm i\tilde{\nu}\nu, \langle \eta, \nu, \eta \rangle, 2\iota \rangle \equiv \langle \langle i\tilde{\nu}\nu, \eta, \nu \rangle, \eta, 2\iota \rangle + \langle i\tilde{\nu}\nu, \eta, \langle \nu, \eta, 2\iota \rangle \rangle \equiv \langle \langle i\tilde{\nu}\nu, \eta, \nu \rangle, \eta, 2\iota \rangle \bmod 2\pi_{17}^S(P^3)$. $\langle i\tilde{\nu}\nu, \eta, \nu \rangle \supset i\tilde{\nu}\langle \nu, \eta, \nu \rangle = i\tilde{\nu}\bar{\nu} \in -i\langle \eta, \nu, \bar{\nu} \rangle \ni 0 \bmod \pi_{12}^S(P^3) \circ \nu = \{t_2\omega_2\nu^2\}$ by Proposition 2.2. $\langle t_2\omega_2\nu^2, \eta, 2\iota \rangle \supset t_2\omega_2\nu\langle \nu, \eta, 2\iota \rangle \ni 0 \bmod 2\pi_{17}^S(P^3)$. Therefore we have $\langle \gamma_3, \nu^3, 2\iota \rangle \ni 0 \bmod 2\pi_{17}^S(P^3)$. This leads us to the first.

We have $2\lambda \in -i\langle \gamma_3, \eta^2\sigma, 2\iota \rangle$ and $p\langle \gamma_3, \eta^2\sigma, 2\iota \rangle \subset \langle \eta, \eta^2\sigma, 2\iota \rangle \ni \zeta \bmod 2\zeta$. So, by Proposition 4.6, we have $2\lambda \equiv \pm i\tilde{\zeta}'' \bmod i\rho$. $8\langle \gamma_3, \eta^2\sigma, 2\iota \rangle \subset \langle \gamma_3, \eta^2\sigma, 16\iota \rangle \supset$

$\langle \gamma_3 \eta^2, \sigma, 16\iota \rangle \bmod \gamma_3 \pi_{10}^S(S^0) + 16\pi_{17}^S(P^3) = \{4\tilde{\zeta}'', 16i\rho\}$ by (8). By Lemmas 1.4 and 4.5, we have $\langle \gamma_3 \eta^2, \sigma, 16\iota \rangle \subset \langle 4t_2\omega_2, \sigma, 16\iota \rangle + \langle 8i\sigma, \sigma, 16\iota \rangle \ni 4\tilde{\zeta}'' + 8i\rho \bmod 16i\rho$. Therefore, by (8), we have $16\lambda = 4i\tilde{\zeta}'' \pm 8i\rho = \pm 8i\rho$, $\# \lambda = 64$ and $2\lambda = \pm i\tilde{\zeta}'' \pm i\rho$. This completes the proof.

By (8) and Lemma 4.7, we have $8\lambda = 4i\tilde{\zeta}'' \pm 4i\rho = \pm 12i\rho$ and $2\gamma_3\beta_1 = \pm i\zeta_2\nu\beta_1 = \pm i\zeta_2\alpha_1\beta_1$. So, by (6)₃, (8) and Proposition 4.6, we have the following

PROPOSITION 4.8. $\pi_{17}^S(P^4) = \mathbf{Z}_{64}\{\lambda\} \oplus \mathbf{Z}_4\{\pm 3i\rho - 2\lambda\} \oplus \mathbf{Z}_2\{\tilde{\nu}^3\} \oplus \mathbf{Z}_{189}\{8i\zeta''\} \oplus \mathbf{Z}_5\{96i\rho\}$.

We consider the following exact sequence:

$$\pi_{18}^S(S^{10}) \xrightarrow{\omega_2^*} \pi_{18}^S(Q_2) \xrightarrow{i_*} \pi_{18}^S(Q_3) \xrightarrow{p_*} \pi_{18}^S(S^{11}) \xrightarrow{\omega_2^*} \dots$$

Let $\tilde{\sigma} \in \langle i, \omega_2, \sigma \rangle \subset \pi_{18}^S(Q_3)$ be a coextension of σ . By Lemma 4.5, we have $16\tilde{\sigma} \in -i\langle \omega_2, \sigma, 16\iota \rangle \ni -ix\zeta \bmod 16i\rho$. So, by Lemma 4.1 and by suitable choice of $\tilde{\sigma}$, we have the following

LEMMA 4.9. $16\tilde{\sigma} = i\tilde{\zeta}$ and ${}^2\pi_{18}^S(Q_3) = \mathbf{Z}_{128}\{\tilde{\sigma}\} \oplus \mathbf{Z}_{32}\{i\rho\}$.

We set $\tilde{\sigma}'' = t_3\tilde{\sigma}$. Then, by use of (6)₄, (8) and Lemma 4.8, we have $16\tilde{\sigma}'' = i\tilde{\zeta}''$, $64\tilde{\sigma}'' = 4i\tilde{\zeta}'' = 16i\rho = 32i'\lambda$ and ${}^2\pi_{17}^S(P^5) = \mathbf{Z}_{128}\{\tilde{\sigma}''\} \oplus \mathbf{Z}_{32}\{i'\lambda - 2\tilde{\sigma}''\} \oplus \mathbf{Z}_2\{i\tilde{\nu}^3\}$.

Let $\tilde{\alpha}_2 \in \langle i, \gamma_4, \alpha_2 \rangle \subset {}^3\pi_{17}^S(P^5)$ be a coextension of α_2 . Then, by Lemma 1.4, $40\gamma_4 = -i\alpha_2$ and $6\tilde{\alpha}_2 \in 2i'\langle i\alpha_2, \alpha_2, 3\iota \rangle \ni i\alpha_4 \bmod 0$. Therefore we have $3\tilde{\alpha}_2 = i\alpha_4 = 9i\tilde{\alpha}'_3$ and ${}^3\pi_{17}^S(P^5) \cong \mathbf{Z}_{27} \oplus \mathbf{Z}_3$. In the 5-component, we have $4\gamma_4\alpha_1 = 8i\alpha_1\alpha_1 = 0$ for $\alpha_1 = 48\sigma$, by Lemma 1.4. So, for $\tilde{\alpha}'_1 \in \langle i, \gamma_4, \alpha_1 \rangle$, we have $10\tilde{\alpha}'_1 \in -2i\langle \alpha_1, \alpha_1, 5\iota \rangle \ni -i\alpha_2 \bmod 10i_*\pi_{15}^S(S^0)$. So we have the following

PROPOSITION 4.10. $\pi_{17}^S(P^5) = \mathbf{Z}_{3200}\{\tilde{\sigma}''\} \oplus \mathbf{Z}_{32}\{i'\lambda - 2\tilde{\sigma}''\} \oplus \mathbf{Z}_2\{i\tilde{\nu}^3\} \oplus \mathbf{Z}_{189}\{8i\tilde{\zeta}''\} \oplus \mathbf{Z}_3\{\tilde{\alpha}_2 - 3i\tilde{\alpha}'_3\}$.

By use of (6)₅ and Lemma 2.9, we have $\pi_{17}^S(P^6) \cong \pi_{17}^S(P^5)$.

LEMMA 4.11. $8i\tilde{\sigma} = 8a\omega_4 + 16bi\rho$ and ${}^2\pi_{18}^S(Q_4) = \mathbf{Z}_{128}\{\omega_4\} \oplus \mathbf{Z}_8\{i\tilde{\sigma} - a\omega_4 - 2bi\rho\} \oplus \mathbf{Z}_{32}\{i\rho\}$, where a is odd.

PROOF. First we consider a commutative diagram

$$\begin{array}{ccccccc} \pi_{19}(Q_3, Q_2) & \xrightarrow{\partial} & \pi_{18}(Q_2) & \xrightarrow{i_*} & \pi_{18}(Q_3) & & \\ \downarrow p_* & & \downarrow j_{2*} & & \downarrow j_{3*} & & \\ \pi_{19}(S^{11}) & \xrightarrow{\partial'} & \pi_{18}(Sp_2) & \xrightarrow{i'_*} & \pi_{18}(Sp_3) & \xrightarrow{p'_*} & \pi_{18}(S^{11}) \xrightarrow{\partial'} \dots \end{array}$$

By [10] and [11], ${}^2\pi_{18}(Sp_2) = \mathbf{Z}_8\{[\zeta_7]\} \oplus \mathbf{Z}_2\{i\bar{\varepsilon}_3\}$ and $\pi_{18}(Sp_3) \cong \mathbf{Z}_{91/24}$. We have $\partial'\sigma_{11} = j_{2*}\omega_2\sigma_{10}$, $4p'_*(j_{2*}\omega_2\sigma_{10}) = 12\nu_7\sigma_{10} \neq 0$ and $8\omega_2\sigma_{10} = 0$. By Lemma 4.2,

$\partial'(\eta_{11}\sigma_{12}) = j_2\omega_2\eta_{10}\sigma_{11} = 0$ and $\partial'\varepsilon_{11} = j_2\omega_2\varepsilon_{10} = i\bar{\varepsilon}_3$. So we have ${}^2\pi_{18}(Sp_3) = \mathbf{Z}_{16}\{[8\sigma_{11}]\}$ and $2[8\sigma_{11}] = i'[\zeta_7]$. By (4), $\tilde{\zeta}_6 \in \{i, \nu', \zeta_6\}$ and $\delta \in \{i, \omega_2, 8\sigma_{10}\}$ hit $[\zeta_7]$ and $[8\sigma_{11}]$ by j_{2*} and j_{3*} , respectively. So, by the James splitting, we have $2\delta = i\tilde{\zeta}$ in the stable range.

Next we consider a commutative diagram

$$\begin{array}{ccccccc} \pi_{18}(Q_3) & \xrightarrow{i_*} & \pi_{18}(Q_4) & & & & \\ \downarrow j_{3*} & & \downarrow j_{4*} & & & & \\ 0 & \longrightarrow & \pi_{18}(Sp_3) & \xrightarrow{i'_*} & \pi_{18}(Sp_4) & \xrightarrow{p'_*} & \pi_{18}(S^{15}). \end{array}$$

By [11], $\pi_{18}(Sp_4) = \mathbf{Z}_{9!}\{j_4\omega_4\}$. So we have $8j_{4*}\omega_4 = i'_*[8\sigma_{11}] = j_{4*}i_*\delta$. By the James splitting, we have $8\omega_4 = i\delta$. Therefore $16\omega_4 = 2i\delta = i\tilde{\zeta}$.

Since $\delta \in \langle i, \omega_2, 8\sigma \rangle \supset \langle i, \omega_2, \sigma \rangle \circ 8\iota \ni 8\tilde{\sigma} \bmod \{i\tilde{\zeta}, i\rho\}$ and $2\delta = i\tilde{\zeta} = 16\tilde{\sigma}$ by Lemma 4.9, we have $\delta = 8\tilde{\sigma} + 4xi\tilde{\zeta} + 16yi\rho$ for integers x and y . So we have $8\omega_4 = i\delta = 8i\tilde{\sigma} + 64xi\tilde{\sigma} + 16yi\rho$ and the first assertion. This gives a splitting of the following exact sequence

$$0 \longrightarrow \pi_{18}^S(Q_3) \xrightarrow{i_*} \pi_{18}^S(Q_4) \xrightarrow{p_*} \pi_{18}^S(S^{15}) \longrightarrow 0.$$

This completes the proof.

By Lemma 4.11, we have $8i\tilde{\sigma}'' = 8at_4\omega_4 + 16bi\rho = 8at_4\omega_4 + 64bi\tilde{\sigma}''$. So there exists an odd integer a' such that $8i\tilde{\sigma}'' = 8a't_4\omega_4$. Therefore, by use of (6)₆, we have the following

PROPOSITION 4.12. $\pi_{17}^S(P^7) = \mathbf{Z}_{9!/5}\{5t_4\omega_4\} \oplus \mathbf{Z}_8\{i\tilde{\sigma}'' - a't_4\omega_4\} \oplus \mathbf{Z}_{32}\{i'\lambda - 2i\tilde{\sigma}''\} \oplus \mathbf{Z}_2\{i\tilde{\nu}^3\} \oplus \mathbf{Z}_3\{i''(\tilde{\alpha}_2 - 3i\tilde{\alpha}'_3)\} \oplus \mathbf{Z}_{25}\{i\alpha'_1\}$.

Finally we consider the exact sequence (6)₇:

$$\pi_{17}^S(S^{15}) \xrightarrow{\gamma\tau^*} \pi_{17}^S(P^7) \xrightarrow{i_*} \pi_{17}^S(P^8) \xrightarrow{p_*} \pi_{17}^S(S^{16}) \xrightarrow{\gamma\tau^*} \dots$$

Let $\tilde{\eta}'' \in \langle i, \nu, \eta \rangle$, $\tilde{\nu}'' \in \langle i, \nu, \bar{\nu} \rangle$, $\tilde{\nu}^{3''} \in \langle i, \nu, \nu^3 \rangle$ and $\tilde{\mu}'' \in \langle i, \nu, \mu \rangle$ be elements of $\pi_*^S(\mathbf{HP}^2)$ such that $h_2(i\tilde{\nu}^3) = \tilde{\nu}^{3''}$, where $h_2 : \mathbf{CP}^5 \rightarrow \mathbf{HP}^2$ is the mapping in (5).

LEMMA 4.13. i) $\pi_{16}^S(\mathbf{HP}^2) = \mathbf{Z}_2\{\tilde{\nu}''\} \oplus \mathbf{Z}_2\{\tilde{\eta}''\sigma\}$, $\pi_{17}^S(\mathbf{HP}^2) = \mathbf{Z}_2\{\tilde{\nu}^{3''}\} \oplus \mathbf{Z}_2\{\tilde{\eta}''\eta\sigma\} \oplus \mathbf{Z}_2\{\tilde{\mu}''\}$ and $\tilde{\nu}^{3''} = \tilde{\nu}''\eta$.

ii) $\gamma\tau\eta^2 = 4t_4\omega_4 + 2ai\tilde{\sigma}'' + bi\lambda + i\tilde{\nu}^3$, where a and b are integers satisfying $a \equiv b \bmod 2$.

PROOF. i) is easily obtained.

By Lemma 1.4 and Proposition 4.10, we have $\gamma_7\eta^2 = 4t_4\omega_4 + xi\tilde{\sigma}'' + yi\lambda + zi\tilde{\nu}^3$ for integers x, y and z . Since $64t_4\omega_4 = 64i\tilde{\sigma}'' = 32i\lambda = 16i\rho$ and $\#i\tilde{\nu}^3 = 2$, we can set $x = 2a$ and $y = b$ satisfying $16(a+b)i\rho = 0$. We have $\gamma_3(\mathbf{H})\eta^2 \in \langle i, \gamma_2(\mathbf{H}), \nu \rangle \circ \eta^2 = -i\langle \gamma_2, (\mathbf{H}), \nu, \eta \rangle \circ \eta \ni i\tilde{\nu}''\eta \bmod i\tilde{\eta}''\eta\sigma$ since $p_*\langle \gamma_2(\mathbf{H}), \nu, \eta \rangle \subset \langle 2\nu, \nu, \eta \rangle \ni \bar{\nu} \bmod \eta\sigma$ and $\text{Ker } p_* = i_*\pi_{16}^S(S^4) = 0$. This leads us to $z = 1$ and completes the proof.

By Proposition 4.12, Lemma 4.13 and by (6)₇, we have the following

PROPOSITION 4.14. i) $\pi_{17}^S(P^8) = \mathbf{Z}_{91/5}\{5\gamma_8\} \oplus \mathbf{Z}_8\{i\tilde{\sigma}'' + a'\gamma_8\} \oplus \mathbf{Z}_{32}\{i'\lambda - 2i\tilde{\sigma}''\} \oplus \mathbf{Z}_3\{i''(\tilde{\alpha}_2 - i\tilde{\alpha}'_3)\} \oplus \mathbf{Z}_{25}\{i\tilde{\alpha}'_1\}$.

We have $16i\rho = 64i\tilde{\sigma}'' = 64\gamma_8$ and $2(i'\lambda - 2i\tilde{\sigma}'') = 2i'\lambda - 4i\tilde{\sigma}'' = \pm i\tilde{\zeta}'' \pm i\rho - 4i\tilde{\sigma}'' = \pm i\rho + 4b'i\tilde{\sigma}''$ for odd b' by Lemma 4.7. So we have $2i'\lambda \pm i\rho = 8ci\tilde{\sigma}'' = -8ca'\gamma_8$, where $1 + b' = 2c$ and a' is an odd integer. Hence we have $2i'\lambda = \pm i\rho$ in $\pi_{17}^S(P^9)$. This and (6)₈ lead us to the following

PROPOSITION 4.15. $2i'\lambda = \pm i\rho$ and $\pi_{17}^S(P^9) = \mathbf{Z}_8\{i\tilde{\sigma}''\} \oplus \mathbf{Z}_{32}\{i'\lambda\} \oplus \mathbf{Z}_3\{i''(\tilde{\alpha}_2 - 3i\tilde{\alpha}'_3)\} \oplus \mathbf{Z}_5\{i\tilde{\alpha}'_1\}$.

Next we shall determine $\pi_{18}^S(P)$. By (6)_n = (6)_{n,18} for $n = 1$, we have $\pi_{18}^S(P^2) = \mathbf{Z}_4\{\tilde{\sigma}^2\}$, where $\tilde{\sigma}^2 \in \langle i, \eta, \sigma^2 \rangle$ and $2\tilde{\sigma}^2 = i\eta^*$. Here we recall that $\eta^* \in \langle \eta, \sigma^2, 2\iota \rangle$.

By use of (6)₂, we have $\pi_{18}^S(P^3) = \mathbf{Z}_4\{i\tilde{\sigma}^2\}$.

In (6)₃, we have $\gamma_3\eta\mu \neq 0$ and $\gamma_3\beta_1 = -2\gamma_3\beta_1 = \pm i\zeta_2\alpha_1\beta_1 \neq 0$. By Proposition 4.6, we have $\gamma_3\beta_1 = -2\gamma_3\beta_1 = \pm i\zeta_2\alpha_1\beta_1 \neq 0$. $\gamma_3\zeta \in -i\langle \gamma_2, \eta, \zeta \rangle \supset \pm 2i\langle \nu, \eta, \zeta \rangle$.

By an argument parallel to the proof of Lemma 10.1 of [13], we have $\langle \nu, \eta, \zeta \rangle \ni 0 \bmod 0$ and $\langle \eta, \nu, \zeta \rangle \ni 0 \bmod \eta\rho$. Since $\tilde{\nu}\zeta \in -i\langle \eta, \nu, \zeta \rangle \ni 0 \bmod i\eta\rho = 0$, we have $\gamma_3\zeta \equiv 0 \bmod i_*\pi_7^S(P^2) \circ \zeta = i_*\{\tilde{\nu}\zeta\} = 0$. Therefore we have the following

PROPOSITION 4.16. $\pi_{18}^S(P^n) = \mathbf{Z}_4\{i\tilde{\sigma}^2\}$ for $n = 3$ and 4.

In (6)₄, by Lemma 4.2, we have $p_5(\tilde{\sigma}''\eta) = \sigma\eta$, $\gamma_4\bar{\nu} = it_2\omega_2\bar{\nu} = i\eta\kappa = 0$ and $\gamma_4\eta^2\sigma = \gamma_4\nu^3 = 0$. By the proof of Lemma 10.1 of [13], we have $\langle 2\nu, \nu, \mu \rangle = \langle \zeta, \eta, \nu \rangle \ni 0 \bmod \eta\rho$. So we have $\gamma_4\mu = it_2\omega_2\mu \in -i\langle 2\nu, \nu, \mu \rangle \ni 0 \bmod i\eta\rho = 0$. Let $\tilde{\bar{\nu}}'' \in \langle i, \gamma_4, \bar{\nu} \rangle$ be a coextension of $\bar{\nu}$. Then $2\tilde{\bar{\nu}}'' \in -i\langle \gamma_4, \bar{\nu}, 2\iota \rangle \supset i\langle t_2\omega_2, \bar{\nu}, 2\iota \rangle \bmod 2i\tilde{\sigma}^2 = i\eta^*$. By the Jacobi identity of Toda brackets, $\langle t_2\omega_2, \bar{\nu}, 2\iota \rangle \equiv \langle \langle i, 2\nu, \nu \rangle, \bar{\nu}, 2\iota \rangle \equiv \langle i, \langle 2\nu, \nu, \bar{\nu} \rangle, 2\iota \rangle \pm \langle i, 2\nu, \langle \nu, \bar{\nu}, 2\iota \rangle \rangle \bmod 2\pi_{18}^S(P^3) + i_*\pi_{16}^S(S^0) = \{i\eta^*\}$. We have $\langle \nu, \bar{\nu}, 2\iota \rangle \subset \pi_{12}^S(S^0) = 0$, $\langle 2\nu, \nu, \bar{\nu} \rangle \equiv \langle \nu, \bar{\nu}, 2\nu \rangle \pm \langle \bar{\nu}, 2\nu, \nu \rangle \bmod 0$, $\langle \nu, \bar{\nu}, 2\nu \rangle \subset \langle \nu, 0, \nu \rangle = 0$ and $\langle \bar{\nu}, 2\nu, \nu \rangle \ni \eta\kappa \bmod 0$. So we have $\langle t_2\omega_2, \bar{\nu}, 2\iota \rangle \equiv \langle i, \eta\kappa, 2\iota \rangle \supset \langle i, \eta, 0 \rangle \ni 0 \bmod i\eta^*$ and $2\tilde{\bar{\nu}}'' \equiv 0 \bmod i\eta^* = 2i\tilde{\sigma}^2$. Therefore we can take $\tilde{\bar{\nu}}''$ satisfying $\#\tilde{\bar{\nu}}'' = 2$ and we have the following

PROPOSITION 4.17. $\pi_{18}^S(P^5) = \mathbf{Z}_4\{i\tilde{\sigma}^2\} \oplus \mathbf{Z}_2\{\tilde{\bar{\nu}}''\} \oplus \mathbf{Z}_2\{\tilde{\sigma}''\eta\}$.

PROPOSITION 4.18. $\gamma_5\sigma \equiv \tilde{\sigma}''\eta \bmod i\eta^*$, $2\tilde{\nu}''\nu \equiv i\tilde{\nu}'' \bmod i\eta^*$ and $\pi_{18}^S(P^7) \cong \pi_{18}^S(P^6) = \mathbf{Z}_4\{i\tilde{\sigma}^2\} \oplus \mathbf{Z}_4\{\tilde{\nu}''\nu\}$.

PROOF. By (6)₆, we have the above isomorphism. In (6)₅, we have $p_6(\tilde{\nu}''\nu) = \nu^2$ and $\gamma_5\sigma = \tilde{\sigma}''\eta + ai\tilde{\sigma}^2$ for an integer a . By Lemma 3.6, (5) and (1), we have $h_2\gamma_5\sigma = h_2\tilde{\sigma}''\eta = h_2t_3\tilde{\sigma}\eta = 0$ and $h_2i\tilde{\sigma}^2 = iq_1i\tilde{\sigma}^2 = i\sigma^2 \neq 0$ in $\pi_{18}^S(\mathbf{HP}^2)$. So we have $a = 2a'$ and $\gamma_5\sigma = \tilde{\sigma}''\eta + a'i\eta^*$.

We have $2\tilde{\nu}''\nu \in -i\langle\gamma_5, \nu, 2\nu\rangle$ and $p_5\langle\gamma_5, \nu, 2\nu\rangle \subset \langle\eta, \nu, 2\nu\rangle \ni \bar{\nu} \bmod \eta\sigma$. We have $\langle\gamma_5, \nu, 2\nu\rangle = \langle\gamma_5, \nu, \langle\eta, 2\iota, \eta\rangle\rangle \equiv \langle\langle\gamma_5, \nu, \eta\rangle, 2\iota, \eta\rangle + \langle\gamma_5, \langle\nu, \eta, 2\iota\rangle, \eta\rangle \equiv \langle\tilde{\nu}^2, 2\iota, \eta\rangle \bmod \{\gamma_5\sigma, i\eta^*\} + \pi_{17}^S(P^5) \circ \eta$, because $\langle\nu, \eta, 2\iota\rangle = 0$, $\langle\gamma_5, \nu, \eta\rangle \ni \tilde{\nu}^2 \bmod \{i\sigma^2, i\kappa\}$ by Proposition 3.7, $\langle i\sigma^2, 2\iota, \eta\rangle \ni i\eta^*$ and $\langle i\kappa, 2\iota, \eta\rangle \supset i\langle\kappa, 2\iota, \eta\rangle \ni 0$ [15]. By Propositions 4.6 and 4.7, we have $\tilde{\nu}^2\eta \in 2i_*\pi_{17}^S(P^3)$ since $p_4(\tilde{\nu}^2\eta) = 0$ and $2\tilde{\nu}^2\eta = 0$. So we have $2\langle\tilde{\nu}^2, 2\iota, \eta\rangle = \tilde{\nu}^2\eta^2 = 0$. Therefore we have $\langle\gamma_5, \nu, 2\nu\rangle \ni \tilde{\nu}'' \bmod \{\gamma_5\sigma, i\eta^*\}$ and $2\tilde{\nu}''\nu \equiv i\tilde{\nu}'' \bmod i\eta^*$. This leads us to the last assertion and completes the proof.

PROPOSITION 4.19. $\gamma_7\nu \equiv i\tilde{\nu}''\nu \bmod i\eta^*$, $\pi_{18}^S(P^8) = \mathbf{Z}_4\{i\tilde{\sigma}^2\}$ and $\pi_{18}^S(P^9) = \mathbf{Z}\{\zeta_9\} \oplus \mathbf{Z}_4\{i\tilde{\sigma}^2\}$.

PROOF. By Lemma 3.4, we have $\gamma_7\nu = i\tilde{\nu}''\nu + ai2\tilde{\sigma}'\nu + bi\tilde{\mu}\nu + \tilde{\eta}'''\nu$. $\tilde{\mu}\nu \in -i\langle\gamma_2, \mu, \nu\rangle \supset \pm 2i\langle\nu, \mu, \nu\rangle \ni 0 \bmod i_*\pi_{15}^S(P^2) \circ \nu = \{i\tilde{\zeta}\nu\} = 0$ and $2\tilde{\sigma}'\nu \in -i\langle\gamma_3, 2\sigma, \nu\rangle \supset \pm i\zeta_2\langle\nu, \sigma, \nu\rangle \ni i\zeta_2\sigma^2 = i\eta^* \bmod i_*\pi_{15}^S(P^3) \circ \nu = \{i\tilde{\zeta}\nu\} + \{\tilde{\mu}\nu\} = 0$. By Lemmas 2.6 and 3.2, $\tilde{\eta}'''\nu \in -t_4i\langle\omega_3, \eta, \nu\rangle \supset i'(i\tilde{\sigma}' + 2\tilde{\nu}''')\langle 2\iota, \eta, \nu\rangle \ni 0 \bmod it_3\pi_{16}^S(Q_3) \circ \nu = \{i\tilde{\mu}\nu\} = 0$. So we have the first. In (6)₇, $\gamma_7\eta^2 \neq 0$. Therefore we have the second. By (6)₈, we have the last. This completes the proof.

§5. Determination of the image of the S^1 -transfer

First we recall the definition of the S^{d-1} -transfer map $g_n(F) : \Sigma^{dn}\tilde{Q}_n(F) \rightarrow S^{dn}$. $g_n(F) \mid \Sigma^{dn}\tilde{Q}_{n-1}(F)$ is homotopic to $\Sigma^d g_{n-1}(F)$ up to sign. So, by (3), we have

$$(9) \quad \pm g_n(F) \in \{\Sigma^d g_{n-1}(F), \Sigma^{dn}\tilde{\omega}_{n-1}(F), \Sigma^{dn-1}\tilde{p}_n(F)\}_1.$$

Let $H : [\Sigma X, \Sigma Y] \rightarrow [\Sigma X, \Sigma(Y \wedge Y)]$ be the Hopf homomorphism. By an argument parallel to the proof of Theorem 4.1 of [18], we have

$$(10) \quad H(g_n(F)) = \pm \Sigma^{dn}\tilde{p}_n(F).$$

REMARK 3. In the proof of Theorem 4.1 of [18], we excluded the case that $F = \mathbf{R}$ and $n = 4$ or 8 . But (10) holds in this case, because $H(g_n(\mathbf{R}) \circ \Sigma^n \gamma_{n-1}(\mathbf{R})) = H(g_n(\mathbf{R})) \circ \Sigma^n \gamma_{n-1}(\mathbf{R})$ and $H([\iota_n, \iota_n]) = \pm 2\iota_{2n-1}$.

By (10) and by use of EHP-sequences, we have the following

- LEMMA 5.1. i) $g_2 \equiv \pm \nu_4 \pmod{\Sigma \nu'}$.
 ii) $g_4 \equiv \pm \sigma_8 \circ \Sigma^9 p_3 \pmod{\Sigma \pi^7(\Sigma^8 P^3)}$.
 iii) *Except for $n = 2$ or 4 , we have*

$$2g_n \equiv \pm[\iota_{2n}, \iota_{2n}] \circ \Sigma^{2n+1} p_{n-1} \pmod{\Sigma \pi^{2n-1}(\Sigma^{2n} P^{n-1})}.$$

By (9) and Lemma 5.1, $\pm g_3 \in \{\nu_6, \eta_9, \Sigma^5 p_2\}$. So g_3 can be taken as an extension $\bar{\nu}'_6$ of ν_6 .

By (3), $\Sigma g_3 \circ \Sigma^8 \gamma_2 = \pm[\iota_7, \iota_7] = 0$. So we define an extension $\bar{\bar{\nu}}_7$ of $\bar{\nu}'_7 = \Sigma g_3$ by $\bar{\bar{\nu}}_7 \in \{\bar{\nu}'_7, \Sigma^8 \gamma_2, \Sigma^7 p_3\}$. We set $\bar{\bar{\nu}}_n = \Sigma^{n-7} \bar{\bar{\nu}}_7$ for $n \geq 7$ and $\bar{\bar{\nu}} = \Sigma^\infty \bar{\bar{\nu}}_7$.

- LEMMA 5.2. i) $\pi^7(\Sigma^8 P^3) = \mathbf{Z}_{360}\{\bar{\bar{\nu}}_7\} \oplus \mathbf{Z}_8\{\sigma' \circ \Sigma^8 p_3\}$.
 ii) $g_4 = \pm \sigma_8 \circ \Sigma^9 p_3 \pm \bar{\bar{\nu}}_8$.

PROOF. As is easily seen, we have $\pi^7(\Sigma^8 P^2) = \mathbf{Z}_{24}\{\bar{\nu}'_7\}$, $\pi^7(\Sigma^9 P^2) = \mathbf{Z}_2\{\nu_7^2 \circ \Sigma^9 p_2\}$ and a short exact sequence

$$0 \longleftarrow \pi^7(\Sigma^8 P^2) \xleftarrow{\Sigma^8 i^*} \pi^7(\Sigma^8 P^3) \xleftarrow{\Sigma^8 p_3^*} \pi_{14}(S^7) \longleftarrow 0.$$

On the 2-component, we have $8\bar{\bar{\nu}}_7 \in -\{8\iota_7, \bar{\nu}'_7, \Sigma^8 \gamma_2\} \circ \Sigma^8 p_3 \subset -\{8\iota_7, \nu_7, 2\nu_{10}\} \circ \Sigma^8 p_3 = 0$. For $\{8\iota_7, \nu_7, 2\nu_{10}\} \supset \{8\iota_7, \nu_7, \bar{\eta}_{10}\} \circ \bar{\eta}_{12} \subset [M^{13}, S^7] \circ \bar{\eta}_{12} = \{\nu_7^2 p \circ \bar{\eta}_{12}\} = \{\nu_7^2 \eta_{13}\} = 0 \pmod{8\pi_{14}(S^7) + \pi_{11}(S^7) \circ 2\nu_{11}} = 0$. So the above sequence splits on the 2-component.

For the odd components, it suffices to consider the stable case.

$3\bar{\bar{\nu}} \in -\langle 3\iota, \bar{\nu}', \gamma_2 \rangle p_3 \subset -\langle 3\iota, \alpha_1, 2\alpha_1 \rangle p_3 \ni -\alpha_2 p_3 \pmod{0}$, where $\bar{\nu}' = \Sigma^\infty \bar{\nu}'_7$. This leads us to i).

By i) and Lemma 5.1, we have ii). This completes the proof.

Now we recall the following [16]: $B_k = 0$ if $k = 1, 2, 4$ or 5 ; $B_k = \pi_k^S(S^0)$ if $k = 3, 6$ or 11 ; $B_7 = \mathbf{Z}_{120}\{2\sigma\}$; $B_8 = \mathbf{Z}_2\{\bar{\nu}\}$; $B_9 = \mathbf{Z}_2\{\nu^3\} \oplus \mathbf{Z}_2\{\eta^2 \sigma\}$; $B_{10} = \mathbf{Z}_3\{\beta_1\}$. We also recall that the image of the stable complex J -homomorphism $J_C : \pi_{2k-1}^S(SU) \rightarrow \pi_{2k-1}^S(S^0)$ is generated by $g_k \zeta_{k-1}$. So we have the following facts: $g_8 \zeta_7 = 2\rho$; $g_9 \zeta_8 = \eta^2 \rho$; $g_{10} \zeta_9 = \bar{\zeta}$.

REMARK 4. i) The order of $g_n \in \pi_S^{-1}(P^{n-1})$ is given by Theorem 7.28 of [5] as follows: $\#g_n = \prod_{p \geq 2} p^{[n/(p-1)]a_n}$, where $[x]$ is the integral part of x and $a_n = 2$ or 1 according as $n \equiv 2 \pmod{4}$ or not.

ii) We know elements which are not in B_* [5]: $\mu_r, \eta\mu_r$ for $r \geq 0$; $\rho_r, \eta\rho_r$ for $r \geq 1$, where $\mu_r \in {}^2\pi_{8r+1}(S^0)$ is an element of the μ -series and ρ_r is a generator of ${}^2J_R(\pi_{8r-1}^S(SO))$.

LEMMA 5.3. $B_{12} = 0$, $B_{13} = \pi_{13}^S(S^0)$, $B_{14} = \pi_{14}^S(S^0)$ and $B_{15} = \mathbf{Z}_{240}\{2\rho\} \oplus \mathbf{Z}_2\{\eta\kappa\}$.

PROOF. The first is trivial. By Proposition 2.3, $g_7(i\beta_1) = \alpha_1\beta_1$. So we have the second.

We recall the elements $\tilde{\sigma} \in \langle i, 2\nu, \sigma \rangle$ and $\tilde{\sigma}' = t_2\tilde{\sigma}$ appearing in §2. By (2), $g_8i\tilde{\sigma}' = g_4\tilde{\sigma}' = g_2(\mathbf{H})\tilde{\sigma}$. Since $2\nu_{10}\sigma_{13} = 0$, we can take an element $\tilde{\sigma}_{13} \in \{i, 2\nu_{10}, \sigma_{13}\}$ such that $\Sigma^\infty\tilde{\sigma}_{13} = \tilde{\sigma}$. By (10), $H(g_2(\mathbf{H}) \circ \Sigma\tilde{\sigma}_{13}) = \pm\sigma_{15}$. So we have $g_2(\mathbf{H}) \circ \Sigma\tilde{\sigma}_{13} = \pm\sigma_8^2 \bmod \{\Sigma\sigma'\sigma_{15}, \kappa_8\}$. Hence we have $g_4\tilde{\sigma}' = \sigma^2$.

By the definition and by (2), we have $g_8(i\widetilde{2\nu''}) = g_6t_3(\widetilde{2\nu'} - 2i\tilde{\sigma}) = g_3(\mathbf{H})\widetilde{2\nu'}$. By use of the cellular decomposition of Q_3 , we can take $g_3(\mathbf{H}) \in \langle \nu, \widetilde{2\nu}, p \rangle$, where $p : Q_3 \rightarrow Q_2^3$ is the projection. So we have $g_3(\mathbf{H})\widetilde{2\nu'} \in \langle \nu, \widetilde{2\nu}, \widetilde{2\nu} \rangle \ni \kappa \bmod \sigma^2$. We note that the secondary composition $\langle \nu, \widetilde{2\nu}, \widetilde{2\nu} \rangle$ becomes a tertiary composition $\langle \nu, 2\nu, \nu, 2\nu \rangle$ [8].

In Remark 2, we proved that $g_8i\tilde{\nu} = g_2(\mathbf{H})\tilde{\nu}' = \eta\kappa$. This completes the proof.

According to Lemma 12.15 of [25], there exists an element $\tilde{\omega}_{14} \in \pi_{30}(S^{14})$ such that $H(\tilde{\omega}_{14}) = \nu_{27}$. By Proposition 12.20 of [25], $\tilde{\omega} \equiv \eta * \bmod \eta\rho$. Here we modify the notation of [25], because we used the notation ω_n to denote $\omega_n(\mathbf{H})$.

LEMMA 5.4. $B_{16} = \mathbf{Z}_2\{\omega^*\}$ and $B_{17} = \mathbf{Z}_2\{\eta\omega^*\} \oplus \mathbf{Z}_2\{\eta^2\rho\} \oplus \mathbf{Z}_2\{\nu\kappa\}$, where $\omega^* = \eta^*$ or $\eta^* + \eta\rho$.

PROOF. By Lemma 5.2, $\bar{\nu}\gamma_3 = g_4\gamma_3 + \sigma\eta = \eta\sigma$. Since $p_3\langle \gamma_3, 8\sigma, 2\iota \rangle \subset \langle \eta, 8\sigma, 2\iota \rangle \ni \mu \bmod \{\eta^2\sigma, \eta\bar{\nu}\}$, we have $\langle \gamma_3, 8\sigma, 2\iota \rangle \ni \tilde{\mu} \bmod \{\gamma_3\eta\sigma, \gamma_3\bar{\nu}\} + i_*\pi_{15}^S(P^2) = \{i\zeta, \gamma_3\eta\sigma, \gamma_3\bar{\nu}\}$ by Proposition 3.5. $g_4i\zeta = g_3\zeta \in \langle \nu, \eta, \zeta \rangle = 0$, $g_4\tilde{\mu} = \sigma\mu + \bar{\nu}\tilde{\mu}$ and $\bar{\nu}\tilde{\mu} \in \bar{\nu}\langle \gamma_3, 8\sigma, 2\iota \rangle \subset \langle \eta\sigma, 8\sigma, 2\iota \rangle \ni \eta\rho = \sigma\mu \bmod 0$. So we have $g_9i\tilde{\mu} = g_4\tilde{\mu} = 0$ and $g_4\pi_{15}^S(P^3) = 0$ by Proposition 3.5.

We have $g_9i\tilde{\nu}'' = g_7\tilde{\nu}'' \in \langle g_6, \gamma_5, \nu \rangle \bmod g_6\pi_{15}^S(P^5) + \pi_{13}^S(S^0) \circ \nu = \{g_5\widetilde{2\sigma'}, g_4\tilde{\mu}\} \equiv 0 \bmod \eta\rho$ by Proposition 3.5, because $g_5\widetilde{2\sigma'} \in \langle g_4, \gamma_3, 2\sigma \rangle \subset \langle g_4, \pm i\zeta_2\nu, \sigma \rangle \supset \langle 0, \nu, \sigma \rangle \ni 0 \bmod \mu\sigma$. We have $\langle g_6, \gamma_5, \nu \rangle \supset \Sigma^\infty\{\Sigma^2g_6, \Sigma^{15}\gamma_5, \nu_{26}\}_1$ and $H\{\Sigma^2g_6, \Sigma^{15}\gamma_5, \nu_{26}\}_1 = -\Delta^{-1}(\Sigma g_6 \circ \Sigma^{14}\gamma_5) \circ \nu_{27} \ni \nu_{27}$. So we have $\{\Sigma^2g_6, \Sigma^{15}\gamma_5, \nu_{26}\}_1 \ni \tilde{\omega}_{14} \bmod \Sigma^2g_6 \circ \pi_{30}(\Sigma^{15}P^5) + \pi_{27}(S^{14}) \circ \nu_{27} + \Sigma\pi_{29}(S^{13})$. Hence we have $g_7\tilde{\nu}'' \equiv \tilde{\omega} \bmod \eta\rho$. We can set $g_7\tilde{\nu}'' = \omega^*$, because $\tilde{\omega} \equiv \eta^* \bmod \eta\rho$ [25]. This leads us to the first.

We have $g_9\zeta_8 = \eta^2\rho$, $g_9i\sigma^2 = \nu\sigma^2 = 0$ and $g_9i\kappa = \nu\kappa$.

From the definition, $\tilde{\nu}''\eta \in -i\langle \gamma_5, \nu, \eta \rangle$. $\langle \gamma_5, \nu, \eta \rangle \equiv \langle \langle i, \gamma_4, \eta \rangle, \nu, \eta \rangle \equiv \langle i, \langle \gamma_4, \eta, \nu \rangle, \eta \rangle + \langle i, \gamma_4, \langle \eta, \nu, \eta \rangle \rangle \ni i\tilde{\nu}^2 \bmod i_*\pi_{16}^S(P^4) + \pi_{15}^S(P^5) \circ \eta$ by Lemma 3.1. $i_*\pi_{16}^S(P^4) = \{i\sigma^2, i\kappa\}$ and $\pi_{15}^S(P^5) \circ \eta = \{i\widetilde{2\sigma'}\eta, i\tilde{\mu}\eta\} \subset i_*\pi_{16}^S(P^3) = \{i\sigma^2, i\kappa\}$, because $\widetilde{2\sigma'}\eta \in -i\langle \gamma_3, 2\sigma, \eta \rangle \subset i_*\pi_{16}^S(P^3)$ and $\tilde{\mu}\eta \in \pi_{16}^S(P^3)$. Hence we have $g_9i\tilde{\nu}^2 \equiv g_7\tilde{\nu}''\eta = \omega^*\eta \bmod i\nu\kappa$. This completes the proof.

REMARK 5. i) By Proposition 2.10, by Lemmas 5.3, 5.4 and their proofs, $\widetilde{2\nu''}\eta = i\tilde{\nu}$ and $\tilde{\nu}''\eta \equiv i\tilde{\nu}^2 \bmod \{i\sigma^2, i\kappa\}$.

ii) According to Morisugi [12], Mahowald's elements are in the image of the

S^3 -transfer. So, by (2), they are in B_* .

LEMMA 5.5. $B_{18} = \mathbf{Z}_8\{\nu^*\}$ and $B_{19} = \pi_{19}^S(S^0)$.

PROOF. By (2) and Proposition 4.15, $g_{10}i\tilde{\sigma}'' = g_6\tilde{\sigma}'' = g_6t_3\tilde{\sigma} = g_3(\mathbf{H})\tilde{\sigma} \in \langle g_2(\mathbf{H}), \omega_2, \sigma \rangle$. On the other hand, we have $\tilde{\sigma}'' \in \langle i, \gamma_4, \sigma \rangle$, $g_6\tilde{\sigma}'' \in \langle g_5, \gamma_4, \sigma \rangle \supset \Sigma^\infty\{\Sigma^2g_5, \Sigma^{13}\gamma_4, \sigma_{22}\}_1$ and $H\{\Sigma^2g_5, \Sigma^{13}\gamma_4, \sigma_{22}\}_1 \ni \sigma_{23}$. So, by Lemma 12.14 of [25] and Proposition 4.8, we have $\{\Sigma^2g_5, \Sigma^{13}\gamma_4, \sigma_{22}\}_1 \ni x\xi_{12} \bmod \Sigma^2g_5 \circ \pi_{30}(\Sigma^{13}P^4) + \pi_{23}(S^{12}) \circ \sigma_{23} + \Sigma\pi_{29}(S^{11})$ and $g_6\tilde{\sigma}'' \in \langle g_5, \gamma_4, \sigma \rangle \ni x\xi = -x\nu^* \bmod \{g_5\lambda, g_5\tilde{\nu}^3, 2\xi, \eta\bar{\mu}\}$ for odd x . By Lemma 1.4, we have $g_5\lambda \in \langle g_4, \gamma_3, \eta^2\sigma \rangle \subset \langle g_4, \gamma_3\eta^2, \sigma \rangle = 4\langle g_4, t_2\omega_2, \sigma \rangle + \langle g_4, 8i\sigma, \sigma \rangle \supset 4\langle g_2(\mathbf{H}), \omega_2, \sigma \rangle + \langle \nu, 8\sigma, \sigma \rangle \ni 4\nu^* + 4\nu^* = 0 \bmod g_4\pi_{17}^S(P^3) = \{g_2(\mathbf{H})\tilde{\zeta}\} = 0$. For $g_2(\mathbf{H})\tilde{\zeta} \in \langle \nu, 2\nu, \zeta \rangle = \langle \nu, \bar{\eta}\tilde{\eta}, \zeta \rangle \supset \langle 0, \tilde{\eta}, \zeta \rangle \ni 0 \bmod 0$. Hence we have $g_5\lambda = 0$.

$g_5\tilde{\nu}^3 \in \langle g_4, \gamma_3, \nu^3 \rangle \supset \Sigma^\infty\{\Sigma^2g_4, \Sigma^{11}\gamma_3, \nu_{18}^3\}_1 \bmod g_4\pi_{17}^S(P^3) + \pi_9^S(S^0) \circ \nu^3 = 0$. By Lemma 12.19 of [25], $H(\lambda'') = \eta_{19}\varepsilon_{20} + a\eta_{19}^2\sigma_{21}$ and $H(\xi'') = \eta_{19}^2\sigma_{21}$, where $a = 0$ or 1. So we have $\{\Sigma^2g_4, \Sigma^{11}\gamma_3, \nu_{18}^3\}_1 \ni \lambda'' + (1+a)\xi'' \bmod \Sigma^2g_4 \circ \pi_{28}(\Sigma^{11}P^3) + \pi_{19}(S^{10}) \circ \nu_{19}^3 + \Sigma\pi_{27}(S^9)$. This shows $g_5\tilde{\nu}^3 \equiv 0 \bmod \{4\nu^*, \eta\bar{\mu}\}$. Hence we have $g_6\tilde{\sigma}'' \equiv x\nu^* \bmod \eta\bar{\mu}$.

We know $\eta^2\bar{\mu} = 4\bar{\zeta}$, $\eta\nu^* = 0$ and $\nu\eta^* = 0$. We have $\eta\langle g_2(\mathbf{H}), \omega_2, \sigma \rangle \subset \langle \langle \eta, \nu, 2\nu \rangle \circ p, \omega_2, \sigma \rangle \subset \langle \langle 2\nu, \nu, \eta \rangle, \nu, \sigma \rangle$. $\langle \langle 2\nu, \nu, \eta \rangle, \nu, \sigma \rangle \equiv \langle 2\nu, \langle \nu, \eta, \nu \rangle, \sigma \rangle + \langle 2\nu, \nu, \langle \eta, \nu, \sigma \rangle \rangle \equiv \langle 2\nu, \bar{\nu}, \sigma \rangle \supset \nu\langle 2\nu, \bar{\nu}, \sigma \rangle = 0 \bmod \langle 2\nu, \nu, \eta \rangle \pi_{11}^S(S^0) + \pi_{12}^S(S^0) \circ \sigma = 0$. This leads us to the first.

By Proposition 4.19, we have $g_{10}i\tilde{\sigma}^2 = g_3\tilde{\sigma}^2 \in \langle \nu, \eta, \sigma^2 \rangle \subset \langle \nu, \eta\sigma, \sigma \rangle \ni \bar{\sigma} \bmod 0$. This completes the proof.

By the above proof, we have $\langle g_2(\mathbf{H}), \omega_2, \sigma \rangle \equiv \langle \nu, 2\nu, \nu, \sigma \rangle \ni \xi \bmod \eta\bar{\mu}$ (cf. (6.1) of [7]).

Lemmas 5.3, 5.4 and 5.5 lead us to Theorem 2.

§6. The group structures of $M(P^n)$ and $\varepsilon(P^n)$ for $n \leq 7$

First we recall the result of [6]. $M(P^n)$ is naturally isomorphic to $\tilde{K}(P^n) \oplus H$, where H is a finite group. Let $D(n, \mathbf{C}) = \text{Im}\{d(g) = (d_1(g), \dots, d_n(g)) \mid g \in M(P^n)\}$, where $d_k(g)$ stands for the degree in homology on g_* in dimension $2k$. Then $D(n, \mathbf{C})$ is a subgroup of $\mathbf{Z}^n$ having a basis $\{(q, \dots, q^n) \mid 1 \leq q \leq n\}$ and $D(n, \mathbf{C}) \cong \tilde{K}(P^n)$. $H = \text{Ker } d$ and $D(n, \mathbf{C})$ becomes a subring of $\mathbf{Z}^n$. We denote by $\varepsilon_n(q) \in [P^n, P^n]$ an element realizing $(q, \dots, q^n)$ which corresponds to $q \in \mathbf{Z} \cong H^2(P^n; \mathbf{Z})$.

We denote by $M'(P^n)$ the torsion-free subgroup of $M(P^n)$ generated by $\{\varepsilon_n(q) \mid 1 \leq q \leq n\}$. $M'(P^n)$ is endowed with a commutative ring structure induced from $D(n, \mathbf{C})$, because $\varepsilon_n(p) \circ \varepsilon_n(q) = \varepsilon_n(pq) = \varepsilon_n(q) \circ \varepsilon_n(p)$. Obviously there is a split epimorphism $M'(P^n) \rightarrow M'(P^{n-1})$ which is induced by the restriction of self

maps.

We set ${}_m C_k = m! / \{(m-k)!k!\}$ and $\iota'_n = \varepsilon_n(1)$. We denote by θ_n an element corresponding to $(-1, 1, \dots, (-1)^n) \in D(n, \mathbf{C})$.

LEMMA 6.1. i) $\varepsilon(P^n)$ contains $\mathbf{Z}_2\{-\iota'_n\} \times \mathbf{Z}_2\{\theta_n\}$ as a direct summand.
 ii) $\theta_n = \sum_{k=1}^n (-1)^k {}_n C_{k+1} \varepsilon_n(k)$.

PROOF. To prove i) it suffices to show that $D(n, \mathbf{C})^* \cong \mathbf{Z}_2 \times \mathbf{Z}_2$, where G^* is the unit group of G . Let $a = (a_1, \dots, a_n) \in \mathbf{Z}^n$ be a unit of $D(n, \mathbf{C})$, where $a_k^2 = 1$ for $1 \leq k \leq n$. Then the following is obtained from the elementary matrix theory: A system of Diophantine equations $x_1 + 2^k x_2 + \dots + n^k x_n = a_k$ for $1 \leq k \leq n$ has a solution if and only if $a_k = 1, -1, (-1)^k$ and $-(-1)^k$ for $1 \leq k \leq n$. So we have i).

By use of the Cramer's rule, we have ii). This completes the proof.

LEMMA 6.2. $M(P^n)$ has no torsion for $n \leq 6$.

PROOF. The assertion holds for $n = 1$ since $M(S^2) \cong \mathbf{Z}$. We consider an exact sequence induced by (6):

$$\{P^{n-1}, P^n\} \xleftarrow{i^*} M(P^n) \xleftarrow{p^*} \pi_{2n}^S(P^n) \xleftarrow{\Sigma \gamma_{n-1}^*} \{\Sigma P^{n-1}, P^n\}.$$

If $n = 2, 3$ and 5 , $\pi_{2n}^S(P^n) \cong \mathbf{Z}$ by Remark 1. So the assertion holds for $n = 2$ and 3 . Since $\pi_8^S(P^4) = \mathbf{Z}\{\zeta_4\} \oplus \mathbf{Z}_2\{i\nu^2\}$ and $i\nu^2 = i\nu q_1 \gamma_3$ by (1), we have the assertion for $n = 4$ and 5 .

By Proposition 2.3, $\pi_{12}^S(P^6) = \mathbf{Z}\{\zeta_6\} \oplus \mathbf{Z}_3\{i\beta_1\}$. By (1), $\alpha_1 q_1 \gamma_3 = \alpha_1 \alpha_1 = 0$. So $\alpha_1 q_1 \in \{P^3, S^1\} = \pi_5^1(P^3)$ is extendable on P^4 . Since $\pi_k^S(S^0)$ is a 2-primary torsion group for $k = 8$ and 9 , there exists an extension $\overline{\alpha_1 q_1} \in \pi_5^1(P^5)$ of $\alpha_1 q_1$. By Proposition 2.1 and its proof, we have $i\overline{\alpha_1 q_1} \circ 4\gamma_5 \in 2i\langle \alpha_1 q_1, \gamma_3, 4\nu \rangle \subset i\langle \alpha_1, \nu, 8\nu \rangle \supset i\langle \alpha_1, \alpha_1, \alpha_1 \rangle \ni i\beta_1 \bmod 0$. So we have $p^* \pi_{12}^S(P^6) = \mathbf{Z}\{\zeta_6 p\}$. This leads us to the assertion for $n = 6$ and completes the proof.

By use of (6) and (1), we have the following

LEMMA 6.3. $\pi_S^{-1}(P^2) = \mathbf{Z}_{24}\{\bar{\nu}'\}$, $\pi_S^{-2}(P^2) = \mathbf{Z}_2\{\nu^2 p_2\}$ and $\pi_S^{-2}(P^3) = \mathbf{Z}_2\{\nu^2 q_1\} \oplus \mathbf{Z}_2\{\eta \sigma p_3\} \oplus \mathbf{Z}_2\{\bar{\nu} p_3\}$.

We set $\alpha = \nu^2 q_1 + \bar{\nu} p_3$. We can define an extension $\bar{\alpha} \in \langle \alpha, \gamma_3, p_4 \rangle$ of α since $\alpha \gamma_3 = \nu^3 + \bar{\nu} \eta = 0$ by (1). $\bar{\alpha}$ is extendable over P^5 since $\bar{\alpha} \gamma_4 = \nu^2 q_1 t_2 \omega_2 + \bar{\nu} \nu = 0$. This extension is denoted by $\bar{\alpha} \in \langle \bar{\alpha}, \gamma_4, p_5 \rangle$.

LEMMA 6.4. i) $\pi_S^{-3}(P^3) = \mathbf{Z}_{240}\{\sigma q_1\} \oplus \mathbf{Z}_2\{\mu p_3\} \oplus \mathbf{Z}_2\{\eta^2 \sigma p_3\} \oplus \mathbf{Z}_2\{\nu^3 p_3\}$.

ii) $\pi_S^{-2}(P^4) = \mathbf{Z}_2\{\bar{\alpha}\} \oplus \mathbf{Z}_3\{\beta_1 p_4\}$ and $\pi_S^{-2}(P^5) = \mathbf{Z}_2\{\bar{\alpha}\} \oplus \mathbf{Z}_3\{\beta_1 q_2\}$.

iii) $\gamma_6^* : \pi_S^{-2}(P^6) \rightarrow \pi_{15}^S(S^0)$ is trivial and $p_7^* : \pi_{15}^S(S^0) \rightarrow \pi_S^{-1}(P^7)$ is a monomorphism.

PROOF. By use of a cofibre sequence starting with the inclusion $S^2 \hookrightarrow P^3$, we have i).

By use of (6), we have an exact sequence:

$$\pi_9^S(S^0) \xleftarrow{\gamma_3^*} \pi_9^{-2}(P^3) \xleftarrow{i^*} \pi_9^{-2}(P^4) \xleftarrow{P_4^*} \pi_{10}^S(S^0) \xleftarrow{\gamma_3^*} \pi_9^{-3}(P^3).$$

By i), $\mu p_3 \gamma_3 = \eta \mu$ and $\eta^2 \sigma p_3 \gamma_3 = \nu^3 p_3 \gamma_3 = \sigma q_1 \gamma_3 = 0$. So, by Lemma 6.3, we have the first of ii).

By an argument parallel to the above and by (1), we have the second of ii).

We have $\beta_1 q_2 \gamma_5 = 2\beta_1 \nu = \pm \alpha_1 \beta_1$ by (1) and $\bar{\alpha} \gamma_5 = 0$. For $0 = 2\bar{\alpha} \gamma_5 \in \pi_{13}^S(S^0) \cong \mathbf{Z}_3$. So, by use of (6), we have $\pi_9^{-2}(P^6) = \{\bar{\alpha}', \sigma^2 p_6, \kappa p_6\}$, where $\bar{\alpha}' \in \langle \bar{\alpha}, \gamma_5, p_6 \rangle$. We have $\bar{\alpha}' \gamma_6 = \bar{\alpha} t_3 \omega_3 \in \langle \bar{\alpha}, \gamma_4, 4\nu \rangle \supset \langle \alpha, t_2 \omega_2, 4\nu \rangle \subset \langle \nu^2 q_1, t_2 \omega_2, 4\nu \rangle + \langle \bar{\nu} p_3, t_2 \omega_2, 4\nu \rangle \subset \langle \nu^2, 0, 4\nu \rangle + \langle \bar{\nu}, \nu, 4\nu \rangle \ni 0 \pmod{\bar{\alpha} \pi_{13}^S(P^4)} = \{\alpha \tilde{\sigma}', \alpha i \tilde{\eta} \varepsilon\} = 0$. For $\langle \bar{\nu}, \nu, 4\nu \rangle \supset 2\langle \bar{\nu}, \nu, 2\nu \rangle \ni 2\eta \kappa = 0 \pmod{0, \alpha \tilde{\sigma}'} = \alpha t_2 \tilde{\sigma} = \nu^2 q_1 t_2 \tilde{\sigma} + \bar{\nu} \sigma = 0$ and $\alpha i \tilde{\eta} \varepsilon = \nu^2 \eta \varepsilon = 0$. Hence we have iii). This completes the proof.

By remark 2 and Lemma 6.4, we have $g_8 i \tilde{\nu} p_7 = \eta \kappa p_7 \neq 0$. Hence, by Proposition 2.10, Lemmas 6.1 and 6.2, we have the following

LEMMA 6.5. i) $M(P^n) \cong D(n, \mathbf{C})$ for $2 \leq n \leq 6$ and $M(P^7) \cong D(7, \mathbf{C}) \oplus \mathbf{Z}_2\{i \tilde{\nu} p_7\}$.
 ii) $\varepsilon(P^n) = \mathbf{Z}_2\{-\iota'_n\} \oplus \mathbf{Z}_2\{\theta_n\}$ for $2 \leq n \leq 6$ and $\varepsilon(P^7) = \mathbf{Z}_2\{-\iota'_7\} \oplus \mathbf{Z}_2\{\theta_7\} \oplus \mathbf{Z}_2\{\iota'_7 - i \tilde{\nu} p_7\}$.

By Lemmas 6.1 and 6.5, we have completed the proof of Theorem 3.

REMARK 6. By [1] and [22], we have an exact sequence for $n \geq 3$:

$$i_* \pi_{2n}^S(P^{n-1}) \xrightarrow{\lambda} \varepsilon(P^n) \xrightarrow{j^*} \varepsilon(P^{n-1}).$$

Here j^* is the restriction and $\lambda(i\alpha) = \iota'_n + i\alpha p$ for $\alpha \in \pi_{2n}^S(P^{n-1})$. By use of this exact sequence, we have ii) of Lemma 6.5.

REMARK 7. By inspecting the proof of Lemma 6.1, we have the following: The assertion of Lemma 6.1 still holds for the unstable case, i.e., for the group of self-homotopy equivalence classes of $\Sigma \mathbf{C}P^n$.

Finally we state an additional remark. The first was proved by Morisugi. ii) and iii) are obtained from using the James splitting theorem.

REMARK 8. i) $\zeta_{n-1} \eta_{2n-1} = (n!/2) \Sigma \gamma_{n-1}$ for $n \geq 2$.

ii) $6i \zeta_{n-1} \nu = \zeta_n \eta$ for $n \geq 3$.

iii) $\zeta_n p_n = \sum_{k=1}^n (-1)^{n+k} {}_n C_k \varepsilon_n(k)$.

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