Vector valued Siegel modular forms and their L -functions; Application of a differential operator

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Introduction

As for automorphic L -functions attached to holomorphic Siegel modular forms for $Sp(n, \mathbb{Z})$ ($n \geq 2$) in the sense of Langlands [18, 19], it is known so far that two types of them, namely the spinor type and the standard type, can be continued analytically to the whole complex plane. They are L -functions corresponding to the spinor representation of $SO(2n+1, \mathbb{C})$ and to the standard representation of $SO(2n+1, \mathbb{C})$ respectively.

Spinor L -functions were introduced by Langlands [18] and Andrianov [1] and for scalar valued cases of $n = 2$, Andrianov [1] proved their meromorphic continuation and the functional equations. Moreover, their poles were studied by Evdokimov [10] and Oda [23]. For vector valued cases of $n = 2$, Arakawa [4] and Sugano [26] have discovered that the spinor L -function is continued analytically as an entire function.

On the other hand, standard L -functions were also introduced by Langlands [18] and Andrianov [2]. For scalar valued cases, Andrianov-Kalinin [3] (in special cases) and Böcherer [7] (in general; see also Piatetski-Shapiro and Rallis [24]) proved their meromorphic continuation and the functional equations. Recently Mizumoto [21] has obtained results on their poles and residues. We will get the functional equations and poles of standard L -functions explicitly for a certain vector valued case (cf. Piatetski-Shapiro and Rallis [24], Weissauer [28]).

Suppose n is a positive integer, k is a positive even integer, s is a complex number such that $k + 2\operatorname{Re}(s) > n + 1$, and Z belongs to the Siegel upper half space $\mathfrak{H}_n$ (of degree n). The way of constructing integral representation of the standard L -function by the use of Eisenstein series $E_k^n(Z, s)$ (defined in §1) is based on Böcherer [7] and the method of determining poles by the use of its integral representation is based on Mizumoto [21]. We use the differential operator $L^{k,l}$ (defined in §2) introduced by Böcherer-Satoh-Yamazaki [9] to transform the scalar valued Eisenstein series into a vector valued function.

In the process of constructing the integral representation, we use the Garrett's pullback formula [12] to decompose $(L^{k,l}E_k^{2n})\left(\begin{pmatrix} Z & 0 \\ 0 & W \end{pmatrix}, s\right)$, $Z, W \in \mathfrak{H}_n$, into functions for Z and for W . The pullback sends the Eisenstein series $E_k^{m+n}\left(\begin{pmatrix} Z & 0 \\ 0 & W \end{pmatrix}, s\right)$, $Z \in \mathfrak{H}_m$, $W \in \mathfrak{H}_n$, to modular forms for Z and for W and it is studied by Garrett [12] for the holomorphic case (i.e., $s = 0$) and by Böcherer [7] for real analytic cases. Moreover Böcherer-Satoh-Yamazaki [9] deals with the pullback formula of the vector valued function $(L^{k,l}E_k^{m+n})\left(\begin{pmatrix} Z & 0 \\ 0 & W \end{pmatrix}, 0\right)$.

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NOTATIONS.

1°. As usual, $\mathbb{Z}$ is the ring of rational integers, $\mathbb{Q}$ the field of rational numbers, $\mathbb{R}$ the field of real numbers, $\mathbb{C}$ the field of complex numbers.

2°. Let $m, n \in \mathbb{Z}$, $m, n > 0$. If A is an $m \times n$ -matrix, then we write it also as $A^{(m,n)}$, and as $A^{(m)}$ if $m = n$. The identity matrix of size n is denoted by 1_n .

3°. For $m, n \in \mathbb{Z}$, $m, n > 0$, and a commutative ring R containing 1, let $R^{(m,n)}$ (resp. $R^{(n)}$) be the R -module of all $m \times n$ (resp. $n \times n$) matrices with entries in R .

4°. For a real symmetric positive definite matrix S , $S^{1/2}$ is the unique real symmetric positive definite matrix such that $(S^{1/2})^2 = S$.

5°. For matrices $A^{(m)}$, $B^{(m,n)}$, we define $A[B] := {}^t\bar{B}AB$, where tB is the transpose of B and $\bar{B}$ is the complex conjugate of B .

6°. For $n \in \mathbb{Z}$, $n > 0$, we put

$$\mathbb{T}^{(n)} := \left\{ T = \begin{pmatrix} t_1 & & & 0 \\ & t_2 & & \\ & & \ddots & \\ 0 & & & t_n \end{pmatrix} \in \mathbb{Z}^{(n)} \right. \\ \left. \left| \begin{array}{l} t_i > 0 \text{ for each } i \in \mathbb{Z} \text{ with } 1 \leq i \leq n, \\ t_1 | \cdots | t_n \end{array} \right. \right\}.$$

7°. For $n \in \mathbb{Z}$, $n > 0$, let $\Gamma^n := Sp(n, \mathbb{Z})$ be the Siegel modular group of degree n and let $\mathfrak{H}_n$ be the Siegel upper half space of degree n , that is,

$$\mathfrak{H}_n := \{Z = X + iY \in \mathbb{C}^{(n)} \mid {}^tZ = Z, Y > 0\}.$$

For $N \in \mathbb{Z}$, $N > 0$, we put

$$\Gamma_0^n(N) := \left\{ \begin{pmatrix} A^{(n)} & B^{(n)} \\ C^{(n)} & D^{(n)} \end{pmatrix} \in \Gamma^n \mid C \equiv 0 \pmod{N} \right\}.$$

For each $r \in \mathbb{Z}$ with $0 \leq r \leq n$, we put

$$P_{n,r} := \left\{ \begin{pmatrix} * & * \\ C^{(n)} & D^{(n)} \end{pmatrix} \in \Gamma^n \mid C = \begin{pmatrix} 0 & 0 \\ 0 & C_4^{(r)} \end{pmatrix}, D = \begin{pmatrix} * & 0 \\ * & D_4^{(r)} \end{pmatrix} \right\}.$$

All these are subgroups of Γ^n .

8°. For $l \in \mathbb{Z}$, $l \geq 0$, we put

$$(a)_l := \begin{cases} a(a+1) \cdots (a+l-1), & \text{if } l > 0, \\ 1, & \text{if } l = 0. \end{cases}$$

The symbol $(\)_l$ is called the Pochhammer symbol. Note $(a)_l = (-1)^l (-a-l+1)_l$. We also have

$$(a+b)_l = \sum_{r=0}^l \binom{l}{r} (a)_r (b)_{l-r},$$

where $\binom{l}{r} = \frac{l!}{r!(l-r)!}$ is the binomial coefficient.

9°. For $n \in \mathbb{Z}$, $n \geq 0$, we put

$$\Gamma_n(s) := \prod_{j=1}^n \Gamma\left(s - \frac{j-1}{2}\right), \quad \Gamma_{\mathbb{R}}(s) := \pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right), \quad \Gamma_{\mathbb{C}}(s) := 2(2\pi)^{-s} \Gamma(s)$$

and

$$\xi(s) := \Gamma_{\mathbb{R}}(s) \zeta(s) = \xi(1-s),$$

where $\Gamma(s)$ is the gamma function and $\zeta(s)$ is the Riemann zeta function. Throughout the paper we understand that the empty product is equal to 1.

§1. Preliminary

1.1. Vector valued Siegel modular forms

Let ρ be a finite-dimensional representation of $GL(n, \mathbb{C})$ with a representation space $\mathcal{V}$. By definition, $\mathcal{V}$ -valued C^∞ -Siegel modular forms of weight ρ are C^∞ -functions from $\mathfrak{H}_n$ to $\mathcal{V}$ satisfying

$$(f|M)(Z) = f(Z)$$

for all $Z \in \mathfrak{H}_n$ and $M = \begin{pmatrix} A^{(n)} & B^{(n)} \\ C^{(n)} & D^{(n)} \end{pmatrix} \in \Gamma^n$, where

$$(f|M)(Z) := \rho((CZ + D)^{-1})f(M\langle Z \rangle) \text{ and } M\langle Z \rangle := (AZ + B)(CZ + D)^{-1}.$$

The space of all such functions is denoted by $M_\rho^n(\mathcal{V})^\infty$.

A function f from $\mathfrak{H}_n$ to $\mathcal{V}$ is called a $\mathcal{V}$ -valued Siegel modular form of weight ρ if it satisfies the following properties:

- (i) holomorphic on $\mathfrak{H}_n$
- (ii) for all $M \in \Gamma^n$, $(f|M)(Z) = f(Z)$
- (iii) holomorphic at the cusps if $n = 1$.

The space of $\mathcal{V}$ -valued Siegel modular forms of weight ρ is denoted by $M_\rho^n(\mathcal{V})$.

We define the Siegel operator Φ on $M_\rho^n(\mathcal{V})$ by

$$(\Phi f)(Z) := \lim_{t \rightarrow \infty} f\left(\begin{pmatrix} Z & 0 \\ 0 & it \end{pmatrix}\right)$$

for $Z \in \mathfrak{H}_{n-1}$. Let $\mathcal{V}'$ be the subspace of $\mathcal{V}$ generated by the values of Φf for all $f \in M_\rho^n(\mathcal{V})$. Then $\mathcal{V}'$ is invariant under the transformation

$$\rho\left(\begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix}\right), \quad g \in GL(n-1, \mathbb{C}).$$

If we assume $\mathcal{V}' \neq \{0\}$, we get the representation ρ' of $GL(n-1, \mathbb{C})$ with the representation space $\mathcal{V}'$. Thus the operator Φ defines the map

$$\Phi: M_\rho^n(\mathcal{V}) \longrightarrow M_{\rho'}^{n-1}(\mathcal{V}').$$

Suppose $f \in M_\rho^n(\mathcal{V})$. Then it is called a cusp form if it satisfies $\Phi f = 0$, and we put

$$S_\rho^n(\mathcal{V}) := \{f \in M_\rho^n(\mathcal{V}) \mid f \text{ is a cuspform}\}.$$

Each $f \in M_\rho^n(\mathcal{V})$ has a Fourier expansion

$$f(Z) = \sum_{H \geq 0} a(H) \exp(2\pi i \operatorname{trace}(HZ)), \quad a(H) \in \mathcal{V},$$

with H running over all symmetric positive semi-definite semi-integral matrices of size n . If $f \in S_\rho^n(\mathcal{V})$, H runs over all symmetric positive definite semi-integral matrices.

Let $\mathcal{V} \cong \oplus \mathcal{V}_i$ be a decomposition of $\mathcal{V}$ into a direct sum of irreducible representations. Then we get a decomposition

$$M_\rho^n(\mathcal{V}) \cong \oplus M_{\rho_i}^n(\mathcal{V}_i),$$

where for each i , ρ_i is a representation of $GL(n, \mathbb{C})$ with the representation space $\mathcal{V}_i$.

If ρ is an irreducible rational representation, ρ is equivalent to an irreducible rational representation $\tilde{\rho}$ satisfying the following condition: Let $\tilde{\mathcal{V}}$ be the representation space of $\tilde{\rho}$. Then, there exists a unique one-dimensional vector subspace $\mathbb{C}\tilde{v}$ of $\tilde{\mathcal{V}}$ such that for any upper triangular matrix of $GL(n, \mathbb{C})$,

$$\tilde{\rho} \left(\begin{pmatrix} g_{11} & & * \\ & \ddots & \\ 0 & & g_{nn} \end{pmatrix} \right) \tilde{v} = \left(\prod_{j=1}^n g_{jj}^{\lambda_j} \right) \tilde{v},$$

where $(\lambda_1, \lambda_2, \dots, \lambda_n) \in \mathbb{Z}^n$ and $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n$.

Then we write

$$(1.1) \quad \rho \sim (\lambda_1, \lambda_2, \dots, \lambda_n).$$

Suppose $\rho \sim (\lambda_1, \lambda_2, \dots, \lambda_n)$. We note that $M_\rho^n(\mathcal{V}) = \{0\}$ if $\lambda_n < 0$ and that $M_\rho^n(\mathcal{V})^\infty = \{0\}$ if $\lambda_1 + \dots + \lambda_n \not\equiv 0 \pmod{2}$.

Let $k, l \in \mathbb{Z}$, $k > 0$, $l \geq 0$. For a vector space W , we denote by $\text{sym}^l(W)$ its l -th symmetric tensor product. We identify $\text{sym}^0(W)$ with $\mathbb{C}$. Let $x = (x_1, \dots, x_n)$ be a row vector consisting of n indeterminates. Then we put $V = \mathbb{C}x_1 \oplus \dots \oplus \mathbb{C}x_n$. We identify $\text{sym}^l(V)$ with the subset of $\mathbb{C}[x_1, \dots, x_n]$ which consists of the homogeneous polynomials of degree l . Let ρ be a representation $\det^k \otimes \text{sym}^l$ of $GL(n, \mathbb{C})$ with a representation space $\text{sym}^l(V)$. Then for each $g \in GL(n, \mathbb{C})$, $\rho(g)$ acts on $\text{sym}^l(V)$ by

$$\rho(g)(v(x)) = \det g^k v(xg), \quad v = v(x) \in \text{sym}^l(V).$$

We note that $\rho \sim (k+l, k, \dots, k)$ and $\dim(\text{sym}^l(V)) = \binom{n+l-1}{l}$.

From now on, we put $\rho = \det^k \otimes \text{sym}^l$ unless stated otherwise.

We write $M_{k,l}^n(\text{sym}^l(V))^\infty$, $M_{k,l}^n(\text{sym}^l(V))$ and $S_{k,l}^n(\text{sym}^l(V))$ for $M_\rho^n(\text{sym}^l(V))^\infty$, $M_\rho^n(\text{sym}^l(V))$ and $S_\rho^n(\text{sym}^l(V))$, respectively.

REMARKS. (i) For $n = 2$, any finite-dimensional rational representation of $GL(n, \mathbb{C})$ is equivalent to a direct sum of representations $\rho = \det^k \otimes \text{sym}^l$ for $k, l \in \mathbb{Z}$.

(ii) If $nk \not\equiv l \pmod{2}$, $M_{k,l}^n(\text{sym}^l(V))^\infty = \{0\}$.

As above, let $V = \mathbb{C}x_1 \oplus \dots \oplus \mathbb{C}x_n$, $x = (x_1, \dots, x_n)$.

For $\sum_{j=1}^n a_j x_j, \sum_{j=1}^n b_j x_j \in V$, we define an inner product of them by

$$\left\langle \sum_{j=1}^n a_j x_j, \sum_{j=1}^n b_j x_j \right\rangle := \sum_{j=1}^n a_j \bar{b}_j.$$

The inner product induces an inner product of $\text{sym}^l(V)$ defined by

$$\langle v_1 \cdots v_l, w_1 \cdots w_l \rangle := \frac{1}{l!} \sum_{\tau \in \mathfrak{S}_l} \prod_{j=1}^l \langle v_{\tau(j)}, w_j \rangle,$$

where $v_j, w_j \in V$. It is also denoted by $\langle \cdot, \cdot \rangle$. Then, for $v = v(x), w = w(x) \in \text{sym}^l(V)$, the inner product has the following properties:

- (i) $\langle v(x), w(x) \rangle = \overline{\langle w(x), v(x) \rangle},$
- (ii) $\begin{aligned} \langle \rho(g)v(x), \rho(g')w(x) \rangle &= \langle \rho({}^t \bar{g}' g)v(x), w(x) \rangle \\ &= \langle v(x), \rho({}^t \bar{g} g')w(x) \rangle \end{aligned}$

for $g, g' \in GL(n, \mathbb{C})$, and

- (iii) $\langle \rho(U)v(x), \rho(U)w(x) \rangle = \langle v(x), w(x) \rangle$

for any $U \in U(n, \mathbb{C})$.

Suppose $f, g \in M_{k,l}^n(\text{sym}^l(V))^\infty$. The Petersson inner product of f and g is defined by

$$\begin{aligned} (f, g) &:= \int_{\Gamma^n \backslash \mathfrak{H}_n} \left\langle \rho\left(\sqrt{\text{Im}(W)}\right) f(W), \rho\left(\sqrt{\text{Im}(W)}\right) g(W) \right\rangle \\ &\quad \times \det(\text{Im}(W))^{-n-1} dX dY \end{aligned}$$

if the right-hand side is convergent. Here $W = X + iY$ with real matrices $X = (x_{jh})$ and $Y = (y_{jh})$;

$$(1.2) \quad dX := \prod_{j \leq h} dx_{jh}, \quad dY := \prod_{j \leq h} dy_{jh};$$

the integral is taken over a fundamental domain of $\Gamma^n \backslash \mathfrak{H}_n$. We write $dW = dX dY$ when there is no fear of confusion.

1.2. Hecke algebra and Hecke operator

We put

$$\begin{aligned}\Delta^n &:= G^+ Sp(n, \mathbb{Q}) \\ &= \left\{ M \in GL(2n, \mathbb{Q}) \mid {}^t M \begin{pmatrix} 0 & 1_n \\ -1_n & 0 \end{pmatrix} M = \mu(M) \begin{pmatrix} 0 & 1_n \\ -1_n & 0 \end{pmatrix}, \right. \\ &\quad \left. \mu(M) > 0 \right\},\end{aligned}$$

and for a prime number p

$$\Delta_p^n := \Delta^n \cap GL\left(2n, \mathbb{Z} \begin{bmatrix} 1 \\ p \end{bmatrix}\right).$$

Let $\mathcal{H}$ (resp. $\mathcal{H}_p$) be a free $\mathbb{C}$ -module generated by the double cosets $\Gamma^n g \Gamma^n$, $g \in \Delta^n$ (resp. Δ_p^n). Then $\mathcal{H}$ is a commutative algebra and $\mathcal{H} = \otimes_p \mathcal{H}_p$. Moreover, the structure of $\mathcal{H}_p$ is known: For $1 \leq j \leq n$, let w_j be an automorphism of $\mathbb{C}[X_0^{\pm 1}, \dots, X_n^{\pm 1}]$ such that

$$w_j(X_0) = X_0 X_j, \quad w_j(X_j) = X_j^{-1} \text{ and } w_j(X_i) = X_i \quad (1 \leq i \leq n, i \neq j).$$

The automorphisms w_j ($1 \leq j \leq n$) and the symmetric group $\mathfrak{S}_n$ consisting of permutations of X_i ($1 \leq i \leq n$) generate a finite group W . We denote by $\mathbb{C}[X_0^{\pm 1}, \dots, X_n^{\pm 1}]^W$ the W -invariant subalgebra of $\mathbb{C}[X_0^{\pm 1}, \dots, X_n^{\pm 1}]$. Then

$$\mathcal{H}_p \cong \mathbb{C}[X_0^{\pm 1}, \dots, X_n^{\pm 1}]^W,$$

where the isomorphism can be written explicitly.

For $g \in \Delta_n$, let $\Gamma^n g \Gamma^n = \bigcup_{i=1}^r \Gamma^n g_i$ be the decomposition of $\Gamma^n g \Gamma^n$ into left cosets.

For $f \in M_{k,l}^n(\text{sym}^l(V))$ (resp. $S_{k,l}^n(\text{sym}^l(V))$, $M_{k,l}^n(\text{sym}^l(V))^\infty$), we put

$$f \mapsto f|(\Gamma^n g \Gamma^n) := \sum_{i=1}^r f|g_i.$$

Then we get a homomorphism

$$\mathcal{H} \longrightarrow \text{End}(M_{k,l}^n(\text{sym}^l(V)))$$

(resp. $\text{End}(S_{k,l}^n(\text{sym}^l(V)))$, $\text{End}(M_{k,l}^n(\text{sym}^l(V))^\infty)$).

Suppose $f \in M_{k,l}^n(\text{sym}^l(V))$ is an eigenform, i.e., a non-zero common eigenfunction of the Hecke algebra. It follows from

$$\Gamma^n g \Gamma^n \mapsto f|(\Gamma^n g \Gamma^n) = \lambda(\Gamma^n g \Gamma^n) f$$

that a homomorphism

$$\lambda : \mathcal{H}_p \xrightarrow{\cong} \mathbb{C}[X_0^{\pm 1}, \dots, X_n^{\pm 1}]^W \xrightarrow{(X_0, \dots, X_n) \mapsto (\alpha_{0,p}, \dots, \alpha_{n,p})} \mathbb{C}$$

with $(\alpha_{0,p}, \dots, \alpha_{n,p}) \in (\mathbb{C}^{*n+1})^W$, is defined. The numbers $\alpha_{0,p}, \dots, \alpha_{n,p}$ are uniquely determined modulo W and they are called the Satake p -parameters of f .

1.3. Standard L -function

Let $f \in S_{k,l}^n(\mathrm{sym}^l(V))$ be an eigenform. We define the standard L -function by

$$(1.3) \quad D_f(s) := \prod_p \left\{ (1 - p^{-s}) \prod_{i=1}^n (1 - \alpha_{i,p}^{-1} p^{-s}) (1 - \alpha_{i,p} p^{-s}) \right\}^{-1},$$

where p runs over all prime numbers. The right-hand side of (1.3) converges absolutely and locally uniformly for $\mathrm{Re}(s) > n + 1$.

We also define the following series:

$$(1.4) \quad L(s, f) := \sum_{T \in \mathbb{T}^{(n)}} \lambda(f, T) \det(T)^{-s},$$

where $\lambda(f, T)$ is an eigenvalue on f of the Hecke operator $\Gamma^n \begin{pmatrix} T & 0 \\ 0 & T^{-1} \end{pmatrix} \Gamma^n$, $T \in \mathbb{T}^{(n)}$. The right-hand side of (1.4) also converges absolutely and locally uniformly for $\mathrm{Re}(s) > 2n + 1$. By Böcherer [8], we have:

$$(1.5) \quad \zeta(s) \prod_{i=1}^n \zeta(2s - 2i) L(s, f) = D_f(s - n).$$

Here we mention a fact on poles of standard L -functions, (1.3.1) below:

Let $H_{m,n}(\rho)$ be the (finite-dimensional) $\mathbb{C}$ -vector space of polynomial maps

$$P : \mathbb{C}^{(m,n)} \longrightarrow \mathrm{sym}^l(V)$$

satisfying the following two conditions:

$$(i) \quad \det g^{m/2} P(Xg) = \rho({}^t g) P(X) \text{ for all } g \in GL(n, \mathbb{C}), X \in \mathbb{C}^{(m,n)},$$

$$(ii) \quad \sum_{j=1}^m \frac{\partial}{\partial x_{ji}} \frac{\partial}{\partial x_{jk}} P(X) = 0 \text{ for } X = (x_{ij}).$$

Such polynomials are called pluriharmonic.

Let $\mathcal{S}_m$ be the set of all symmetric positive definite even integral unimodular matrices of size m . As is well-known, $\mathcal{S}_m \neq \emptyset$ if and only if $m \equiv 0 \pmod{8}$. For each $P \in H_{m,n}(\rho)$ and $S \in \mathcal{S}_m$ with $m \equiv 0 \pmod{8}$, we attach the theta series

$$\vartheta_{S,P}(Z) := \sum_{G \in \mathbb{Z}^{(m,n)}} P(S^{1/2}G) \exp(\pi i \operatorname{trace}(S[G]Z))$$

which converges for all $Z \in \mathfrak{H}_n$ and belongs to $M_{k,l}^n(\operatorname{sym}^l(V))$. We denote by $B_{k,l}^n(m)$ the subspace of $M_{k,l}^n(\operatorname{sym}^l(V))$ of all finite sums of theta series $\vartheta_{S,P}$. The space $B_{k,l}^n(m)$ is invariant under the action of the Hecke algebra. We note that $B_{k,l}^n(m) = \{0\}$ if $m \not\equiv 0 \pmod{8}$ or $m > 2k$.

Under the notation above, we have the following by Weissauer [28]:

(1.3.1) For $n, k, l \in \mathbb{Z}$, $n, k, l > 0$, let $f \in S_{k,l}^n(\operatorname{sym}^l(V))$ be an eigenform. Suppose $k > n$. Then $D_f(s)$ has a simple pole at $s = 1$ if and only if $f \in B_{k,l}^n(2n) \cap S_{k,l}^n(\operatorname{sym}^l(V))$.

1.4. Eisenstein series

For scalar valued cases (i.e., $l = 0$), we write M_k^∞ , M_k^n and S_k^n for $M_{k,0}^n(\operatorname{sym}^0(V))^\infty$, $M_{k,0}^n(\operatorname{sym}^0(V))$ and $S_{k,0}^n(\operatorname{sym}^0(V))$, respectively.

For $k \in 2\mathbb{Z}$, $k > 0$, $s \in \mathbb{C}$ and $Z \in \mathfrak{H}_n$, we define the Eisenstein series by

$$E_k^n(Z, s) := \sum_{M = \begin{pmatrix} * & * \\ C^{(n)} & D^{(n)} \end{pmatrix} \in P_{n,0} \backslash \Gamma^n} \det(CZ + D)^{-k} \det(\operatorname{Im}(M\langle Z \rangle))^s.$$

Then $E_k^n(Z, s) \in M_k^\infty$. The function $E_k^n(Z, s) \det(\operatorname{Im}(Z))^{-s}$ converges absolutely and locally uniformly for $k + 2\operatorname{Re}(s) > n + 1$. Moreover we have the following by Mizumoto [21, 22] (see also Andrianov-Kalinin [3], Kalinin [15], Langlands [20]):

(1.4.1) Let $n, k \in \mathbb{Z}$, $n, k > 0$, k : even. Then for $Z \in \mathfrak{H}_n$,

$$(1.6) \quad \mathbb{E}_k^n(Z, s) := \frac{\Gamma_n(s + \frac{k}{2})}{\Gamma_n(s)} \xi(2s) \prod_{i=1}^{\lfloor \frac{n}{2} \rfloor} \xi(4s - 2i) E_k^n\left(Z, s - \frac{k}{2}\right)$$

is invariant under $s \mapsto \frac{n+1}{2} - s$ and it is an entire function in s .

It is also known that every partial derivative (in z_{jh} 's) of the Eisenstein series $E_k^n(Z, s)$ is slowly increasing (locally uniformly in s). That is, by Mizumoto [22] we have:

(1.4.2) Let $n, k \in \mathbb{Z}$, $n, k > 0$, k : even.

For each $s_0 \in \mathbb{C}$, we can take $d \in \mathbb{Z}$, $d > 0$ and a suitable neighborhood $\mathcal{U}$ of s_0 depending only on n , k and s_0 such that $(s - s_0)^d E_k^n(Z, s)$ is holomorphic in s on $\mathcal{U}$. Then, for $s \in \mathcal{U}$, $l \in \mathbb{Z}$, $l \geq 0$, $\text{Im}(Z) \geq \varepsilon 1_n$ ($\varepsilon > 0$), there exist positive constants α , β depending only on n , k , l , ε , s_0 , d and $\mathcal{U}$ such that

$$\left| (s - s_0)^d \frac{\partial^l}{\partial z_{j_1 h_1} \cdots \partial z_{j_l h_l}} E_k^n(Z, s) \right| \leq \alpha \det(\text{Im}(Z))^\beta$$

$$(1 \leq j_\nu, h_\nu \leq n).$$

Here we summarize some facts on standard L -functions attached to scalar valued Siegel modular forms.

Let $f \in S_k^n$ be an eigenform. Then we put

$$(1.7) \quad \Lambda(s, f) := \Gamma_{\mathbb{R}}(s + \varepsilon) \prod_{j=1}^n \Gamma_{\mathbb{C}}(s + k - j) D_f(s)$$

with

$$(1.8) \quad \varepsilon := \begin{cases} 0 & \text{for } n \text{ even,} \\ 1 & \text{for } n \text{ odd.} \end{cases}$$

Moreover we put

$$(1.9) \quad \gamma(s) := \begin{cases} \frac{\Gamma_n\left(\frac{s+n}{2}\right)}{\Gamma_n\left(\frac{s}{2}\right)} & \text{for } n \text{ even,} \\ \frac{\Gamma_{n-1}\left(\frac{s+n}{2}\right)}{\Gamma_{n-1}\left(\frac{s-1}{2}\right)} & \text{for } n \text{ odd.} \end{cases}$$

We note

$$(1.10) \quad \gamma(s) = \gamma(1 - s).$$

Then we have the following ([3], [7], [24]):

(1.4.3) Let $f \in S_k^n$ be an eigenform with a positive integer n and a positive even integer k .

(i) We obtain the integral representation

$$(1.11) \quad \left(f, \mathbb{E}_k^{2n} \left(\begin{pmatrix} -\bar{Z} & 0 \\ 0 & * \end{pmatrix}, \frac{\bar{s} + n}{2} \right) \right) = 2i^{nk} \pi^{-\frac{1}{2}n^2 + kn + \frac{1}{2}\varepsilon} \gamma(s) \Lambda(s, f) f(Z).$$

(ii) $\Lambda(s, f)$ has a meromorphic continuation to the whole s -plane and satisfies the functional equation

$$(1.12) \quad \Lambda(s, f) = \Lambda(1 - s, f).$$

To talk about poles of $\Lambda(s, f)$ we write $H_{m,n}(k)$ and $B_k^n(m)$ for $H_{m,n}(\det^k \otimes \text{sym}^0)$ and $B_{k,0}^n(m)$, where we must replace the condition (ii) of $H_{m,n}(\rho)$ in §1.3 with the condition

$$(ii') \quad \sum_{i=1}^m \sum_{j=1}^n \frac{\partial^2}{\partial x_{ij}^2} P(X) = 0 \text{ for } X = (x_{ij}).$$

By Mizumoto [21], we have:

(1.4.4) For $n, k \in \mathbb{Z}$, $n, k > 0$, let $f \in S_k^n$ be an eigenform. Suppose $k \geq n$. Then $\Lambda(s, f)$ is holomorphic except for possible simple poles at $s = 0$ and $s = 1$; it has a pole at $s = 0$ if and only if $f \in B_k^n(2n) \cap S_k^n$.

§2. Differential operator and Pullback formula

2.1. Differential operator

In what follows, we put

$$\begin{aligned} V &= \mathbb{C}x_1 \oplus \cdots \oplus \mathbb{C}x_{2n}, & x &= (x_1, \dots, x_{2n}), \\ V_1 &= \mathbb{C}x_1 \oplus \cdots \oplus \mathbb{C}x_n, & e_1 &= (x_1, \dots, x_n), \\ V_2 &= \mathbb{C}x_{n+1} \oplus \cdots \oplus \mathbb{C}x_{2n}, & e_2 &= (x_{n+1}, \dots, x_{2n}). \end{aligned}$$

Let ι be an isomorphism from V_1 to V_2 defined by $\iota(x_i) = x_{n+i}$ ($1 \leq i \leq n$). It induces an isomorphism (also denoted by ι) from $\text{sym}^l(V_1)$ to $\text{sym}^l(V_2)$. For $f \in C^\infty(\mathfrak{H}_n, \text{sym}^l(V_1))$ we define $\iota(f)$ by

$$(\iota(f))(Z) := \iota(f(Z)).$$

Let $Z = (z_{jh})$ be a variable on $\mathfrak{H}_{2n}$. For an integer $l \geq 0$ and a function $f \in C^\infty(\mathfrak{H}_{2n}, \text{sym}^l(V))$, we put

$$Df := \left(\frac{\partial}{\partial Z} f \right) [{}^t x] \quad \text{with} \quad \frac{\partial}{\partial Z} = \left(\frac{1 + \delta_{jh}}{2} \frac{\partial}{\partial z_{jh}} \right)_{1 \leq j, h \leq 2n}.$$

Here, for $z_{jh} = x_{jh} + iy_{jh}$ we put

$$(2.1) \quad \frac{\partial}{\partial z_{jh}} = \frac{1}{2} \left(\frac{\partial}{\partial x_{jh}} - i \frac{\partial}{\partial y_{jh}} \right), \quad \frac{\partial}{\partial \bar{z}_{jh}} = \frac{1}{2} \left(\frac{\partial}{\partial x_{jh}} + i \frac{\partial}{\partial y_{jh}} \right).$$

Then Df is a $V^{(l+2)}$ -valued function. We also put $D_{\uparrow}f := Df|_{e_2=0}$, $D_{\downarrow}f := Df|_{e_1=0}$ and $D_0 := D - D_{\uparrow} - D_{\downarrow}$. For a function f on $\mathfrak{H}_{2n}$, $\begin{pmatrix} Z^{(n)} & {}^tU^{(n)} \\ U^{(n)} & W^{(n)} \end{pmatrix} \in \mathfrak{H}_{2n}$, we define the pullback d^* by

$$(d^*f) \left(\begin{pmatrix} Z & {}^tU \\ U & W \end{pmatrix} \right) := f \left(\begin{pmatrix} Z & 0 \\ 0 & W \end{pmatrix} \right).$$

For $k, l \in \mathbb{Z}$, $k, l \geq 0$, we define an operator

$$L^{k,l} : C^\infty(\mathfrak{H}_{2n}, \mathbb{C}) \longrightarrow C^\infty(\mathfrak{H}_n \times \mathfrak{H}_n, \text{sym}^{2l}(V))$$

by

$$(2.2) \quad L^{k,l} := \frac{1}{(2\pi i)^l (k)_l} d^* \sum_{\nu=0}^{\lfloor \frac{l}{2} \rfloor} \frac{1}{\nu! (l-2\nu)! (2-k-l)_\nu} (D_{\uparrow} D_{\downarrow})^\nu D_0^{l-2\nu}.$$

We also define an operator

$$L_*^{k,l} : C^\infty(\mathfrak{H}_{2n}, \mathbb{C}) \longrightarrow C^\infty(\mathfrak{H}_{2n}, \text{sym}^{2l}(V))$$

by

$$(2.3) \quad L_*^{k,l} := \frac{1}{(2\pi i)^l (k)_l} \sum_{\nu=0}^{\lfloor \frac{l}{2} \rfloor} \frac{1}{\nu! (l-2\nu)! (2-k-l)_\nu} (D_{\uparrow} D_{\downarrow})^\nu D_0^{l-2\nu}.$$

Then we get the following by [9]:

(2.1.1) (i) For $k, l \in \mathbb{Z}$, $k, l \geq 0$, $L^{k,l}$ satisfies

$$\begin{aligned} & d^* \sum_{l=0}^{\infty} \frac{t^l}{(2\pi i)^l l! (k)_l} D^l \\ &= \sum_{l=0}^{\infty} \left(\sum_{\lambda=0}^{\infty} \frac{t^\lambda}{(2\pi i)^\lambda \lambda! (k+l)_\lambda} D_{\uparrow}^\lambda \right) \left(\sum_{\lambda=0}^{\infty} \frac{t^\lambda}{(2\pi i)^\lambda \lambda! (k+l)_\lambda} D_{\downarrow}^\lambda \right) t^l L^{k,l}. \end{aligned}$$

(ii) The operator $L^{k,l}$ is an operator which sends modular forms to modular forms:

$$L^{k,l} : M_k^{2n\infty} \longrightarrow M_{k,l}^n(\text{sym}^l(V_1))^\infty \otimes M_{k,l}^n(\text{sym}^l(V_2))^\infty.$$

Moreover, $L^{k,l}$ is a holomorphic operator and it satisfies

$$L^{k,l} : M_k^{2n} \longrightarrow M_{k,l}^n(\text{sym}^l(V_1)) \otimes M_{k,l}^n(\text{sym}^l(V_2)).$$

REMARK. On C^∞ -functions the operators d^* , $D_\uparrow$, $D_\downarrow$ are commutative and D , D_0 , $D_\uparrow$, $D_\downarrow$ are also commutative. But d^* and D , d^* and D_0 are not commutative, respectively. Therefore, $L^{k,l}$ is commutative with $D_\uparrow$ and $D_\downarrow$, and not commutative with D or D_0 .

The main results of this section is the following:

PROPOSITION 1. Let $k, l \in 2\mathbb{Z}$, $k, l > 0$, $s \in \mathbb{C}$ and $k + 2\operatorname{Re}(s) > 2n + 1$. For $\mathfrak{Z} = \begin{pmatrix} Z^{(n)} & {}^t U^{(n)} \\ U^{(n)} & W^{(n)} \end{pmatrix} \in \mathfrak{H}_{2n}$, $\mathfrak{Z}_0 = \begin{pmatrix} Z^{(n)} & 0 \\ 0 & W^{(n)} \end{pmatrix} \in \mathfrak{H}_{2n}$, we get

$$(L^{k,l} E_k^{2n})(\mathfrak{Z}, s) = \sum_M \frac{1}{(2\pi i)^l} \det(\mathfrak{C}\mathfrak{Z}_0 + \mathfrak{D})^{-k} \\ \times \det(\operatorname{Im}(M\langle\mathfrak{Z}_0\rangle))^s \left\{ \sum_{\nu=0}^{\frac{l}{2}} a(l, \nu, k, s) Q_0^{l-2\nu} (P_0 - P'_0)^\nu (R_0 - R'_0)^\nu \right\},$$

where $M = \begin{pmatrix} \mathfrak{C}^{(2n)*} & \mathfrak{D}^{(2n)*} \end{pmatrix}$ runs over a complete system of representatives of $P_{2n,0} \backslash \Gamma^{2n}$,

$$(2.4) \quad a(l, \nu, k, s) = \sum_{h=\nu}^{\lfloor \frac{l}{2} \rfloor} (-1)^{h-\nu} \binom{h}{\nu} b(l, h, k, s),$$

$$(2.5) \quad b(l, \nu, k, s) = (-1)^l \frac{(2k-2+2\nu)_{l-2\nu}}{\nu!(l-2\nu)!(k-1+\nu)_{l-\nu}} \frac{(-s)_\nu (k+s)_{l-\nu}}{(k)_l} \\ (0 \leq \nu \leq \lfloor \frac{l}{2} \rfloor),$$

$$P_0 = ((\mathfrak{C}\mathfrak{Z}_0 + \mathfrak{D})^{-1} \mathfrak{C}) \left[\begin{pmatrix} {}^t e_1 \\ 0 \end{pmatrix} \right], \quad P'_0 = \left(\frac{1}{2i} (\operatorname{Im}(\mathfrak{Z}_0))^{-1} \right) \left[\begin{pmatrix} {}^t e_1 \\ 0 \end{pmatrix} \right], \\ Q_0 = (e_1 \quad 0) ((\mathfrak{C}\mathfrak{Z}_0 + \mathfrak{D})^{-1} \mathfrak{C}) \begin{pmatrix} 0 \\ {}^t e_2 \end{pmatrix}, \quad Q'_0 = (e_1 \quad 0) \left(\frac{1}{2i} (\operatorname{Im}(\mathfrak{Z}_0))^{-1} \right) \begin{pmatrix} 0 \\ {}^t e_2 \end{pmatrix} \\ = 0, \\ R_0 = ((\mathfrak{C}\mathfrak{Z}_0 + \mathfrak{D})^{-1} \mathfrak{C}) \left[\begin{pmatrix} 0 \\ {}^t e_2 \end{pmatrix} \right], \quad R'_0 = \left(\frac{1}{2i} (\operatorname{Im}(\mathfrak{Z}_0))^{-1} \right) \left[\begin{pmatrix} 0 \\ {}^t e_2 \end{pmatrix} \right].$$

Now we put

$$S_0 = ((\mathfrak{C}\mathfrak{Z}_0 + \mathfrak{D})^{-1} \mathfrak{C}) [{}^t x], \quad \delta = \det(\mathfrak{C}\mathfrak{Z} + \mathfrak{D}), \quad d^* \delta = \delta_0 = \det(\mathfrak{C}\mathfrak{Z}_0 + \mathfrak{D}),$$

$$S_0' = \left(\frac{1}{2i} (\text{Im}(\mathfrak{Z}_0))^{-1} \right) [{}^t x], \quad \sigma = \det(\text{Im}(\mathfrak{Z})), \quad d^* \sigma = \sigma_0 = \det(\text{Im}(\mathfrak{Z}_0)).$$

We note

$$S_0 = P_0 + 2Q_0 + R_0, \quad S_0' = P_0' + 2Q_0' + R_0' = P_0' + R_0'.$$

Since

$$\det(\text{Im}(M(\mathfrak{Z}))) = |\det(\mathfrak{C}\mathfrak{Z} + \mathfrak{D})|^{-2} \det(\text{Im}(\mathfrak{Z})), \quad \text{Im}(\mathfrak{Z}) = \frac{1}{2i}(\mathfrak{Z} - \bar{\mathfrak{Z}}),$$

we have only to prove Lemma 1 below so as to prove Proposition 1:

LEMMA 1. For $k, l \in \mathbb{Z}$, $k, l > 0$, $s \in \mathbb{C}$, and $M = \begin{pmatrix} * & * \\ \mathfrak{C}^{(2n)} & \mathfrak{D}^{(2n)} \end{pmatrix} \in \Gamma^{2n}$, we get

$$L^{k,l}(\delta^{-k-s}\sigma^s) = \frac{1}{(2\pi i)^l} \delta_0^{-k-s} \sigma_0^s \sum_{\nu=0}^{[\frac{l}{2}]} a(l, \nu, k, s) Q_0^{l-2\nu} (P_0 - P_0')^\nu (R_0 - R_0')^\nu,$$

where notations are the same as those in Proposition 1.

In order to prove Lemma 1 inductively, we need the following Lemma:

LEMMA 2. For $k, l \in \mathbb{Z}$, $k > 0$, $l \geq 0$, we have

$$L^{k,l} = \frac{1}{(2\pi i)l(k+l-1)} L^{k,l-1} D_0 + \frac{(k)_{l-2}(4-2k-l)}{(2\pi i)^2 l(k)_l (2-k-l)_2} D_{\uparrow} D_{\downarrow} L^{k,l-2}.$$

Since $L^{k,l-1} D_0 \neq D_0 L^{k,l-1}$ ($= 0$), we must consider what $L^{k,l-1} D_0 (\delta^{-k-s} \sigma^s)$ is in a different way. We note $L^{k,l-1} D_0 = d^* L_*^{k,l-1} D_0 = d^* D_0 L_*^{k,l-1} \dots$

LEMMA 3. For $k, l \in \mathbb{Z}$, $k > 0$, $l \geq 2$, $s \in \mathbb{C}$, we obtain

$$\begin{aligned} L^{k,l-1} D_0 (\delta^{-k-s} \sigma^s) &= d^* D_0 \left(L_*^{k,l-1} \Big|_{Q'=0} \right) (\delta^{-k-s} \sigma^s) \\ &\quad + \frac{s}{(2\pi i)k} (d^* (D_0^2 \sigma)) \cdot L^{k+1,l-2} (\delta^{-k-s} \sigma^{s-1}). \end{aligned}$$

PROOF OF LEMMA 2. We prove it for even l . The assertion for odd l is proved in the same way.

Let $l = 2m$. Then,

$$\begin{aligned}
 & L^{k,2m} - \frac{1}{2m(k+2m-1)(2\pi i)} L^{k,2m-1} D_0 \\
 &= \frac{1}{(2\pi i)^l (k)_{2m}} d^* \left\{ \sum_{\nu=1}^{m-1} \left(\frac{1}{\nu! (2m-2\nu)! (2-k-2m)_\nu} \right. \right. \\
 &\quad \left. \left. - \frac{1}{\nu! 2m(2m-1-2\nu)! (3-k-2m)_\nu} \right) (D_\uparrow D_\downarrow)^\nu D_0^{2m-2\nu} \right\} \\
 &\quad + \frac{1}{(2\pi i)^l (k)_{2m} m! (2-k-2m)_m} d^* (D_\uparrow D_\downarrow)^m.
 \end{aligned}$$

Here, we note

$$\begin{aligned}
 & \frac{1}{\nu! (2m-2\nu)! (2-k-2m)_\nu} - \frac{1}{\nu! 2m(2m-1-2\nu)! (3-k-2m)_\nu} \\
 &= \frac{4-2k-2m}{2m(\nu-1)! (2m-2\nu)! (2-k-2m)_{\nu+1}}.
 \end{aligned}$$

Therefore,

$$\begin{aligned}
 & L^{k,2m} - \frac{1}{2m(k+2m-1)(2\pi i)} L^{k,2m-1} D_0 \\
 &= \frac{1}{(2\pi i)^{2m} (k)_{2m}} D_\uparrow D_\downarrow d^* \sum_{\nu=1}^m \frac{4-2k-2m}{(\nu-1)! 2m(2m-2\nu)! (2-k-2m)_{\nu+1}} \\
 &\quad \times (D_\uparrow D_\downarrow)^{\nu-1} D_0^{2m-2\nu} \\
 &= \frac{4-2k-2m}{(2\pi i)^{2m} 2m(k)_{2m}} D_\uparrow D_\downarrow d^* \sum_{\nu=0}^{m-1} \frac{1}{\nu! ((2m-2)-2\nu)! (2-k-2m)_{\nu+2}} \\
 &\quad \times (D_\uparrow D_\downarrow)^\nu D_0^{(2m-2)-2\nu} \\
 &= \frac{(k)_{2m-2} (4-2k-2m)}{(2\pi i)^2 (k)_{2m} 2m(2-k-2m)_2} D_\uparrow D_\downarrow L^{k,2m-2}.
 \end{aligned}$$

□

We introduce some formulae before we prove Lemma 1 and Lemma 3.

When we replace $\mathfrak{z}_0$ in $P_0, P'_0, \dots, S_0, S'_0$ with $\mathfrak{z}$, we write $P, P', \dots, S, S'$ for $P_0, P'_0, \dots, S_0, S'_0$, respectively. We note $d^*P = P_0, d^*P' = P'_0, \dots, d^*S = S'_0$ and $d^*S' = S'_0$. Then we get the following table:

	δ	S	P	Q	R
D	δS	$-S^2$	$-(P+Q)^2$	$-(P+Q)(R+Q)$	$-(R+Q)^2$
$D_{\uparrow}$	δP	$-(P+Q)^2$	$-P^2$	$-PQ$	$-Q^2$
D_0	$2\delta Q$	$-2(P+Q)(R+Q)$	$-2PQ$	$-PR-Q^2$	$-2RQ$
$D_{\downarrow}$	δR	$-(R+Q)^2$	$-Q^2$	$-RQ$	$-R^2$

Moreover, we get

$$D_{\uparrow}D_0\delta = D_0D_{\uparrow}\delta = 0, \quad D_{\downarrow}D_0\delta = D_0D_{\downarrow}\delta = 0, \quad D_0^3\delta = 0,$$

$$D_0^2\delta = D_0(2\delta Q) = -2\delta(PR - Q^2), \quad D_{\uparrow}D_{\downarrow}\delta = \delta(PR - Q^2).$$

If we replace $\delta, S, \dots, R$ with $\sigma, S', \dots, R'$ respectively, we obtain formulae for $\sigma, S', \dots, R'$ in the same way. We also get tables for δ_0 and for σ_0 if we only note $D_0d^* = 0$ and $Q_0' = 0$.

PROOF OF LEMMA 3. We recall $L^{k,l-1}D_0(\delta^{-k-s}\sigma^s) = d^*D_0L^{k,l-1}(\delta^{-k-s}\sigma^s)$. Then we have

$$L^{k,l-1}(\delta^{-k-s}\sigma^s) = \delta^{-k-s}\sigma^s H,$$

where H is a polynomial in P, Q, R, P', Q' and R' . Moreover, we have an expansion

$$L^{k,l-1}(\delta^{-k-s}\sigma^s) = \delta^{-k-s}\sigma^s \sum_{r=0}^h H_r Q'^r,$$

where H_r is a polynomial in P, Q, R, P' and R' , and $h < \infty$. By the table above and so on, we obtain

$$d^*D_0L^{k,l-1}(\delta^{-k-s}\sigma^s) = d^*D_0(\delta^{-k-s}\sigma^s(H_0 + H_1Q'))$$

and

$$d^*D_0(\delta^{-k-s}\sigma^s H_0) = d^*D_0\left(L^{k,l-1}\big|_{Q'=0}\right)(\delta^{-k-s}\sigma^s).$$

On the other hand, $d^*D_0(\delta^{-k-s}\sigma^s H_1Q')$ is given in the following way:

We consider $\delta^{-k-s}\sigma^s$ a composite function of δ^{-k-s} and σ^s . Once a factor D_0 of $(D_{\uparrow}D_{\downarrow})^{\nu}D_0^{(l-1)-2\nu}$ ($0 \leq \nu \leq \lfloor \frac{l-1}{2} \rfloor$) in $L^{k,l-1}$ acts on σ^s , nothing acts on $D_0\sigma$ ($= 2\sigma Q'$) which comes from $D_0(\sigma^s) = s\sigma^{s-1}(D_0\sigma)$. That is,

$$\begin{aligned}
& d^* D_0(\delta^{-k-s} \sigma^s H_1 Q') \\
&= d^* D_0 \left\{ \left(\frac{1}{(2\pi i)^{l-1} (k)_{l-1}} \sum_{\nu=0}^{\lfloor \frac{l}{2} \rfloor - 1} \frac{1}{\nu! (l-1-2\nu)! (2-k-(l-1))_\nu} \right. \right. \\
&\quad \left. \left. \times (D_\uparrow D_\downarrow)^\nu D_0^{l-2-2\nu} (\delta^{-k-s} \sigma^{s-1}) \right) \binom{l-1-2\nu}{1} s(D_0 \sigma) \right\} \\
&= \left\{ \frac{1}{(2\pi i)^{l-1} (k)_{l-1}} d^* \sum_{\nu=0}^{\lfloor \frac{l-2}{2} \rfloor} \frac{1}{\nu! (l-1-2\nu)! (2-k-(l-1))_\nu} \right. \\
&\quad \left. \times (D_\uparrow D_\downarrow)^\nu D_0^{l-2-2\nu} (\delta^{-k-s} \sigma^{s-1}) \right\} \binom{l-1-2\nu}{1} s(d^*(D_0^2 \sigma)) \\
&= \left\{ \frac{1}{(2\pi i)^{l-2} (k+1)_{l-2}} d^* \sum_{\nu=0}^{\lfloor \frac{l-2}{2} \rfloor} \frac{1}{\nu! (l-2-2\nu)! (2-(k+1)-(l-2))_\nu} \right. \\
&\quad \left. \times (D_\uparrow D_\downarrow)^\nu D_0^{l-2-2\nu} (\delta^{-k-s} \sigma^{s-1}) \right\} \frac{s}{(2\pi i)k} (d^*(D_0^2 \sigma)) \\
&= \frac{s}{(2\pi i)k} (d^*(D_0^2 \sigma)) \cdot L^{k+1, l-2} (\delta^{-k-s} \sigma^{s-1}).
\end{aligned}$$

REMARK. For odd l , $L_*^{k, l-1}$ has a term $(D_\uparrow D_\downarrow)^{\frac{l-1}{2}}$, but $(D_\uparrow D_\downarrow)^{\frac{l-1}{2}}$ has no relation to $d^* D_0(\delta^{-k-s} \sigma^s H_1 Q')$ because $d^* D_0(D_\uparrow D_\downarrow)^{\frac{l-1}{2}} = (D_\uparrow D_\downarrow)^{\frac{l-1}{2}} d^* D_0$. $\square$

PROOF OF LEMMA 1. We use induction on l . For $l = 1$ and for $l = 2$, Lemma 1 holds.

Let $l > 2$. By Lemma 2, we have

$$\begin{aligned}
& L^{k, l}(\delta^{-k-s} \sigma^s) \\
&= \frac{1}{(2\pi i)l(k+l-1)} L^{k, l-1} D_0(\delta^{-k-s} \sigma^s) \\
&\quad + \frac{(k)_{l-2} (4-2k-l)}{(2\pi i)^2 (k)_l l (2-k-l)_2} D_\uparrow D_\downarrow L^{k, l-2}(\delta^{-k-s} \sigma^s)
\end{aligned}$$

by Lemma 3,

$$\begin{aligned}
&= \frac{1}{(2\pi i)l(k+l-1)} d^* D_0 \left(L_*^{k,l-1} \Big|_{Q'=0} \right) (\delta^{-k-s} \sigma^s) \\
&\quad + \frac{s}{(2\pi i)^2 l(k+l-1)k} (d^* (D_0^2 \sigma)) L^{k+1,l-2} (\delta^{-k-s} \sigma^{s-1}) \\
&\quad + \frac{(k)_{l-2}(4-2k-l)}{(2\pi i)^2 (k)_l l(2-k-l)_2} D_{\uparrow} D_{\downarrow} L^{k,l-2} (\delta^{-k-s} \sigma^s)
\end{aligned}$$

since the induction hypothesis is valid for $\left(L_*^{k,l-1} \Big|_{Q'=0} \right) (\delta^{-k-s} \sigma^s)$,

$$\begin{aligned}
&= \frac{1}{(2\pi i)^l l(k+l-1)} \\
&\quad \times d^* D_0 \left(\sum_{\nu=0}^{\lfloor \frac{l-1}{2} \rfloor} \delta^{-k-s} \sigma^s a(l-1, \nu, k, s) Q^{l-1-2\nu} (P-P')^\nu (R-R')^\nu \right) \\
&\quad + \frac{-2s}{(2\pi i)^l l(k+l-1)k} \\
&\quad \times \left\{ \sum_{\nu=0}^{\lfloor \frac{l}{2} \rfloor - 1} \delta_0^{-k-s} \sigma_0^s a(l-2, \nu, k+1, s-1) Q_0^{l-2-2\nu} (P_0-P'_0)^\nu (R_0-R'_0)^\nu \right\} P'_0 R'_0 \\
&\quad + \frac{(k)_{l-2}(4-2k-l)}{(2\pi i)^l (k)_l l(2-k-l)_2} \\
&\quad \times D_{\uparrow} D_{\downarrow} \left(\sum_{\nu=0}^{\lfloor \frac{l}{2} \rfloor - 1} \delta_0^{-k-s} \sigma_0^s a(l-2, \nu, k, s) Q_0^{l-2-2\nu} (P_0-P'_0)^\nu (R_0-R'_0)^\nu \right).
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
&d^* D_0 (\delta^{-k-s} \sigma^s Q^{l-1-2\nu} (P-P')^\nu (R-R')^\nu) \\
&= -(2k+2s+l-1-2\nu) \delta_0^{-k-s} \sigma_0^s Q_0^{l-2\nu} (P_0-P'_0)^\nu (R_0-R'_0)^\nu \\
&\quad - 4\nu \delta_0^{-k-s} \sigma_0^s Q_0^{l-2\nu} P_0 R_0 (P_0-P'_0)^{\nu-1} (R_0-R'_0)^{\nu-1} \\
&\quad + 2\nu \delta_0^{-k-s} \sigma_0^s Q_0^{l-2\nu} (P_0 R'_0 + P'_0 R_0) (P_0-P'_0)^{\nu-1} (R_0-R'_0)^{\nu-1} \\
&\quad - (l-1-2\nu) \delta_0^{-k-s} \sigma_0^s Q_0^{l-2-2\nu} P_0 R_0 (P_0-P'_0)^\nu (R_0-R'_0)^\nu \\
&\hspace{25em} (0 \leq \nu \leq \lfloor \frac{l-1}{2} \rfloor),
\end{aligned}$$

and

$$\begin{aligned}
& D_{\uparrow} D_{\downarrow} (\delta_0^{-k-s} \sigma_0^s Q_0^{l-2-2\nu} (P_0 - P'_0)^\nu (R_0 - R'_0)^\nu) \\
&= \nu^2 \delta_0^{-k-s} \sigma_0^s Q_0^{l-2\nu+2} (P_0 - P'_0)^{\nu-1} (R_0 - R'_0)^{\nu-1} \\
&\quad + (k+s+l-2-\nu) \delta_0^{-k-s} \sigma_0^s Q_0^{l-2\nu} (P_0 - P'_0)^\nu (R_0 - R'_0)^\nu \\
&\quad + \nu(2k+2s+2l-2\nu-3) \delta_0^{-k-s} \sigma_0^s Q_0^{l-2\nu} P_0 R_0 (P_0 - P'_0)^{\nu-1} (R_0 - R'_0)^{\nu-1} \\
&\quad - \nu(k+2s+l-2\nu-1) \delta_0^{-k-s} \sigma_0^s Q_0^{l-2\nu} (P_0 R'_0 + P'_0 R_0) (P_0 - P'_0)^{\nu-1} \\
&\quad \quad \quad \times (R_0 - R'_0)^{\nu-1} \\
&\quad + \nu(2s-2\nu+1) \delta_0^{-k-s} \sigma_0^s Q_0^{l-2\nu} P'_0 R'_0 (P_0 - P'_0)^{\nu-1} (R_0 - R'_0)^{\nu-1} \\
&\quad + (k+s+l-2-\nu)^2 \delta_0^{-k-s} \sigma_0^s Q_0^{l-2\nu-2} P_0 R_0 (P_0 - P'_0)^\nu (R_0 - R'_0)^\nu \\
&\quad - (k+s+l-2-\nu)(s-\nu) \delta_0^{-k-s} \sigma_0^s Q_0^{l-2\nu-2} (P_0 R'_0 + P'_0 R_0) (P_0 - P'_0)^\nu \\
&\quad \quad \quad \times (R_0 - R'_0)^\nu \\
&\quad + (s-\nu)^2 \delta_0^{-k-s} \sigma_0^s Q_0^{l-2\nu-2} P'_0 R'_0 (P_0 - P'_0)^\nu (R_0 - R'_0)^\nu \\
&\quad \quad \quad (0 \leq \nu \leq [\frac{l}{2}] - 1).
\end{aligned}$$

Thus we have only to prove the following equations:

(1) coefficient of Q_0^l

$$\begin{aligned}
a(l, 0, k, s) &= \frac{1}{l(k+l-1)} \left\{ -(2k+2s+l-1)a(l-1, 0, k, s) \right. \\
&\quad + \frac{(k)_{l-2}(4-2k-l)}{(k)_l l(2-k-l)_2} \left\{ (k+s+l-2)a(l-2, 0, k, s) \right. \\
&\quad \quad \quad \left. + a(l-2, 1, k, s) \right\}
\end{aligned}$$

(2) coefficient of $Q_0^{l-2\nu} P_0 R_0 (P_0 - P'_0)^{\nu-1} (R_0 - R'_0)^{\nu-1}$ ($1 \leq \nu \leq [\frac{l}{2}] - 2$)

$$\begin{aligned}
a(l, \nu, k, s) &= \frac{1}{l(k+l-1)} \left\{ -(2k+2s+l-1+2\nu)a(l-1, \nu, k, s) \right. \\
&\quad \quad \quad \left. - (l+1-2\nu)a(l-1, \nu-1, k, s) \right\} \\
&\quad + \frac{(k)_{l-2}(4-2k-l)}{(k)_l l(2-k-l)_2} \left\{ (\nu+1)^2 a(l-2, \nu+1, k, s) \right. \\
&\quad \quad \quad + (k+s+l-2-\nu)a(l-2, \nu, k, s) \\
&\quad \quad \quad + \nu(2k+2s+2l-2\nu-3)a(l-2, \nu, k, s) \\
&\quad \quad \quad \left. + (k+s+l-1-\nu)^2 a(l-2, \nu-1, k, s) \right\}.
\end{aligned}$$

- (3) coefficient of $-Q_0^{l-2\nu}(P_0R'_0 + P'_0R_0)(P_0 - P'_0)^{\nu-1}(R_0 - R'_0)^{\nu-1}$
 $(1 \leq \nu \leq [\frac{l}{2}] - 2)$

$$\begin{aligned} a(l, \nu, k, s) = & \frac{1}{l(k+l-1)} \left\{ -(2k+2s+l-1)a(l-1, \nu, k, s) \right\} \\ & + \frac{(k)_{l-2}(4-2k-l)}{(k)_l l(2-k-l)_2} \left\{ (\nu+1)^2 a(l-2, \nu+1, k, s) \right. \\ & + (k+s+l-2-\nu)a(l-2, \nu, k, s) \\ & + \nu(k+2s+l-2\nu-1)a(l-2, \nu, k, s) \\ & \left. + (k+s+l-1-\nu)(s-\nu+1)a(l-2, \nu-1, k, s) \right\}. \end{aligned}$$

- (4) coefficient of $Q_0^{l-2\nu}P'_0R'_0(P_0 - P'_0)^{\nu-1}(R_0 - R'_0)^{\nu-1}$ $(1 \leq \nu \leq [\frac{l}{2}] - 2)$

$$\begin{aligned} a(l, \nu, k, s) = & \frac{1}{l(k+l-1)} \left\{ -(2k+2s+l-1-2\nu)a(l-1, \nu, k, s) \right. \\ & \left. - \frac{2s}{k}a(l-2, \nu-1, k+1, s-1) \right\} \\ & + \frac{(k)_{l-2}(4-2k-l)}{(k)_l l(2-k-l)_2} \left\{ (\nu+1)^2 a(l-2, \nu+1, k, s) \right. \\ & + (k+s+l-2-\nu)a(l-2, \nu, k, s) \\ & \left. + \nu(2s-2\nu+1)a(l-2, \nu, k, s) + (s-\nu+1)^2 a(l-2, \nu-1, k, s) \right\}. \end{aligned}$$

- (5) coefficient of $Q_0^{l-2\nu}P_0R_0(P_0 - P'_0)^{\nu-1}(R_0 - R'_0)^{\nu-1}$ $(\nu = [\frac{l}{2}] - 1)$

$$\begin{aligned} a(l, \nu, k, s) = & \frac{1}{l(k+l-1)} \left\{ -(2k+2s+l-1+2\nu)a(l-1, \nu, k, s) \right. \\ & \left. - (l+1-2\nu)a(l-1, \nu-1, k, s) \right\} \\ & + \frac{(k)_{l-2}(4-2k-l)}{(k)_l l(2-k-l)_2} \left\{ (k+s+l-2-\nu)a(l-2, \nu, k, s) \right. \\ & + \nu(2k+2s+2l-2\nu-3)a(l-2, \nu, k, s) \\ & \left. + (k+s+l-1-\nu)^2 a(l-2, \nu-1, k, s) \right\}. \end{aligned}$$

- (6) coefficient of $-Q_0^{l-2\nu}(P_0R'_0 + P'_0R_0)(P_0 - P'_0)^{\nu-1}(R_0 - R'_0)^{\nu-1}$
 $(\nu = [\frac{l}{2}] - 1)$

$$\begin{aligned}
 a(l, \nu, k, s) = & \frac{1}{l(k+l-1)} \left\{ -(2k+2s+l-1)a(l-1, \nu, k, s) \right\} \\
 & + \frac{(k)_{l-2}(4-2k-l)}{(k)_l l(2-k-l)_2} \left\{ (k+s+l-2-\nu)a(l-2, \nu, k, s) \right. \\
 & \quad + \nu(k+2s+l-2\nu-1)a(l-2, \nu, k, s) \\
 & \quad \left. + (k+s+l-1-\nu)(s-\nu+1)a(l-2, \nu-1, k, s) \right\}. \\
 (7) \quad & \text{coefficient of } Q_0^{l-2\nu} P'_0 R'_0 (P_0 - P'_0)^{\nu-1} (R_0 - R'_0)^{\nu-1} \quad (\nu = [\frac{l}{2}] - 1)
 \end{aligned}$$

$$\begin{aligned}
 a(l, \nu, k, s) = & \frac{1}{l(k+l-1)} \left\{ -(2k+2s+l-1-2\nu)a(l-1, \nu, k, s) \right. \\
 & \quad \left. - \frac{2s}{k}a(l-2, \nu-1, k+1, s-1) \right\} \\
 & + \frac{(k)_{l-2}(4-2k-l)}{(k)_l l(2-k-l)_2} \left\{ (k+s+l-2-\nu)a(l-2, \nu, k, s) \right. \\
 & \quad + \nu(2s-2\nu+1)a(l-2, \nu, k, s) \\
 & \quad \left. + (s-\nu+1)^2 a(l-2, \nu-1, k, s) \right\}.
 \end{aligned}$$

For even l , we put $l = 2m$.

$$(8) \quad \text{coefficient of } P_0 R_0 (P_0 - P'_0)^{m-1} (R_0 - R'_0)^{m-1}$$

$$\begin{aligned}
 a(l, m, k, s) = & \frac{1}{l(k+l-1)} \left\{ -a(l-1, m-1, k, s) \right\} \\
 & + \frac{(k)_{l-2}(4-2k-l)}{(k)_l l(2-k-l)_2} \left\{ (k+s+m-1)^2 a(l-2, m-1, k, s) \right\}.
 \end{aligned}$$

$$(9) \quad \text{coefficient of } -(P_0 R'_0 + P'_0 R_0) (P_0 - P'_0)^{m-1} (R_0 - R'_0)^{m-1}$$

$$a(l, m, k, s)$$

$$= \frac{(k)_{l-2}(4-2k-l)}{(k)_l l(2-k-l)_2} \left\{ (k+s+m-1)(s-m+1)a(l-2, m-1, k, s) \right\}.$$

$$(10) \quad \text{coefficient of } P'_0 R'_0 (P_0 - P'_0)^{m-1} (R_0 - R'_0)^{m-1}$$

$$\begin{aligned}
 a(l, m, k, s) = & \frac{1}{l(k+l-1)} \left\{ -\frac{2s}{k}a(l-2, m-1, k+1, s-1) \right\} \\
 & + \frac{(k)_{l-2}(4-2k-l)}{(k)_l l(2-k-l)_2} \left\{ (s-m+1)^2 a(l-2, m-1, k, s) \right\}.
 \end{aligned}$$

For odd l , we put $l = 2m + 1$.

(8') coefficient of $Q_0 P_0 R_0 (P_0 - P'_0)^{m-1} (R_0 - R'_0)^{m-1}$

$$\begin{aligned} a(l, m, k, s) = & \frac{1}{l(k+l-1)} \left\{ -(2k+2s+4m)a(l-1, m, k, s) \right. \\ & \left. - 2a(l-1, m-1, k, s) \right\} \\ & + \frac{(k)_{l-2}(4-2k-l)}{(k)_l l(2-k-l)_2} \left\{ (k+s+m)^2 a(l-2, m-1, k, s) \right\}. \end{aligned}$$

(9') coefficient of $-Q_0(P_0 R'_0 + P'_0 R_0)(P_0 - P'_0)^{m-1}(R_0 - R'_0)^{m-1}$

$$\begin{aligned} a(l, m, k, s) = & \frac{1}{l(k+l-1)} \left\{ -(2k+2s+2m)a(l-1, m, k, s) \right\} \\ & + \frac{(k)_{l-2}(4-2k-l)}{(k)_l l(2-k-l)_2} \left\{ (k+s+m)(s-m+1)a(l-2, m-1, k, s) \right\}. \end{aligned}$$

(10') coefficient of $Q_0 P'_0 R'_0 (P_0 - P'_0)^{m-1} (R_0 - R'_0)^{m-1}$

$$\begin{aligned} a(l, m, k, s) = & \frac{1}{l(k+l-1)} \left\{ -(2k+2s)a(l-1, m, k, s) \right. \\ & \left. - \frac{2s}{k} a(l-2, m-1, k+1, s-1) \right\} \\ & + \frac{(k)_{l-2}(4-2k-l)}{(k)_l l(2-k-l)_2} \left\{ (s-m+1)^2 a(l-2, m-1, k, s) \right\}. \end{aligned}$$

By properties of $a(l, \nu, k, s)$, it is easy to prove (5), (6), (7), (8), (9), (10), (8'), (9'), (10). Therefore, we have only to prove (1), (2), (3), (4). But we only prove (2) for even l , here. The assertions of (1), (3), (4) and of (2) for odd l are proved in the same way.

Let $l = 2m$. Then, by (2.4), the right-hand side of (2) is:

$$\begin{aligned} & \frac{1}{2m(k+2m-1)} \left\{ -(2k+2s+2m-1+2\nu) \sum_{h=\nu}^{m-1} (-1)^{h-\nu} \binom{h}{\nu} b(2m-1, h, k, s) \right. \\ & \quad \left. - (2m+1-2\nu) \sum_{h=\nu-1}^{m-1} (-1)^{h+1-\nu} \binom{h}{\nu-1} b(2m-1, h, k, s) \right\} \\ & - \frac{(k)_{2m-2}(2k+2m-4)}{(k)_{2m} 2m(k+2m-3)_2} \left\{ (\nu+1)^2 \sum_{h=\nu+1}^{m-1} (-1)^{h-\nu+1} \binom{h}{\nu+1} b(2m-2, h, k, s) \right. \end{aligned}$$

$$\begin{aligned}
& + (k + s + 2m - \nu - 1)^2 \sum_{h=\nu-1}^{m-1} (-1)^{h+1-\nu} \binom{h}{\nu-1} b(2m-2, h, k, s) \\
& + ((k + s + 2m - \nu - 2) + \nu(2k + 2s + 4m - 2\nu - 3)) \\
& \quad \times \sum_{h=\nu}^{m-1} (-1)^{h-\nu} \binom{h}{\nu} b(2m-2, h, k, s) \Big\} \\
(2.6) \quad & = \frac{1}{2m(k + 2m - 1)} \left\{ \sum_{h=\nu-1}^{m-1} (-1)^{h-\nu} \left(\binom{h}{\nu-1} (2k + 2s + 4m) \right. \right. \\
& \quad \left. \left. - \binom{h+1}{\nu} (2k + 2s + 2m + 2\nu - 1) \right) b(2m-1, h, k, s) \right\} \\
& + \frac{(k)_{2m-2} (2k + 2m - 4)}{(k)_{2m} 2m (k + 2m - 3)_2} \left\{ -(k + s + 2m - \nu - 1)^2 b(2m-2, \nu-1, k, s) \right. \\
& + \sum_{h=\nu}^{m-1} (-1)^{h-\nu} \left(\binom{h}{\nu-1} (k + s + 2m - \nu - 1)(k + s + 2m - h - 2) \right. \\
& \quad \left. \left. - \binom{h}{\nu} (\nu + 1)(k + s + 2m - h - 2) \right) b(2m-2, h, k, s) \right\}.
\end{aligned}$$

We note

$$\begin{aligned}
(2.7) \quad & \left(\binom{h}{\nu-1} (2k + 2s + 4m) - \binom{h+1}{\nu} (2k + 2s + 2m + 2\nu - 1) \right) \\
& = \binom{h+1}{\nu} (2m - 2h - 1) - 2 \binom{h}{\nu} (k + s + 2m - h - 1),
\end{aligned}$$

$$\begin{aligned}
(2.8) \quad & \left(\binom{h}{\nu-1} (k + s + 2m - \nu - 1) - \binom{h}{\nu} (\nu + 1) \right) \\
& = \binom{h}{\nu-1} (k + s + 2m - h - 1) + \binom{h}{\nu-1} (h - \nu) - \binom{h}{\nu} (\nu + 1) \\
& = \binom{h}{\nu-1} (k + s + 2m - h - 1) - \binom{h+1}{\nu}
\end{aligned}$$

and

$$(2.9) \quad (k + s + 2m - \nu - 1)^2 = (k + s + 2m - \nu - 1)_2 - (k + s + 2m - \nu - 1).$$

By (2.7), (2.8), (2.9), the formula (2.6) is:

$$\begin{aligned} & \frac{1}{2m(k+2m-1)} \left\{ \sum_{h=\nu-1}^{m-1} (-1)^{h-\nu} \binom{h+1}{\nu} (2m-2h-1)b(2m-1, h, k, s) \right. \\ & \quad \left. - \sum_{h=\nu}^{m-1} (-1)^{h-\nu} \binom{h}{\nu} 2(k+s+2m-h-1)b(2m-1, h, k, s) \right\} \\ & + \frac{(k)_{2m-2}(2k+2m-4)}{(k)_{2m}2m(k+2m-3)_2} \left\{ \sum_{h=\nu-1}^{m-1} (-1)^{h-\nu} \left(\binom{h}{\nu-1} (k+s+2m-h-2)_2 \right. \right. \\ & \quad \left. \left. - \binom{h+1}{\nu} (k+s+2m-h-2) \right) b(2m-2, h, k, s) \right\} \end{aligned}$$

by (2.5)

$$\begin{aligned} & = - \sum_{h=\nu-1}^{m-1} (-1)^{h-\nu} \binom{h+1}{\nu} \frac{(2k+2h-2)_{2m-2h-1}}{2m(h)!(2m-2h-2)!(k+h-1)_{2m-h-1}} \\ & \quad \times \frac{(-s)_h(k+s)_{2m-h-1}}{(k)_{2m}} \\ & + \sum_{h=\nu}^{m-1} (-1)^{h-\nu} \binom{h}{\nu} \frac{(2k+2h-2)_{2m-2h-1}}{m(h)!(2m-2h-1)!(k+h-1)_{2m-h-1}} \\ & \quad \times \frac{(-s)_h(k+s)_{2m-h}}{(k)_{2m}} \\ (2.10) \quad & + \sum_{h=\nu-1}^{m-1} (-1)^{h-\nu} \binom{h}{\nu-1} \frac{(2k+2h-2)_{2m-2h-1}}{2m(h)!(2m-2h-2)!(k+h-1)_{2m-h}} \\ & \quad \times \frac{(-s)_h(k+s)_{2m-h}}{(k)_{2m}} \\ & - \sum_{h=\nu-1}^{m-1} (-1)^{h-\nu} \binom{h+1}{\nu} \frac{(2k+2h-2)_{2m-2h-1}}{2m(h)!(2m-2h-2)!(k+h-1)_{2m-h}} \\ & \quad \times \frac{(-s)_h(k+s)_{2m-h-1}}{(k)_{2m}}. \end{aligned}$$

We also note

$$\begin{aligned}
 (2.11) \quad & \frac{1}{m(h)!(2m-2h-1)!(k+h-1)_{2m-h-1}} \\
 &= \frac{2k+2m-3}{h!(2m-2h)!(k+h-1)_{2m-h}} + \frac{1}{2m(h)!(2m-2h-2)!(k+h-1)_{2m-h}} \\
 &\quad - \frac{2k+2h-3}{m(h-1)!(2m-2h)!(k+h-1)_{2m-h}}.
 \end{aligned}$$

If we add the first term of (2.10) to the fourth term of (2.10) and if we transform the second term of (2.10) according to (2.11), we get:

$$\begin{aligned}
 & \sum_{h=\nu}^{m-1} (-1)^{h-\nu} \binom{h}{\nu} \frac{(2k+2h-2)_{2m-2h}}{h!(2m-2h)!(k+h-1)_{2m-h}} \frac{(-s)_h (k+s)_{2m-h}}{(k)_{2m}} \\
 &+ \sum_{h=\nu}^{m-1} (-1)^{h-\nu} \binom{h}{\nu} \frac{(2k+2h-2)_{2m-2h-1}}{2m(h)!(2m-2h-2)!(k+h-1)_{2m-h}} \frac{(-s)_h (k+s)_{2m-h}}{(k)_{2m}} \\
 &- \sum_{h=\nu}^{m-1} (-1)^{h-\nu} \binom{h}{\nu} \frac{(2k+2h-3)_{2m-2h}}{m(h-1)!(2m-2h)!(k+h-1)_{2m-h}} \\
 &\quad \times \frac{(-s)_h (k+s)_{2m-h}}{(k)_{2m}} \\
 &+ \sum_{h=\nu-1}^{m-1} (-1)^{h-\nu} \binom{h}{\nu-1} \frac{(2k+2h-2)_{2m-2h-1}}{2m(h)!(2m-2h-2)!(k+h-1)_{2m-h}} \\
 &\quad \times \frac{(-s)_h (k+s)_{2m-h}}{(k)_{2m}} \\
 &- \sum_{h=\nu-1}^{m-1} (-1)^{h-\nu} \binom{h+1}{\nu} \frac{(k+2m-1)(2k+2h-2)_{2m-2h-1}}{2m(h)!(2m-2h-2)!(k+h-1)_{2m-h}} \\
 &\quad \times \frac{(-s)_h (k+s)_{2m-h-1}}{(k)_{2m}}
 \end{aligned}$$

$$\begin{aligned}
&= a(2m, \nu, k, s) - (-1)^{m-\nu} \binom{m}{\nu} \frac{1}{m!(k+m-1)_m} \frac{(-s)_m (k+s)_m}{(k)_{2m}} \\
&+ \sum_{h=\nu-1}^{m-1} (-1)^{h-\nu} \binom{h+1}{\nu} \frac{(2k+2h-2)_{2m-2h-1}}{2m(h)!(2m-2h-2)!(k+h-1)_{2m-h}} \\
&\quad \times \frac{(-s)_h (k+s)_{2m-h}}{(k)_{2m}} \\
&- \sum_{h=\nu}^{m-1} (-1)^{h-\nu} \binom{h}{\nu} \frac{(2k+2h-3)_{2m-2h}}{m(h-1)!(2m-2h)!(k+h-1)_{2m-h}} \\
&\quad \times \frac{(-s)_h (k+s)_{2m-h}}{(k)_{2m}} \\
&- \sum_{h=\nu-1}^{m-1} (-1)^{h-\nu} \binom{h+1}{\nu} \frac{(k+2m-1)(2k+2h-2)_{2m-2h-1}}{2m(h)!(2m-2h-2)!(k+h-1)_{2m-h}} \\
&\quad \times \frac{(-s)_h (k+s)_{2m-h-1}}{(k)_{2m}} \\
&= a(2m, \nu, k, s) - (-1)^{m-\nu} \binom{m}{\nu} \frac{1}{m!(k+m-1)_m} \frac{(-s)_m (k+s)_m}{(k)_{2m}} \\
&- \sum_{h=\nu}^{m-1} (-1)^{h-\nu} \binom{h}{\nu} \frac{(2k+2h-3)_{2m-2h}}{m(h-1)!(2m-2h)!(k+h-1)_{2m-h}} \\
&\quad \times \frac{(-s)_h (k+s)_{2m-h}}{(k)_{2m}} \\
&+ \sum_{h=\nu-1}^{m-1} (-1)^{h+1-\nu} \binom{h+1}{\nu} \frac{(2k+2h-1)_{2m-2h-2}}{m(h)!(2m-2h-2)!(k+h)_{2m-h-1}} \\
&\quad \times \frac{(-s)_{h+1} (k+s)_{2m-h-1}}{(k)_{2m}} \\
&= a(2m, \nu, k, s).
\end{aligned}$$

□

2.2. Pullback formula

We consider $\Gamma^n \times \Gamma^n$ imbedded in Γ^{2n} by

$$\begin{pmatrix} A^{(n)} & B^{(n)} \\ C^{(n)} & D^{(n)} \end{pmatrix} \times \begin{pmatrix} A'^{(n)} & B'^{(n)} \\ C'^{(n)} & D'^{(n)} \end{pmatrix} \mapsto \begin{pmatrix} A & 0 & B & 0 \\ 0 & A' & 0 & B' \\ C & 0 & D & 0 \\ 0 & C' & 0 & D' \end{pmatrix},$$

and when convenient will identify $\Gamma^n \times \Gamma^n$ with its image in Γ^{2n} .

In this section we decompose $(L^{k,l}E_k^{2n})(3, s)$, $3 = \begin{pmatrix} Z^{(n)} & {}^tU^{(n)} \\ U^{(n)} & W^{(n)} \end{pmatrix} \in \mathfrak{H}_{2n}$ into functions for Z and for W . The main tool is the coset decomposition by Garrett [12]:

(2.2.1). (i) *The double coset $P_{2n,0} \backslash \Gamma^{2n} / \Gamma^n \times \Gamma^n$ has an irredundant set of coset representatives*

$$g_{\tilde{T}} = \begin{pmatrix} 1_n & 0 & 0 & 0 \\ 0 & 1_n & 0 & 0 \\ 0 & \tilde{T}^{(n)} & 1_n & 0 \\ \tilde{T}^{(n)} & 0 & 0 & 1_n \end{pmatrix},$$

where $\tilde{T} = \begin{pmatrix} 0 & 0 \\ 0 & T^{(r)} \end{pmatrix}$, $T \in \mathbb{T}^{(r)}$ ($0 \leq r \leq n$).

(ii) *The left coset $P_{2n,0} \backslash P_{2n,0} g_{\tilde{T}}(\Gamma^n \times \Gamma^n)$ has an irredundant set of coset representatives $g_{\tilde{T}} \tilde{g}_1 g_2 \tilde{g}'_1 g'_2$,*

$$\tilde{g}_1 \in G_{n,r}, \quad g_2 \in P_{n,r} \backslash \Gamma^n, \quad \tilde{g}'_1 \in \Gamma^r(T) \backslash G_{n,r}, \quad g'_2 \in P_{n,r} \backslash \Gamma^n,$$

where

$$(2.12) \quad G_{n,r} := \left\{ \tilde{g} = \begin{pmatrix} \tilde{A}^{(n)} & \tilde{B}^{(n)} \\ \tilde{C}^{(n)} & \tilde{D}^{(n)} \end{pmatrix} = \begin{pmatrix} 1_{n-r} & 0 & 0 & 0 \\ 0 & A^{(r)} & 0 & B^{(r)} \\ 0 & 0 & 1_{n-r} & 0 \\ 0 & C^{(r)} & 0 & D^{(r)} \end{pmatrix} \in \Gamma^n \right. \\ \left. \left| g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma^r \right\} \right.$$

and for $T \in \mathbb{T}^{(r)}$,

$$(2.13) \quad \Gamma^r(T) := \left\{ g \in \Gamma^r \mid \begin{pmatrix} 0 & T^{-1} \\ T & 0 \end{pmatrix} g \begin{pmatrix} 0 & T^{-1} \\ T & 0 \end{pmatrix} \in \Gamma^r \right\}.$$

Then the pullback formula is given by the following:

PROPOSITION 2. *Let $k, l \in 2\mathbb{Z}$, $k, l > 0$, $s \in \mathbb{C}$ and $k + 2\operatorname{Re}(s) > 2n + 1$. For $3 = \begin{pmatrix} Z^{(n)} & {}^tU^{(n)} \\ U^{(n)} & W^{(n)} \end{pmatrix} \in \mathfrak{H}_{2n}$, we have*

$$(2.14) \quad (L^{k,l}E_k^{2n})(3, s) = \sum_{\nu=0}^{\frac{l}{2}} \left(-\frac{1}{4}\right)^{\nu} \frac{a(l, \nu, k, s)}{(2\pi i)^l} \sum_{r=1}^n \sum_{T \in \mathbb{T}^{(r)}} \mathcal{P}_{\nu} \left(Z, W, \begin{pmatrix} 0 & 0 \\ 0 & T^{(r)} \end{pmatrix}, s \right),$$

with

(2.15)

$$\begin{aligned}
& \mathcal{P}_\nu \left(Z, W, \begin{pmatrix} 0 & 0 \\ 0 & T \end{pmatrix}, s \right) \\
&:= \sum_{g_2 \in P_{n,r} \setminus \Gamma^n} \sum_{g'_2 \in P_{n,r} \setminus \Gamma^n} \sum_{\tilde{g}_1 \in G_{n,r}} \sum_{\tilde{g}'_1 \in \Gamma^r(T) \setminus G_{n,r}} \\
& \left\{ \det(\operatorname{Im}(\tilde{g}_1 g_2 \langle Z \rangle))^s \det(\operatorname{Im}(\tilde{g}'_1 g'_2 \langle W \rangle))^s \right. \\
& \times \left| \det(1_n - \tilde{T}(\tilde{g}'_1 g'_2 \langle W \rangle) \tilde{T}(\tilde{g}_1 g_2 \langle Z \rangle)) \right|^{-2s} \\
& \times \rho_1 \left((C_2 Z + D_2)^{-1} (\tilde{C}_1(g_2 \langle Z \rangle) + \tilde{D}_1)^{-1} (1_n - \tilde{T}(\tilde{g}'_1 g'_2 \langle W \rangle) \tilde{T}(\tilde{g}_1 g_2 \langle Z \rangle))^{-1} \right) \\
& \times \rho \left((C'_2 W + D'_2)^{-1} (\tilde{C}'_1(g'_2 \langle W \rangle) + \tilde{D}'_2)^{-1} \right) \left(e_1 \tilde{T} {}^t e_2 \right)^{l-2\nu} \\
& \times \left(e_1 (1_n - \tilde{T}(\tilde{g}'_1 g'_2 \langle W \rangle) \tilde{T}(\tilde{g}_1 g_2 \langle \bar{Z} \rangle)) (\operatorname{Im}(\tilde{g}_1 g_2 \langle Z \rangle))^{-1} \right. \\
& \quad \times {}^t (1_n - \tilde{T}(\tilde{g}'_1 g'_2 \langle W \rangle) \tilde{T}(\tilde{g}_1 g_2 \langle Z \rangle)) {}^t e_1 \Big)^{\nu} \\
& \times \left(e_2 (1_n - \tilde{T}(\tilde{g}_1 g_2 \langle Z \rangle) \tilde{T}(\tilde{g}'_1 g'_2 \langle W \rangle))^{-1} (1_n - \tilde{T}(\tilde{g}_1 g_2 \langle Z \rangle) \tilde{T}(\tilde{g}'_1 g'_2 \langle \bar{W} \rangle)) \right. \\
& \quad \times \left. \left. (\operatorname{Im}(\tilde{g}'_1 g'_2 \langle W \rangle))^{-1} {}^t e_2 \right)^{\nu} \right\},
\end{aligned}$$

where $\rho_1 = \det^k \otimes \operatorname{sym}^l$ (resp. $\rho = \det^k \otimes \operatorname{sym}^l$) is the representation of $GL(n, \mathbb{C})$ with the representation space $\operatorname{sym}^l(V_1)$ (resp. $\operatorname{sym}^l(V_2)$), $\tilde{g}_1 = \begin{pmatrix} \tilde{A}_1^{(n)} & \tilde{B}_1^{(n)} \\ \tilde{C}_1^{(n)} & \tilde{D}_1^{(n)} \end{pmatrix}$, $\tilde{g}'_1 = \begin{pmatrix} \tilde{A}'_1^{(n)} & \tilde{B}'_1^{(n)} \\ \tilde{C}'_1^{(n)} & \tilde{D}'_1^{(n)} \end{pmatrix}$ are of the form (2.12), $g_2 = \begin{pmatrix} A_2^{(n)} & B_2^{(n)} \\ C_2^{(n)} & D_2^{(n)} \end{pmatrix}$, $g'_2 = \begin{pmatrix} A'_2^{(n)} & B'_2^{(n)} \\ C'_2^{(n)} & D'_2^{(n)} \end{pmatrix}$, $\tilde{T}^{(n)} = \begin{pmatrix} 0 & 0 \\ 0 & T^{(r)} \end{pmatrix}$.

PROOF. By Proposition 1, we have only to prove

$$\begin{aligned}
& \sum_M \det(\mathfrak{C}\mathfrak{Z}_0 + \mathfrak{D})^{-k} \det(\operatorname{Im}(M \langle \mathfrak{Z}_0 \rangle))^s Q_0^{l-2\nu} (P_0 - P'_0)^\nu (R_0 - R'_0)^\nu \\
&= \left(-\frac{1}{4} \right)^\nu \sum_{r=1}^n \sum_{T \in \mathbb{T}^{(r)}} \mathcal{P}_\nu \left(Z, W, \begin{pmatrix} 0 & 0 \\ 0 & T \end{pmatrix}, s \right) \quad \left(0 \leq \nu \leq \frac{l}{2} \right),
\end{aligned}$$

where the notations are the same as those in Proposition 1.

We put $g = \begin{pmatrix} A^{(n)} & B^{(n)} \\ C^{(n)} & D^{(n)} \end{pmatrix} = \tilde{g}_1 g_2$, $g' = \begin{pmatrix} A'^{(n)} & B'^{(n)} \\ C'^{(n)} & D'^{(n)} \end{pmatrix} = \tilde{g}'_1 g'_2$. Then a coset representative $M = \begin{pmatrix} \mathfrak{A}^{(2n)} & \mathfrak{B}^{(2n)} \\ \mathfrak{C}^{(2n)} & \mathfrak{D}^{(2n)} \end{pmatrix}$ of $P_{2n,0} \backslash \Gamma^{2n}$ can be written in the following form:

$$(2.16) \quad M = g_{\tilde{T}} g g' = \begin{pmatrix} A & 0 & B & 0 \\ 0 & A' & 0 & B' \\ C & \tilde{T}A' & D & \tilde{T}B' \\ \tilde{T}A & C' & \tilde{T}B & D' \end{pmatrix}.$$

By $(\text{Im}(\mathfrak{Z}_0))^{-1} = (\mathfrak{C}\mathfrak{Z}_0 + \mathfrak{D})^{-1}(\text{Im}(M\langle\mathfrak{Z}_0\rangle))^{-1} {}^t(\mathfrak{C}\mathfrak{Z}_0 + \mathfrak{D})^{-1} + 2i(\mathfrak{C}\mathfrak{Z}_0 + \mathfrak{D})^{-1}\mathfrak{C}$, we have

$$P_0 - P'_0 = \left(-\frac{1}{2i}\right) ((\mathfrak{C}\mathfrak{Z}_0 + \mathfrak{D})^{-1}(\mathfrak{C}\bar{\mathfrak{Z}}_0 + \mathfrak{D})(\text{Im}(\mathfrak{Z}_0))^{-1}) \left[\begin{pmatrix} {}^t e_1 \\ 0 \end{pmatrix} \right],$$

$$R_0 - R'_0 = \left(-\frac{1}{2i}\right) ((\mathfrak{C}\mathfrak{Z}_0 + \mathfrak{D})^{-1}(\mathfrak{C}\bar{\mathfrak{Z}}_0 + \mathfrak{D})(\text{Im}(\mathfrak{Z}_0))^{-1}) \left[\begin{pmatrix} 0 \\ {}^t e_2 \end{pmatrix} \right].$$

On the other hand, we have

$$\mathfrak{C}\mathfrak{Z}_0 + \mathfrak{D} = \begin{pmatrix} CZ + D & \tilde{T}(A'W + B') \\ \tilde{T}(AZ + B) & C'W + D' \end{pmatrix}$$

and

$$(\mathfrak{C}\mathfrak{Z}_0 + \mathfrak{D})^{-1} = \begin{pmatrix} (CZ + D)^{-1} & -(CZ + D)^{-1} \\ \times (1_n - \tilde{T}g'\langle W \rangle \tilde{T}g\langle Z \rangle)^{-1} & \times (1_n - \tilde{T}g'\langle W \rangle \tilde{T}g\langle Z \rangle)^{-1} \tilde{T}g'\langle W \rangle \\ -(C'W + D')^{-1} & (C'W + D')^{-1} \\ \times (1_n - \tilde{T}g\langle Z \rangle \tilde{T}g'\langle W \rangle)^{-1} \tilde{T}g\langle Z \rangle & \times (1_n - \tilde{T}g\langle Z \rangle \tilde{T}g'\langle W \rangle)^{-1} \end{pmatrix}.$$

If we put $(\mathfrak{C}\mathfrak{Z}_0 + \mathfrak{D})^{-1}(\mathfrak{C}\bar{\mathfrak{Z}}_0 + \mathfrak{D})(\text{Im}(\mathfrak{Z}_0))^{-1} = \begin{pmatrix} \mathcal{A}^{(n)} & * \\ * & \mathcal{D}^{(n)} \end{pmatrix}$ and $(\mathfrak{C}\mathfrak{Z}_0 + \mathfrak{D})^{-1}\mathfrak{C} = \begin{pmatrix} * & \mathcal{B}^{(n)} \\ * & * \end{pmatrix}$, we get

$$\begin{aligned} \mathcal{A} &= (CZ + D)^{-1}(1_n - \tilde{T}g'\langle W \rangle \tilde{T}g\langle Z \rangle)^{-1}(1_n - \tilde{T}g'\langle W \rangle \tilde{T}g\langle \bar{Z} \rangle) \\ &\quad \times (C\bar{Z} + D)(\text{Im}(Z))^{-1}, \\ \mathcal{D} &= (C'W + D')^{-1}(1_n - \tilde{T}g\langle Z \rangle \tilde{T}g'\langle W \rangle)^{-1}(1_n - \tilde{T}g\langle Z \rangle \tilde{T}g'\langle \bar{W} \rangle) \\ &\quad \times (C'\bar{W} + D')(\text{Im}(W))^{-1}, \\ \mathcal{B} &= (CZ + D)^{-1}(1_n - \tilde{T}g'\langle W \rangle \tilde{T}g\langle Z \rangle)^{-1} \tilde{T} {}^t(C'W + D')^{-1}. \end{aligned}$$

Thus, if we note

$$\begin{aligned} g\langle Z \rangle &= \tilde{g}_1 g_2 \langle Z \rangle, & CZ + D &= (\tilde{C}_1(g_2 \langle Z \rangle) + \tilde{D}_1)(C_2 Z + D_2), \\ g'\langle W \rangle &= \tilde{g}'_1 g'_2 \langle W \rangle, & C'W + D' &= (\tilde{C}'_1(g'_2 \langle W \rangle) + \tilde{D}'_1)(C'_2 W + D'_2), \end{aligned}$$

we get

$$\begin{aligned} Q_0 &= e_1(C_2 Z + D_2)^{-1}(\tilde{C}_1(g_2 \langle Z \rangle) + \tilde{D}_1)^{-1}(1_n - \tilde{T}(\tilde{g}'_1 g'_2 \langle W \rangle)\tilde{T}(\tilde{g}_1 g_2 \langle Z \rangle))^{-1} \\ &\quad \times \tilde{T}({}^t(\tilde{C}'_1(g'_2 \langle W \rangle) + \tilde{D}'_1)^{-1}({}^t(C'_2 W + D'_2))^{-1}{}^t e_2, \end{aligned}$$

$$\begin{aligned} P_0 - P'_0 &= \left(-\frac{1}{2i}\right) e_1(C_2 Z + D_2)^{-1}(\tilde{C}_1(g_2 \langle Z \rangle) + \tilde{D}_1)^{-1} \\ &\quad \times (1_n - \tilde{T}(\tilde{g}'_1 g'_2 \langle W \rangle)\tilde{T}(\tilde{g}_1 g_2 \langle Z \rangle))^{-1} \\ &\quad \times (1_n - \tilde{T}(\tilde{g}'_1 g'_2 \langle W \rangle)\tilde{T}(\tilde{g}_1 g_2 \langle \bar{Z} \rangle))(\tilde{C}_1(g_2 \langle \bar{Z} \rangle) + \tilde{D}_1) \\ &\quad \times (C_2 \bar{Z} + D_2)(\text{Im}(Z))^{-1}{}^t e_1, \end{aligned}$$

$$\begin{aligned} R_0 - R'_0 &= \left(-\frac{1}{2i}\right) e_2(C'_2 W + D'_2)^{-1}(\tilde{C}'_1(g'_2 \langle W \rangle) + \tilde{D}'_1)^{-1} \\ &\quad \times (1_n - \tilde{T}(\tilde{g}_1 g_2 \langle Z \rangle)\tilde{T}(\tilde{g}'_1 g'_2 \langle W \rangle))^{-1} \\ &\quad \times (1_n - \tilde{T}(\tilde{g}_1 g_2 \langle Z \rangle)\tilde{T}(\tilde{g}'_1 g'_2 \langle \bar{W} \rangle))(\tilde{C}'_1(g'_2 \langle \bar{W} \rangle) + \tilde{D}'_1) \\ &\quad \times (C'_2 \bar{W} + D'_2)(\text{Im}(W))^{-1}{}^t e_2 \end{aligned}$$

and

$$\begin{aligned} &\det(\mathfrak{C}\mathfrak{Z}_0 + \mathfrak{D})^{-k} \det(\text{Im}(M\langle \mathfrak{Z}_0 \rangle))^s \\ &= \det(\mathfrak{C}\mathfrak{Z}_0 + \mathfrak{D})^{-k} |\det(\mathfrak{C}\mathfrak{Z}_0 + \mathfrak{D})|^{-2s} \det(\text{Im}(\mathfrak{Z}_0))^s \\ &= \det(\tilde{C}_1(g_2 \langle Z \rangle) + \tilde{D}_1)^{-k} \det(C_2 Z + D_2)^{-k} \det(\tilde{C}'_1(g'_2 \langle W \rangle) + \tilde{D}'_1)^{-k} \\ &\quad \times \det(C'_2 W + D'_2)^{-k} \det(1_n - \tilde{T}(\tilde{g}'_1 g'_2 \langle W \rangle)\tilde{T}(\tilde{g}_1 g_2 \langle Z \rangle))^{-k} \\ &\quad \times \left| \det(1_n - \tilde{T}(\tilde{C}'_1(g'_2 \langle W \rangle) + \tilde{D}'_1)\tilde{T}(\tilde{g}_1 g_2 \langle Z \rangle)) \right|^{-2s} \det(\text{Im}(\tilde{g}_1 g_2 \langle Z \rangle))^s \det(\text{Im}(\tilde{g}'_1 g'_2 \langle W \rangle))^s. \end{aligned}$$

Combining these formulae and using the representations ρ_1, ρ , we obtain Proposition 2. $\square$

§3. Analytic properties of standard L -functions

3.1. Integral representation

We first prove the following:

THEOREM 1. *Let $k, l \in 2\mathbb{Z}$, $k, l > 0$, $s \in \mathbb{C}$ and $k + 2\operatorname{Re}(s) > 2n + 1$. For an eigenform $f \in S_{k,l}^n(\operatorname{sym}^l(V_2))$ and each $T \in \mathbb{T}^{(r)}$ ($1 \leq r \leq n$), the Petersson inner product of f and $\mathcal{P}_\nu \left(Z, W, \begin{pmatrix} 0 & 0 \\ 0 & T \end{pmatrix}, s \right)$ ($0 \leq \nu \leq \frac{l}{2}$) is convergent and then,*

(i) for $T \in \mathbb{T}^{(r)}$ ($1 \leq r < n$),

$$\left(f, \mathcal{P}_\nu \left(-\bar{Z}, *, \begin{pmatrix} 0 & 0 \\ 0 & T \end{pmatrix}, \bar{s} \right) \right) = 0,$$

(ii) for $T \in \mathbb{T}^{(n)}$,

$$\begin{aligned} & \left(f, \left(-\frac{1}{4} \right)^\nu \mathcal{P}_\nu \left(-\bar{Z}, *, T, \bar{s} \right) \right) \\ &= \lambda(f, T) \det(T)^{-k-2s} 2^{n(n+1-k-2s)-l+1} i^{nk+l} \pi^{\frac{n(n+1)}{2}} (\iota^{-1}(f))(Z) \\ & \quad \times \frac{(-1)^{\nu\nu!}}{(k+s+l-\nu-1)_{\nu+1}} \sum_{j=1}^{n-1} \frac{\Gamma(2k+2s-2n-1+2j)(2k+2s-n-2+j)_l}{(k+s-n-1+j)\Gamma(2k+2s+l-n-1+j)}. \end{aligned}$$

PROOF. It follows from (1.4.2) that $\left(f, (L^{k,l} E_k^{2n}) \left(\begin{pmatrix} -\bar{Z}^{(n)} & 0 \\ 0 & * \end{pmatrix}, \bar{s} \right) \right)$ converges absolutely and locally uniformly for $k + 2\operatorname{Re}(s) > 2n + 1$. Then, by Proposition 2, $\left(f, \mathcal{P}_\nu \left(-\bar{Z}, *, \begin{pmatrix} 0 & 0 \\ 0 & T \end{pmatrix}, \bar{s} \right) \right)$ is convergent.

The assertion (i) is proved in the same way as that by Klingen [17, Satz 2]:

We can write

$$\begin{aligned} & \mathcal{P}_\nu \left(Z, W, \begin{pmatrix} 0 & 0 \\ 0 & T \end{pmatrix}, s \right) \\ &= \sum_{g_2 \in P_{n,r} \backslash \Gamma^n} \sum_{g'_2 \in P_{n,r} \backslash \Gamma^n} \sum_{\tilde{g}_1 \in G_{n,r}} \sum_{\tilde{g}'_1 \in \Gamma^r(T) \backslash G_{n,r}} \left\{ \det(\operatorname{Im}(Z))^s \det(\operatorname{Im}(W))^s \right. \\ & \quad \times \left| \det(1_n - \tilde{T}W\tilde{T}Z) \right|^{-2s} \rho_1 \left((1_n - \tilde{T}W\tilde{T}Z)^{-1} \right) \left(e_1 \tilde{T} {}^t e_2 \right)^{l-2\nu} \\ & \quad \times \left(e_1 (1_n - \tilde{T}W\tilde{T}\bar{Z}) \operatorname{Im}(Z)^{-1} {}^t (1_n - \tilde{T}W\tilde{T}Z) {}^t e_1 \right)^\nu \end{aligned}$$

$$\times \left(e_2(1_n - \tilde{T}Z\tilde{T}W)^{-1}(1_n - \tilde{T}Z\tilde{T}\bar{W})\mathrm{Im}(W)^{-1} {}^t e_2 \right)^\nu \Bigg\}$$

$$\left| (\tilde{g}'_1)_W \right| (\tilde{g}_1)_Z \left| (g'_2)_W \right| (g_2)_Z,$$

where $(\)_Z$ (resp. $(\)_W$) denotes the action on Z (resp. W). Then we put

$$G_\nu \left(Z, W, \begin{pmatrix} 0 & 0 \\ 0 & T \end{pmatrix}, s \right)$$

$$:= \det(\mathrm{Im}(Z))^s \det(\mathrm{Im}(W))^s \left| \det(1_n - \tilde{T}W\tilde{T}Z) \right|^{-2s}$$

$$\times \rho_1 \left((1_n - \tilde{T}W\tilde{T}Z)^{-1} \right) \left(e_1 \tilde{T} {}^t e_2 \right)^{l-2\nu}$$

$$\times \left(e_1(1_n - \tilde{T}W\tilde{T}\bar{Z})\mathrm{Im}(Z)^{-1} {}^t(1_n - \tilde{T}W\tilde{T}Z) {}^t e_1 \right)^\nu$$

$$\times \left(e_2(1_n - \tilde{T}Z\tilde{T}W)^{-1}(1_n - \tilde{T}Z\tilde{T}\bar{W})\mathrm{Im}(W)^{-1} {}^t e_2 \right)^\nu.$$

Now, we put

$$W = X + iY = \begin{pmatrix} W_1^{(n-r)} & {}^t W_2^{(n-r,r)} \\ W_2^{(r,n-r)} & W_*^{(r)} \end{pmatrix},$$

$$X = \begin{pmatrix} X_1^{(n-r)} & {}^t X_2^{(n-r,r)} \\ X_2^{(r,n-r)} & X_*^{(r)} \end{pmatrix}, \quad Y = \begin{pmatrix} Y_1^{(n-r)} & 0 \\ 0 & Y_*^{(r)} \end{pmatrix} \left[\begin{pmatrix} 1_{n-r} & 0 \\ Y_2^{(r,n-r)} & 1_r \end{pmatrix} \right].$$

Let F_n be a fundamental domain of $\Gamma^n \backslash \mathfrak{H}_n$ and $F_{n,r}$ be a fundamental domain of $P_{n,r} \backslash \mathfrak{H}_n$, that is,

$$F_{n,r} := \{ W \in \mathfrak{H}_n \mid W_* \in F_r, Y_1 \in M_{n-r}, X \text{ and } Y_2 \text{ are reduced mod } 1 \},$$

where M_{n-r} is the Minkowski reduction domain of the positive definite quadratic forms of degree $n-r$. Then we obtain

$$\left(f, \mathcal{P}_\nu \left(-\bar{Z}, *, \begin{pmatrix} 0 & 0 \\ 0 & T \end{pmatrix}, \bar{s} \right) \right)$$

$$= \int_{F_n} \left\langle \rho \left(\sqrt{\mathrm{Im}(W)} \right) f(W), \rho \left(\sqrt{\mathrm{Im}(W)} \right) \mathcal{P}_\nu \left(-\bar{Z}, W, \begin{pmatrix} 0 & 0 \\ 0 & T \end{pmatrix}, \bar{s} \right) \right\rangle$$

$$\times \det(\mathrm{Im}(W))^{-n-1} dW$$

$$\begin{aligned}
&= 2^{-1} \int_{F_{n,r}} \left\langle \rho \left(\sqrt{\operatorname{Im}(W)} \right) f(W), \right. \\
&\quad \sum_{g'_2} \sum_{\tilde{g}_1} \sum_{\tilde{g}'_1} \rho \left(\sqrt{\operatorname{Im}(W)} \right) G_\nu \left(-\bar{Z}, W, \begin{pmatrix} 0 & 0 \\ 0 & T \end{pmatrix}, \bar{s} \right) \left| (\tilde{g}'_1)_W \right| (\tilde{g}_1)_Z \left| (g'_2)_W \right\rangle \\
&\quad \times \det(\operatorname{Im}(W))^{-n-1} dW.
\end{aligned}$$

Here we recall that $\tilde{g}'_1$ is of the form

$$\begin{pmatrix} \tilde{A}'_1^{(n)} & \tilde{B}'_1^{(n)} \\ \tilde{C}'_1^{(n)} & \tilde{D}'_1^{(n)} \end{pmatrix} = \begin{pmatrix} 1_{n-r} & 0 & 0 & 0 \\ 0 & A'_1{}^{(r)} & 0 & B'_1{}^{(r)} \\ 0 & 0 & 1_{n-r} & 0 \\ 0 & C'_1{}^{(r)} & 0 & D'_1{}^{(r)} \end{pmatrix} \quad \text{with} \quad g'_1 = \begin{pmatrix} A'_1 & B'_1 \\ C'_1 & D'_1 \end{pmatrix}.$$

Since

$$\tilde{g}'_1 \langle W \rangle = \begin{pmatrix} W_1 + {}^t W_2 (C'_1 W_* + D'_1)^{-1} C'_1 W_2 & {}^t W_2 (C'_1 W_* + D'_1)^{-1} \\ A'_1 W_2 + (g'_1 \langle W_* \rangle) C'_1 W_2 & g'_1 \langle W_* \rangle \end{pmatrix},$$

each of $\tilde{T}(\tilde{g}'_1 \langle W \rangle) \tilde{T}$, $\tilde{T}(\tilde{g}'_1 \langle W \rangle)$, $\tilde{C}'_1 W + \tilde{D}'_1$ as a function on W does not depend on W_1 . Therefore,

$$\sum_{g'_2} \sum_{\tilde{g}_1} \sum_{\tilde{g}'_1} \rho \left(\sqrt{\operatorname{Im}(W)} \right) G_\nu \left(-\bar{Z}, W, \begin{pmatrix} 0 & 0 \\ 0 & T \end{pmatrix}, \bar{s} \right) \left| (\tilde{g}'_1)_W \right| (\tilde{g}_1)_Z \left| (g'_2)_W \right|$$

is a constant function on X_1 . On the other hand, since f is a cusp form, its Fourier expansion has no constant terms in X_1 . If we consider an integral of the integrand above on a unit cube in X_1 , we find that it vanishes. Thus the assertion (i) is proved.

PROOF OF (ii). By $\Gamma^n \begin{pmatrix} T & 0 \\ 0 & T^{-1} \end{pmatrix} \Gamma^n = \bigcup_{g' \in \Gamma^n(T) \setminus \Gamma^n} \Gamma^n \begin{pmatrix} T & 0 \\ 0 & T^{-1} \end{pmatrix} g'$, we get

$$\mathcal{P}_\nu(Z, W, T, s) = \left(\mathcal{P}_\nu(Z, W, 1_n, s) \left| \left(\Gamma^n \begin{pmatrix} T & 0 \\ 0 & T^{-1} \end{pmatrix} \Gamma^n \right)_W \right. \right) \det(T)^{-k-2s},$$

where

$$\begin{aligned}
&\mathcal{P}_\nu(Z, W, 1_n, s) \\
&:= \sum_{g \in \Gamma^n} \left\{ \det(\operatorname{Im}(Z))^s \det(\operatorname{Im}(W))^s |\det(W + Z)|^{-2s} \rho_1((W + Z)^{-1}) \right.
\end{aligned}$$

$$\begin{aligned} & \times (e_1 \ ^t e_2)^{l-2\nu} (e_1(W + \bar{Z})\text{Im}(Z)^{-1}(W + Z)^t e_1)^\nu \\ & \times (e_2(W + Z)^{-1}(\bar{W} + Z)\text{Im}(W)^{-1} \ ^t e_2)^\nu \Big| (g)_Z. \end{aligned}$$

Since the Hecke operator is an Hermitian operator and f is an eigenform, we have

$$\begin{aligned} (3.1) \quad & \left(f, \left(-\frac{1}{4} \right)^\nu \mathcal{P}_\nu(-\bar{Z}, *, T, \bar{s}) \right) \\ & = \lambda(f, T) \det(T)^{-k-2s} (-1)^\nu 2^{-2\nu} (f, \mathcal{P}_\nu(-\bar{Z}, *, 1_n, \bar{s})) \\ & = \lambda(f, T) \det(T)^{-k-2s} (-1)^\nu 2^{-2\nu+1} \\ & \quad \times \int_{\mathfrak{H}_n} \left\langle \rho \left(\sqrt{\text{Im}(W)} \right) f(W), \rho \left(\sqrt{\text{Im}(W)} (W - \bar{Z})^{-1} \right) (e_1 \ ^t e_2)^{l-2\nu} \right. \\ & \quad \times (e_1(W - \bar{Z})^{-1}(W - Z)\text{Im}(Z)^{-1} \ ^t e_1)^\nu \\ & \quad \times (e_2(\bar{W} - \bar{Z})\text{Im}(W)^{-1}(W - \bar{Z})^t e_2)^\nu \Big\rangle \\ & \quad \times |\det(W - \bar{Z})|^{-2s} \det(\text{Im}(Z))^s \det(\text{Im}(W))^{s-n-1} dW. \end{aligned}$$

We compute the integral (3.1) according to Klingen [16, §1]:

For $Z \in \mathfrak{H}_n$, there exists $F \in GL(n, \mathbb{R})$ such that $\text{Im}(Z) [^t F] = 1_n$. Then, let $\mathcal{L}_Z$ be a biholomorphic transformation from $\mathfrak{H}_n$ onto

$$S^n := \left\{ S \in \mathbb{C}^{(n)} \mid S = {}^t S, 1_n - \bar{S}S > 0 \right\}$$

such that for $W \in \mathfrak{H}_n$,

$$\mathcal{L}_Z(W) = S := F(W - Z)(W - \bar{Z})^{-1}F^{-1}.$$

We note

$$\text{Im}(W) = 2^{-2}(\bar{W} - Z)^t F(1_n - \bar{S}S)F(W - \bar{Z}),$$

$$dW = 2^{-n(n+1)} \det(\text{Im}(Z))^{-(n+1)} |\det(W - \bar{Z})|^{2(n+1)} dS,$$

where $dS = dX_S dY_S$ ($S = X_S + iY_S$) is of the form (1.2). We also put

$$\widehat{f}(S) := \rho(W - \bar{Z}) f(W).$$

Then, the integral (3.1) is:

$$\begin{aligned}
& \lambda(f, T) \det(T)^{-k-2s} (-1)^\nu 2^{n(n+1-2s)-2(nk+l)+1} \\
& \quad \times \int_{S^n} \left\langle \widehat{f}(S), \rho({}^t F(1_n - \overline{S}S)F) (e_1 {}^t e_2)^{l-2\nu} (e_1 {}^t F S F {}^t e_1)^\nu \right. \\
& \quad \left. \times (e_2 F^{-1} \overline{S}(1_n - S\overline{S})^{-1} {}^t F^{-1} {}^t e_2)^\nu \right\rangle \det(1_n - \overline{S}S)^{s-n-1} dS \\
& = \lambda(f, T) \det(T)^{-k-2s} (-1)^\nu 2^{n(n+1-2s)-2(nk+l)+1} \\
& \quad \times \int_{S^n} \left\langle \rho(Z - \overline{Z}) f(Z), \rho({}^t F(1_n - \overline{S}S)F) (e_1 {}^t e_2)^{l-2\nu} (e_1 {}^t F S F {}^t e_1)^\nu \right. \\
& \quad \left. \times (e_2 F^{-1} \overline{S}(1_n - S\overline{S})^{-1} {}^t F^{-1} {}^t e_2)^\nu \right\rangle \det(1_n - \overline{S}S)^{s-n-1} dS \\
& \text{by } \frac{\partial}{\partial S} \det(1_n - S\overline{S}) = -\det(1_n - \overline{S}S) (\overline{S}(1_n - S\overline{S})^{-1}) \quad \text{with } \frac{\partial}{\partial S} = \\
& \quad \left(\frac{1 + \delta_{jh}}{2} \frac{\partial}{\partial s_{jh}} \right)_{1 \leq j, h \leq n}, \text{ where } \frac{\partial}{\partial s_{jh}} \text{ is of the form (2.1) ,} \\
& = \lambda(f, T) \det(T)^{-k-2s} 2^{n(n+1-k-2s)-l+1} i^{nk+l} \\
& \quad \times \int_{S^n} \left\langle \rho(1_n - \overline{S}S) \rho({}^t F^{-1}) f(Z), \rho_1({}^t F) (e_1 {}^t e_2)^{l-2\nu} (e_1 S {}^t e_1)^\nu \right. \\
& \quad \left. \times \left(e_2 \left(\frac{\partial}{\partial S} \det(1_n - S\overline{S}) \right) {}^t e_2 \right)^\nu \right\rangle \det(1_n - \overline{S}S)^{s-n-1-\nu} dS.
\end{aligned}$$

Moreover, there exists a linear map $\psi = \psi(n, l, \nu, k + s - n - 1) \in GL(\text{sym}^l(V_1))$ such that

$$\begin{aligned}
& \rho_1({}^t \overline{F}) \psi \rho_1({}^t F^{-1}) (\iota^{-1}(f))(Z) \\
& = \int_{S^n} \left\langle \rho(1_n - \overline{S}S) \rho({}^t F^{-1}) f(Z), \rho_1({}^t F) (e_1 {}^t e_2)^{l-2\nu} (e_1 S {}^t e_1)^\nu \right. \\
& \quad \left. \times \left(e_2 \left(\frac{\partial}{\partial S} \det(1_n - S\overline{S}) \right) {}^t e_2 \right)^\nu \right\rangle \det(1_n - \overline{S}S)^{s-n-1-\nu} dS.
\end{aligned}$$

For any unitary matrix $U \in U(n, \mathbb{C})$, we have

$$\begin{aligned}
& \rho_1({}^t \overline{F}) \rho_1(U^{-1}) \psi \rho_1(U) \rho_1({}^t F^{-1}) (\iota^{-1}(f))(Z) \\
& = \rho_1({}^t (\overline{UF})) \psi (\iota^{-1}(\rho(U {}^t F^{-1}) f))(Z)
\end{aligned}$$

$$= \int_{S^n} \left\langle \rho(1_n - \bar{S}S) \rho(U^t F^{-1}) f(Z), \rho_1({}^t(UF)) (e_1 {}^t e_2)^{l-2\nu} (e_1 S {}^t e_1)^\nu \right. \\ \left. \times \left(e_2 \left(\frac{\partial}{\partial S} \det(1_n - S\bar{S}) \right) {}^t e_2 \right)^\nu \right\rangle \det(1_n - \bar{S}S)^{s-n-1-\nu} dS$$

which equals, with the change of the variable S to ${}^tUSU = S'$,

$$= \int_{S^n} \left\langle \rho(U) \rho(1_n - \bar{S}'S') \rho({}^tF^{-1}) f(Z), \rho(U) \rho_1({}^tF) (e_1 {}^t e_2)^{l-2\nu} (e_1 S' {}^t e_1)^\nu \right. \\ \left. \times \left(e_2 \left(\frac{\partial}{\partial S'} \det(1_n - S'\bar{S}') \right) {}^t e_2 \right)^\nu \right\rangle \det(1_n - \bar{S}'S')^{s-n-1-\nu} dS' \\ = \rho_1({}^t\bar{F}) \psi \rho_1({}^tF^{-1}) (\iota^{-1}(f))(Z),$$

that is, $\rho_1(U^{-1})\psi\rho_1(U) = \psi$. Since $\rho_1 = \det^k \otimes \text{sym}^l$ is an irreducible representation of $U(n, \mathbb{C})$, ψ is a homothety by Schur's lemma.

Thus we get

$$\left(f, \left(-\frac{1}{4} \right)^\nu \mathcal{P}_\nu(-\bar{Z}, *, T, \bar{s}) \right) \\ = \lambda(f, T) \det(T)^{-k-2s} 2^{n(n+1-k-2s)-l+1} i^{nk+l} \psi(\iota^{-1}(f))(Z)$$

and we have only to prove

$$(3.2) \quad \psi = \pi^{\frac{n(n+1)}{2}} \frac{(-1)^\nu \nu!}{(k+s+l-\nu-1)_{\nu+1}} \\ \times \prod_{j=1}^{n-1} \frac{\Gamma(2k+2s-2n+2j-1)(2k+2s-n-2+j)_l}{(k+s-n-1+j)\Gamma(2k+2s+l+j-n-1)}.$$

Here, we can write

$$\psi(n, l, \nu, k+s-n-1) \\ = \int_{S^n} \det(1_n - S\bar{S})^{k+s-n-1-\nu} ((1_n - S\bar{S}) [{}^t p_n])^l (\bar{S} [{}^t p_n])^\nu \\ \times \left(\left(\frac{\partial}{\partial \bar{S}} \det(1_n - S\bar{S}) \right) [{}^t p_n] \right)^\nu dS,$$

where $p_n^{(1,n)} = (1, 0, \dots, 0)$.

Let $\mu = k+s-n-1$.

We put $S^{(n)} = \begin{pmatrix} S_1^{(n-1)} & {}^t v^{(n-1,1)} \\ {}_{v(1,n-1)} & z \end{pmatrix}$. By $1_{n-1} - S_1 \bar{S}_1 > 0$, there exists $g \in GL(n-1, \mathbb{C})$ such that $1_{n-1} - S_1 \bar{S}_1 = g \bar{g}$. If we put $v = u^t g$, we get $dv = |\det g|^2 du = \det(1_{n-1} - S_1 \bar{S}_1) du$. Moreover, we put

$$a = -\frac{1}{1 - \bar{u}^t u} (< 0), \quad b = -\frac{u^t g \bar{S}_1 {}^t \bar{g}^{-1} {}^t u}{1 - \bar{u}^t u},$$

$$c = 1 - u^t g \bar{g}^t \bar{u} - u^t g \bar{S}_1 {}^t \bar{g}^{-1} g^{-1} S_1 \bar{g}^t \bar{u} - \frac{|u^t g \bar{S}_1 {}^t \bar{g}^{-1} {}^t u|^2}{|1 - \bar{u}^t u|}.$$

We note $|b|^2 - ac = 1$. Thus the condition $1_n - S \bar{S} > 0$ is equivalent to the conditions

$$1_{n-1} - S_1 \bar{S}_1 > 0, \quad 1 - \bar{u}^t u > 0, \quad c + b\bar{z} + \bar{b}z + az\bar{z} > 0,$$

and we get

$$\det(1_n - S \bar{S}) = \det(1_{n-1} - S_1 \bar{S}_1)(1 - \bar{u}^t u)(c + b\bar{z} + \bar{b}z + az\bar{z}),$$

$$dS = \det(1_{n-1} - S_1 \bar{S}_1) dS_1 du \, dz.$$

Therefore, we get

$$\begin{aligned} & \psi(n, l, \nu, \mu) \\ &= \int \left\{ \det(1_{n-1} - S_1 \bar{S}_1)^{\mu-\nu+1} (1 - \bar{u}^t u)^\mu \right. \\ & \quad \times ((g(1_{n-1} - {}^t u \bar{u})^t \bar{g}) [{}^t p_{n-1}])^l (\bar{S}_1 [{}^t p_{n-1}])^\nu \\ & \quad \times \left(\left(\frac{\partial}{\partial \bar{S}_1} \det(1_{n-1} - S_1 \bar{S}_1) \right) [{}^t p_{n-1}] \right)^\nu \\ & \quad \times \left. \left\{ \iint (c + b\bar{z} + \bar{b}z + az\bar{z})^\mu dz \right\} \right\} dS_1 du. \end{aligned}$$

By Hua [14, §2.3], we have

$$\iint (c + b\bar{z} + \bar{b}z + az\bar{z})^\mu dz = (1 - \bar{u}^t u)^{\mu+2} \frac{\pi}{\mu+1}.$$

Using the equation above, we get

$$\begin{aligned} & \psi(n, l, \nu, \mu) \\ &= \frac{\pi}{\mu+1} \int_{S^{n-1}} \left\{ \det(1_{n-1} - S_1 \bar{S}_1)^{\mu-\nu+1} (\bar{S}_1 [{}^t p_{n-1}])^\nu \right. \\ & \quad \times \left(\left(\frac{\partial}{\partial \bar{S}_1} \det(1_{n-1} - S_1 \bar{S}_1) \right) [{}^t p_{n-1}] \right)^\nu \\ & \quad \times \left. \left\{ \int_{1 - \bar{u}^t u > 0} (1 - \bar{u}^t u)^{2\mu+2} ((g(1_{n-1} - {}^t u \bar{u})^t \bar{g}) [{}^t p_{n-1}])^l du \right\} \right\} dS_1. \end{aligned}$$

By [9, Proposition 3.1], we have

$$\begin{aligned} & \int (1 - \bar{u} {}^t u)^{2\mu+2} ((g(1_{n-1} - {}^t u \bar{u}) {}^t \bar{g}) [{}^t p_{n-1}])^l du \\ &= d(n-1, l, 2\mu+2) ((1_{n-1} - S_1 \bar{S}_1) [{}^t p_{n-1}])^l, \end{aligned}$$

where

$$d(n, l, \mu) = \pi^n \frac{\Gamma(\mu+1)}{\Gamma(\mu+l+n+1)} (n+\mu)_l.$$

Thus, we get

$$\begin{aligned} \psi(n, l, \nu, \mu) &= \frac{\pi}{\mu+1} \psi(n-1, l, \nu, \mu+1) d(n-1, l, 2\mu+2) \\ &= \psi(1, l, \nu, \mu+n-1) \prod_{j=1}^{n-1} \frac{\pi}{(\mu+j)} d(n-j, l, 2\mu+2j) \\ &= \psi(1, l, \nu, \mu+n-1) \pi^{\frac{n(n+1)}{2}-1} \prod_{j=1}^{n-1} \frac{\Gamma(2\mu+2j+1)(n+2\mu+j)_l}{(\mu+j)\Gamma(2\mu+j+l+n+1)}, \end{aligned}$$

where

$$\begin{aligned} \psi(1, l, \nu, \mu+n-1) &= \iint_{S^1} (1 - s\bar{s})^{\mu+n-1-\nu} (1 - s\bar{s})^l (\bar{s})^\nu \left(\frac{\partial}{\partial \bar{s}} (1 - s\bar{s}) \right)^\nu ds \\ &= (-1)^\nu \frac{\nu!}{(\mu+n+l-\nu)_{\nu+1}} \pi. \end{aligned}$$

□

Now, by Proposition 2 and Theorem 1, we have

$$\begin{aligned} & \left(f, (L^{k,l} E_k^{2n}) \left(\begin{pmatrix} -\bar{Z} & 0 \\ 0 & * \end{pmatrix}, \bar{s} \right) \right) \\ &= \sum_{\nu=0}^{\frac{l}{2}} \frac{a(l, \nu, k, s)}{(2\pi i)^l} \sum_{r=1}^n \sum_{T \in \mathbb{T}^{(r)}} \left(f, \left(-\frac{1}{4} \right)^\nu \mathcal{P}_\nu \left(-\bar{Z}, *, \begin{pmatrix} 0 & 0 \\ 0 & T \end{pmatrix}, \bar{s} \right) \right) \\ &= \sum_{\nu=0}^{\frac{l}{2}} \frac{a(l, \nu, k, s)}{(2\pi i)^l} \sum_{T \in \mathbb{T}^{(n)}} \left(f, \left(-\frac{1}{4} \right)^\nu \mathcal{P}_\nu (-\bar{Z}, *, T, \bar{s}) \right). \end{aligned}$$

By (1.4), it equals

$$\begin{aligned} & 2^{n(n+1-k-2s)-2l+1} i^{nk} \pi^{\frac{n(n+1)}{2}-l} \prod_{j=1}^{n-1} \frac{\Gamma(2k+2s-2n-1+2j)(2k+2s-n-2+j)_l}{(k+s-n-1+j)\Gamma(2k+2s+l-n-1+j)} \\ & \times \left\{ \sum_{\nu=0}^{\frac{l}{2}} a(l, \nu, k, s) \frac{(-1)^\nu \nu!}{(k+s+l-\nu-1)_{\nu+1}} \right\} L(k+2s, f)(\iota^{-1}(f))(Z). \end{aligned}$$

On the other hand, by (2.4), we get

$$\sum_{\nu=0}^{\frac{l}{2}} a(l, \nu, k, s) \frac{(-1)^\nu \nu!}{(k+s+l-\nu-1)_{\nu+1}} = \sum_{\nu=0}^{\frac{l}{2}} b(l, \nu, k, s) \frac{(-1)^\nu}{k+s+l-\nu-1}.$$

Thus, by (1.5) and (2.5), we get

$$\begin{aligned} & \left(f, (L^{k,l} E_k^{2n}) \left(\begin{pmatrix} -\bar{Z} & 0 \\ 0 & * \end{pmatrix}, \bar{s} \right) \right) \\ &= 2^{n(n+1-k-2s)-2l+1} i^{nk} \pi^{\frac{n(n+1)}{2}-l} \frac{\Gamma(k+s-n)\Gamma(2k+2s+l-n-1)}{\Gamma(k+s-1)\Gamma(2k+2s+l-2)} \\ & \quad \times \prod_{j=1}^{n-1} \frac{\Gamma(2k+2s+2j-2n-1)}{\Gamma(2k+2s+j-n-2)} \frac{1}{(k)_l} \frac{\Gamma(k+s+\frac{l}{2}-1)}{\Gamma(k+s)} \\ & \quad \times \left\{ \sum_{\nu=0}^{\frac{l}{2}} \frac{(-1)^\nu (2k-2+2\nu)_{l-2\nu}}{\nu!(l-2\nu)!(k-1+\nu)_{l-\nu}} (-s)_\nu (k+s+\frac{l}{2}-1)_{\frac{l}{2}-\nu} \right\} \\ & \quad \times \zeta(2s+k)^{-1} \prod_{j=1}^n \zeta(4s+2k-2j)^{-1} D_f(2s+k-n)(\iota^{-1}(f))(Z). \end{aligned}$$

Here, we prove the following:

LEMMA 4. Let $k, l \in \mathbb{Z}$, $k, l > 0$. For an indeterminate X ,

$$\begin{aligned} (3.3) \quad & \sum_{\nu=0}^{\lfloor \frac{l}{2} \rfloor} \frac{(-1)^\nu (2k-2+2\nu)_{l-2\nu}}{\nu!(l-2\nu)!(k-1+\nu)_{l-\nu}} \left(-X+k-\frac{1}{2}\right)_\nu \left(X+\left[\frac{l+1}{2}\right]-\frac{1}{2}\right)_{\lfloor \frac{l}{2} \rfloor - \nu} \\ &= \frac{2^l}{l!} X_{\lfloor \frac{l}{2} \rfloor}. \end{aligned}$$

PROOF. We follow Zagier's method in [9, lemma 4.1].

We denote the left-hand side of (3.3) by $P_{k,l}(X)$.

When we consider $P_{k,l}(X)$ a polynomial in X , the coefficient of the highest power of X is

$$\sum_{\nu=0}^{\lfloor \frac{l}{2} \rfloor} \frac{(2k-2+2\nu)_{l-2\nu}}{\nu!(l-2\nu)!(k-1+\nu)_{l-\nu}}.$$

Then we get

$$\begin{aligned}
 & \binom{k+l-2}{l} \sum_{\nu=0}^{\lfloor \frac{l}{2} \rfloor} \frac{(2k-2+2\nu)_{l-2\nu}}{\nu!(l-2\nu)!(k-1+\nu)_{l-\nu}} \\
 &= \frac{1}{l!} \sum_{\nu=0}^{\lfloor \frac{l}{2} \rfloor} \binom{2k+l-3}{l-2\nu} \binom{k+\nu-2}{k-2} \\
 &= \frac{1}{l!} \operatorname{Res}_{x=0} \left[\frac{1}{(1-x^2)^{k-1}} (1+x)^{2k+l-3} \frac{1}{x^{l+1}} dx \right]
 \end{aligned}$$

by putting $t = \frac{x}{1+x}$,

$$\begin{aligned}
 &= \frac{1}{l!} \operatorname{Res}_{t=0} \left[(1-2t)^{-(k-1)} \frac{1}{t^{l+1}} dt \right] \\
 &= \frac{2^l}{l!} \binom{k+l-2}{l}.
 \end{aligned}$$

Thus, we have only to prove $P_{k,l}(-j) = 0$ for $j \in \mathbb{Z}$, $0 \leq j \leq \lfloor \frac{l}{2} \rfloor - 1$. We use induction on l and j . We first prove $P_{k,l}(0) = 0$:

$$P_{k,l}(0) = \frac{(2k-2)_l}{(k-1)_l} \sum_{\nu=0}^{\lfloor \frac{l}{2} \rfloor} \frac{(-1)^\nu}{\nu!(l-2\nu)!2^{2\nu}} \left(\lfloor \frac{l+1}{2} \rfloor - \frac{1}{2} \right)_{\lfloor \frac{l}{2} \rfloor - \nu}.$$

For even l , by

$$(l-2\nu)! = \frac{2^{l-2\nu} \left(\frac{l}{2} - \nu \right)! \left(\frac{1}{2} \right)_{\frac{l}{2}}}{\left(\frac{1-l}{2} \right)_\nu (-1)^\nu},$$

we have

$$\begin{aligned}
 P_{k,l}(0) &= \frac{(2k-2)_l}{(k-1)_l 2^l \left(\frac{1}{2} \right)_{\frac{l}{2}}} \sum_{\nu=0}^{\frac{l}{2}} \frac{1}{\nu! \left(\frac{l}{2} - \nu \right)! \left(\frac{1-l}{2} \right)_\nu \left(\frac{l-1}{2} \right)_{\frac{l}{2}-\nu}} \\
 &= 0.
 \end{aligned}$$

For odd l , by

$$(l-2\nu)! = \frac{2^{l-2\nu} \left(\frac{l-1}{2} - \nu \right)! \left(\frac{1}{2} \right)_{\frac{l+1}{2}}}{\left(-\frac{l}{2} \right)_\nu (-1)^\nu},$$

we have

$$P_{k,l}(0) = \frac{(2k-2)_l}{(k-1)_l 2^l \left(\frac{1}{2}\right)_{\frac{l+1}{2}}} \sum_{\nu=0}^{\frac{l-1}{2}} \frac{1}{\nu! \left(\frac{l-1}{2} - \nu\right)!} \left(-\frac{l}{2}\right)_\nu \left(\frac{l}{2}\right)_{\frac{l-1}{2}-\nu} = 0.$$

Next, we suppose $P_{k,l'}(-j') = 0$ for any k, j', l' such that $j' < j, l' < l$.

For even l , if we note

$$\left(k - \frac{1}{2} + j\right)_\nu = \nu \left(k - \frac{1}{2} + j\right)_{\nu-1} + \left(k - \frac{1}{2} + j - 1\right)_\nu$$

and

$$\left(\frac{l-1}{2} - j\right)_{\frac{l}{2}-\nu} = -\frac{l-2\nu}{2} \left(\frac{l-1}{2} - (j-1)\right)_{\frac{l}{2}-\nu-1} + \left(\frac{l-1}{2} - (j-1)\right)_{\frac{l}{2}-\nu},$$

we have

$$\begin{aligned} & P_{k,l}(-j) \\ &= \sum_{\nu=0}^{\frac{l}{2}} \frac{(-1)^\nu (2k-2+2\nu)_{l-2\nu}}{\nu! (l-2\nu)! (k-1+\nu)_{l-\nu}} \left(k - \frac{1}{2} + j - 1\right)_\nu \left(\frac{l-1}{2} - (j-1)\right)_{\frac{l}{2}-\nu} \\ &+ \sum_{\nu=1}^{\frac{l}{2}} \frac{(-1)^\nu (2k-2+2\nu)_{l-2\nu}}{(\nu-1)! (l-2\nu)! (k-1+\nu)_{l-\nu}} \left(k - \frac{1}{2} + j\right)_{\nu-1} \left(\frac{l-1}{2} - (j-1)\right)_{\frac{l}{2}-\nu} \\ &- \sum_{\nu=0}^{\frac{l}{2}-1} \frac{(-1)^\nu (2k-2+2\nu)_{l-2\nu}}{2(\nu)! (l-1-2\nu)! (k-1+\nu)_{l-\nu}} \left(k - \frac{1}{2} + j\right)_\nu \left(\frac{l-1}{2} - (j-1)\right)_{\frac{l}{2}-\nu-1} \\ &= P_{k,l}(-(j-1)) \\ &- \sum_{\nu=0}^{\frac{l}{2}-1} \left\{ \frac{(-1)^\nu (2k+2\nu)_{l-2\nu-2}}{2(\nu)! (l-1-2\nu)! (k-1+\nu)_{l-\nu}} \left\{ 2(k+\nu-1)(l-1-2\nu) \right. \right. \\ &\quad \left. \left. + (2k+2\nu-2)(2k+2\nu-1) \right\} \left(k - \frac{1}{2} + j\right)_\nu \left(\frac{l+1}{2} - j\right)_{\frac{l}{2}-\nu-1} \right\} \\ &= P_{k,l}(-(j-1)) - P_{k+1,l-1}(-(j-1)) \\ &= 0. \end{aligned}$$

In the same way, for odd l , we have

$$P_{k,l}(-j) = P_{k,l}(-(j-1)) - \frac{2}{l-2j} P_{k+1,l-1}(-(j-1)) + \frac{1}{l-2j} P_{k,l}(-j).$$

Thus, Lemma 4 is proved. $\square$,

By Lemma 4, we get

$$\begin{aligned}
 & \left(f, (L^{k,l} E_k^{2n}) \left(\begin{pmatrix} -\bar{Z} & 0 \\ 0 & * \end{pmatrix}, \bar{s} \right) \right) \\
 &= \frac{1}{(k)_l l!} 2^{n(n+1-k-2s)-l+1} i^{nk} \pi^{\frac{n(n+1)}{2}-l} \prod_{j=1}^{n-1} \frac{\Gamma(2k+2s+2j-2n-1)}{\Gamma(2k+2s+j-n-2)} \\
 & \quad \times \frac{\Gamma(k+s+\frac{l}{2}-1) \Gamma(k+s+\frac{l}{2}-\frac{1}{2}) \Gamma(k+s-n) \Gamma(2k+2s+l-n-1)}{\Gamma(k+s) \Gamma(k+s-\frac{1}{2}) \Gamma(k+s-1) \Gamma(2k+2s+l-2)} \\
 & \quad \times \zeta(2s+k)^{-1} \prod_{j=1}^n \zeta(4s+2k-2j)^{-1} D_f(2s+k-n)(\iota^{-1}(f))(Z).
 \end{aligned}$$

Combining this with (1.6), we obtain

$$\begin{aligned}
 & \left(f, (L^{k,l} \mathbb{E}_k^{2n}) \left(\begin{pmatrix} -\bar{Z} & 0 \\ 0 & * \end{pmatrix}, \frac{\bar{s}+n}{2} \right) \right) \\
 &= \frac{1}{(k)_l l!} 2^{1-l} i^{nk} \pi^{-\frac{1}{2}n^2+kn+\frac{1}{2}\varepsilon} \gamma(s) \\
 & \quad \times \Gamma_{\mathbb{R}}(s+\varepsilon) \Gamma_{\mathbb{C}}(s+k+l-1) \prod_{j=2}^n \Gamma_{\mathbb{C}}(s+k-j) D_f(s)(\iota^{-1}(f))(Z),
 \end{aligned}$$

where ε and $\gamma(s)$ are of the form (1.8) and (1.9), respectively. Here, we put

$$(3.4) \quad \Lambda(s, f) := \Gamma_{\mathbb{R}}(s+\varepsilon) \Gamma_{\mathbb{C}}(s+k+l-1) \prod_{j=2}^n \Gamma_{\mathbb{C}}(s+k-j) D_f(s).$$

On the other hand, it follows from (1.4.1) and (1.4.2) that

$\left(f, (L^{k,l} \mathbb{E}_k^{2n}) \left(\begin{pmatrix} -\bar{Z} & 0 \\ 0 & * \end{pmatrix}, \bar{s} \right) \right)$ is invariant under $s \mapsto \frac{n+1}{2} - s$ and that it is an entire function in s . Thus, we have:

THEOREM 2. *Let $k, l \in 2\mathbb{Z}$, $k > 0$, $l \geq 0$. If $f \in S_{k,l}^n(\text{sym}^l(V_2))$ is an eigen-*

(i)

$$\begin{aligned}
 (3.5) \quad & \left(f, (L^{k,l} \mathbb{E}_k^{2n}) \left(\begin{pmatrix} -\bar{Z} & 0 \\ 0 & * \end{pmatrix}, \frac{\bar{s}+n}{2} \right) \right) \\
 &= \frac{1}{(k)_l l!} 2^{1-l} i^{nk} \pi^{-\frac{1}{2}n^2+kn+\frac{1}{2}\varepsilon} \gamma(s) \Lambda(s, f)(\iota^{-1}(f))(Z)
 \end{aligned}$$

or equivalently,

(3.6)

$$\begin{aligned} \alpha_{k,l}^n(s) \zeta(s+n) \prod_{j=0}^{n-1} \zeta(2s+2j) \left(f, (L^{k,l} E_k^{2n}) \left(\begin{pmatrix} -\bar{Z} & 0 \\ 0 & * \end{pmatrix}, \frac{\bar{s}+n-k}{2} \right) \right) \\ = \Lambda(s, f)(\iota^{-1}(f))(Z), \end{aligned}$$

where

$$\begin{aligned} \alpha_{k,l}^n(s) &:= (k)_l l! 2^{\frac{n^2-3n}{2}+sn-1+l} i^{nk} \pi^{-(n+\frac{1}{2})s-nk-\frac{n+\varepsilon}{2}} \\ &\quad \times \Gamma_n \left(\frac{s+n+k}{2} \right) \Gamma_n \left(\frac{s+k}{2} \right) \Gamma \left(\frac{s+\varepsilon}{2} \right). \end{aligned}$$

(ii) $\Lambda(s, f)$ has a meromorphic continuation to the whole s -plane and satisfies the functional equation

$$\Lambda(s, f) = \Lambda(1-s, f).$$

REMARK. For $(\lambda_1, \lambda_2, \dots, \lambda_n) \in \mathbb{Z}^n$ and $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n \geq 0$, let $\rho \sim (\lambda_1, \lambda_2, \dots, \lambda_n)$ be an irreducible rational representation of $GL(n, \mathbb{C})$ with a representation space $\mathcal{V}$. Suppose that $f \in S_\rho^n(\mathcal{V})$ is an eigenform. Then, it is expected that completed Dirichlet series

$$\Lambda(s, f) := \Gamma_{\mathbb{R}}(s+\varepsilon) \prod_{j=1}^n \Gamma_{\mathbb{C}}(s+\lambda_j-j) D_f(s)$$

should satisfy a functional equation.

3.2. Poles of standard L -functions

THEOREM 3. Let $k, l \in 2\mathbb{Z}$, $k > 0$, $l \geq 0$ and $k > n$. If $f \in S_{k,l}^n(\text{sym}^l(V_2))$ is an eigenform, $\Lambda(s, f)$ is holomorphic except for possible simple poles at $s = 0$ and $s = 1$; it has a pole at $s = 0$ (or equivalently, $s = 1$) if and only if $f \in B_{k,l}^n(2n) \cap S_{k,l}^n(\text{sym}^l(V_2))$.

COROLLARY. Under the assumption of Theorem 3, suppose $n \not\equiv 0 \pmod{4}$. Then $\Lambda(s, f)$ is entire.

PROOF OF THEOREM 3. Theorem 3 is proved in the same way as that by Mizumoto [21, Theorem 1]:

Let Γ_* be a finite-index subgroup of Γ^n . A function g from $\mathfrak{H}_n$ to $\mathbb{C}$ is called a C^∞ -modular form of weight k (i.e., $\rho = \det^k \otimes \text{sym}^0$) with respect to Γ_* if it is a C^∞ -function and satisfies

$$(g|M)(Z) := \det(CZ + D)^{-k} g(M(Z)) = g(Z)$$

for all $M = \begin{pmatrix} * & * \\ C^{(n)} & D^{(n)} \end{pmatrix} \in \Gamma_*$. Note that this notation $|$ is compatible with the one in §1.1.

Let Γ_j ($j = 1, 2$) be finite-index subgroups of Γ^n such that $\Gamma_1 \subset \Gamma_2$. Suppose g is a C^∞ -modular form of weight k with respect to Γ_1 . Then

$$\mathrm{Tr}(\Gamma_1, \Gamma_2; g)(Z) := \sum_{\gamma \in \Gamma_1 \backslash \Gamma_2} (g|\gamma)(Z)$$

is a C^∞ -modular form of weight k with respect to Γ_2 .

For $n, N \in \mathbb{Z}$, $k \in 2\mathbb{Z}$, $n, N, k > 0$ and $Z \in \mathfrak{H}_n$, we put

$$E_k^n(Z, s, N) := \sum_{M = \begin{pmatrix} * & * \\ C^{(n)} & D^{(n)} \end{pmatrix} \in P_{n,0} \backslash \Gamma_0^n(N)} \det(CZ + D)^{-k} \det(\mathrm{Im}(M\langle Z \rangle))^s$$

for $k + 2\mathrm{Re}(s) > n + 1$, which has a meromorphic continuation to the whole s -plane by [20]. In particular,

$$E_k^n(Z, s, 1) = E_k^n(Z, s).$$

By Shimura [25, Proposition 2.1],

$$\mathrm{Tr}(\Gamma_0^n(N), \Gamma^n; E_k^n(Z, s, N)) = E_k^n(Z, s),$$

so

$$\begin{aligned} (3.7) \quad E_k^{2n} \left(3, \frac{s+n-k}{2} \right) \\ = \mathrm{Tr} \left(J^{-1} \Gamma_0^{2n}(N) J, \Gamma^{2n}; E_k^{2n} \left(*, \frac{s+n-k}{2}, N \right) \middle| J \right) (3) \end{aligned}$$

where $3 \in \mathfrak{H}_{2n}$ and $J := \begin{pmatrix} 0 & 1_{2n} \\ -1_{2n} & 0 \end{pmatrix}$.

Suppose $k > n$ and N : even. Using the same notation as in [11], [21], let

$$\begin{aligned} (3.8) \quad D^{2n} \left(3, \frac{s+n-k}{2}; k, 1, N \right) \\ := \beta_k^n(s) \zeta_N(s+n) \prod_{j=0}^{n-1} \zeta_N(2s+2j) \left(E_k^{2n} \left(*, \frac{s+n-k}{2}, N \right) \middle| J \right) (3), \end{aligned}$$

where

$$\zeta_N(s) := \prod_{\substack{p \mid N \\ p: \text{prime}}} (1 - p^{-s}) \zeta(s),$$

and

$$\beta_k^n(s) := 2^{s-1} \pi^{-\frac{1}{2}} \Gamma\left(\frac{s+\varepsilon}{2}\right) \Gamma\left(\frac{s+n+k}{2}\right) \Gamma\left(\frac{s+k-n+1}{2}\right) \\ \times \prod_{j=1}^{n-1} \Gamma(s+k+n-2j).$$

Now, $\alpha_{k,l}^n(s)$ in Theorem 2 (i) and

$$\beta_k^n(s) \prod_{p \mid N} \left\{ (1-p^{-s-n}) \prod_{j=0}^{n-1} (1-p^{-2s-2j}) \right\}$$

are holomorphic and non-zero in the region $\operatorname{Re}(s) > 0$. So

$$g(s) := \left(\beta_k^n(s) \prod_{p \mid N} \left\{ (1-p^{-s-n}) \prod_{j=0}^{n-1} (1-p^{-2s-2j}) \right\} \right)^{-1}$$

is also holomorphic and non-zero in the region $\operatorname{Re}(s) > 0$.

Hence, by (3.7) and (3.8), we get

$$(3.9) \quad \zeta(s+n) \prod_{j=0}^{n-1} \zeta(2s+2j) E_k^{2n} \left(3, \frac{s+n-k}{2} \right) \\ = g(s) \operatorname{Tr}_J^N \left(D^{2n} \left(*, \frac{s+n-k}{2}; k, 1, N \right) \right) (3),$$

where

$$\operatorname{Tr}_J^N(*) := \operatorname{Tr}(J^{-1} \Gamma_0^{2n}(N) J, \Gamma^{2n}; *).$$

Thus (3.6) may also be written as follows:

(3.10)

$$\Lambda(s, f)(\iota^{-1}(f))(Z) \\ = \alpha_{k,l}^n(s) g(s) \left(f, \left(L^{k,l} \operatorname{Tr}_J^N \left(D^{2n} \left(*, \frac{\bar{s}+n-k}{2}; k, 1, N \right) \right) \right) \begin{pmatrix} -\bar{Z} & 0 \\ 0 & * \end{pmatrix} \right).$$

A result of Feit [11, Theorem 9.1] tells us that

$$(3.11) \quad D^{2n} \left(*, \frac{s+n-k}{2}; k, 1, N \right)$$

for even N is holomorphic in s except for a possible simple pole at $s = 1$; moreover, it is entire if n is odd.

On the other hand, it follows from (3.9) and (1.4.2) that convergence of

$$\left(f, \left(L^{k,l} \text{Tr}_J^N \left(D^{2n} \left(*, \frac{\bar{s} + n - k}{2}; k, 1, N \right) \right) \right) \begin{pmatrix} -\bar{Z} & 0 \\ 0 & * \end{pmatrix} \right)$$

in s is the same as that of (3.11) in s . Hence the integral representation (3.10) shows that $\Lambda(s, f)$ is holomorphic for $\text{Re}(s) > 0$ except for a possible simple pole at $s = 1$ and that the pole does not appear if n is odd. By (1.3.1), $\Lambda(s, f)$ has a pole at $s = 1$ exactly when $n \equiv 0 \pmod{4}$ and $f \in B_{k,l}^n(2n) \cap S_{k,l}^n(\text{sym}^l(V_2))$. Combining these facts with the functional equation in Theorem 2 (ii), we obtain Theorem 3. $\square$

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