Real and complex symplectic structures. Application to concentration in degree for microfunctions at the boundary

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§1. Symplectic forms and Levi forms

Let X be a complex manifold of dimension n , $\pi : T^*X \rightarrow X$ the cotangent bundle, ∂ the holomorphic differential on X . Let $X^{\mathbf{R}}$ denote the real underlying manifold to X and $d = \partial + \bar{\partial}$ its real differential. We shall always identify $T^*(X^{\mathbf{R}}) \cong (T^*X)^{\mathbf{R}}$. (If ϕ is a C^1 function on $X^{\mathbf{R}}$, the identification is given by $d\phi(x) \mapsto \partial\phi(x)$.) Let ω be the (complex) canonical 1-form on T^*X , $\sigma = \partial\omega$ the canonical 2-form, $H : T^*T^*X \rightarrow TT^*X$ the corresponding Hamiltonian isomorphism. We shall consider the induced one-forms $\omega^{\mathbf{R}} = 2\operatorname{Re}\omega$, $\omega^{\mathbf{I}} = 2\operatorname{Im}\omega$ and two-forms $\sigma^{\mathbf{R}} = 2\operatorname{Re}\sigma$, $\sigma^{\mathbf{I}} = 2\operatorname{Im}\sigma$ on $T^*X^{\mathbf{R}}$.

Let M be a real analytic hypersurface of X defined by the equation $\phi(z) = 0$ at $z_0 \in M$. Let $p = (z_0, \partial\phi(z_0))$, and set:

$$\begin{aligned} T_{z_0}M &= \{v \in T_{z_0}X; \operatorname{Re}\langle v, \partial\phi \rangle = 0\} \\ T_{z_0}^{\mathbf{C}}M &= T_{z_0}M \cap iT_{z_0}M = \{v \in T_{z_0}X; \langle v, \partial\phi \rangle = 0\} \\ \lambda_M(p) &= T_pT_M^*X \\ \lambda_0(p) &= T_p\pi^{-1}\pi(p) \\ v(p) &= \mathbf{C}H(\omega(p)). \end{aligned}$$

The morphism

$$\begin{aligned} M &\rightarrow T_M^*X \\ z &\mapsto (z, \partial\phi(z)), \end{aligned}$$

induces the morphism

$$\begin{aligned} \psi : T_{z_0}M &\rightarrow \lambda_M(p) \\ v &\mapsto (v, \partial\langle \partial\phi, v \rangle + \partial\langle \bar{\partial}\phi, \bar{v} \rangle) \end{aligned}$$

by which $T_{z_0}^{\mathbf{C}}M$ is identified to $\{(v, \partial\langle\bar{\partial}\phi, \bar{v}\rangle); \langle\partial\phi, v\rangle = 0\}$. Let L_M be the Levi form of M at p . Recall that, if $(z_1, \dots, z_n)$ is a local system of coordinates at z_0 , L_M is the Hermitian form on $T_{z_0}^{\mathbf{C}}M$ represented by the matrix $(\partial_i\bar{\partial}_j\phi(z_0))_{i,j}$. One immediately checks that:

$$\psi^*(\sigma|_{\psi(T_{z_0}^{\mathbf{C}}M)}) = L_M.$$

Let S be a real analytic submanifold of M with $\text{codim}_X M = 1$, $\text{codim}_M S = r$. We shall set:

$$\gamma(S, p) = \dim_{\mathbf{C}}(\lambda_S(p) \cap i\lambda_S(p) \cap \lambda_0(p)).$$

One says that $S \times_M T_M^*X$ is regular when $\omega|_{S \times_M T_M^*X} (= \frac{i}{2}\omega^{\mathbf{I}}|_{S \times_M T_M^*X}) \neq 0$.

PROPOSITION 1.1. *The following assertions are equivalent:*

$$(1.1) \quad S \times_M T_M^*X \text{ is regular at } p;$$

$$(1.2) \quad \dim^{\mathbf{R}}(\lambda_S(p) \cap \nu(p)) = 1.$$

PROOF. Let S be locally defined by the equations $\phi_i = 0$ ($i = 1, \dots, r+1$), where $\phi_1 \cong \phi$ is a local equation for M , and recall that $p = (z_0, \partial\phi(z_0))$. We have:

$$\psi^*(\omega^{\mathbf{I}}) = -\text{Re}(i\partial\phi).$$

Thus:

$$\begin{aligned} S \times_M T_M^*X \text{ is regular} &\Leftrightarrow i\frac{\partial\phi}{\partial z}(z_0) \notin (T_S^*X)_{z_0} \\ &\Leftrightarrow \dim_{\mathbf{R}}(\lambda_S(p) \cap \nu(p)) = 1. \end{aligned}$$

□

REMARK 1.2. Let $\text{codim}_X S = 2$, $\text{codim}_X M = 1$. Then $S \times_M T_M^*X$ is regular iff $\gamma(S, p) = 0$ (i.e. S is "generic").

We shall assume from now on that

$$(1.3) \quad T_M^*X \text{ is I-symplectic}$$

i.e. that $\sigma^{\mathbf{I}}|_{\lambda_M}$ is non-degenerate. This is equivalent to $\lambda_M(p) \cap i\lambda_M(p) = \{0\}$ or else to

$$L_M \text{ is non-degenerate,}$$

(cf [S1]). In fact via ψ one may easily identify $\lambda_M(p) \cap i\lambda_M(p)$ with the null-space of L_M (cf [D'A-Z1]).

Corresponding to the non-degenerate forms $\sigma^{\mathbf{I}}$ and L_M we may thus consider two Hamiltonian isomorphisms $H^{\mathbf{I}}$ and $\tilde{H}^{\mathbf{I}}$:

$$\begin{array}{ccc} T^*T_M^*X & \xrightarrow{H^{\mathbf{I}}} & TT_M^*X \\ \downarrow \iota\psi & & \uparrow \psi \\ (T^{\mathbf{C}}M)^* & \xrightarrow{\tilde{H}^{\mathbf{I}}} & T^{\mathbf{C}}M. \end{array}$$

Consider the morphism $\pi_M^* : T_{z_0}^*M \rightarrow T_p^*T_M^*X$ induced by the projection $\pi_M : T_M^*X \rightarrow M$.

PROPOSITION 1.3. For $\theta \in T_{z_0}^*M$, we have

$$(1.4) \quad H^{\mathbf{I}}(\pi_M^*(\theta))|_{\psi(T^{\mathbf{C}}M)} = \psi(v)$$

(in the identification $\lambda_M(p) \xrightarrow{\sim} \lambda_M^*(p)$ given by $w \mapsto \sigma(w, \cdot)$), where $v \in T_{z_0}^{\mathbf{C}}M$ is the unique solution of

$$(1.5) \quad \partial\langle\bar{\partial}\phi, \bar{v}\rangle = \frac{i}{2}\theta|_{T^{\mathbf{C}}M}$$

(in the identification $T^{\mathbf{C}}M \xrightarrow{\sim} T^{\mathbf{C}}M^*$ given by $v \mapsto \langle v, \cdot \rangle$).

PROOF. The vectors v which satisfy (1.4) must verify for every $u \in T_{z_0}^{\mathbf{C}}M$:

$$\begin{aligned} \frac{1}{2}[\langle\theta, u\rangle + \langle\bar{\theta}, \bar{u}\rangle] &= \sigma(\psi(v), \psi(u)) \\ &= -[i\langle\partial\langle\bar{\partial}\phi, \bar{v}\rangle, u\rangle - i\langle\partial\langle\bar{\partial}\phi, \bar{u}\rangle, v\rangle] \\ &= -[i\langle\partial\langle\bar{\partial}\phi, \bar{v}\rangle, u\rangle + \overline{i\langle\partial\langle\bar{\partial}\phi, \bar{v}\rangle, u\rangle}], \end{aligned}$$

hence:

$$\operatorname{Re}\langle\frac{i}{2}\theta, \cdot\rangle|_{T^{\mathbf{C}}M} = \operatorname{Re}\langle\partial\langle\bar{\partial}\phi, \bar{v}\rangle, \cdot\rangle|_{T^{\mathbf{C}}M}$$

Reasoning in the same way for iu we get the conclusion. $\square$

Let us remark now that (when (1.3) holds):

$$\dim_{\mathbf{R}}(\lambda_S(p) \cap \nu(p)) = 1 \Leftrightarrow iH^{\mathbf{I}}(\pi^*(p)) \notin \lambda_S(p),$$

where $\pi^* : T_{z_0}^*X \rightarrow T_p^*T^*X$ denotes the map associated to $\pi : T^*X \rightarrow X$. We may then correspondingly rephrase the equivalent conditions of Proposition 1.1. We also have

PROPOSITION 1.4. *Let $S \subset M \subset X$ with $\text{codim}_X M = 1$ and with L_M being positive or negative definite at p . Then*

$$(1.6) \quad S \times_M T_M^* X \text{ is regular involutive} \Rightarrow \gamma(S, p) = 0.$$

PROOF. Let S be locally defined by the equations $\phi_i = 0$ ($i = 1, \dots, r+1$), where $\phi_1 \cong \phi$ is a local equation for M . We have already seen that:

$$(1.7) \quad \begin{aligned} S \times_M T_M^* X \text{ is regular} &\Leftrightarrow i\partial\phi \notin \mathbf{R}\partial\phi_1 + \dots + \mathbf{R}\partial\phi_{r+1} \\ &\Leftrightarrow \partial\phi_2|_{T^{\mathbf{C}}M}, \dots, \partial\phi_{r+1}|_{T^{\mathbf{C}}M} \text{ are } \mathbf{R}\text{-independent.} \end{aligned}$$

Let v_i solve (1.4) for $\theta_i = \partial\phi_i$. Let χ and ψ be functions on M which vanish on S and let $u, v \in \bigoplus_i \mathbf{R}v_i$ be such that: $\psi(u) = H^{\mathbf{I}}(\pi_M^*(d\chi))$, $\psi(v) = H^{\mathbf{I}}(\pi_M^*(d\psi))$. Owing to Proposition 1.3, we have:

$$\{\chi, \psi\}_{\mathbf{I}} = -iL_M(\bar{u} \wedge v),$$

($\{\cdot, \cdot\}_{\mathbf{I}}$ denoting the Poisson bracket). Hence:

$$(1.8) \quad \{\chi, \psi\}_{\mathbf{I}} = 0 \quad \forall \chi, \psi \quad (\text{i.e. } S \times_M T_M^* X \text{ is involutive}) \Rightarrow u \neq iv \quad \forall u, v.$$

(In fact if $v = iu$, then $\sigma(Hu, Hv) = i\sigma^{\mathbf{I}}(Hu, iHu) = -2iL_M(\bar{u} \wedge u) \neq 0$ since L_M is definite (> 0 or < 0).) We also remark that

$$v \mapsto \langle \partial(\bar{\partial}\phi, \bar{v}), \cdot \rangle|_{T^{\mathbf{C}}M} \text{ is injective,}$$

due to (1.4). By (1.7), (1.8), this implies that $\partial\phi_i|_{T^{\mathbf{C}}M}$, $i \geq 2$ are $\mathbf{C}$ -independent. This is in turn equivalent to the fact that $\partial\phi_i$, $i \geq 1$ are $\mathbf{C}$ -independent.

§2. Degenerate R-Lagrangian submanifolds

Let X be a complex manifold of dimension n , and let $\bar{X}$ be the complex conjugate manifold to X . We shall identify $X^{\mathbf{R}}$ to the diagonal $X \times_X \bar{X}$. The manifold $X \times \bar{X}$ is then a natural complexification of $X^{\mathbf{R}}$.

Let M be a C^ω -hypersurface of $X^{\mathbf{R}}$ defined at $z_0 \in M$ by the equation $\phi(z) = 0$, and set $p = (z_0, \partial\phi(z_0))$. We shall identify $\mathbf{C} \otimes_{\mathbf{R}} T_{z_0} M$ with $\{(u, \bar{v}) \in T_{z_0} X \times T_{z_0} \bar{X}; \langle \partial\phi, u \rangle + \langle \bar{\partial}\phi, \bar{v} \rangle = 0\}$ and consider:

$$(2.1) \quad \begin{aligned} \psi^{\mathbf{C}} : \mathbf{C} \otimes_{\mathbf{R}} T_{z_0} M &\rightarrow \lambda_M(p) + i\lambda_M(p) \\ (u, \bar{v}) &\mapsto (u; \partial\langle \partial\phi, u \rangle + \partial\langle \bar{\partial}\phi, \bar{v} \rangle). \end{aligned}$$

Let $S \subset M$ be a submanifold with $\text{codim}_M S = r$. Consider the null space

$$NS(L_M |_{T_{z_0}^{\mathbb{C}} S}) = \{v \in T_{z_0}^{\mathbb{C}} S; \partial \langle \bar{\partial} \phi, \bar{v} \rangle \in \pi^*((T_S^* X)_{z_0})\}.$$

Using $TX \hookrightarrow TX \times_{TX} T\bar{X}$ and ψ , we get

PROPOSITION 2.1. *We have an identification*

$$(2.2) \quad \lambda_S(p) \cap i\lambda_S(p) \cong NS(L_M |_{T_{z_0}^{\mathbb{C}} S}) \oplus (\lambda_S(p) \cap i\lambda_S(p) \cap \lambda_0(p)).$$

PROOF. We shall often omit the indices z_0 and p in the following. The projection $\mathcal{R} : H\pi^*(T_S^* X + iT_S^* X) \rightarrow H\pi^*(T_S^* X)$ is well defined modulo $\lambda_S \cap i\lambda_S \cap \lambda_0$. It is easy to check that (cf also [D'A-Z1]):

$$\begin{aligned} & \frac{\lambda_S \cap i\lambda_S}{\lambda_S \cap i\lambda_S \cap \lambda_0} \\ &= \{\psi^{\mathbb{C}}(v) - 2\mathcal{R}(\partial \langle \bar{\partial} \phi, \bar{v} \rangle); v \in T^{\mathbb{C}} S, \partial \langle \bar{\partial} \phi, \bar{v} \rangle \in H\pi^*(T_S^* X + iT_S^* X)\}. \end{aligned}$$

Then (2.2) follows at once. $\square$

We suppose all through this section that $T_M^* X$ is $\mathbf{I}$ -symplectic and that $S \times_M T_M^* X$ is a regular involutive submanifold of $T_M^* X$. We also set $\Sigma = S \times_M T_M^* X$ and define $\tilde{\Sigma}$ to be the union of the complexifications of the bicharacteristic leaves of Σ . This is a germ of $\mathbf{R}$ -Lagrangian submanifold of $T^* X^{\mathbf{R}}$.

PROPOSITION 2.2. *We have an identification*

$$(2.3) \quad T_p \tilde{\Sigma} \cap iT_p \tilde{\Sigma} \cong \psi(\{v \in T_{z_0} S + iT_{z_0} S; \partial \langle \bar{\partial} \phi, \bar{v} \rangle \in H\pi^*(T_S^* X_{z_0} + iT_S^* X_{z_0})\}).$$

PROOF. We have by definition

$$(2.4) \quad T\tilde{\Sigma} = T\Sigma + iT\Sigma^{\perp},$$

(where $\cdot^{\perp}$ denotes the symplectic orthogonal in $(T_M^* X, \sigma^{\mathbf{I}})$). But

$$(2.5) \quad T\Sigma^{\perp} = \psi(\{u \in T^{\mathbb{C}} M \cap TS; \partial \langle \bar{\partial} \phi, \bar{u} \rangle |_{T^{\mathbb{C}} M} \in iH\pi^*(T_S^* X) |_{T^{\mathbb{C}} M}\}).$$

Then (2.3) follows immediately. $\square$

PROPOSITION 2.3. *Under the above assumptions, we have*

$$(2.6) \quad \dim_{\mathbf{C}}(T_p \tilde{\Sigma} \cap iT_p \tilde{\Sigma}) = r.$$

PROOF. We have $T\Sigma \cap iT\Sigma = 0$ (due to the fact that T_M^*X is I-symplectic). It follows:

$$\begin{aligned} T\tilde{\Sigma} \cap iT\tilde{\Sigma} &= (T\Sigma + iT\Sigma^\perp) \cap (T\Sigma^\perp + iT\Sigma) \\ &= T\Sigma^\perp + iT\Sigma^\perp = \mathbf{C} \otimes_{\mathbf{R}} T\Sigma^\perp. \end{aligned} \quad \square$$

For $v \in T\tilde{\Sigma} + iT\tilde{\Sigma}$ (resp. $v \in T\Sigma + iT\Sigma$, resp. $v \in \lambda_S + i\lambda_S$) let us denote by $v^{c_{T\tilde{\Sigma}}}$ (resp. $v^{c_{T\Sigma}}$, resp. $v^{c_{\lambda_S}}$) the “conjugate” with respect to $T\tilde{\Sigma}$ (resp. $T\Sigma$, resp. λ_S). This is well defined modulo $T\tilde{\Sigma} \cap iT\tilde{\Sigma} = (T\tilde{\Sigma} + iT\tilde{\Sigma})^\perp$ (resp. $T\Sigma \cap iT\Sigma = \{0\}$, resp. $\lambda_S \cap i\lambda_S = (\lambda_S + i\lambda_S)^\perp$). We remark now that $T\tilde{\Sigma} + iT\tilde{\Sigma} = T\Sigma + iT\Sigma$ and thus

$$(2.7) \quad \sigma(v, w^{c_{T\tilde{\Sigma}}})|_{v, w \in (T\tilde{\Sigma} + iT\tilde{\Sigma}) \cap \lambda_0} \sim \sigma(v, w^{c_{T\Sigma}})|_{v, w \in (T\Sigma + iT\Sigma) \cap \lambda_0}$$

(where “ $\sim$ ” means equivalence in signature and rank). On the other hand:

$$\begin{cases} \psi^{\mathbf{C}}(\overline{T^{\mathbf{C}}S}) + (\lambda_S \cap \lambda_0)^{\mathbf{C}} = (\lambda_S + i\lambda_S) \cap \lambda_0 \\ \psi^{\mathbf{C}}(\overline{T^{\mathbf{C}}S}) \cap (\lambda_S \cap \lambda_0)^{\mathbf{C}} = \lambda_S \cap i\lambda_S \cap \lambda_0 \end{cases}$$

and

$$(T\Sigma + iT\Sigma) \cap \lambda_0 = \psi^{\mathbf{C}}(\overline{T^{\mathbf{C}}S}) \oplus (\lambda_S \cap \lambda_0)^{\mathbf{C}}$$

(where $\cdot^{\mathbf{C}} = \cdot + i \cdot$ and where we make the identification $\overline{T^{\mathbf{C}}S} \hookrightarrow \{0\} \times T\overline{X}$). It follows:

$$(2.8) \quad \sigma(v, w^{c_{T\Sigma}})|_{v, w \in (T\Sigma + iT\Sigma) \cap \lambda_0} \sim \sigma(v, w^{c_{\lambda_S}})|_{v, w \in (\lambda_S + i\lambda_S) \cap \lambda_0}.$$

Let τ denote the inertia index for a triple of Lagrangian planes in the sense of [K-S1] and [K-S2]. We recall that $\frac{1}{2}\tau(T\tilde{\Sigma}, iT\tilde{\Sigma}, \lambda_0)$ and $n - \dim_{\mathbf{R}}(T\tilde{\Sigma} \cap \lambda_0) + 2\dim_{\mathbf{C}}(T\tilde{\Sigma} \cap iT\tilde{\Sigma} \cap \lambda_0) - \dim_{\mathbf{C}}(T\tilde{\Sigma} \cap iT\tilde{\Sigma})$ are respectively the signature and the rank of $\sigma(v, w^{c_{T\tilde{\Sigma}}})|_{v, w \in (T\tilde{\Sigma} + iT\tilde{\Sigma}) \cap \lambda_0}$. In the same way $\frac{1}{2}\tau(\lambda_S, i\lambda_S, \lambda_0)$ and $n - 1 - r + 2\gamma(S) - \dim_{\mathbf{C}}(\lambda_S \cap i\lambda_S)$ are signature and rank of $\sigma(v, w^{c_{\lambda_S}})|_{v, w \in \lambda_S \cap i\lambda_S \cap \lambda_0}$ due to [D'A-Z1]. By (2.7), (2.8) the above signatures and ranks have to coincide. In particular

$$\begin{aligned} (2.9) \quad & \dim_{\mathbf{R}}(T\tilde{\Sigma} \cap \lambda_0) + \dim_{\mathbf{C}}(T\tilde{\Sigma} \cap iT\tilde{\Sigma}) - 2\dim_{\mathbf{C}}(T\tilde{\Sigma} \cap iT\tilde{\Sigma} \cap \lambda_0) \\ &= 1 + r + \dim_{\mathbf{C}}(\lambda_S \cap i\lambda_S) - 2\gamma(S). \end{aligned}$$

We also notice that

$$\begin{aligned} (2.10) \quad & T\tilde{\Sigma} \cap iT\tilde{\Sigma} \cap \lambda_0 = H\pi^*(T_S^*X \cap iT_S^*X)|_{T^{\mathbf{C}}M} \\ & \cong H\pi^*(T_S^*X \cap iT_S^*X) \\ & = \lambda_S \cap i\lambda_S \cap \lambda_0. \end{aligned}$$

Thus by using (2.10) and (2.6) in (2.9) we get at once the following statement.

THEOREM 2.4. *We have*

$$(2.11) \quad \dim_{\mathbf{R}}(T_p \tilde{\Sigma} \cap \lambda_0) = 1 + \dim_{\mathbf{C}}(\lambda_S(p) \cap i\lambda_S(p)).$$

§3. A vanishing theorem for generalized microfunctions at the boundary

Let X be an open subset of $\mathbf{C}^n$, and let M be a C^ω -hypersurface of X . We denote by $D^b(X)$ the derived category of the category of complexes of sheaves with bounded cohomology. We denote by $\mathcal{O}_X$ the sheaf of holomorphic functions on X and we will consider its microlocalization along M , $\mu_M(\mathcal{O}_X)$ (μ_M being the Sato microlocalization functor). For any C^ω -submanifold W of $X^{\mathbf{R}}$, we shall similarly define $\mu_W(\mathcal{O}_X)$.

Let M be defined in local coordinates by $\phi(z) = 0$ and let $p = (z_0, \partial\phi(z_0))$ with $z_0 \in M$. Let $s^\pm(M, p)$ denote the numbers of eigenvalues ≥ 0 for the Levi form L_M . Let S be a C^ω -submanifold of M with $\text{codim}_M S = r$. We shall denote by $s^\pm(S, p)$ the corresponding numbers of eigenvalues for $L_M|_{T^*S}$ (cf [D'A-Z1]). Set now $\Lambda = T_M^*X$, $\Sigma = S \times_M T_M^*X$ and assume that Λ is $\mathbf{I}$ -symplectic and Σ is regular involutive in Λ . Let us recall that the latter hypothesis implies that $\dim_{\mathbf{R}}(\lambda_S(p) \cap \nu(p)) = 1$ (recall that ν denotes the Euler vector field), and moreover, if M is the boundary of a strictly pseudoconvex or pseudoconcave domain, it implies that $\gamma(S, p) := \dim_{\mathbf{C}}(\lambda_S(p) \cap i\lambda_S(p) \cap \lambda_0(p)) = 0$.

Define $\tilde{\Sigma}$ to be the union of the complexifications of the bicharacteristic leaves of Σ . Let us assume

$$(3.1) \quad \delta(S, p) := \dim_{\mathbf{C}}(\lambda_S(p) \cap i\lambda_S(p)) \quad \text{is constant with respect to } p.$$

Let us remark that if (3.1) holds, then by (2.11) there exists W with $S \subset W \subset X$ and with $\text{codim}_X W = 1 + \delta(S, p)$ such that $\tilde{\Sigma} = T_W^*X$.

We also know from [K-S1, Proposition 11.3.5] that

$$(3.2) \quad \mu_M(\mathcal{O}_X) \quad \text{is concentrated in degree } 1 + s^-(M, p).$$

(In fact the assumptions of that Proposition are fulfilled since, T_M^*X being $\mathbf{I}$ -symplectic, $s^-(M, p)$ is constant.) Let $S \subset M \subset X$, $\text{codim}_X M = 1$, $\text{codim}_M S = r$. Our result goes as follows.

THEOREM 3.1. *Let T_M^*X be $\mathbf{I}$ -symplectic, let $S \times_M T_M^*X$ be regular involutive in T_M^*X , and assume (3.1) to hold. Then*

$$(3.3) \quad H^j \mu_W(\mathcal{O}_X) = 0 \quad \text{for } j \neq 1 + \delta(S, p) + s^-(S, p) - \gamma(S, p).$$

PROOF. (We shall often omit the indices p or $z_0 = \pi(p)$ in the following.) We perform a contact transformation χ in T^*X such that

$$(3.4) \quad \begin{cases} T_M^*X \rightarrow T_{M'}^*X & \text{codim}_X M' = 1, s^-(M') = 0 \\ \Sigma \rightarrow S' \times_{M'} T_{M'}^*X & \text{codim}_{M'} S' = r, \gamma(S') = 0 \\ T_W^*X(=\tilde{\Sigma}) \rightarrow T_{W'}^*X & \text{codim}_X W' = 1. \end{cases}$$

(The fact that $\gamma(S') = 0$ in the second line follows from Proposition 1.4. This, together with $NS(L_{M'})|_{T^{\mathbf{C}}S'} = \{0\}$ (cf. §2), gives $\delta(S') = 0$ due to Proposition 2.1. This implies $\text{codim} W' = 1$ in the third line of (3.4), due to Theorem 2.4.) Let $s^0(S) := \delta(S) - \gamma(S)$; thus $s^+(S) + s^-(S) + s^0(S) = n - 1 - r - \gamma(S) = \text{codim}_{T^{\mathbf{C}}M} T^{\mathbf{C}}S$. We have

$$(3.5) \quad s^+(W') \equiv n - 1 - r, \quad s^0(W') \equiv r$$

(since $\delta(W') = \dim_{\mathbf{C}}(\tilde{\Sigma} \cap i\tilde{\Sigma}) = r$). In particular $s^-(W') = 0$ and hence $T_{W'}^*X$ is the (exterior) conormal to the boundary of a (weakly) pseudoconvex domain and therefore by the results of [H1]:

$$(3.6) \quad \mu_{W'}(\mathcal{O}_X) \quad \text{is concentrated in degree 1.}$$

By (3.5) we have $\frac{1}{2}\tau(\lambda_{W'}, i\lambda_{W'}, \lambda_0) = n - 1 - r$. We recall from §2:

$$(3.7) \quad \begin{aligned} \sigma(v, w^{c\lambda_W})|_{v, w \in (\lambda_W + i\lambda_W) \cap \lambda_0} &\sim \sigma(v, w^{cT\Sigma})|_{(T\Sigma + iT\Sigma) \cap \lambda_0} \\ &\sim \sigma(v, w^{c\lambda_S})|_{(\lambda_S + i\lambda_S) \cap \lambda_0}, \end{aligned}$$

$$(3.8) \quad \dim_{\mathbf{C}}(T\tilde{\Sigma} \cap iT\tilde{\Sigma}) = r,$$

$$(3.9) \quad \dim_{\mathbf{C}}(T\tilde{\Sigma} \cap iT\tilde{\Sigma} \cap \lambda_0) = \gamma(S).$$

In particular $s^{\pm}(W) = s^{\pm}(S)$. It follows

$$(3.10) \quad \begin{aligned} \frac{1}{2}\tau(\lambda_W, i\lambda_W, \lambda_0) &= s^+(W) + s^-(W) - 2s^-(W) \\ &= n - 1 - \delta(S) - r + 2\gamma(S) - 2s^-(S). \end{aligned}$$

Let

$$(3.11) \quad \begin{aligned} d_{W, W'} &= \frac{1}{2} \left[\text{codim} W' - \text{codim} W - \frac{1}{2}\tau(\lambda_{W'}, i\lambda_{W'}, \lambda_0) \right. \\ &\quad \left. + \frac{1}{2}\tau(\lambda_W, i\lambda_W, \lambda_0) \right]. \end{aligned}$$

We know from [K-S1, Proposition 11.2.8] that the contact transformation χ can be quantized to an isomorphism

$$(3.12) \quad \chi_*(\mu_W(\mathcal{O}_X)) \cong \mu_{W'}(\mathcal{O}_X)[d_{W,W'}].$$

But

$$\begin{aligned} d_{W,W'} &= \frac{1}{2}[1 - (1 + \delta(S)) - (n - 1 - r) + (n - 1 - \delta(S) + 2\gamma(S) - 2s^-(W))] \\ &= -\delta(S) + \gamma(S) - s^-(S). \quad \square \end{aligned}$$

LEMMA 3.2. *The following are equivalent*

$$(3.13) \quad s^-(M, p) - s^-(S, p) = \delta(S) - \gamma(S, p)$$

$$(3.14) \quad s^+(M, p) - s^+(S, p) = r - \gamma(S, p).$$

PROOF. We have $(s^+(M) + s^-(M)) - (s^+(S) + s^-(S)) = r + \delta(S) - 2\gamma(S)$. $\square$

Let $S \subset M \subset X$, $\text{codim}_X M = 1$, $\text{codim}_M S = r$.

PROPOSITION 3.3. *Let T_M^*X be I -symplectic, let $S \times_M T_M^*X$ be regular involutive in T_M^*X , let (3.1) hold, and let*

$$(3.15) \quad s^-(M, p) - s^-(S, p) = \delta(S, p) - \gamma(S, p).$$

We may then define a natural morphism

$$(3.16) \quad \mu_{W'}(\mathcal{O}_X) \rightarrow \mu_M(\mathcal{O}_X).$$

PROOF. We may find a contact transformation on T^*X

$$\begin{cases} T_M^*X \rightarrow T_{\mathbf{R}^n}^*\mathbf{C}^n, \\ \Sigma \rightarrow W' \times_{\mathbf{R}^n} T_{\mathbf{R}^n}^*\mathbf{C}^n, \quad W' \subset \mathbf{R}^n, \end{cases}$$

and hence $T_{W'}^*X \rightarrow T_{W'}^*\mathbf{C}^n$. Thus the natural morphism

$$\mu_{W'}(\mathcal{O}_X) \rightarrow \mu_{\mathbf{R}^n}(\mathcal{O}_X)$$

induces (3.16) (via quantization of χ as in (3.11), (3.12)) if and only if $d_{W',W} = d_{\mathbf{R}^n,M}$ i.e.

$$1 + \delta(S) - (n + r) - \frac{1}{2}\tau(\lambda_W, i\lambda_W, \lambda_0) = 1 - n - \frac{1}{2}\tau(\lambda_M, i\lambda_M, \lambda_0)$$

i.e.

$$\delta(S) - r - (n - 1 - \delta(S) - r + 2\gamma(S)) + 2s^-(S) = -(n - 1) + 2s^-(M).$$

The latter is in turn equivalent to (3.15). $\square$

We define now $\mu_{\Lambda \backslash \tilde{\Sigma}}(\mathcal{O}_X)$ in $D^b(X)$ to be the third term of a distinguished triangle:

$$(3.17) \quad \mu_W(\mathcal{O}_X) \rightarrow \mu_M(\mathcal{O}_X) \rightarrow \mu_{\Lambda \backslash \tilde{\Sigma}}(\mathcal{O}_X)^{+1}.$$

Let $S \subset M \subset X$, $\text{codim}_X M = 1$, $\text{codim}_M S = r$.

THEOREM 3.4. *Let T_M^*X be I -symplectic, let $S \times_M T_M^*X$ be regular involutive, let (3.1) hold, and assume:*

$$(3.18) \quad s^-(M, p) - s^-(S, p) = \delta(S, p) - \gamma(S, p).$$

Then

$$(3.19) \quad H^j \mu_{\Lambda \backslash \tilde{\Sigma}}(\mathcal{O}_X) = 0 \quad \text{for } j \neq 1 + s^-(M, p).$$

PROOF. We have $1 + \delta(S) - \gamma(S) + s^-(S) = 1 + s^-(M)$. Hence by Proposition 3.1, $\mu_W(\mathcal{O}_X)$ and $\mu_M(\mathcal{O}_X)$ are both concentrated in degree $1 + s^-(M)$. Moreover owing to Proposition 3.3 we have a morphism

$$H^{1+s^-(M)} \mu_W(\mathcal{O}_X) \rightarrow H^{1+s^-(M)} \mu_M(\mathcal{O}_X),$$

which is injective due to [S2].

REMARK 3.5. We will give in [D'A-M-Z] the following extension of the previous results. Let $\mu(p) := \lambda_M(p) \cap i\lambda_M(p)$ and $\delta(M, p) := \dim_{\mathbb{C}}(\mu(p))$. The preceding results can be extended to the following situation:

$$(3.20) \quad \left\{ \begin{array}{l} M \text{ is generic} \\ \delta(M, p) \text{ is constant for } p \in T_M^*X \\ T_p \Sigma \supset \mu(p) \\ T_p \Sigma^{\mu(p)} \text{ is regular involutive in } \lambda_M^{\mu(p)}(p). \end{array} \right.$$

In this frame, the results on involutivity and “genericity” of §1 still hold, and we also have

$$\begin{aligned} \dim_{\mathbb{C}}(T\tilde{\Sigma} \cap iT\tilde{\Sigma}) &= r + \delta(M), \\ \dim_{\mathbb{C}}(T\tilde{\Sigma} \cap iT\tilde{\Sigma} \cap \lambda_0) &= \dim_{\mathbb{C}}(\lambda_S \cap i\lambda_S \cap \lambda_0), \\ \dim_{\mathbb{R}}(T\tilde{\Sigma} \cap \lambda_0) &= 1 + \delta(S) - \delta(M). \end{aligned}$$

In particular if $\delta(S)$ is constant in p , then there exists $W \subset X$ with $\tilde{\Sigma} = T_W^*X$. We still have

$$\begin{aligned} s^\pm(W) &= s^\pm(S), \\ s^0(W) &= r + \delta(M) - \gamma(S) \\ &= s^0(M) + \text{codim}_W S. \end{aligned}$$

Thus repeating step by step the proof of Theorem 3.1 we get the following statement:

Assume that (3.20) is fulfilled and that $\delta(S)$ is constant in p (and hence $\tilde{\Sigma} = T_W^*X$). Then

- (i) $H^j \mu_W(\mathcal{O}_X) = 0$ for $j \neq 1 + s^-(S) + \delta(S) - \delta(M) - \gamma(S)$
- (ii) *If moreover $s^-(M) - s^-(S) = \delta(S) - \delta(M) - \gamma(S)$ then there is a natural morphism*

$$\mu_W(\mathcal{O}_X) \rightarrow \mu_M(\mathcal{O}_X)$$

and if one defines $\mu_{\Lambda \setminus \tilde{\Sigma}}(\mathcal{O}_X)$ as in (3.17), one has

$$H^j \mu_{\Lambda \setminus \tilde{\Sigma}}(\mathcal{O}_X) = 0 \quad \text{for } j \neq 1 + s^-(M).$$

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