On the homotopy type of some subgroups of Diff \((M^3)\)

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Introduction

The purpose of this paper is to determine the homotopy type of some subgroups of the diffeomorphism group of a closed oriented 3-dimensional manifold.

Let \(M\) be a closed oriented \(n\)-dimensional manifold and \(\mathcal{F}\) be a codimension one foliation on \(M\) of class \(C^r (r \geq 2)\). A pair \((M, \mathcal{F})\) is called a generalized Reeb foliated manifold if \((M, \mathcal{F})\) is decomposed as \((M, \mathcal{F}) = \bigcup_{i=1}^l (M_i, \mathcal{F}_i)\), where \((M_i, \mathcal{F}_i)\) is a generalized Reeb component for each \(i\) (see §1 for definition). Let \(\text{FDiff}_0 (M, \mathcal{F})\) denote the identity component of the topological group of foliation preserving diffeomorphisms of \((M, \mathcal{F})\).

Then our result is

**Theorem 4.2.** Let \((M, \mathcal{F})\) be a generalized Reeb foliated 3-dimensional manifold. Then \(\text{FDiff}_0 (M, \mathcal{F})\) has the same homotopy type as an \(\ell\)-dimensional torus \(T^\ell (0 \leq \ell \leq \lambda + \mu)\), where \(\lambda\) is the number of generalized Reeb components and \(\mu\) is the number of compact leaves homeomorphic to \(T^2\).

Let \((M, \mathcal{J})\) be a closed \(n\)-dimensional manifold with a spinnable structure \(\mathcal{J}\). Let \(\text{SDiff}_0 (M, \mathcal{J})\) denote the identity component of the topological group of spinnable structure preserving diffeomorphisms of \((M, \mathcal{J})\). In the case of \(n=3\), the axis of \(\mathcal{J}\) is some union of circles. Hence we can construct a codimension one foliation on \(M\) from this spinnable structure (Lawson [7]). We denote this foliation by \(\mathcal{F}_\varphi\). \(\mathcal{F}_\varphi\) is a typical example of generalized Reeb foliations.

Our main result is as follows:

**Theorem (Theorems 5.6 and 6.1).** Let \((M, \mathcal{J})\) be a closed oriented 3-dimensional manifold with a spinnable structure \(\mathcal{J}\). Then \(\text{SDiff}_0 (M, \mathcal{J})\) is homotopy equivalent to \(\text{FDiff}_0 (M, \mathcal{F}_\varphi)\) and has the same homotopy type as a point or a circle \(S^1\) or a 2-dimensional torus \(T^2\).

The paper is organized as follows. In §1, we introduce the notion of...
a generalized Reeb foliation and prove the fibration lemma (Lemma 1.13) which is fundamental for our results and is valid in the general dimensions. In §2, we study the subspace \( \text{LDiff} (M, \mathcal{F}) \) of \( \text{FDiff}_0 (M, \mathcal{F}) \). In §3, we prepare the results which will be needed in §4, 5 and 6. We prove Theorem 4.2 in §4 and prove the main theorem in §5 and 6.

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§ 1. Generalized Reeb foliation and fibration lemma

Let \( M \) be a closed oriented \( n \)-dimensional manifold and \( \mathcal{F} \) a codimension one foliation on \( M \) of class \( C^r \) (\( r \geq 2 \)).

**Definition 1.1.** An orientation preserving diffeomorphism \( f: M \rightarrow M \) is called a foliation preserving diffeomorphism (resp. a leaf preserving diffeomorphism) if for each point \( x \) of \( M \), the leaf through \( x \) is mapped to the leaf through \( f(x) \) (resp. \( x \)), that is, \( f(L_x) = L_{f(x)} \) (resp. \( f(L_x) = L_x \)), where \( L_x \) is the leaf that contains \( x \). It is clear that a foliation preserving diffeomorphism (resp. a leaf preserving diffeomorphism) \( f \) induces a homeomorphism \( \bar{f} \) (resp. \( \text{id} \)) of the leaf space \( M/\mathcal{F} \) such that the diagram commutes,

\[
\begin{array}{ccc}
M & \xrightarrow{id} & M \\
\downarrow & & \downarrow \\
M/\mathcal{F} & \xrightarrow{\bar{f}} & M/\mathcal{F}
\end{array}
\quad \text{resp.} \quad
\begin{array}{ccc}
M & \xrightarrow{f} & M \\
\downarrow & & \downarrow \\
M/\mathcal{F} & \xrightarrow{\text{id}} & M/\mathcal{F}
\end{array}
\]

where vertical arrows are canonical projections (see Reeb [11]). Let \( \text{FDiff}^r (M, \mathcal{F}) \) or \( \text{FDiff} (M, \mathcal{F}) \) (resp. \( \text{LDiff}^r (M, \mathcal{F}) \) or \( \text{LDiff} (M, \mathcal{F}) \)) denote the space of all foliation (resp. leaf) preserving diffeomorphisms of \( (M, \mathcal{F}) \) of class \( C^r \). It is clear that \( \text{LDiff} (M, \mathcal{F}) \subseteq \text{FDiff} (M, \mathcal{F}) \subseteq \text{Diff} (M) \). Topologies of the spaces are induced by the \( C^r \) topology of \( \text{Diff} (M) \). Then it is well known that these spaces are topological groups. There is an exact sequence of topological groups;

\[
1 \rightarrow \text{LDiff}^r (M, \mathcal{F}) \rightarrow \text{FDiff}^r (M, \mathcal{F}) \rightarrow \text{Homeo} (M/\mathcal{F}) \rightarrow 1
\]

where the second arrow is the inclusion map and the map \( \pi \) is defined by \( \pi(f) = \bar{f} \).

**Definition 1.2.** A compact foliated manifold \((M, \mathcal{F})(\partial M = \emptyset)\) is called a generalized Reeb component if the following three conditions are satisfied; (1) all leaves in \( \text{Int} M \) are non-compact and proper, (2) the holonomy groups
of all leaves in \( \text{Int} \, M \) are trivial and (3) each of the elements of the holonomy group of each compact leaf of \( \mathcal{F} \) can be represented by a local diffeomorphism of \( R_+=[0, \infty) \), leaving fixed 0, which is \( C^r \)-tangent to identity at 0 and whose second derived function is non-negative or non-positive in some neighborhood of 0.

The structure of a generalized Reeb component was studied by Imanishi-Yagi [6]. Our definition is slightly different from that in [6]. A generalized Reeb component in [6] means a compact foliated manifold \((M, \mathcal{F}) (\partial M \equiv \emptyset)\) satisfying above (1), (2). In the first part of this section, we recall some properties of a generalized Reeb component. See [6; §2] for more details.

**DEFINITION 1.3.** A vector field \( X \) on \( M \) transverse to \( \mathcal{F} \) is called a nice vector field if \( X \) has a closed orbit \( C \) such that \( C \cap L = \{ \text{one point} \} \) for any leaf \( L \) in \( \text{Int} \, M \). Such a closed orbit \( C \) is called a nice orbit.

**PROPOSITION 1.4 [6; Proposition 2.1].** Let \((M, \mathcal{F})\) be a generalized Reeb component. Then there exists a nice vector field \( X \) on \( M \).

We identify \( S^1 \) with the nice orbit \( C \) in Proposition 1.4. Let \( p: \text{Int} \, M \to S^1 \) be a map defined by \( p(x)=C \cap L_x \). Then we see that \( p \) is a locally trivial fibration. Let \( dt \) be the natural one form on \( S^1=R^1/Z \) and \( w=p^*dt \). Then there exists a positive function \( g \) on \( \text{Int} \, M \) such that \( w(gX) \equiv 1 \). Let \( \phi_t \) denote the flow associated to \( gX \). \( \phi_t \) is the foliation preserving flow on \( \text{Int} \, M \) and \( \phi_n(L)=L \) for any leaf \( L \) in \( \text{Int} \, M \) and any integer \( n \).

**REMARK 1.5.** By putting \( \phi_t(z)=z \) for \( z \in \partial M \), we may show from Lemma 1.8 below and Definition 1.2(3), that \( \phi_t \) is a foliation preserving flow of class \( C^r \) on \( M \) and is \( C^r \)-tangent to identity at \( \partial M \).

**LEMMA 1.6 [6; Lemma 2.5].** Let \( V \) be a component of \( \partial M \) and \( z \) a point of \( V \). Let \( T \) be the maximal solution curve of \( X \) which contains \( z \) and \( y_0 \) be a point of \( L_{y_0} \cap T \). Then \( L_{y_0} \cap T = \{ y_n=\phi_n(y_0), n \in Z \} \), and if \( X \) is outward normal at \( z \), \( \lim_{n \to \infty} y_n = z \).

To describe the structure of \( \mathcal{F} \) near \( V \), we define a foliated manifold \( V(N, h) \) as follows. Let \( N \) be a codimension one submanifold of \( V \) such that \( V-N \) is connected and the manifold \( V_N \) obtained from \( V \) by cutting along \( N \) has two boundary components \( N_1 \) and \( N_2 \) which are copies of \( N \). Let \( h \) be a contracting diffeomorphism of \( [0, \varepsilon), \varepsilon>0 \). \( V(N, h) \) is obtained from \( V_N \times [0, \varepsilon) \) by identifying \( (x, t) \in N_1 \times [0, \varepsilon) \) with \( (x, h(t)) \in N_2 \times [0, \varepsilon) \). There exists a dually foliated structure on \( V(N, h) \) which is induced from the
product structure $V_N \times [0, \varepsilon)$. The dual structure of $\mathcal{F}$ is defined by $X$.

**Lemma 1.7** [6; Lemma 2.6]. There exist a submanifold $N$ and a diffeomorphism $h$ satisfying above conditions. There exists an embedding $j$ of $V(N, h)$ into $M$ which preserves the dually foliated structures, satisfying $j(x, 0) = x$ for $x \in V$.

**Lemma 1.8** [6; Lemma 2.7]. Let $j: V(N, h) \to M$ be as above. We identify $(x, \tau) \in (V_N - N) \times [0, \varepsilon)$ with a point of $V(N, h)$. For $t \geq 0$ we define $\phi_t'(x, \tau) = j^{-1} \circ \phi_t \circ j(x, \tau)$, then $\phi_t'$ preserves the foliated structure on $V(N, h)$ and we have $\phi_t'(x, 0) = (x, h'(\tau))$.

For any $f$ in $\text{FDiff}(M, \mathcal{F})$, there exists a diffeomorphism $\tilde{f}$ of $S^1$ such that the diagram commutes,

\[
\begin{array}{ccc}
\text{Int } M & \xrightarrow{\text{Int } M} & \text{Int } M \\
\downarrow p & & \downarrow p \\
S^1 & \xrightarrow{\tilde{f}} & S^1
\end{array}
\]

Let $\text{FDiff}_0(M, \mathcal{F})$ be the identity component of $\text{FDiff}(M, \mathcal{F})$. Let $\pi: \text{FDiff}_0(M, \mathcal{F}) \to \text{Diff}_0(S^1)$ be a map defined by $\pi(f) = \tilde{f}$. Clearly this map is the continuous homomorphism.

**Lemma 1.9.** $\text{Im } \pi = \text{SO}(2)$.

**Proof.** That $\text{Im } \pi$ contains the rotation group $\text{SO}(2)$, is easily proved by using the foliation preserving flow $\phi_t$. Let us prove that $\text{Im } \pi$ is contained in $\text{SO}(2)$. Suppose for some $f$ in $\text{FDiff}_0(M, \mathcal{F})$, $\pi(f) \not\in \text{SO}(2)$. We shall deduce a contradiction from this assumption.

A point $x_0$ in the nice orbit $C$ corresponds to a point $x_0$ in $S^1$. We can assume $\pi(f)(x_0) = \tilde{f}(x_0) = x_0, f = \text{id}$, by composing a relevant rotation which is induced by the foliation preserving flow $\phi_t$.

**Assertion 1.10.** There exists a leaf preserving diffeomorphism $g$ such that $g \circ f$ preserves each orbit of $X$ in some small neighborhood of $\partial M$.

**Proof.** Let $V$ be a component of $\partial M$ and $f^{-1}|_V: V \to V$ be the diffeomorphism restricted to $V$ of $f^{-1}$, which is contained in the identity component of the space of diffeomorphisms of $V$, $\text{Diff}_0(V)$. Take a smooth path $h_t$ from $\text{id}_V$ to $f^{-1}|_V$ in $\text{Diff}_0(V)$, $h_0 = \text{id}_V$, $h_1 = f^{-1}|_V$. Let $H: V \times I \to V \times I$ be a map defined by $H(x, t) = (h_t(x), t)$. Consider a vector field defined by $\left(\frac{\partial h_t}{\partial t}, 1\right)$ on $V \times I$ in $M \times I$. Take small tubular neighborhoods $N_1, N_2$ of $V \times I$ in
$M \times I, N_1 \supset N_2$ and the vector field, which is denoted by $v(x, t)$, on $M \times I$ such that it is tangent to the leaves and the derivative $dp_1$ of the projection $p_1: M \times I \to M$ maps $v(x, t)$ to the zero vector outside $N_1$, and the derivative $dp_2$ of the projection $p_2: M \times I \to I$ maps $v(x, t)$ to the unit vector $\frac{\partial}{\partial t}$, and that in $N_2$ it commutes with the differential map of the projection of $N_1$ to $V$ along the orbits of $X$. Then integrating the vector field $v(x, t)$, we obtain an element $g_1$ of $\text{LDiff}(M, \mathcal{F})$ which is the extension of $f^{-1}|_V$. Note that $\pi(g_1 \circ f) = \tilde{f}$ and $g_1 \circ f|_V = \text{id}_V$. We can find a relevant leaf preserving diffeomorphism $g_2$ such that $g_2|_V = \text{id}_V$, $g_2|_{\text{outside of } N_1} = \text{id}$ and $g_2 \circ g_1 \circ f$ has a required property. Put $g = g_2 \circ g_1$. $g$ is a leaf preserving diffeomorphism. It is similar for the case of other component of $\partial M$. Q.E.D.

Again we denote such $g \circ f$ by $f$ for simplicity.

**Assertion 1.11.** Under Lemma 1.6, there exists a unique integer $m$ such that $f(y_n) = y_{n+m}$ for a sufficiently large integer $n$.

**Proof.** From Assertion 1.10, there is a commutative diagram in $T$,

$$
\begin{array}{ccc}
[y_n, y_{n+1}] & \xrightarrow{f} & [y_{n+m}, y_{n+m+1}]
\end{array}
$$

\[
\begin{array}{ccc}
p & \searrow & p \\
\downarrow & & \downarrow \\
S^1 & \xrightarrow{f} & S^1
\end{array}
\]

where $p(y_n) = \tilde{x}_0$ for each $n$ and $[y_n, y_{n+1}] = \bigcup_{0 \leq t \leq 1} \phi_t(y_n)$ in $T$. Therefore $m' = m \pm 1$. Since $\tilde{f}$ is the orientation preserving diffeomorphism, we have $m' = m + 1$. Q.E.D.

**Proof of Lemma 1.9 Continued.** By composing the foliation preserving diffeomorphism induced from $\phi_{-m}$, we may assume $f(y_n) = y_n$ for a large positive integer $n$. Let $T_n$ denote a set $\{\bigcup_{t \in \mathbb{R}} \phi_t(y_n)\} \cup \{z\}$ and $f|_{T_n}: T_n \to T_n$ be the restriction of $f$ to $T_n$. We can assume that $T_n$ is parametrized by the interval $[0, \eta]$ such that $z$ corresponds to 0. Put $f_0 = f|_{[y_n, y_{n+k}]}$. The diffeomorphism $f|_{T_n}$ is described by $f_0$ as follows:

$$f(x) = \begin{cases} 
    \text{h' \circ f_0 \circ h}(x), & \text{for } x \in [y_n+(t-1)k, y_n+tk] \\
    \text{x}, & \text{for } x = z,
\end{cases}$$

where $h$, which is that in Lemmas 1.7 and 1.8, is a contracting diffeomorphism of $T_n = [0, \eta]$. Note that the second derived function $h'' \leq 0$ in some neighborhood of 0 from Definition 1.2(3). From the assumption, $f_0 \approx \text{id}$, there is $x_0 \in [y_n, y_{n+k}]$ that satisfies the following 1) or 2):

1) $x_0 \approx f_0(x_0)$ and $f_0'(x_0) > 1$,
2) \( x_0 \leq f(x_0) \) and \( f'(x_0) < 1 \).

Let \( x_n = h^n(x_0) \) (\( n = 1, 2, \ldots \)). When \( x_0 \) satisfies the condition 1),

\[
\frac{f'(x_n)}{f'(x_0)} = \frac{(h^n)'(f(x_0))}{(h^n)'(x_0)} \geq f'(x_0) > 1.
\]

Hence \( f'(x_n) \) cannot converge to 1. This fact and \( f(y_n) = y_n \) lead to a contradiction. It is similarly proved when \( x_0 \) satisfies the condition 2).

Q.E.D.

**DEFINITION 1.12.** \( \mathcal{F} \) is called a **generalized Reeb foliation** on a closed oriented manifold \( M \) if there is a decomposition of \((M, \mathcal{F})\) such that \((M, \mathcal{F}) = \bigcup_{i=1}^i (M_i, \mathcal{F}_i)\), where \((M_i, \mathcal{F}_i)\) denotes a generalized Reeb component.

Let \( f_i \) be the restriction of \( f \) to \((M_i, \mathcal{F}_i)\) for any \( f \) in \( \text{FDiff}_0(M, \mathcal{F}) \).

From Lemma 1.9, we can define a map \( \varphi: \text{FDiff}_0(M, \mathcal{F}) \rightarrow \text{SO}(2) \times \cdots \times \text{SO}(2) \) by \( \varphi(f) = (f_1, \ldots, f_i) \).

**LEMMA 1.13 (fibration lemma).** \( \varphi \) is a locally trivial fibration.

**PROOF.** We define a foliation preserving flow \( \phi_t \) on \( M \) to be a union of the foliation preserving flows on generalised Reeb components. From Remark 1.5, \( \phi_t \) is well defined and of class \( C^r \). Hence it is easily proved by using this flow \( \phi_t \).

Q.E.D.

Let \( \text{LDiff}(M, \mathcal{F}) \) denote the fiber of this fibration \( \pi \). Note that this space is the space \( \text{LDiff}(M, \mathcal{F}) \cap \text{FDiff}_0(M, \mathcal{F}) \).

**COROLLARY 1.14.** \( \text{FDiff}_0(M, \mathcal{F}) / \text{LDiff}(M, \mathcal{F}) \) is homeomorphic to \( S^1 \times \cdots \times S^1 \).

Let \( \text{LDiff}_0(M, \mathcal{F}) \) denote the identity component of \( \text{LDiff}(M, \mathcal{F}) \).

Since \( \text{LDiff}(M, \mathcal{F}) \) is a closed subgroup of \( \text{FDiff}_0(M, \mathcal{F}) \) and the natural map \( \text{FDiff}_0(M, \mathcal{F}) \rightarrow \text{FDiff}_0(M, \mathcal{F}) / \text{LDiff}(M, \mathcal{F}) \) has a local section, we use “the bundle structure theorem” (Steenrod [13; p. 30]).

**PROPOSITION 1.15.** Let \( p: \text{FDiff}_0(M, \mathcal{F}) / \text{LDiff}_0(M, \mathcal{F}) \rightarrow \text{FDiff}_0(M, \mathcal{F}) / \text{LDiff}(M, \mathcal{F}) \) be the map induced by the inclusion of cosets. Then we can assign a bundle structure to \( \text{FDiff}_0(M, \mathcal{F}) / \text{LDiff}_0(M, \mathcal{F}) \) relative to \( p \).

The fiber of the bundle is \( \text{LDiff}(M, \mathcal{F}) / \text{LDiff}_0(M, \mathcal{F}) \).

**COROLLARY 1.16.** \( \text{FDiff}_0(M, \mathcal{F}) / \text{LDiff}_0(M, \mathcal{F}) \) is homeomorphic to a \( \ell \)-dimensional manifold which has the same homotopy type as an \( \ell \)-torus.
REMARK 1.17. Leslie [8] has proved "Let \((M, \mathcal{F})\) be a compact foliated \(n\)-dimensional manifold of codimension \(k\), and of class \(C^\infty\). If \(F\) has a finite number of leaves \(L_1, \ldots, L_e\) such that \(L_1 \cup \cdots \cup L_e = M\), then \(\text{FDiff}_0(M, \mathcal{F})/\text{LDiff}_0(M, \mathcal{F})\) is a Lie group of dimension \(\leq e \cdot k\)."

\[\text{§ 2. On the space \(\text{LDiff}(M, \mathcal{F})\)}\]

Let \((M, \mathcal{F})\) be a generalized Reeb foliated manifold and \(V_i (i=1, 2, \ldots, v)\) its compact leaves. Let \(L\) denote the subspace of \(\text{LDiff}(M, \mathcal{F})\) consisting of leaf preserving diffeomorphisms such that in some tubular neighborhood \(N(V_i)\) of \(V_i (1 \leq i \leq v)\), the following diagram commutes:

\[
\begin{array}{c}
N(V_1) \cup \cdots \cup N(V_v) \xrightarrow{f} f(N(V_1) \cup \cdots \cup N(V_v)) \subset M \\
\downarrow q \\
V_1 \cup \cdots \cup V_v \\
\end{array}
\]

where \(q: N(V_i) \rightarrow V_i\) is a map defined by \(q(y) = \lim_{t \to \infty} f^t(y)\) (see Lemma 1.6).

**LEMMA 2.1.** The inclusion map \(L \hookrightarrow \text{LDiff}(M, \mathcal{F})\) is a weak homotopy equivalence.

**PROOF.** Let \(K\) be any compact set and \(\psi: K \rightarrow \text{LDiff}(M, \mathcal{F})\) be a continuous map. We will make a homotopy \(h\), such that \(h_0 = \psi, h_1(K) \subset L\). We may choose distinguished open neighborhoods \(A_i, B_i, C_i (k=1, 2, \ldots, v; i=1, 2, \ldots, n_k)\) in \(M\) such that \(1) A_i \supset B_i, B_i \supset C_i, 2) \bigcup_{i=1}^{n_k} C_i \supset V_k (k=1, 2, \ldots, v)\) and \(3) A_i \cap A_j = \emptyset\) for \(k \neq \ell\). At first we construct a homotopy in \(A_1\). The local coordinate of a distinguished neighborhood \(A_1\) is \((u, x_1, \ldots, x_{n-1})\). Each orbit of \(X\) in \(A_1\) is considered as \((u, x_1, \ldots, x_{n-1})\), where \(x_i=\text{constant}\) for each \(i\). For any \(a \in K\), the orbits of \(d\psi(a)(X)|_{A_1}\) are described as follows:

\((u, y_1(a, u, x), y_2(a, u, x), \ldots, y_{n-1}(a, u, x))\), where \(y_i\) is a differentiable function of class \(C^{r-1}\) and \(y_i(a, 0, x) = x_i\) for each \(i\). Since \(K\) is compact, we may assume that for \((u, x) \in C_1, (u, y_1(a, u, x), \ldots, y_{n-1}(a, u, x)) \in B_1\). Let \(h: M \rightarrow [0, 1]\) be a smooth function such that \(h|_{C_1} = 1\) and \(h|_{\text{outside of } B_1} = 0\), and \(\chi: [0, 1] \rightarrow [0, 1]\) be a smooth function such that \(\chi(0) = 0, \chi(1) = 1\). Let \(\psi_t: K \times [0, 1] \rightarrow \text{LDiff}(M, \mathcal{F})\) be a homotopy of \(\psi\) defined by

\[
\psi_t(a, t)(\psi(a)(p)) =
\begin{cases}
(u, y_1(a, (1-h(u, x) \cdot \chi(t))u, x), \ldots, \\
y_{n-1}(a, (1-h(u, x) \cdot \chi(t))u, x)) & \text{for } \psi(a)(p) \in A_1; \\
\psi(a)(p) & \text{for } \psi(a)(p) \in A_1.
\end{cases}
\]
Note that $\psi|(a, 1)$ preserves the orbits of $X$ in $C_1^a$. Next, in $A_1^a$ we construct a homotopy of $\psi|_{X \times [1]}$ by the same way. After iterating this process finite times, we obtain the required homotopy of $\psi$. Q.E.D.

The space $\mathcal{L}$ is included in $\text{FDiff}_0(M, \mathcal{F})$, hence the restriction to each $V_i$ belong to the identity component $\text{Diff}_0(V_i)$ of $\text{Diff}(V_i)$. Let $\text{res}: \mathcal{L} \to \text{Diff}_0(V_1) \times \cdots \times \text{Diff}_0(V_v)$ be the restriction map, i.e., $\text{res}(f) = (f|_{V_1}, \ldots, f|_{V_v})$.

**Lemma 2.2.** There is an exact sequence:

$$1 \to \mathcal{I} \to \mathcal{L} \to \text{Diff}_0(V_1) \times \cdots \times \text{Diff}_0(V_v) \to 1,$$

where $\mathcal{I}$ is the kernel of $\text{res}$, and $\text{res}$ is a locally trivial fibration.

**Proof.** This is proved by the same way as in [4; Lemma 3].

Let $(M, \mathcal{F})$ and $(M, \mathcal{F}_2)$ be generalized Reeb components with $V_i$ as a component of boundary. For $f$ in $\mathcal{I}$, $\rho_i(f)$ is a pair of integers $(k, k')$, where $k$ and $k'$ are the integer $m$ in Assertion 1.11 for $(M_1, \mathcal{F}_1$, $(M_2, \mathcal{F}_2)$ respectively.

**Lemma 2.3.** $\rho_i \oplus \cdots \oplus \rho_v: \mathcal{I} \to (\mathbb{Z} \oplus \mathbb{Z}) \oplus \cdots \oplus (\mathbb{Z} \oplus \mathbb{Z})$ is a homomorphism.

**Proof.** It is easily proved by the following commutative diagram;

$$\begin{array}{ccc}
[y_n, y_{n+1}] & \xrightarrow{f} & [y_{n+m}, y_{n+m+1}] \\
\downarrow p & \ & \downarrow p \\
S^1 & \xrightarrow{\bar{f}} & S^1 \\
\end{array}$$

where $n$ is a sufficiently large positive integer.

**Remark 2.4.** Clearly $\rho_i \oplus \cdots \oplus \rho_v$ is a locally trivial fibration over the image $\rho_i \oplus \cdots \oplus \rho_v$.

§ 3. The homotopy type of the space of diffeomorphisms of a 2-dimensional manifold and its application

Let $M_g^2$ be a closed oriented 2-dimensional manifold of genus $g$ and $D_1^2 \cup D_2^2 \cup \cdots \cup D_v^2$ be 2-discs embedded in $M_g$. Let $\text{Diff}^r(M_g)$ be the space of orientation preserving diffeomorphisms of $M_g$ of class $C^r$ with $C^r$ topology and $\text{Diff}_0(M_g)$ be its identity component. By $\text{Diff}^r(M_g; D_1^2 \cup \cdots \cup D_v^2)$ we denote the subgroup of $\text{Diff}(M_g)$ consisting of the diffeomorphisms whose restriction to $D_1 \cup \cdots \cup D_v$ are identity.

**Proposition 3.1.** $\text{Diff}^r_0(M_g; D_1^2 \cup \cdots \cup D_v^2)$ is contractible for any $g$ and
any positive integer \( \ell \).

**Lemma 3.2.** Let \( V \) be a compact oriented 2-dimensional manifold with boundary. Then \( \text{res} : \text{Diff}_0^+ (V) \to \text{Diff}_0^+ (\partial V) \) is a locally trivial fibration, where \( \text{res} \) is the restriction map.

**Proof.** It is easy to see that \( \text{res} \) is surjective. Let \( U(\text{id}) \) be a neighborhood of \( \text{id} \) in \( \text{Diff}_0^+ (\partial V) \). We may consider \( U(\text{id}) \) as the set consisting of sections \( s \) of the tangent bundle \( T(\partial V) \) of \( \partial V \) such that the norm of \( s \), \( ||s|| < \varepsilon \) for a small positive number \( \varepsilon \). To prove Lemma 3.2, in the following diagram

\[
\begin{array}{ccc}
T(\partial V) & \to & (TV) \\
\downarrow s & & \downarrow S^i \\
\partial V & \hookrightarrow & V,
\end{array}
\]

we have only to extend the section \( s \) of \( T(\partial V) \) to \( T(V) \). Let \( N \) be a tubular neighborhood of \( \partial V \) in \( V \), which is diffeomorphic to \( \partial V \times [0, 1) \). Since \( T(V)|_N = N \times \mathbb{R}^2 \), for any section \( s \) in \( U(\text{id}) \), we define a section of \( T(V) \), \( S : N \to T(V)|_N \) by \( S(v, t) = (v, t, \chi(t) \cdot s(v), 0) \), where \( \chi : [0, 1) \to [0, 1] \) is a smooth function such that \( \chi[0, 1/3] = 1, \chi[2/3, 1) = 0 \). Q.E.D.

**Proof of Proposition 3.1.** Let \( V \) be a compact oriented 2-dimensional manifold (with or without boundary) which is not diffeomorphic to a 2-sphere \( S^2 \), a 2-torus \( T^2 \), a 2-disc \( D^2 \) and a cylinder \( C^2 (= S^1 \times [0, 1]) \). The group \( \text{Diff}_0^+ (V) \) is contractible (see Gramain [5]). Note that the fiber of the fibration in Lemma 3.2 is \( \text{Diff}_0^+ (V; \partial V) \). Hence \( \text{Diff}_0^+ (V; \partial V) \) is contractible. It is well known that \( \text{Diff}_0^+ (D^2; \partial D^2) \) is contractible (Smale [12]). For the case of \( V = C^2 \), we easily see that \( \text{Diff}_0^+ (C^2; \partial C^2) \) is contractible.

Next, we consider the non-compact case. Let \( L \) be a non-compact oriented 2-dimensional manifold. By \( \text{Diff}^{c+r} (L) \) we denote the subgroup of \( \text{Diff}^{c+r} (L) \) consisting of diffeomorphisms with compact support.

**Proposition 3.3.** \( \pi_i (\text{Diff}^{c+r} (L)) = 0 \) for each positive integer \( i \).

**Proof.** Let \( S^i \) be a \( i \)-dimensional sphere with base point \( s_0 \) (\( i \geq 1 \)). Let \( \varphi : (S^i, s_0) \to (\text{Diff}^{c+r} (L), \text{id}) \) be any continuous map. Since \( S^i \) is compact, there exists a compact submanifold \( K \) of \( L \) such that \( \varphi(S^i) \) restricted to \( L-K \) is identity. Hence the image of \( \varphi \) is contained in \( \text{Diff} (K; \partial K) \). From the contractibility of identity component of \( \text{Diff} (K; \partial K) \), there exists a homotopy \( \Phi : S^i \times [0, 1] \to \text{Diff} (K; \partial K) \) such that \( \Phi(s, 0) = \varphi(s) \) and \( \Phi(s, 1) = \text{id} \). Q.E.D.

Let \( E^2 \) be the total space of a fibration over \( S^1 \) with \( L^2 \) as fiber, that is,
$E^3 = L \times I/(x, 0) \sim (h(x), 1)$, where $L$ is a non-compact oriented 2-manifold and $h: L \to L$ is an orientation preserving diffeomorphism. Then we study the homotopy type of the space \{$f \in \text{Diff}^c(E); \pi \circ f = \pi$, where $\pi$ is the fibration map\}, denoted by $P^h(\text{Diff}^c(L))$. This space is identified with the space \{$\varphi: I \to \text{Diff}^c(L)$, differentiable map; $\varphi(0) = h^{-1} \circ \varphi(1) \circ h$\}. Furthermore, this space is homotopy equivalent to the space \{$\varphi: I \to \text{Diff}^c(L)$, continuous map; $\varphi(0) = h^{-1} \circ \varphi(1) \circ h$\} with C-O topology, which is also denoted by $P^h(\text{Diff}^c(L))$. Let $q: P^h(\text{Diff}^c(L)) \to \text{Diff}^c_0(L)$ be a map defined by $q(\varphi) = \varphi(0)$.

**Lemma 3.4.** $q$ is a locally trivial fibration.

**Proof.** First we show $q$ is surjective. For any $f$ in $\text{Diff}^c_0(L)$, take a smooth path $f_t$ from identity to $f$ in $\text{Diff}^c_0(L)$. $h^{-1} \circ f_t \circ h$ is a smooth path from identity to $h^{-1} \circ f \circ h$ in $\text{Diff}^c_0(L)$. Let $g_t$ be a homotopy defined by

$$g_t = \begin{cases} f_{1-t} & \text{for } 0 \leq t \leq 1/2, \\ h^{-1} \circ f_{1-1} \circ h & \text{for } 1/2 \leq t \leq 1. \end{cases}$$

$g_t$ is a path connecting $f$ and $h^{-1} \circ f \circ h$.

Next we show that $q$ has a local section. Let $U_0(id)$ be a neighborhood of $id$ in $\text{Diff}^c_0(L)$, which is homeomorphic to the set \{$s \in \Gamma_c(T(L)); \|s\| < \varepsilon$\} (by a coordinate mapping (Eells [3])), where $\Gamma_c(T(L))$ is the space of sections of the tangent bundle $T(L)$ of $L$ whose restrictions to outside of the compact set are zero-sections. Because of the continuity of a map $f \mapsto h^{-1} \circ f \circ h$, there exists a neighborhood $U_0(id)$ such that for any $f$ in $U_0(id)$, $h^{-1} \circ f \circ h$ is contained in $U_0(id)$. Put $U = U_0(id) \cap U_0(id)$. Let $\psi_{id}: U \to P^h(\text{Diff}^c_0(L))$ be a map defined by

$$\psi_{id}(f)(t) = \begin{cases} (1-2t)s_f & \text{for } 0 \leq t \leq 1/2, \\ (2t-1)s_{h^{-1} \circ f \circ h} & \text{for } 1/2 \leq t \leq 1, \end{cases}$$

where $s_f$ in $\Gamma_c(T(L))$ corresponds to $f$ in $U$ by a coordinate mapping. $\psi_{id}$ is a continuous map and $q \circ \psi_{id}(f) = f$. Hence $\psi_{id}$ is a local section. Let $U_0(f)$ be a neighborhood of $f$ in $\text{Diff}^c_0(L)$ which is homeomorphic to the set \{$s \in \Gamma_c(f^*T(L)); \|s\| < \varepsilon$\}, and $U(h^{-1} \circ f \circ h)$ be a neighborhood of $h^{-1} \circ f \circ h$ in $\text{Diff}^c_0(L)$. Let $\ell_t$ be a smooth path connecting $f$ and $h^{-1} \circ f \circ h$. Let $U$ be a small neighborhood of $f$ such that $U \subset U_0(f)$, and any $f$ in $U$, $h^{-1} \circ f \circ h$ is contained in $U_0(h^{-1} \circ f \circ h)$. Let $\psi_f: U \to P^h(\text{Diff}^c_0(L))$ be a map defined by

$$\psi_f(f')(t) = \begin{cases} (1-3t)s_f & \text{for } 0 \leq t \leq 1/3, \\ \ell_{1/3} & \text{for } 1/3 \leq t \leq 2/3, \\ (3t-2)s_{h^{-1} \circ f \circ h} & \text{for } 2/3 \leq t \leq 1. \end{cases}$$
The fiber of the fibration $q$ is the space of based loops in $\text{Diff}_0^c(L)$, which is denoted by $\Omega(\text{Diff}_0^c(L))$. Consider the homotopy exact sequence of the fibration $q$. Then we have

**Proposition 3.5.** $\pi_i(\text{Ph}(\text{Diff}_0^c(L)))=0$ for each $i \geq 0$.

**Corollary 3.6.** $\text{Ph}(\text{Diff}_0^c(L))$ is a connected component of $\text{Ph}(\text{Diff}^c(L))$.

§ 4. Proof of Theorem 4.2

Note that the kernel of $\rho \oplus \cdots \oplus \rho$, in Lemma 2.3 is some union of connected components of the space $\text{Ph}^\ell(\text{Diff}^c(L_1)) \times \cdots \times \text{Ph}^\mu(\text{Diff}^c(L_\mu))$. Consider the homotopy exact sequence of the fibration $\text{res}$ in Lemma 2.2,

$$\ldots \rightarrow \pi_i(\mathcal{F}) \rightarrow \pi_i(\mathcal{L}) \rightarrow \pi_i(\text{Diff}_0^c(V_1) \times \cdots \times \text{Diff}_0^c(V_\mu)) \xrightarrow{\Delta}$$

$$\ldots \rightarrow \pi_1(\mathcal{L}) \rightarrow \pi_1(\text{Diff}_0^c(V_1) \times \cdots \times \text{Diff}_0^c(V_\mu)) \rightarrow \pi_0(\mathcal{F}) \rightarrow \pi_0(\mathcal{L}) \rightarrow \ldots$$

Let $\mu$ be the number of $V_i$ homeomorphic to a 2-torus $T^2$. By the result of Earle and Eells [2], Gramain [5], $\text{Diff}_0^c(T^2)$ is homotopy equivalent to $T^2$, and the other group $\text{Diff}_0^c(V)$ is contractible. (Note that $V_i$ is not diffeomorphic to $S^2$.) Thus the map $\Delta: \pi_i(\text{Diff}_0^c(V_1) \times \cdots \times \text{Diff}_0^c(V_\mu)) \rightarrow \pi_0(\mathcal{F})$ reduces to the map $\Delta: (\mathbb{Z} \oplus \mathbb{Z}) \oplus \cdots \oplus (\mathbb{Z} \oplus \mathbb{Z}) \rightarrow \pi_0(\mathcal{F})$. By considering the holonomy around each 2-torus $T^2$, we may assume that $\Delta|_{\oplus \mathbb{Z} \oplus \cdots \oplus \mathbb{Z} \oplus \mathbb{Z}}$ is injective. Therefore combining Lemma 2.1, we have

**Theorem 4.1.**

$$\pi_i(\text{LDiff}^c(M, \mathcal{F}))=\begin{cases} 0 & \text{for } i \geq 2, \\ \bigoplus \mathbb{Z} & \text{for } i=1, 0 \leq \ell \leq \mu. \end{cases}$$

**Theorem 4.2.** Let $(M, \mathcal{F})$ be a generalized Reeb foliated 3-dimensional manifold. Then $\text{FDiff}_0^c(M, \mathcal{F})$ has the same homotopy type as an $\ell$-dimensional torus $T^\ell(0 \leq \ell \leq \lambda + \mu)$, where $\lambda$ is the number of generalized Reeb components and $\mu$ is the number of compact leaves homeomorphic to $T^2$.

**Proof.** Consider the homotopy exact sequence of the fibration in Lemma 1.13,

$$\ldots \rightarrow \pi_i(\text{LDiff}(M, \mathcal{F})) \rightarrow \pi_i(\text{FDiff}_0^c(M, \mathcal{F})) \rightarrow \pi_i(S^1 \times \cdots \times S^\ell) \rightarrow \pi_{i-1}(\text{LDiff}(M, \mathcal{F})) \rightarrow \pi_{i-1}(\text{FDiff}_0^c(M, \mathcal{F})) \rightarrow \ldots$$
Since $\text{FDiff}_0(M, F)$ is a topological group, $\pi_i(\text{FDiff}_0(M, F))$ is an abelian group. Therefore we have

$$\pi_i(\text{FDiff}_0(M, F)) = \begin{cases} 0 & \text{for } i \geq 2, \\ \mathbb{Z} & \text{for } i = 1, 0 \leq \ell \leq \lambda + \mu. \end{cases}$$

Hence $\text{FDiff}_0(M, F)$ is weak homotopy equivalent to an $\ell$-dimensional torus $T^\ell$. By a result of Palais [10], $\text{FDiff}_0(M, F)$ is homotopy equivalent to $T^\ell$ for $0 \leq \ell \leq \lambda + \mu$. Q.E.D.

Let $F$ be a codimension one foliation on $S^1 \times S^2$ such that $F|_{S^1 \times D^2_i}$ ($i = 1, 2$) is a Reeb component, where $S^1 \times S^2 = S^1 \times D^2_1 \cup_{id} S^1 \times D^2_2$.

**Example 4.3.** $\text{FDiff}_0(S^1 \times S^2, F)$ is homotopy equivalent to $T^2$.

**Proof.** First we consider about the homotopy type of $\text{LDiff}(S^1 \times S^2, F)$, which is the fiber of the fibration $\pi$ in Lemma 1.13. From the contractibility of $\text{Diff}(D^2; \partial D^2)$, we see that $J$ is homotopy equivalent to $\mathbb{Z} \oplus \mathbb{Z}$ (see Lemma 2.3. In this case, $\rho$ is an epimorphism.). Consider the homotopy exact sequence of the fibration $\text{res}$ in Lemma 2.2,

$$\cdots \rightarrow \pi_1(F) \rightarrow \pi_1(D^0(T^2)) \rightarrow \pi_0(J) \rightarrow \pi_0(S^2) \rightarrow 1.$$

From the structure of the foliation $F$, $J|_{S^1 \times S^2}$ is an injection. Combining Lemma 2.1, we have

$$\pi_i(\text{LDiff}(S^1 \times S^2, F); id) = \begin{cases} 0 & \text{for } i \geq 2, \\ \mathbb{Z} & \text{for } i = 0, 1. \end{cases}$$

Next, consider the homotopy exact sequence of the fibration $\pi$ in Lemma 1.13,

$$\cdots \rightarrow \pi_1(S^1 \times S^2) \rightarrow \pi_1(\text{LDiff}(S^1 \times S^2, F)) \rightarrow \pi_0(\text{FDiff}_0(S^1 \times S^2, F)) \rightarrow 1.$$

Hence we have

$$\pi_i(\text{FDiff}_0(S^1 \times S^2, F); id) = \begin{cases} \mathbb{Z} \oplus \mathbb{Z} & \text{for } i = 1, \\ 0 & \text{for } i \geq 2. \end{cases}$$ Q.E.D.
§ 5. The space of diffeomorphisms which preserve a spinnable structure

A closed 3-dimensional manifold $M$ is called \textit{spinnable} if there exists a 1-dimensional submanifold $X$, which is a finite union of circles, called an \textit{axis}, satisfying the following conditions: 1) the normal bundle of $X$ is trivial, 2) let $X \times D^2$ be a tubular neighborhood of $X$, then $M - X \times \text{Int} D^2$ is the total space of a fiber bundle $\xi$ over a circle $S^1$, and 3) let $p : M - X \times \text{Int} D^2 \rightarrow S^1$ be the projection of $\xi$, then the diagram

\[
X \times S^1 \xrightarrow{\iota} M - X \times \text{Int} D^2 \xrightarrow{p'} \xrightarrow{p} S^1
\]

commutes, where $\iota$ denotes the inclusion and $p'$ denotes the projection onto the second factor. The fiber $L$ of $\xi$ is called a \textit{generator} and the pair $(X, \xi)$ is called a \textit{spinnable structure} on $M$.

\begin{remark}[Alexander [1]]
Every closed orientable 3-dimensional manifold has a spinnable structure.
\end{remark}

Let $(M, \mathcal{S})$ be a closed orientable 3-dimensional manifold with a spinnable structure $\mathcal{S} = (X, \xi)$.

\begin{definition}
An orientation preserving diffeomorphism $f : M \rightarrow M$ is called a \textit{spinnable structure preserving diffeomorphism} if $f(X) = X$ and $f|_{M - X}$ is a bundle equivalence of $\xi$, i.e., there is a commutative diagram:

\[
\begin{array}{ccc}
M - X & \xrightarrow{f|_{M - X}} & M - X \\
\downarrow p & & \downarrow p \\
S^1 & \xrightarrow{f} & S^1
\end{array}
\]

Let $\text{SDiff}^r (M, \mathcal{S})$ or $\text{SDiff} (M, \mathcal{S})$ denote the space of spinnable structure preserving diffeomorphisms of $(M, \mathcal{S})$ of class $C^r$ ($r \geq 1$). As in § 1, a topology of $\text{SDiff} (M, \mathcal{S})$ is induced from the $C^r$ topology of $\text{Diff} (M)$. Let $\text{SDiff}^r_0 (M, \mathcal{S})$ denote the identity component of $\text{SDiff} (M, \mathcal{S})$. From Definition 5.2, we have a map $\pi : \text{SDiff}^r_0 (M, \mathcal{S}) \rightarrow \text{Diff} (S^1)$ defined by $\pi(f) = \tilde{f}$ for $f$ in $\text{SDiff}^r_0 (M, \mathcal{S})$. Note that this map is the continuous homomorphism.

\begin{lemma}
$\text{Im} \, \pi = SO(2)$, where $SO(2)$ denotes the rotation group of $S^1$.
\end{lemma}

\begin{proof}
This proof is analogue to the proof of Lemma 1.7. That $\text{Im} \, \pi$
contains the rotation group \( SO(2) \), is easily proved. Let us prove that \( \text{Im } \pi \) is contained in \( SO(2) \). Suppose \( \pi(f) = \tilde{f} \in SO(2) \) for some \( f \) in \( \text{SDiff}_0(M, \mathcal{F}) \). For a point \( x_0 \) in \( X \), we can assume \( f(x_0 \times D^2) = x_0 \times D^2 \), \( f|_{x_0 \times D^2} = \tilde{f} \in SO(2) \), by composing a relevant diffeomorphism. Since \( \tilde{f} \) is in \( \text{SDiff}_0(M, \mathcal{F}) \), \( \tilde{f}(t, y) = (f_1(t, y), f_2(y)) \), where \( (t, y) \in D^2 \), \( t \) is the radius and \( y \) is the polar angle mod 1. Note that \( f_2(y) = \tilde{f}(y) \). From \( \tilde{f} \in SO(2) \), \( f \) is not differentiable at the origin of \( D^2 \). This is a contradiction. Q.E.D.

**Lemma 5.4.** \( \pi \) is a locally trivial fibration.

Let \( \mathcal{B} \) denote the fiber of \( \pi \) and \( \lambda \) be the number of connected components of the axis of \( \mathcal{F} \).

**Lemma 5.5.** Let \( \text{res}: \mathcal{B} \to \text{Diff}_0(S^1) \times \cdots \times \text{Diff}_0(S^1) \) be the restriction map defined by \( \text{res}(f) = f|_X \) for \( f \) in \( \mathcal{B} \). Then \( \text{res} \) is a locally trivial fibration.

**Proof.** First we show \( \text{res} \) is surjective. It is sufficient to prove this lemma for the case \( \lambda = 1 \). For any \( f|_X \) in \( \text{Diff}_0(S^1) \), take a smooth path \( h_t(0 \leq t \leq 1) \) from \( f|_X \) to \( \text{ids}_{S^1} \) in \( \text{Diff}_0(S^1) \). We define a map \( H: S^1 \times D^2 \to S^1 \times D^2 \) as follows:

\[
H(x, y, t) = \begin{cases} 
(h_t(x), y, t) & \text{for } 0 \leq t \leq 1/2, \\
(h_{2t-1}(x), y, t) & \text{for } 1/2 \leq t \leq 1,
\end{cases}
\]

where \( (x, y, t) \in S^1 \times D^2 \), \( t \) is the radius and \( y \) is the polar angle mod 1. Note that \( H \) is the diffeomorphism of \( S^1 \times D^2 \) such that \( H|_{S^1 \times \{0\}} = f|_X \) and \( H|_{S^1 \times \{D^2\}} = \text{id} \). Put \( H|_{\text{outside of } S^1 \times D^2} = \text{id} \). Then this map is a spinnable structure preserving diffeomorphism of \((M, \mathcal{F})\).

That \( \text{res} \) has a local section, is proved by the same way as in Lemma 3.2. Q.E.D.

Let \( \mathcal{F} \) denote the fiber of \( \text{res} \) which is some union of connected components of the space \( P^n(\text{Diff}(L, \partial L)) \) in §3, where \( L \) is the generator of \( \mathcal{F} \). From Proposition 3.5, note that a connected component of \( \mathcal{F} \) is contractible.

Consider the homotopy exact sequence of the fibration \( \text{res} \),

\[
\cdots \to \pi_i(\mathcal{F}) \to \pi_i(\mathcal{B}) \to \pi_i(\text{Diff}_0(S^1) \times \cdots \times \text{Diff}_0(S^1)) \to \\
\to \pi_0(\mathcal{F}) \to \pi_0(\mathcal{B}) \to 1.
\]

From now we investigate properties of the map \( \Delta \) with respect to the topology of \( L \) and the number \( \lambda \).

(a) For the case \( \lambda = 1 \) and \( L = D^2 \).
The map $A: \pi_1(\text{Diff}_0(S^1)) \to \pi_0(\mathcal{F})$ is trivial since $\mathcal{F} = \text{Ph}( \text{Diff}(D^3, \partial D^3))$ is contractible. Hence we have $\pi_1(\mathcal{F}) \cong \mathbb{Z}$, $\pi_0(\mathcal{F}) = 0$.

(b) For the case $\lambda = 2$ and $L = C^2$ (the S1×[0,1]). We can prove that $\text{Diff}(C^2, \partial C^2)$ is homotopy equivalent to $\mathbb{Z}$. Hence the space $\text{Ph}( \text{Diff}(C^2, \partial C^2))$ is also homotopy equivalent to $\mathbb{Z}$. The map $A: \pi_1(\text{Diff}_0(S^1) \times \text{Diff}_0(S^1)) \to \pi_0(\mathcal{F})$ reduces to the map $A: \mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z}$. And we can prove that $A|_{\mathbb{Z} \oplus \mathbb{Z}}$ is injective. Hence we have $\pi_1(B) \cong \mathbb{Z}$, $\pi_0(B) = 0$.

(c) For the case $\lambda = 1$ and $L \neq D^2$.

The map $A: \pi_1(\text{Diff}_0(S^1)) \to \pi_0(\mathcal{F})$ reduces to the map $A: \mathbb{Z} \to \pi_0(\mathcal{F})$. For $n \neq m$ in $\mathbb{Z}$, $A(n)$ is not isotopic to $A(m)$ in $\mathcal{F}$. In fact, if $A(n)$ is isotopic to $A(m)$, $A(n)^* = A(m)^*$ as a homomorphism of $\pi_1(L)$. $A(n)^*(a) = ban$ and $A(m)^*(a) = bam$, where $a, b$ are elements of $\pi_1(L)$ as in Fig. 1. This is a contradiction. Hence $A$ is injective and we have $\pi_1(\mathcal{F}) = 0$.

(d) For the case $\lambda = 2$ and $L \neq C^2$.

The map $A: \pi_1(\text{Diff}_0(S^1) \times \text{Diff}_0(S^1)) \to \pi_0(\mathcal{F})$ reduces to the map $A: \mathbb{Z} \oplus \mathbb{Z} \to \pi_0(\mathcal{F})$. As in case (c), we can prove that $A$ is injective. Hence we have $\pi_1(\mathcal{F}) = 0$.

(e) For the case $\lambda \geq 3$.

The map $A: \pi_1(\text{Diff}_0(S^1) \times \cdots \times \text{Diff}_0(S^1)) \to \pi_0(\mathcal{F})$ reduces to the map $A: \mathbb{Z} \oplus \cdots \oplus \mathbb{Z} \to \pi_0(\mathcal{F})$. As in case (c), we can prove that $A$ is injective. Hence we have $\pi_1(\mathcal{F}) = 0$.

Combining these results and the homotopy exact sequence of the fibration $\pi$, we have

**Theorem 5.6.** $\text{SDiff}_g(M, \mathcal{F})$ is homotopy equivalent to a point or a circle $S^1$ or a 2-dimensional torus $T^2$. In fact, if $(M, \mathcal{F})$ is in case (a) and (b), $\text{SDiff}_g(M, \mathcal{F}) \cong T^2$, and if $(M, \mathcal{F})$ is in case (c), (d) and (e), $\text{SDiff}_g(M, \mathcal{F})$...
Consider the polynomial \( f(z_1, z_2) = z_1^p + z_2^q \) in two complex variables, with a critical point at the origin. Assume that the integer \( p, q \) are relatively prime and \( \geq 2 \). Let \( \mathcal{S} \) be a spinnable structure on \( S^3 \) induced by the polynomial \( f \) (see [9]).

**Example 5.7.** \( SDiff_0(S^3, \mathcal{S}) \) is homotopy equivalent to \( S^1 \).

**Proof.** \( (S^3, \mathcal{S}) \) is in case (c). Hence from Theorem 5.6, \( S Diff_0(S^3, \mathcal{S}) \) \( \cong \{ \text{one point} \} \) or \( S^1 \). Consider the homotopy exact sequence of the fibration \( \pi \).

\[
\cdots \rightarrow \pi_1(\mathcal{B}) \rightarrow \pi_1(SDiff_0(S^3, \mathcal{S})) \rightarrow \pi_1(S^3) \rightarrow \pi_0(\mathcal{B}) \rightarrow 1.
\]

\( \partial(1) \) is a diffeomorphism in \( \mathcal{B} \) induced from the characteristic homeomorphism \( h \), where 1 is a generator of \( \pi_1(S^3) \). The characteristic homeomorphism \( h \) is given by the formula

\[
h(z_1, z_2) = (e^{2\pi i/p} \cdot z_1, e^{2\pi i/q} \cdot z_2)
\]

(see [9; p. 74]). Since \( \partial(pq) \) is in the identity connected component of \( \mathcal{B} \), we have \( \pi_1(SDiff_0(S^3, \mathcal{S})) \cong \mathbb{Z} \). Q.E.D.

**Theorem 5.8.** There exists a closed 3-dimensional manifold \( M^3 \) with a spinnable structure \( \mathcal{S} \) such that \( SDiff_0(M^3, \mathcal{S}) \) is contractible.

**Proof.** Let \( L^2 \) be a compact oriented 2-dimensional manifold of genus two with boundary \( \partial L = S^1 \). We can construct a diffeomorphism \( h : L \to L \) such that \( h \) is identity on some tubular neighborhood of \( \partial L \) and maps a loop \( a \) to a loop \( b \) as in Fig. 2. Let \( \tilde{M} \) denote a space obtained from \( L \times [0, 1] \) by identifying \( (x, 0) \) with \( (h(x), 1) \). \( \tilde{M} \) is a compact oriented 3-dimensional manifold with boundary \( \partial \tilde{M} = T^2 \). Put \( M = \tilde{M} \cup_{id} S^1 \times D^2 \). Then \( M \) has a
spinnable structure naturally induced from $\tilde{M}$. To prove that the space $\text{SDiff}_0(M, \mathcal{S})$ is homotopically trivial, we consider the homotopy exact sequence of the fibration $\pi$ in Lemma 5.4:

$$
\cdots \rightarrow \pi_1(\mathcal{B}) \rightarrow \pi_1(\text{SDiff}_0(M, \mathcal{S})) \rightarrow \pi_1(S^1) \rightarrow \pi_0(\mathcal{B}) \rightarrow 1.
$$

Since $\partial(1)$ is a diffeomorphism in $\mathcal{B}$ induced from the attaching diffeomorphism $h$, $\partial$ is injective. Hence $\pi_1(\text{SDiff}_0(M, \mathcal{S})) = 0$. Q.E.D.

§ 6. Foliations induced from spinnable structures

Let $(M, \mathcal{S})$ be a closed 3-dimensional manifold with a spinnable structure $\mathcal{S}$. The axis $X$ is a finite union of circles, i.e., $\bigcup_{i=1}^{\ell} S_1^i$. Hence we can construct a codimension one foliation on $M$ from this spinnable structure (see [7]). We denote this foliation by $\mathcal{F}_\mathcal{S}$. Note that $\mathcal{F}_\mathcal{S}$ is a generalized Reeb foliation. Thus by Theorem 4.2, we know that $\text{FDiff}_0(M, \mathcal{S})$ has the same homotopy type as $T^{\ell}$ for some integer $\ell$, $0 \leq \ell \leq 2\ell + 1$. In this case we obtain a better information.

**Theorem 6.1.** $\text{FDiff}_r(M, \mathcal{S})$ is homotopy equivalent to $\text{SDiff}_r(M, \mathcal{S})$ ($r \geq 2$).

The proof of Theorem 6.1 will be preceded by some lemmas. Let $(M', \mathcal{F}')$ be a generalized Reeb component of $(M, \mathcal{F}_\mathcal{S})$ corresponding to $\xi = (M - X \times \text{Int } D^2, p, S^1)$. We define a map $\pi : \text{FDiff}_0(M, \mathcal{S}) \rightarrow S^1$ by $\pi(f) = f|_{M'}$ (see § 1).

**Lemma 6.2.** $\pi$ is a locally trivial fibration.

**Proof.** This is a consequence of Lemma 1.13.

Let $\mathcal{C}$ denote the fiber of the fibration $\pi$. Let $N_+(T_i^0)$ be a one-sided tubular neighborhood of $T_i^0 (i = 1, 2, \ldots, \lambda)$ in $M - X \times \text{Int } D^2$. By $\mathcal{C}$ we denote the subspace of $\mathcal{C}$ consisting of diffeomorphisms such that in $N^+(T_i^0)$, the following diagram commutes:

$$
\begin{array}{ccc}
N_+(T_i^0) \cup \cdots \cup N_+(T_i^0) & \xrightarrow{f} & f(N_+(T_i^0) \cup \cdots \cup N_+(T_i^0)) \subset M \\
\downarrow q & & \downarrow q \\
T_i^0 \cup \cdots \cup T_i^0 & \xrightarrow{f|_{T_i^0} \cup \cdots \cup T_i^0} & T_i^0 \cup \cdots \cup T_i^0
\end{array}
$$

where $q : N_+(T_i^0) \rightarrow T_i^0$ is a map defined by $q(y) = \lim_{t \to -\infty} \phi_t(y)$ (see Lemma 1.6).

As in § 2, we have two lemmas.
LEMMA 6.3. The inclusion map $\mathcal{C} \hookrightarrow \mathcal{E}$ is a weak homotopy equivalence.

LEMMA 6.4. $\text{res}: \mathcal{C} \rightarrow \text{FDiff}_0(S^1 \times D^2, \mathcal{F}_a) \times \cdots \times \text{FDiff}_0(S^1 \times D^2, \mathcal{F}_a)$ is a locally trivial fibration, where $\text{res}$ denotes the restriction to each Reeb component $(S^1 \times D^2, \mathcal{F}_a)$.

Let $\mathcal{F}$ denote the fiber of $\text{res}$. Then we see that the space $\mathcal{F}$ is

$$(\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}) \oplus \{\text{some union of connected components of } P^h(\text{Diff}(L, \partial L))\},$$

where $L$ is a generator of $\mathcal{F}$ (cf. Lemma 2.3).

PROOF OF THEOREM 6.1. Consider the homotopy exact sequence of the fibration $\text{res}$:

$$\cdots \rightarrow \pi_1(\mathcal{F}) \rightarrow \pi_1(\mathcal{C}) \rightarrow \pi_1(\text{FDiff}_0(S^1 \times D^2, \mathcal{F}_a) \times \cdots \times \text{FDiff}_0(S^1 \times D^2, \mathcal{F}_a))$$

$$\Delta \rightarrow \pi_0(\mathcal{F}) \rightarrow \pi_0(\mathcal{C}) \rightarrow 1.$$  

From [4], we have $\pi_1(\text{FDiff}_0(S^1 \times D^2, \mathcal{F}_a)) \cong \mathbb{Z} \oplus \mathbb{Z}$. Hence the map $\Delta$ reduces to the map $\Delta: (\mathbb{Z} \oplus \mathbb{Z}) \oplus \cdots \oplus (\mathbb{Z} \oplus \mathbb{Z}) \rightarrow \mathbb{Z} \oplus \cdots \oplus \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$.

Furthermore we see that $\Delta|_{(\mathbb{Z} \oplus \mathbb{Z}) \oplus \cdots \oplus (\mathbb{Z} \oplus \mathbb{Z})}: (\mathbb{Z} \oplus 0) \oplus \cdots \oplus (\mathbb{Z} \oplus 0) \rightarrow \mathbb{Z} \oplus \cdots \oplus \mathbb{Z} \oplus 0$ is bijective and $\Delta|_{(\mathbb{Z} \oplus \mathbb{Z}) \oplus \cdots \oplus (\mathbb{Z} \oplus \mathbb{Z})}: (0 \oplus \mathbb{Z}) \oplus \cdots \oplus (0 \oplus \mathbb{Z}) \rightarrow \pi_0(\text{Diff}(L, \partial L))$ corresponds to that in § 5. This completes the proof. Q.E.D.

References

On the homotopy type of $\text{FDiff}$


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