On some operators for p -adic uniformly differentiable functions

By Katsumi SHIRATANI

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§ 1. Introduction

The purpose of this paper is to give two theorems on the multiplicative theory of p -adic Fourier analysis. Namely we shall treat operators $\mathfrak{M}_\chi$ with Dirichlet characters χ on the p -adic Banach algebra of p -adic uniformly differentiable functions defined on the group U_p of principal units in the rational p -adic number field. Then we discuss several fundamental properties of $\mathfrak{M}_\chi$. Such operators $\mathfrak{M}_\chi$ for the analytic functions on U_p are already defined and investigated by Kubota-Leopoldt in order to introduce their p -adic L -functions [1].

Recently C. F. Woodcock [4], [5] gave an additive theory of p -adic Fourier analysis for p -adic Lipschitz functions. But we need the multiplicative theory to explain both the classical congruences of Kummer for the Bernoulli numbers and the modules of continuity of the p -adic L -functions clearly, and hence to extend our previous results [3] about the Bernoulli numbers to the ones for more general p -adic functions.

§ 2. Preliminaries

Let Q_p be the rational p -adic number field and Z_p the ring of all rational p -adic integers. The ring of rational integers is denoted by Z . We set $q=p$ for the prime $p>2$ and $q=4$ for $p=2$, and we use the normalized exponential valuation ν_p of Q_p such that $\nu_p(p)=1$ and extend it to a valuation on the completion Ω_p of the algebraic closure of Q_p . The ring of integers in Ω_p is written by O_{Ω_p} .

We define a norm for Ω_p -valued continuous functions f defined on Z_p by

$$(1) \quad \nu(f) = \min_{x \in Z_p} \nu_p(f(x)).$$

Then we know that all such functions form a p -adic Banach algebra $\mathcal{C}(Z_p, \Omega_p)$ under the pointwise operations and the above norm.

Now, we set

$$(2) \quad U_p = \{u \in Q_p^* ; \nu_p(u-1) \geq \nu_p(q)\}.$$

Then U_p is the group of all principal units of Q_p ($p > 2$) or its subgroup of index 2 ($p = 2$).

The correspondence

$$(3) \quad u \in U_p \longleftrightarrow x = \frac{\log u}{\log(1+q)} \in Z_p$$

gives an isomorphism of the multiplicative group U_p and the additive group Z_p algebraically and topologically. Hence there is an isomorphism between the p -adic Banach algebra $\mathcal{C}(U_p, \Omega_p)$ under the norm $\nu(f) = \min_{u \in U_p} \nu_p(f(u))$ for $f \in \mathcal{C}(U_p, \Omega_p)$ and the p -adic Banach algebra $\mathcal{C}(Z_p, \Omega_p)$.

By a Lipschitz function $f \in \mathcal{C}(U_p, \Omega_p)$ we mean a function for which there exists a real constant R such that $\nu_p\left(\frac{f(u)-f(v)}{u-v}\right) \geq R$ holds for all $u, v \in U_p, u \neq v$. If we set

$$(4) \quad R(f) = \inf_{\substack{u, v \in U_p \\ u \neq v}} \nu_p\left(\frac{f(u)-f(v)}{u-v}\right)$$

and define a norm $V(f)$ for a Lipschitz function f by

$$(5) \quad V(f) = \min(\nu(f), R(f)),$$

then all Ω_p -valued Lipschitz functions defined on U_p under the pointwise operations and the norm V form a p -adic Banach algebra $\mathcal{Lip}(U_p, \Omega_p)$.

If, for a given function $f: U_p \rightarrow \Omega_p$, there exists a continuous function $\phi_f(u, v): U_p \times U_p \rightarrow \Omega_p$ such that we have for all $u, v \in U_p, u \neq v$

$$(6) \quad \phi_f(u, v) = \frac{f(u)-f(v)}{u-v},$$

then we call f a uniformly differentiable function.

In this case we have

$$(7) \quad R(f) = \inf_{\substack{u, v \in U_p \\ u \neq v}} \nu_p(\phi_f(u, v)) = \nu(\phi_f)$$

and all uniformly differentiable functions make a subalgebra of the algebra $\mathcal{Lip}(U_p, \Omega_p)$. It is denoted by $\mathcal{UD}(U_p, \Omega_p)$.

Now, the dual group of U_p in the sense of Pontrjagin is isomorphic to the p -adic torus $T_p = \varinjlim_n Z/p^n Z$ and if we take the functions $\phi_\chi(u) = \chi(u)$ on

U_p corresponding to the elements χ of T_p , then we obtain, from the additive theory of Woodcock and by the isomorphism (3), a Fourier series expansion of $f \in \mathcal{UD}(U_p, \Omega_p)$

$$(8) \quad f(u) = \sum_{\chi} M_0(f\phi_{\bar{\chi}})\phi_{\chi}(u).$$

Herein we denote by $M_0(f\phi_{\bar{\chi}})$, simply written by $M_{\bar{\chi}}(f)$, the integral

$$(9) \quad \begin{aligned} M_0(f\phi_{\bar{\chi}}) &= I_0(f((1+q)^x)\phi_{\bar{\chi}}((1+q)^x)) \\ &= \lim_{\rho \rightarrow \infty} \frac{1}{p^{\rho}} \sum_{i=0}^{p^{\rho}-1} f((1+q)^i)\phi_{\bar{\chi}}((1+q)^i), \end{aligned}$$

and the sum over χ means the limit of partial sums over all elements χ in the cyclic subgroup $T_p^{(n)}$ of order p^n of T_p under letting n tend to the infinity. Finally $\bar{\chi}$ denotes the inverse character χ^{-1} of χ .

In the above we note that χ can be regarded as a Dirichlet character of the second kind, namely such character of the conductor p -power, that satisfies $\chi(x) = \chi(\langle x \rangle)$ for the canonical decomposition $x = \omega(x)\langle x \rangle$ of $x \in Z$, $(x, p) = 1$ with $\langle x \rangle \in U_p$, where $\omega(x) = \lim_{\rho \rightarrow \infty} x^{p^{\rho}}$ for $p > 2$ or $\omega(x) = \pm 1 \equiv x \pmod{4}$ for $p = 2$. We call any other Dirichlet character to be not of the second kind, namely it does not satisfy these conditions. Furthermore we see at once $\phi_{\chi}(u) \in \mathcal{UD}(U_p, \Omega_p)$ and $\phi'_{\chi}(u) = 0$.

§ 3. The operator $\mathfrak{M}_{\chi}$

Let χ be a primitive Dirichlet character with conductor $\mathfrak{f}_{\chi} = \mathfrak{f}$.

For any $f \in \mathcal{UD}(U_p, \Omega_p)$ and any non-negative rational integer ρ we set

$$(10) \quad \mathfrak{M}_{\chi}^{(\rho)}(f) = \frac{1}{\mathfrak{f}p^{\rho}} \sum_{x=1}^{\mathfrak{f}p^{\rho}*} \chi(x)f(\langle x \rangle),$$

where $*$ denotes to take sum over all integers prime to p in the given range.

First we give an elementary lemma as follows.

LEMMA. If $f \in \mathcal{C}(U_p, \Omega_p)$, then we have

$$\lim_{\rho \rightarrow \infty} \sum_{i=1}^{\mathfrak{f}p^{\rho}*} \chi(i)f(\langle i \rangle) = 0.$$

PROOF. Take a p -power p^{σ} arbitrarily and fix it. By continuity of f there is a bound σ_0 such that $\langle i \rangle \equiv \langle i_0 \rangle \pmod{p^{\sigma}}$ with $\sigma \geq \sigma_0$ yields $f(\langle i \rangle) \equiv f(\langle i_0 \rangle) \pmod{p^{\sigma}}$. Hence, for ρ sufficiently large we see

$$\sum_{i=1}^{\mathfrak{f}p^{\rho}*} \chi(i)f(\langle i \rangle) \equiv \sum_{i_0=1}^{\mathfrak{f}p^{\sigma}*} \chi(i_0)f(\langle i_0 \rangle)N_{\sigma} \pmod{p^{\sigma}},$$

where $N_\sigma = \# \{i \in Z; 1 \leq i \leq \mathfrak{f}p^\sigma, (i, p) = 1, i \equiv i_0 \pmod{\mathfrak{f}p^\sigma}\}$.

Thus we see

$$\sum_{i=1}^{\mathfrak{f}p^\rho} \chi(i) f(\langle i \rangle) \equiv \sum_{i_0=1}^{\mathfrak{f}p^\sigma} \chi(i_0) f(\langle i_0 \rangle) p^{\rho-\sigma} \pmod{p^\mu},$$

from which we have for ρ sufficiently large

$$\sum_{i=1}^{\mathfrak{f}p^\rho} \chi(i) f(\langle i \rangle) \equiv 0 \pmod{p^\mu}.$$

This means our assertion.

Now, from the definition we see readily

$$\begin{aligned} \mathfrak{M}_\chi^{(\rho+1)}(f) &= \frac{1}{\mathfrak{f}p^{\rho+1}} \sum_{x=1}^{\mathfrak{f}p^{\rho+1}} \chi(x) f(\langle x \rangle) \\ (11) \quad &= \frac{1}{\mathfrak{f}p^{\rho+1}} \sum_{x_\rho=1}^{\mathfrak{f}p^\rho} \sum_{y=0}^{p-1} \chi(x_\rho) \{f(\langle x_\rho \rangle + \bar{\omega}(x_\rho) \mathfrak{f}p^\rho y) - f(\langle x_\rho \rangle)\} \\ &\quad + \frac{1}{\mathfrak{f}p^\rho} \sum_{x_\rho=1}^{\mathfrak{f}p^\rho} \chi(x_\rho) f(\langle x_\rho \rangle), \end{aligned}$$

where $\bar{\omega}$ means also the inverse character ω^{-1} of ω .

Therefore we have

$$\begin{aligned} \mathfrak{M}_\chi^{(\rho+1)}(f) - \mathfrak{M}_\chi^{(\rho)}(f) &= \frac{1}{\mathfrak{f}p^{\rho+1}} \sum_{x_\rho=1}^{\mathfrak{f}p^\rho} \sum_{y=0}^{p-1} \chi(x_\rho) \bar{\omega}(x_\rho) \mathfrak{f}p^\rho y \frac{f(\langle x_\rho \rangle + \bar{\omega}(x_\rho) \mathfrak{f}p^\rho y) - f(\langle x_\rho \rangle)}{\bar{\omega}(x_\rho) \mathfrak{f}p^\rho y} \\ (12) \quad &= \frac{1}{p} \sum_{x_\rho=1}^{\mathfrak{f}p^\rho} \chi(x_\rho) \sum_{y=0}^{p-1} \bar{\omega}(x_\rho) y \frac{f(\langle x_\rho \rangle + \bar{\omega}(x_\rho) \mathfrak{f}p^\rho y) - f(\langle x_\rho \rangle)}{\bar{\omega}(x_\rho) \mathfrak{f}p^\rho y}, \end{aligned}$$

and we see

$$\begin{aligned} \mathfrak{M}_\chi^{(\rho+1)}(f) - \mathfrak{M}_\chi^{(\rho)}(f) &= \frac{1}{p} \sum_{x_\rho=1}^{\mathfrak{f}p^\rho} \chi(x_\rho) \sum_{y=0}^{p-1} \bar{\omega}(x_\rho) y \phi_f(\langle x_\rho \rangle + \bar{\omega}(x_\rho) \mathfrak{f}p^\rho y, \langle x_\rho \rangle) \\ (13) \quad &= \frac{1}{p} \sum_{x_\rho=1}^{\mathfrak{f}p^\rho} \chi(x_\rho) \sum_{y=0}^{p-1} \bar{\omega}(x_\rho) y \phi_f(\langle x_\rho \rangle, \langle x_\rho \rangle) \\ &\quad + \frac{1}{p} \sum_{x_\rho=1}^{\mathfrak{f}p^\rho} \chi(x_\rho) \sum_{y=0}^{p-1} \bar{\omega}(x_\rho) y \{\phi_f(\langle x_\rho \rangle + \bar{\omega}(x_\rho) \mathfrak{f}p^\rho y, \langle x_\rho \rangle) - \phi_f(\langle x_\rho \rangle, \langle x_\rho \rangle)\}. \end{aligned}$$

Because we have $\phi_f(u, u), \phi_f(u + \bar{\omega}(x) \mathfrak{f}p^\rho y, u) \in \mathcal{C}(U_p, \Omega_p)$ we conclude by Lemma and uniform continuity of $\phi_f(u, v)$ that $\mathfrak{M}_\chi^{(\rho+1)}(f) - \mathfrak{M}_\chi^{(\rho)}(f) \rightarrow 0$ as $\rho \rightarrow \infty$.

Furthermore we have

$$(14) \quad \nu_p(\mathfrak{M}_x^{(\rho+1)}(f) - \mathfrak{M}_x^{(\rho)}(f)) \geq R(f) - 1$$

for any ρ . It follows from this that the operator $\mathfrak{M}_x$ defined by

$$\mathfrak{M}_x(f) = \lim_{\rho \rightarrow \infty} \mathfrak{M}_x^{(\rho)}(f)$$

is a bounded operator, and hence a continuous operator on $\mathcal{UD}(U_p, \Omega_p)$.

§ 4. Fundamental properties of $\mathfrak{M}_x$

For $c \in Z$, $(c, \mathfrak{f})=1$ and $f \in \mathcal{UD}(U_p, \Omega_p)$ we define $f^c(u) = f(\langle c \rangle u)$. Then we see $f^c \in \mathcal{UD}(U_p, \Omega_p)$. By the condition $cx = x_\rho + \mathfrak{f}p^\rho r_\rho(x_\rho)$ with $1 \leq x_\rho \leq \mathfrak{f}p^\rho$, $1 \leq x \leq \mathfrak{f}p^\rho$ the numbers $r_\rho(x_\rho) \in Z$ depending on c are well determined. We have easily

$$(15) \quad \begin{aligned} \mathfrak{M}_x^{(\rho)}(f^c) &= \bar{\chi}(c) \frac{1}{\mathfrak{f}p^\rho} \sum_{x_\rho=1}^{\mathfrak{f}p^\rho} \chi(x_\rho) f(\langle x_\rho \rangle) + \bar{\omega}(x_\rho) \mathfrak{f}p^\rho r_\rho(x_\rho) \\ &= \bar{\chi}(c) \mathfrak{M}_x^{(\rho)}(f) \\ &\quad + \bar{\chi}(c) \frac{1}{\mathfrak{f}p^\rho} \sum_{x_\rho=1}^{\mathfrak{f}p^\rho} \chi(x_\rho) \mathfrak{f}p^\rho \bar{\omega}(x_\rho) r_\rho(x_\rho) \phi_f(\langle x_\rho \rangle) + \bar{\omega}(x_\rho) \mathfrak{f}p^\rho r_\rho(x_\rho), \langle x_\rho \rangle. \end{aligned}$$

Therefore we obtain from Lemma as before

$$(16) \quad \mathfrak{M}_x(f^c) = \bar{\chi}(c) \mathfrak{M}_x(f) + \bar{\chi}(c) \lim_{\rho \rightarrow \infty} \sum_{x_\rho=1}^{\mathfrak{f}p^\rho} \chi(x_\rho) \bar{\omega}(x_\rho) r_\rho(x_\rho) f'(\langle x_\rho \rangle).$$

In the case $p=2$ we readily see

$$(17) \quad \begin{aligned} &\lim_{\rho \rightarrow \infty} \sum_{x_\rho=1}^{\mathfrak{f}2^\rho} \chi(x_\rho) \bar{\omega}(x_\rho) r_\rho(x_\rho) f'(\langle x_\rho \rangle) \\ &= (1 + \chi(-1)) \lim_{\rho \rightarrow \infty} \sum_{x_\rho=1}^{\mathfrak{f}2^\rho-1} \chi(x_\rho) \bar{\omega}(x_\rho) r_\rho(x_\rho) f'(\langle x_\rho \rangle). \end{aligned}$$

Consequently we obtain to $f \in \mathcal{UD}(U_p, \Omega_p)$ such that $f'(u) \in O_{\Omega_p}$ for any $u \in U_p$

$$(18) \quad \chi(c) \mathfrak{M}_x(f^c) \equiv \mathfrak{M}_x(f) \pmod{p^{-1}q}.$$

If $\mathfrak{f}$ is a p -power, then χ can be extended to a function on Z_p naturally and $\chi(c)$ for $c \in Z_p$, $(c, \mathfrak{f})=1$ is well determined. By the same argument as above, but using $cx = x_\rho + \mathfrak{f}p^\rho r_\rho(x_\rho)$ with $x, x_\rho \in Z$, $r_\rho(x_\rho) \in Z_p$, we conclude the congruence (18) for any such c . Thus, take $c = \zeta_{p-1}$ a primitive $(p-1)$ -th root of unity for $p > 2$ and notice $\mathfrak{M}_x(f^c) = \mathfrak{M}_x(f)$, then we have $(1 - \chi(\zeta_{p-1})) \mathfrak{M}_x(f) \equiv 0 \pmod{p^0}$.

If χ is not of the second kind, then we have $\chi(\zeta_{p-1}) \neq 1$ and $1 - \chi(\zeta_{p-1})$ is

a unit and we see $\mathfrak{M}_\chi(f) \equiv 0 \pmod{p^0}$. In the case $p=2$, if χ is not of the second kind, then $\chi(-1)=-1$ and $\mathfrak{M}_\chi(f)=0$ hold. In either case we have $\mathfrak{M}_\chi(f) \equiv 0 \pmod{p^{-1}q}$ for χ not of the second kind.

Next, if $\mathfrak{f}=\mathfrak{f}_0 p^\mu$, $(\mathfrak{f}_0, p)=1$, $\mathfrak{f}_0 > 1$, then the canonical decomposition $\chi = \chi_0 \chi_1$ holds with $\mathfrak{f}_{\chi_0} = \mathfrak{f}_0$, $\mathfrak{f}_{\chi_1} = p^\mu$.

When we set $x = x_0 \mathfrak{f}_0 + x_1 p^{\mu+\rho} + \mathfrak{f} p^\rho r_\rho(x_0, x_1)$, $1 \leq x_0 \leq p^{\mu+\rho}$, $0 \leq x_1 \leq \mathfrak{f}_0 - 1$, $1 \leq x \leq \mathfrak{f} p^\rho$, the numbers $r_\rho(x_0, x_1) \in Z$ are also well determined. The condition $(x, p)=1$ is equivalent to $(x_0, p)=1$. Then we have similarly as above

$$(19) \quad \langle x \rangle = \langle x_0 \mathfrak{f}_0 \rangle + \bar{\omega}(x_0 \mathfrak{f}_0) (x_1 p^{\mu+\rho} + \mathfrak{f} p^\rho r_\rho(x_0, x_1)),$$

and

$$\begin{aligned} \mathfrak{M}_\chi^{(\rho)}(f) &= \frac{1}{\mathfrak{f} p^\rho} \sum_{x_0=1}^{p^{\mu+\rho}} \sum_{x_1=0}^{\mathfrak{f}_0-1} \chi_0(x_1 p^{\mu+\rho}) \chi_1(x_0 \mathfrak{f}_0) f(\langle x_0 \mathfrak{f}_0 \rangle + \bar{\omega}(x_0 \mathfrak{f}_0) (x_1 p^{\mu+\rho} + \mathfrak{f} p^\rho r_\rho(x_0, x_1))) \\ &= \frac{1}{\mathfrak{f} p^\rho} \sum_{x_1=0}^{\mathfrak{f}_0-1} \chi_0(x_1 p^{\mu+\rho}) \sum_{x_0=1}^{p^{\mu+\rho}} \chi_1(x_0 \mathfrak{f}_0) f(\langle x_0 \mathfrak{f}_0 \rangle + p^{\mu+\rho} \bar{\omega}(x_0 \mathfrak{f}_0) (x_1 + \mathfrak{f}_0 r_\rho(x_0, x_1))) \\ &= \frac{1}{\mathfrak{f} p^\rho} \sum_{x_1=0}^{\mathfrak{f}_0-1} \chi_0(x_1 p^{\mu+\rho}) \sum_{x_0=1}^{p^{\mu+\rho}} \chi_1(x_0 \mathfrak{f}_0) f(\langle x_0 \mathfrak{f}_0 \rangle) \\ (20) \quad &+ \frac{1}{\mathfrak{f}_0} \sum_{x_1=0}^{\mathfrak{f}_0-1} \chi_0(x_1 p^{\mu+\rho}) \sum_{x_0=0}^{p^{\mu+\rho}} \chi_1(x_0 \mathfrak{f}_0) \bar{\omega}(x_0 \mathfrak{f}_0) (x_1 + \mathfrak{f}_0 r_\rho(x_0, x_1)) \\ &\quad \times \phi_f(\langle x_0 \mathfrak{f}_0 \rangle + p^{\mu+\rho} \bar{\omega}(x_0 \mathfrak{f}_0) (x_1 + \mathfrak{f}_0 r_\rho(x_0, x_1)), \langle x_0 \mathfrak{f}_0 \rangle), \\ &= \frac{1}{\mathfrak{f}_0} \sum_{x_0=1}^{p^{\mu+\rho}} \sum_{x_1=0}^{\mathfrak{f}_0-1} \chi_0(x_1 p^{\mu+\rho}) \chi_1(x_0 \mathfrak{f}_0) \bar{\omega}(x_0 \mathfrak{f}_0) (x_1 + \mathfrak{f}_0 r_\rho(x_0, x_1)) \\ &\quad \times \phi_f(\langle x_0 \mathfrak{f}_0 \rangle + p^{\mu+\rho} \bar{\omega}(x_0 \mathfrak{f}_0) (x_1 + \mathfrak{f}_0 r_\rho(x_0, x_1)), \langle x_0 \mathfrak{f}_0 \rangle). \end{aligned}$$

Hence we have

$$(21) \quad \mathfrak{M}_\chi(f) = \lim_{\rho \rightarrow \infty} \sum_{x_1=0}^{\mathfrak{f}_0-1} \chi_0(x_1 p^{\mu+\rho}) \sum_{x_0=1}^{p^{\mu+\rho}} \chi_1(x_0 \mathfrak{f}_0) r_\rho(x_0, x_1) f'(\langle x_0 \mathfrak{f}_0 \rangle).$$

In the case $p=2$ we see moreover

$$(22) \quad \mathfrak{M}_\chi(f) = \lim_{\rho \rightarrow \infty} (1 + \chi(-1)) \sum_{x_1=0}^{\mathfrak{f}_0-1} \chi_0(x_1 p^{\mu+\rho}) \sum_{x_0=1}^{2^{\mu+\rho}-1} \chi_1(x_0 \mathfrak{f}_0) \bar{\omega}(x_0 \mathfrak{f}_0) r_\rho(x_0, x_1) f'(\langle x_0 \mathfrak{f}_0 \rangle).$$

Consequently, if $f'(u) \in O_{\mathfrak{p}}$ for any $u \in U_{\mathfrak{p}}$, then we obtain

$$(23) \quad \mathfrak{M}_\chi(f) \equiv 0 \pmod{p^{-1}q}.$$

For a character χ of the second kind and $p > 2$ we set $x = \zeta_{p-1}^\alpha (1+p)^\beta + p^\rho R_\rho(\alpha, \beta)$, $0 \leq \alpha \leq p-2$, $0 \leq \beta \leq p^{\rho-1}-1$, $1 \leq x \leq p^\rho$ and determine $R_\rho(\alpha, \beta) \in Z_p$.

Then we have

$$(24) \quad \begin{aligned} \mathfrak{M}_\chi^{(\rho)}(f) &= \frac{p-1}{p^\rho} \sum_{\beta=0}^{p^\rho-1-1} \chi((1+p)^\beta) f((1+p)^\beta) \\ &+ \sum_{\alpha=0}^{p-2} \sum_{\beta=0}^{p^\rho-1-1} \zeta_{p^{-1}}^{-\alpha} \chi((1+p)^\beta) R_\rho(\alpha, \beta) \phi_f((1+p)^\beta) + p^\rho \zeta_{p^{-1}}^{-\alpha} R_\rho(\alpha, \beta), (1+p)^\beta. \end{aligned}$$

Hence we have also

$$(25) \quad \begin{aligned} \mathfrak{M}_\chi(f) &= \frac{p-1}{p} I_0(\chi((1+p)^x) f((1+p)^x)) \\ &+ \lim_{\rho \rightarrow \infty} \sum_{\alpha=0}^{p-2} \sum_{\beta=0}^{p^\rho-1-1} \zeta_{p^{-1}}^{-\alpha} \chi((1+p)^\beta) R_\rho(\alpha, \beta) f'((1+p)^\beta). \end{aligned}$$

Thus, for $f \in \mathcal{UD}(U_p, \Omega_p)$ such that $f'(u) \in O_{\mathfrak{a}_p}$ for any $u \in U_p$ we obtain

$$(26) \quad \mathfrak{M}_\chi(f) \equiv \frac{p-1}{p} M_\chi(f) \pmod{p^0}.$$

Similarly in the case $p=2$ we have

$$(27) \quad \begin{aligned} \mathfrak{M}_\chi(f) &= \frac{1}{2} I_0(\chi(5^x) f(5^x)) \\ &+ \lim_{\rho \rightarrow \infty} (1 + \chi(-1)) \sum_{\beta=0}^{2^\rho-2-1} \chi(5^\beta) R_\rho(0, \beta) f'(5^\beta). \end{aligned}$$

Namely, we have for $f'(u) \in O_{\mathfrak{a}_p}$ for any $u \in U_p$

$$(28) \quad \mathfrak{M}_\chi(f) \equiv \frac{1}{2} M_\chi(f) \pmod{2}.$$

In the both cases we obtain finally

$$(29) \quad \mathfrak{M}_\chi(f) \equiv \frac{p-1}{p} M_\chi(f) \pmod{p^{-1}q}.$$

We summarize our results in the following

THEOREM 1. *Let f be any function in $\mathcal{UD}(U_p, \Omega_p)$ such that $f'(u) \in O_{\mathfrak{a}_p}$ for each $u \in U_p$. Then we have the congruences:*

$$(30) \quad \mathfrak{M}_\chi(f) \equiv 0 \pmod{p^{-1}q} \text{ if } \chi \text{ is not of the second kind,}$$

$$(31) \quad \mathfrak{M}_\chi(f) \equiv \frac{p-1}{p} M_\chi(f) \pmod{p^{-1}q} \text{ if } \chi \text{ is of the second kind.}$$

For any $c \in \mathbb{Z}$, $(c, f) = 1$ it holds that

$$(32) \quad \chi(c) \mathfrak{M}_\chi(f^c) \equiv \mathfrak{M}_\chi(f) \pmod{p^{-1}q}.$$

Furthermore, when the function $f(u)$ is multiplicative, i.e. $f(uv) = f(u)f(v)$ for any $u, v \in U_p$, we have the following Theorem 2.

From the definition we have

$$(33) \quad \begin{aligned} M_\chi(f^{1+p}) &= M_\chi(f) + \lim_{\rho \rightarrow \infty} \frac{1}{p^\rho} \{f((1+p)^{p^\rho}) - f(1)\} \\ &= M_\chi(f) + \lim_{\rho \rightarrow \infty} \frac{f((1+p)^{p^\rho}) - f(1)}{(1+p)^{p^\rho} - 1} \frac{(1+p)^{p^\rho} - 1}{p^\rho}. \end{aligned}$$

Therefore we have

$$(34) \quad M_\chi(f^{1+p}) = M_\chi(f) + f'(1) \log(1+p).$$

By virtue of the multiplicative property $M_\chi(f^{1+p}) = \chi(1+p)f(1+p)M_\chi(f)$ it follows that

$$(35) \quad (1 - \chi(1+p)f(1+p))M_\chi(f) = -f'(1) \log(1+p).$$

In the case $p=2$ quite similarly it also holds:

$$(36) \quad (1 - \chi(1+q)f(1+q))M_\chi(f) = -f'(1) \log(1+q).$$

Thus we obtain

THEOREM 2. For any multiplicative function $f \in \mathcal{UD}(U_p, \Omega_p)$ such that $f(u), f'(u) \in O_{\Omega_p}$ for each $u \in U_p$ we have

$$(37) \quad (1 - \chi(1+q)f(1+q))\mathfrak{M}_\chi(f) \equiv 0 \pmod{p^{-1}q}.$$

This is a generalization of determination of the denominators of the Bernoulli numbers.

§ 5. Examples

1) As usual we define a linear difference operator Δ for any sequence $\{a_m\}$ in Ω_p by $\Delta a_m = a_{m+1} - a_m$.

We take the function $f(u) = \frac{1}{q^k} \Delta^k \frac{1}{m} u^m$ for $m \geq 1$, where k denotes an arbitrarily fixed non-negative rational integer. Then the formulas (30), (31) in Theorem 1 reduce simply to the known congruences of Kummer.

In fact we see

$$(38) \quad f'(u) = \frac{1}{q^k} \Delta^k u^{m-1} = \frac{1}{q^k} u^{m-1} (u-1)^k \in Z_p$$

and

$$(39) \quad \mathfrak{M}_\chi(f) = \frac{1}{q^k} \Delta^k \mathfrak{M}_\chi\left(\frac{1}{m} u^m\right) = \frac{1}{q^k} \Delta^k \frac{1}{m} D_{\chi\omega^{-m}}^m,$$

where we put $D_{\chi\omega^{-m}}^m = -(1 - \chi\omega^{-m}(p)p^{m-1})B_{\chi\omega^{-m}}^m$ with the m -th generalized Bernoulli number $B_{\chi\omega^{-m}}^m$ [3].

2) If we take $f(u) = -\frac{1}{1-s} u^{1-s}$, $s \in Z_p$, $s \neq 1$, then Theorem 1 means a determination of the exact modules of continuity of p -adic L -functions of Kubota-Leopoldt. Namely, we see

$$(40) \quad f'(u) = -\frac{1}{1-s} \frac{d}{du} (e^{(1-s) \log u}) = -u^{1-s} u^{-1} = -u^{-s} \in Z_p,$$

and

$$(41) \quad \mathfrak{M}_\chi(f) = -\frac{1}{1-s} \mathfrak{M}_\chi(u^{1-s}) = L_p(s, \chi)$$

by the definition of the p -adic L -functions.

Thus we can know immediately:

$$(42) \quad L_p(s, \chi) \equiv 0 \pmod{p^{-1}q} \quad \text{for } \chi \text{ not of the second kind,}$$

$$(43) \quad L_p(s, \chi) \equiv -\frac{u_p}{1 - \chi(1+q)(1+q)^{1-s}} \pmod{p^{-1}q}$$

with a constant $u_p = -\frac{p-1}{p} \log(1+q)$ for χ of the second kind.

But these facts are already well known [2], [3].

3) When we select $f(u)$ as $f(u) = \phi_{\chi_2}(u) = \chi_2(u)$ with any Dirichlet character χ_2 of the second kind, we have $f'(u) = 0$. Therefore all the congruences in Theorems 1, 2 are automatically equalities as can be seen in the preceding section. Thus we have

$$(44) \quad \mathfrak{M}_\chi(\phi_{\chi_2}) = 0 \quad \text{for } \chi \text{ not of the second kind,}$$

$$(45) \quad \mathfrak{M}_\chi(\phi_{\chi_2}) = \frac{p-1}{p} M_\chi(\phi_{\chi_2}) \quad \text{for } \chi \text{ of the second kind,}$$

$$(46) \quad M_\chi(\phi_{\chi_2}) = I_0(\phi_{\chi_2}) = \begin{cases} 0 & \text{for } \chi \neq \bar{\chi}_2, \\ 1 & \text{for } \chi = \bar{\chi}_2. \end{cases}$$

4) Let χ be a Dirichlet character of the second kind. Multiplying $\chi(u)$ to the both hand sides in (25) and summing over the elements in $T_p^{(n)}$ we have in the case $p \neq 2$

$$\begin{aligned} \sum_{\chi \in T_p^{(n)}} \mathfrak{M}_{\chi}(f) \chi(u) &= \frac{p-1}{p} \sum_{\chi \in T_p^{(n)}} I_0(f((1+p)^x) \phi_{\chi}((1+p)^x)) \phi_{\chi}(u) \\ &\quad + \lim_{\rho \rightarrow \infty} \sum_{\alpha=0}^{p-2} \zeta_{p-1}^{-\alpha} \sum_{\substack{0 \leq \beta \leq p^{\rho-1}-1 \\ (1+p)^{\beta}=u}} p^{\rho} R_{\rho}(\alpha, \beta) f'((1+p)^{\beta}). \end{aligned}$$

Consequently we have

$$(47) \quad \sum_{\chi \in T_p} \mathfrak{M}_{\chi}(f) \phi_{\chi}(u) = \frac{p-1}{p} \sum_{\chi \in T_p} I_0(f((1+p)^x) \phi_{\chi}((1+p)^x)) \phi_{\chi}(u).$$

Note here that we regard $\psi(x) = \phi_{\chi}((1+p)^x)$ as a character of the additive group Z_p . Hence the sum

$$\sum_{\chi \in T_p} I_0(f((1+p)^x) \phi_{\chi}((1+p)^x)) \phi_{\chi}(u)$$

is a Fourier series expansion of the function $g(x) = f((1+p)^x)$.

By making use of Woodcock's theory [4] we obtain

$$(48) \quad \sum_{\chi \in T_p} \mathfrak{M}_{\chi}(f) \phi_{\chi}(u) = \frac{p-1}{p} f(u).$$

In the case $p=2$ we have quite the same.

In particular, if we take $f(u) = -\frac{1}{1-s} u^{1-s}$, then we conclude from the above

$$(49) \quad -\frac{1}{1-s} u^{1-s} = \frac{p}{p-1} \sum_{\chi \in T_p} L_p(s, \chi) \phi_{\chi}(u).$$

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DEPARTMENT OF MATHEMATICS
KYUSHU UNIVERSITY
812 FUKUOKA