Nonoscillation and comparison theorems for a class of higher order elliptic systems

By Norio YOSHIDA

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Introduction

In this paper we are concerned with elliptic systems of differential equations or inequalities involving differential operators of the form

$$(*) \quad \mathcal{L}[u] \equiv \sum_{k=0}^m \mathcal{A}^k(a_k \mathcal{A}^k u),$$

where a_k are $N \times N$ matrix functions,

$$\mathcal{A} = \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_n^2}$$

is the Laplacian, and $\mathcal{A}^k$ is the k -th iterated Laplacian.

First, we present nonoscillation criteria in the sense of Glazman [13]-Kuks [22] for the *linear* system $\mathcal{L}[u]=0$. Next, we derive a generalized Picone identity for a pair of *quasilinear* elliptic operators of the form (*). The derivation is based on some of the identities used to develop nonoscillation criteria and valid also for the quasilinear case. Finally, we apply the Picone identity to establish Sturmian comparison theorems for elliptic systems of differential inequalities involving operators of the form (*). The Picone identity and comparison theorems are extensions of the corresponding results of the present author [39] for single elliptic equations.

§ 1. Nonoscillation theorems

Consider the linear elliptic system

$$(1) \quad \mathcal{L}[u] \equiv \sum_{k=0}^m \mathcal{A}^k(a_k(x) \mathcal{A}^k u) = 0$$

in an unbounded domain Ω in Euclidean n -space R^n . As usual $x=(x_1, \dots, x_n)$ denotes a point in R^n and D_i denotes differentiation with respect to

$x_i, i=1, \dots, n$. For a multi-index $\alpha=(\alpha(1), \dots, \alpha(n))$ with norm $|\alpha|=\sum_{i=1}^n \alpha(i)$ the differential operator D^α is defined by $D^\alpha=D_1^{\alpha(1)} \dots D_n^{\alpha(n)}$. It is assumed that the coefficients a_k are $N \times N$ matrix functions of class $C^{2k}(\bar{\Omega})$, $k=0, 1, \dots, m$. The domain $\mathfrak{D}_l(G)$ of l relative to a subdomain G of Ω is defined as the set of all $N \times 1$ vector functions $u \in C^{4m}(G) \cap C^{2m}(\bar{G})$.

A bounded domain $G \subset \Omega$ is said to be a *nodal domain* for the operator l if there exists a nontrivial solution $u \in \mathfrak{D}_l(G)$ of (1) satisfying $D^\alpha u=0$ on ∂G , $|\alpha| \leq 2m-1$.

Following Glazman [13] and Kuks [22] we say that the system (1) is *nonoscillatory* in Ω if there exists a number $r>0$ such that l has no nodal domain in Ω_r , where

$$\Omega_r = \Omega \cap \{x: |x| > r\}.$$

Let L be the linear matrix differential operator

$$(2) \quad L[V] \equiv \sum_{k=0}^m \Delta^k (A_k(x) \Delta^k V),$$

where the coefficients A_k are $N \times N$ matrix functions of class $C^{2k}(\bar{\Omega})$, $k=0, 1, \dots, m$, and A_m is symmetric and positive semidefinite in Ω . The domain $\mathfrak{D}_L(G)$ of L relative to a subdomain G of Ω is the set of all $N \times N$ matrix functions of class $C^{4m}(G) \cap C^{2m}(\bar{G})$.

The operator L is said to belong to class $\mathfrak{M}[l; \Omega_r]$ if, for every bounded subdomain G of Ω_r , the functional

$$\int_G \left[\sum_{k=0}^m (\Delta^k u^T)(a_k - A_k) \Delta^k u \right] dx$$

is nonnegative for all $N \times 1$ vector functions $u \in C^{2m}(\bar{G})$ such that $D^\alpha u=0$ on ∂G , $|\alpha| \leq 2m-1$, where the superscript T denotes the transpose.

We are now in a position to state and prove our first nonoscillation result which might be viewed as a partial generalization of a theorem of Kuks [22, Theorem 2] for a second order elliptic system.

THEOREM 1.1. *The system (1) is nonoscillatory in Ω if for some $r>0$ there exist an operator L of the form (2) which belongs to class $\mathfrak{M}[l; \Omega_r]$ and matrix functions $(\tilde{\Phi}_j)_{0 \leq j \leq m-1}$, $(\Phi_{k,j})_{0 \leq j \leq k \leq m}$ and $(\Psi_{i,j})_{1 \leq i \leq n, 1 \leq j \leq m-1}$ with the following properties:*

- (i) $\tilde{\Phi}_j$, $1 \leq j \leq m-1$, are nonsingular in $\bar{\Omega}_r$;
- (ii) $A_m \tilde{\Phi}_m$, $\sum_{p=j}^m \Phi_{p,p-j}$, $\sum_{p=j}^m \Phi_{p,p-j} \Psi_{i,j}$, $1 \leq i \leq n$, $1 \leq j \leq m$, and $\sum_{p=j}^m \Phi_{p,p-j} \tilde{\Phi}_j^{-1}$, $1 \leq j \leq m-1$, are symmetric in Ω_r ;
- (iii) the following relations hold:

$$\begin{aligned}
A_m \tilde{\Phi}_m &= \Phi_{m,0} \quad \text{in } \Omega_r, \\
A_j - \Phi_{j,0} \tilde{\Phi}_j^{-1} &\geq 0, \quad 1 \leq j \leq m-1, \quad \text{in } \Omega_r, \\
A_0 - \Phi_{0,0} &\geq 0 \quad \text{in } \Omega_r, \\
\sum_{p=0}^m \Phi_{p,p} &\geq 0 \quad \text{in } \Omega_r, \\
\sum_{p=1}^m \Phi_{p,p-1} &< 0 \quad \text{in } \Omega_r, \\
\sum_{p=j}^m \Phi_{p,p-j} &\leq 0, \quad 2 \leq j \leq m, \quad \text{in } \Omega_r, \\
\sum_{p=j}^m \Phi_{p,p-j} \tilde{\Phi}_j^{-1} &\geq 0, \quad 1 \leq j \leq m-1, \quad \text{in } \Omega_r, \\
\Delta \left(\sum_{p=j}^m \Phi_{p,p-j} \right) + \tilde{\Phi}_j^T \sum_{p=j}^m \Phi_{p,p-j} + 2 \sum_{i=1}^n \sum_{p=j}^m D_i(\Phi_{p,p-j}) \Psi_{i,j} \\
- \sum_{p=j}^m \Phi_{p,p-j+1} \tilde{\Phi}_{j-1}^{-1} &\geq 0, \quad 1 \leq j \leq m, \quad \text{in } \Omega_r, \\
- \sum_{p=j}^m \sum_{i=1}^n \Phi_{p,p-j} D_i \Psi_{i,j} + \tilde{\Phi}_j^T \sum_{p=j}^m \Phi_{p,p-j} - \sum_{i=1}^n \sum_{p=j}^m \Psi_{i,j}^T \Phi_{p,p-j} \Psi_{i,j} &\leq 0, \\
&\quad 1 \leq j \leq m, \quad \text{in } \Omega_r;
\end{aligned}$$

(iv) the relation

$$\nabla u - \begin{bmatrix} \Psi_{1,1} \\ \Psi_{2,1} \\ \vdots \\ \Psi_{n,1} \end{bmatrix} u \neq 0 \quad \text{in } G$$

holds for any bounded subdomain $G \subset \Omega_r$ and for all nontrivial $u \in \mathfrak{D}_i(G)$ such that $D^\alpha u = 0$ on ∂G , $|\alpha| \leq 2m-1$.

Moreover, if the condition $\sum_{p=0}^m \Phi_{p,p} \geq 0$ in (iii) is replaced by the stronger condition $\sum_{p=0}^m \Phi_{p,p} > 0$, the condition (iv) may be omitted.

PROOF. Suppose to the contrary that there exists a nodal domain $G \subset \Omega_r$ and a nontrivial solution u of (1) such that $D^\alpha u = 0$ on ∂G , $|\alpha| \leq 2m-1$. Applying Green's formula and using the hypothesis $L \in \mathfrak{M}[l; \Omega_r]$, we obtain

$$\begin{aligned}
(3) \quad 0 &= \int_G u^T l[u] dx \\
&= \int_G \sum_{k=0}^m (\Delta^k u^T) a_k \Delta^k u dx \\
&\geq \int_G \sum_{k=0}^m (\Delta^k u^T) A_k \Delta^k u dx.
\end{aligned}$$

On the other hand, using the symmetry of $\sum_{p=j}^m \Phi_{p,p-j}$ and $\sum_{p=j}^m \Phi_{p,p-j} \Psi_{i,j}$ ($1 \leq i \leq n, 1 \leq j \leq m$), we have the following identities:

$$(4) \quad \sum_{i=1}^n \sum_{j=1}^m D_i \left[(\Delta^{j-1} u^T) D_i \left(\sum_{p=j}^m \Phi_{p,p-j} \right) \Delta^{j-1} u - 2(\Delta^{j-1} u^T) \left(\sum_{p=j}^m \Phi_{p,p-j} \right) D_i \Delta^{j-1} u \right] \\ = \sum_{j=1}^m \left[(\Delta^{j-1} u^T) \Delta \left(\sum_{p=j}^m \Phi_{p,p-j} \right) \Delta^{j-1} u - 2(\Delta^{j-1} u^T) \left(\sum_{p=j}^m \Phi_{p,p-j} \right) \Delta^j u \right. \\ \left. - 2 \sum_{i=1}^n (D_i \Delta^{j-1} u^T) \left(\sum_{p=j}^m \Phi_{p,p-j} \right) D_i \Delta^{j-1} u \right],$$

$$(5) \quad 2 \sum_{i=1}^n \sum_{j=1}^m D_i \left[(\Delta^{j-1} u^T) \left(\sum_{p=j}^m \Phi_{p,p-j} \Psi_{i,j} \right) \Delta^{j-1} u \right] \\ = \sum_{j=1}^m \left[2(\Delta^{j-1} u^T) \left(\sum_{i=1}^n \sum_{p=j}^m D_i (\Phi_{p,p-j} \Psi_{i,j}) \right) \Delta^{j-1} u \right. \\ \left. + 2(\Delta^{j-1} u^T) \sum_{i=1}^n \sum_{p=j}^m \Phi_{p,p-j} (D_i \Psi_{i,j}) \Delta^{j-1} u \right. \\ \left. + 4(\Delta^{j-1} u^T) \sum_{i=1}^n \left(\sum_{p=j}^m \Phi_{p,p-j} \Psi_{i,j} \right) D_i \Delta^{j-1} u \right].$$

Since $\sum_{p=j}^m \Phi_{p,p-j}$ and $\sum_{p=j}^m \Phi_{p,p-j} \tilde{\Phi}_j^{-1}$ ($1 \leq j \leq m-1$) are symmetric, the following identity holds:

$$(6) \quad \sum_{j=1}^{m-1} (\Delta^j u - \tilde{\Phi}_j \Delta^{j-1} u)^T \left(\sum_{p=j}^m \Phi_{p,p-j} \tilde{\Phi}_j^{-1} \right) (\Delta^j u - \tilde{\Phi}_j \Delta^{j-1} u) \\ = \sum_{j=1}^{m-1} \left[(\Delta^j u^T) \left(\sum_{p=j}^m \Phi_{p,p-j} \tilde{\Phi}_j^{-1} \right) \Delta^j u - 2(\Delta^{j-1} u^T) \left(\sum_{p=j}^m \Phi_{p,p-j} \right) \Delta^j u \right. \\ \left. + (\Delta^{j-1} u^T) \tilde{\Phi}_j^T \left(\sum_{p=j}^m \Phi_{p,p-j} \right) \Delta^{j-1} u \right].$$

We observe moreover that:

$$(7) \quad \sum_{j=1}^{m-1} (\Delta^j u^T) \left(\sum_{p=j}^m \Phi_{p,p-j} \tilde{\Phi}_j^{-1} \right) \Delta^j u \\ = \sum_{j=1}^{m-1} (\Delta^j u^T) \Phi_{j,0} \tilde{\Phi}_j^{-1} \Delta^j u - u^T \left(\sum_{p=1}^m \Phi_{p,p} \right) u \\ + \sum_{j=1}^m (\Delta^{j-1} u^T) \left(\sum_{p=j}^m \Phi_{p,p-j+1} \tilde{\Phi}_{j-1}^{-1} \right) \Delta^{j-1} u.$$

From (6) and (7) we see that

$$\sum_{j=1}^m (\Delta^{j-1} u^T) \left(\sum_{p=j}^m \Phi_{p,p-j+1} \tilde{\Phi}_{j-1}^{-1} \right) \Delta^{j-1} u - 2 \sum_{j=1}^m (\Delta^{j-1} u^T) \left(\sum_{p=j}^m \Phi_{p,p-j} \right) \Delta^j u \\ + \sum_{j=1}^m (\Delta^{j-1} u^T) \tilde{\Phi}_j^T \left(\sum_{p=j}^m \Phi_{p,p-j} \right) \Delta^{j-1} u$$

$$\begin{aligned}
 (8) \quad &= u^T \left(\sum_{p=1}^m \Phi_{p,p} \right) u - \sum_{j=1}^{m-1} (\Delta^j u^T) (\Phi_{j,0} \tilde{\Phi}_j^{-1}) \Delta^j u \\
 &+ \sum_{j=1}^{m-1} (\Delta^j u - \tilde{\Phi}_j \Delta^{j-1} u)^T \left(\sum_{p=j}^m \Phi_{p,p-j} \tilde{\Phi}_j^{-1} \right) (\Delta^j u - \tilde{\Phi}_j \Delta^{j-1} u) \\
 &- 2(\Delta^{m-1} u^T) \Phi_{m,0} \Delta^m u + (\Delta^{m-1} u^T) \tilde{\Phi}_m^T \Phi_{m,0} \Delta^{m-1} u.
 \end{aligned}$$

Since A_m and $A_m \tilde{\Phi}_m$ are symmetric and $A_m \tilde{\Phi}_m = \Phi_{m,0}$, we have

$$\begin{aligned}
 (9) \quad &(\Delta^m u^T) A_m \Delta^m u - 2(\Delta^{m-1} u^T) \Phi_{m,0} \Delta^m u + (\Delta^{m-1} u^T) \tilde{\Phi}_m^T \Phi_{m,0} \Delta^{m-1} u \\
 &= (\Delta^m u - \tilde{\Phi}_m \Delta^{m-1} u)^T A_m (\Delta^m u - \tilde{\Phi}_m \Delta^{m-1} u).
 \end{aligned}$$

Combining the identities (4), (5), (8) and (9), we see that the following integral identity holds:

$$\begin{aligned}
 &\int_G \left[\sum_{k=0}^m (\Delta^k u^T) A_k \Delta^k u \right] dx \\
 &= \int_G \left[(\Delta^m u - \tilde{\Phi}_m \Delta^{m-1} u)^T A_m (\Delta^m u - \tilde{\Phi}_m \Delta^{m-1} u) \right. \\
 &\quad + \sum_{j=1}^{m-1} (\Delta^j u - \tilde{\Phi}_j \Delta^{j-1} u)^T \left(\sum_{p=j}^m \Phi_{p,p-j} \tilde{\Phi}_j^{-1} \right) (\Delta^j u - \tilde{\Phi}_j \Delta^{j-1} u) \\
 &\quad + \sum_{j=1}^{m-1} (\Delta^j u^T) (A_j - \Phi_{j,0} \tilde{\Phi}_j^{-1}) \Delta^j u + u^T \left(A_0 - \Phi_{0,0} + \sum_{p=0}^m \Phi_{p,p} \right) u \Big] dx \\
 (10) \quad &+ \int_G \sum_{j=1}^m (\Delta^{j-1} u)^T \left[\Delta \left(\sum_{p=j}^m \Phi_{p,p-j} \right) + \tilde{\Phi}_j^T \sum_{p=j}^m \Phi_{p,p-j} \right. \\
 &\quad \left. + 2 \sum_{i=1}^n \sum_{p=j}^m D_i (\Phi_{p,p-j}) \Psi_{i,j} - \sum_{p=j}^m \Phi_{p,p-j+1} \tilde{\Phi}_{j-1}^{-1} \right] (\Delta^{j-1} u) dx \\
 &- 2 \int_G \sum_{j=1}^m \left[\sum_{i=1}^n (D_i \Delta^{j-1} u - \Psi_{i,j} \Delta^{j-1} u)^T \left(\sum_{p=j}^m \Phi_{p,p-j} \right) (D_i \Delta^{j-1} u - \Psi_{i,j} \Delta^{j-1} u) \right. \\
 &\quad + (\Delta^{j-1} u^T) \left(- \sum_{p=j}^m \Phi_{p,p-j} \sum_{i=1}^n D_i \Psi_{i,j} + \tilde{\Phi}_j^T \sum_{p=j}^m \Phi_{p,p-j} \right. \\
 &\quad \left. \left. - \sum_{i=1}^n \Psi_{i,j}^T \left(\sum_{p=j}^m \Phi_{p,p-j} \right) \Psi_{i,j} \right) \Delta^{j-1} u \right] dx.
 \end{aligned}$$

The fifth condition of (iii) plus (iv) implies

$$-2 \int_G \left[\sum_{i=1}^n (D_i u - \Psi_{i,1} u)^T \left(\sum_{p=1}^m \Phi_{p,p-1} \right) (D_i u - \Psi_{i,1} u) \right] dx > 0.$$

In view of this inequality and other conditions of (iii) we conclude that the left hand side of (10) is positive. This, however, contradicts the inequality (3). The second statement is obvious. This completes the proof.

DEFINITION. We say that an $N \times N$ matrix function W of class C^{4m-2} is L -prepared if the matrices

$$\begin{aligned}
(\Delta^{j-1}W^T) \sum_{p=j}^m \Delta^{p-j}(A_p \Delta^p W), \quad 1 \leq j \leq m, \\
(D_i \Delta^{j-1}W^T) \sum_{p=j}^m \Delta^{p-j}(A_p \Delta^p W), \quad 1 \leq i \leq n, 1 \leq j \leq m, \\
(\Delta^j W^T) \sum_{p=j}^m \Delta^{p-j}(A_p \Delta^p W), \quad 1 \leq j \leq m
\end{aligned}$$

are all symmetric. This definition also applies to the case when L is quasilinear.

A more feasible nonoscillation criterion is given in the following theorem.

THEOREM 1.2. *The system (1) is nonoscillatory in Ω if for some $r > 0$ there exist an operator L of the form (2) which belongs to class $\mathfrak{M}[l; \Omega_r]$ and an $N \times N$ matrix function W of class $\mathfrak{D}_L(\Omega_r)$ such that:*

- (i) W is an L -prepared matrix in Ω_r ;
- (ii) $\Delta^j W$, $0 \leq j \leq m-1$, are nonsingular in $\bar{\Omega}_r$;
- (iii) the following relations hold:

$$\begin{aligned}
\sum_{p=j}^m \Delta^{p-j}(A_p \Delta^p W)(\Delta^j W)^{-1} &\geq 0, \quad 1 \leq j \leq m-1, \quad \text{in } \Omega_r, \\
\sum_{p=j}^m \Delta^{p-j}(A_p \Delta^p W)(\Delta^{j-1} W)^{-1} &\leq 0, \quad 2 \leq j \leq m, \quad \text{in } \Omega_r, \\
\sum_{p=1}^m \Delta^{p-1}(A_p \Delta^p W)W^{-1} &< 0 [\leq 0] \quad \text{in } \Omega_r, \\
L[W]W^{-1} &\geq 0 [> 0] \quad \text{in } \Omega_r.
\end{aligned}$$

PROOF. Define the matrix functions

$$(\tilde{\Phi}_j)_{0 \leq j \leq m-1}, \quad (\Phi_{k,j})_{0 \leq j \leq k \leq m}, \quad (\Psi_{i,j})_{1 \leq i \leq n, 1 \leq j \leq m-1}$$

as follows:

$$\begin{aligned}
\tilde{\Phi}_0 &= I_N \text{ (the } N \times N \text{ identity matrix),} \\
\tilde{\Phi}_j &= (\Delta^j W)(\Delta^{j-1} W)^{-1}, \quad 1 \leq j \leq m-1, \\
(11) \quad \Phi_{k,j} &= \Delta^j(A_k \Delta^k W)(\Delta^{k-j-1} W)^{-1}, \quad 0 \leq j \leq k-1, 1 \leq k \leq m, \\
\Phi_{j,j} &= \Delta^j(A_j \Delta^j W)W^{-1}, \quad 0 \leq j \leq m, \\
\Psi_{i,j} &= (D_i \Delta^{j-1} W)(\Delta^{j-1} W)^{-1}, \quad 1 \leq i \leq n, 1 \leq j \leq m-1.
\end{aligned}$$

It is easy to verify that the conditions of Theorem 1.1 are satisfied. In fact, (ii) implies (i) of Theorem 1.1, (ii) of Theorem 1.1 is a consequence of (i), and (iii) of Theorem 1.1 is assured by the following relations:

$$\begin{aligned}
(12) \quad & A_j - \Phi_{j,0} \tilde{\Phi}_j^{-1} = 0, \quad 1 \leq j \leq m-1, \\
& A_0 - \Phi_{0,0} = 0, \\
& \sum_{p=0}^m \Phi_{p,p} = L[W]W^{-1}, \\
& \sum_{p=j}^m \Phi_{p,p-j} = \sum_{p=j}^m \Delta^{p-j} (A_p \Delta^p W) (\Delta^{j-1} W)^{-1}, \quad 1 \leq j \leq m, \\
& \sum_{p=j}^m \Phi_{p,p-j} \tilde{\Phi}_j^{-1} = \sum_{p=j}^m \Delta^{p-j} (A_p \Delta^p W) (\Delta^j W)^{-1}, \quad 1 \leq j \leq m-1, \\
& \Delta \left(\sum_{p=j}^m \Phi_{p,p-j} \right) + \tilde{\Phi}_j^T \sum_{p=j}^m \Phi_{p,p-j} + 2 \sum_{i=1}^n \sum_{p=j}^m D_i (\Phi_{p,p-j}) \Psi_{i,j} \\
& \quad - \sum_{p=j}^m \Phi_{p,p-j+1} \tilde{\Phi}_{j-1}^{-1} = 0, \\
& - \sum_{p=j}^m \Phi_{p,p-j} \sum_{i=1}^n D_i \Psi_{i,j} + \tilde{\Phi}_j^T \sum_{p=j}^m \Phi_{p,p-j} - \sum_{i=1}^n \Psi_{i,j}^T \left(\sum_{p=j}^m \Phi_{p,p-j} \right) \Psi_{i,j} = 0, \\
& \nabla u - \begin{bmatrix} \Psi_{1,1} \\ \Psi_{2,1} \\ \vdots \\ \Psi_{n,1} \end{bmatrix} u = \begin{bmatrix} W D_1 (W^{-1} u) \\ W D_2 (W^{-1} u) \\ \vdots \\ W D_n (W^{-1} u) \end{bmatrix}.
\end{aligned}$$

Hence, the conclusion follows from Theorem 1.1.

COROLLARY. The system (1) with l defined by

$$l[u] \equiv \Delta^m (a \Delta^m u) + cu$$

is nonoscillatory in Ω if $n > 2m$ and the matrices a, c are uniformly positive definite in Ω .

PROOF. Let $L[V] \equiv a_0 \Delta^{2m} V + c_0 V$, where a_0 and c_0 are positive constants such that $\xi^T a \xi \geq a_0 |\xi|^2$ and $\xi^T c \xi \geq c_0 |\xi|^2$ for all $\xi \in R^N$, where $|\xi|$ denotes the Euclidean norm of ξ . We easily see that the operator L belongs to class $\mathfrak{M}[l; \Omega]$. We can choose a number s so that

$$\begin{aligned}
(-1)^j \Delta^j (|x|^s) &> 0, \quad 0 \leq j \leq m-1, \\
(-1)^j \Delta^j (|x|^s) &\geq 0, \quad m \leq j \leq 2m-1, \\
\Delta^{2m} (|x|^s) &\geq 0.
\end{aligned}$$

(See Noussair and Yoshida [26, Theorem 5].) Define $W = |x|^s I_N$. Then, W is L -prepared and satisfies

$$\begin{aligned}
\Delta^{m-j} (a_0 I_N \Delta^m W) (\Delta^j W)^{-1} &= a_0 \Delta^{2m-j} (|x|^s) (1 / \Delta^j (|x|^s)) I_N \geq 0, \quad 1 \leq j \leq m-1, \\
\Delta^{m-j} (a_0 I_N \Delta^m W) (\Delta^{j-1} W)^{-1} &= a_0 \Delta^{2m-j} (|x|^s) (1 / \Delta^{j-1} (|x|^s)) I_N \leq 0, \quad 1 \leq j \leq m, \\
L[W]W^{-1} &= (a_0 \Delta^{2m} (|x|^s) + c_0 |x|^s) |x|^{-s} I_N > 0.
\end{aligned}$$

The conclusion now follows from Theorem 1.2.

REMARK. When specialized to the case of a single equation ($N=1$), the above results are closely related to recent results of Noussair and the present author [26]. For other nonoscillation criteria we refer to the papers [13, 15, 16, 19, 22–25, 28, 29, 32, 37, 38].

§ 2. A Picone identity

Let G be a bounded domain in R^n with piecewise smooth boundary ∂G and let H be a domain in R^N containing the origin. We consider the following quasilinear operators

$$(1') \quad l[u] = \sum_{k=0}^m \Delta^k (a_k(x, u) \Delta^k u),$$

$$(2') \quad L[V] = \sum_{k=0}^m \Delta^k (A_k(x, V) \Delta^k V).$$

The coefficients $a_k = a_k(x, \xi)$ are $N \times N$ matrix functions of class $C^{2k}(\bar{G} \times H)$ and the coefficients $A_k = A_k(x, \eta)$ are $N \times N$ matrix functions of class $C^{2k}(\bar{G} \times H^N)$. Furthermore, we suppose that the coefficient A_m is symmetric and positive semidefinite in $G \times H^N$. The domain $\mathfrak{D}_l$ of l is the set $\mathfrak{D}$ of all $N \times 1$ vector functions of class $C^{4m}(G) \cap C^{2m}(\bar{G})$ with range in H and the domain $\mathfrak{D}_L$ of L is the set $\mathfrak{D}^N$ of all $N \times N$ matrix functions whose column vectors belong to $\mathfrak{D}$.

We now present a Picone type integral identity for the operators l and L . The identity is a generalization of the identity we have previously established for single linear elliptic operators of order $4m$ [39].

THEOREM 2.1. Assume ∂G is piecewise smooth. If $u \in \mathfrak{D}_l$ and if $W \in \mathfrak{D}_L$ is an L -prepared matrix such that $\Delta^j W$, $0 \leq j \leq m-1$, are nonsingular in $\bar{G}$, then the following identity holds:

$$\begin{aligned} & \int_{\partial G} \sum_{j=1}^m \left[(\Delta^{j-1} u^T) \frac{\partial}{\partial n} \left(\sum_{p=j}^m \Delta^{p-j} (A_p \Delta^p W) (\Delta^{j-1} W)^{-1} \right) \cdot \Delta^{j-1} u \right. \\ & \quad - (\Delta^{j-1} u^T) \frac{\partial}{\partial n} \left(\sum_{p=j}^m \Delta^{p-j} (a_p \Delta^p u) \right) + \frac{\partial}{\partial n} (\Delta^{j-1} u^T) \cdot \sum_{p=j}^m \Delta^{p-j} (a_p \Delta^p u) \\ & \quad \left. - 2(\Delta^{j-1} u^T) \left(\sum_{p=j}^m \Delta^{p-j} (A_p \Delta^p W) \right) \frac{\partial}{\partial n} ((\Delta^{j-1} W)^{-1} \Delta^{j-1} u) \right] ds \\ & = \int_G \sum_{k=0}^m (\Delta^k u^T) (a_k - A_k) \Delta^k u \, dx \end{aligned}$$

$$\begin{aligned}
& + \int_G \left[(\Delta^m u - (\Delta^m W)(\Delta^{m-1} W)^{-1} \Delta^{m-1} u)^T A_m (\Delta^m u - (\Delta^m W)(\Delta^{m-1} W)^{-1} \Delta^{m-1} u) \right. \\
& \quad + \sum_{j=1}^{m-1} (\Delta^j u - (\Delta^j W)(\Delta^{j-1} W)^{-1} \Delta^{j-1} u)^T \left(\sum_{p=j}^m \Delta^{p-j} (A_p \Delta^p W)(\Delta^j W)^{-1} \right) \\
& \quad \times (\Delta^j u - (\Delta^j W)(\Delta^{j-1} W)^{-1} \Delta^{j-1} u) \Big] dx \\
& - 2 \int_G \sum_{i=1}^n \sum_{j=1}^m (D_i \Delta^{j-1} u - (D_i \Delta^{j-1} W)(\Delta^{j-1} W)^{-1} \Delta^{j-1} u)^T \left(\sum_{p=j}^m \Delta^{p-j} (A_p \Delta^p W) \right. \\
& \quad \times (\Delta^{j-1} W)^{-1} \Big) (D_i \Delta^{j-1} u - (D_i \Delta^{j-1} W)(\Delta^{j-1} W)^{-1} \Delta^{j-1} u) dx \\
& + \int_G (u^T L[W] W^{-1} u - u^T l[u]) dx,
\end{aligned}$$

where $\partial/\partial n$ denotes the exterior normal derivative.

PROOF. Define the matrix functions $(\tilde{\Phi}_j)$, $(\Phi_{k,j})$, $(\Psi_{i,j})$ by (11). It is easy to check that, though L is quasilinear, the relations in (12) and the integral identity (10) still hold true. Using (12) in (10), we obtain

$$\begin{aligned}
& \int_G \sum_{k=0}^m (\Delta^k u^T) A_k \Delta^k u \, dx \\
& = \int_G \left[(\Delta^m u - (\Delta^m W)(\Delta^{m-1} W)^{-1} \Delta^{m-1} u)^T A_m (\Delta^m u - (\Delta^m W)(\Delta^{m-1} W)^{-1} \Delta^{m-1} u) \right. \\
& \quad + \sum_{j=1}^{m-1} (\Delta^j u - (\Delta^j W)(\Delta^{j-1} W)^{-1} \Delta^{j-1} u)^T \left(\sum_{p=j}^m \Delta^{p-j} (A_p \Delta^p W)(\Delta^j W)^{-1} \right) \\
& \quad \times (\Delta^j u - (\Delta^j W)(\Delta^{j-1} W)^{-1} \Delta^{j-1} u) + u^T L[W] W^{-1} u \Big] dx \\
(13) \quad & - 2 \int_G \sum_{i=1}^n \sum_{j=1}^m (D_i \Delta^{j-1} u - (D_i \Delta^{j-1} W)(\Delta^{j-1} W)^{-1} \Delta^{j-1} u)^T \left(\sum_{p=j}^m \Delta^{p-j} (A_p \Delta^p W) \right. \\
& \quad \times (\Delta^{j-1} W)^{-1} \Big) (D_i \Delta^{j-1} u - (D_i \Delta^{j-1} W)(\Delta^{j-1} W)^{-1} \Delta^{j-1} u) dx \\
& - \int_{\partial G} \sum_{i=1}^n \sum_{j=1}^m \left[(\Delta^{j-1} u^T) D_i \left(\sum_{p=j}^m \Delta^{p-j} (A_p \Delta^p W)(\Delta^{j-1} W)^{-1} \right) \Delta^{j-1} u \right. \\
& \quad \left. - 2 (\Delta^{j-1} u^T) \left(\sum_{p=j}^m \Delta^{p-j} (A_p \Delta^p W)(\Delta^{j-1} W)^{-1} \right) D_i \Delta^{j-1} u \right] n_i \, ds \\
& - 2 \int_{\partial G} \sum_{i=1}^n \sum_{j=1}^m \left[(\Delta^{j-1} u^T) \left(\sum_{p=j}^m \Delta^{p-j} (A_p \Delta^p W)(\Delta^{j-1} W)^{-1} \right. \right. \\
& \quad \left. \left. \times (D_i \Delta^{j-1} W)(\Delta^{j-1} W)^{-1} \right) \Delta^{j-1} u \right] n_i \, ds,
\end{aligned}$$

where $n = (n_i)_{1 \leq i \leq n}$ is the unit outward normal vector. The last two surface integrals of the above equality (13) are rewritten as follows:

$$(14) \quad \begin{aligned} & - \int_{\partial G} \sum_{i=1}^n \sum_{j=1}^m \left[(\Delta^{j-1} u^T) D_i \left(\sum_{p=j}^m \Delta^{p-j} (A_p \Delta^p W) (\Delta^{j-1} W)^{-1} \right) \Delta^{j-1} u \right] n_i ds \\ & + 2 \int_{\partial G} \sum_{i=1}^n \sum_{j=1}^m \left[(\Delta^{j-1} u^T) \left(\sum_{p=j}^m \Delta^{p-j} (A_p \Delta^p W) \right) D_i ((\Delta^{j-1} W)^{-1} \Delta^{j-1} u) \right] n_i ds. \end{aligned}$$

On the other hand, an easy calculation yields the identity:

$$(15) \quad \begin{aligned} & \int_{\partial G} \sum_{j=1}^m \left[-(\Delta^{j-1} u^T) \frac{\partial}{\partial n} \left(\sum_{p=j}^m \Delta^{p-j} (a_p \Delta^p u) \right) + \frac{\partial}{\partial n} (\Delta^{j-1} u^T) \sum_{p=j}^m \Delta^{p-j} (a_p \Delta^p u) \right] ds \\ & = \int_G \sum_{k=0}^m (\Delta^k u^T) a_k \Delta^k u dx - \int_G u^T l[u] dx. \end{aligned}$$

Now, the desired identity follows readily from (13), (14) and (15).

REMARK. The above Picone identity is a generalization of that presented by the present author [39] for single elliptic equations. Extensions of the classical Picone identity to elliptic systems have been obtained by various authors; see e.g. the papers [4, 19, 21, 24, 25, 34–36].

§ 3. Comparison theorems

In this section as an application of the Picone identity we establish Sturmian comparison theorems for elliptic systems of differential inequalities involving the quasilinear operators (1') and (2'). We introduce the following notations:

$$\begin{aligned} F[u, W] &= \int_G \sum_{k=0}^m (\Delta^k u^T) A_k(x, W) \Delta^k u dx, \\ J[u, W] &= -2 \int_G \sum_{i=1}^n \sum_{j=1}^m (D_i \Delta^{j-1} u - (D_i \Delta^{j-1} W) (\Delta^{j-1} W)^{-1} \Delta^{j-1} u)^T \\ & \quad \times \left(\sum_{p=j}^m \Delta^{p-j} (A_p(x, W) \Delta^p W) (\Delta^{j-1} W)^{-1} \right) (D_i \Delta^{j-1} u - (D_i \Delta^{j-1} W) \\ & \quad \times (\Delta^{j-1} W)^{-1} \Delta^{j-1} u) dx, \\ Q[u, W] &= \int_G \left[(\Delta^m u - (\Delta^m W) (\Delta^{m-1} W)^{-1} \Delta^{m-1} u)^T A_m(x, W) (\Delta^m u - (\Delta^m W) \right. \\ & \quad \times (\Delta^{m-1} W)^{-1} \Delta^{m-1} u) \\ & \quad + \sum_{j=1}^{m-1} (\Delta^j u - (\Delta^j W) (\Delta^{j-1} W)^{-1} \Delta^{j-1} u)^T \left(\sum_{p=j}^m \Delta^{p-j} (A_p(x, W) \Delta^p W) \right. \\ & \quad \times (\Delta^j W)^{-1} \left. \right) (\Delta^j u - (\Delta^j W) (\Delta^{j-1} W)^{-1} \Delta^{j-1} u) \left. \right] dx, \\ f[u] &= \int_G \sum_{k=0}^m (\Delta^k u^T) a_k(x, u) \Delta^k u dx. \end{aligned}$$

LEMMA 3.1. Let ∂G be piecewise smooth and let u be an $N \times 1$ vector function of class $C^{2m}(\bar{G})$ with range in H and such that $D^\alpha u = 0$ on ∂G , $|\alpha| \leq 2m-2$. Then, every L -prepared matrix $W \in \mathfrak{D}_L$ such that $\Delta^j W$, $0 \leq j \leq m-1$, are nonsingular in $\bar{G}$ satisfies

$$(16) \quad F[u, W] = \int_G u^T L[W] W^{-1} u \, dx + J[u, W] + Q[u, W].$$

LEMMA 3.2. Let ∂G be piecewise smooth and let u be an $N \times 1$ vector function of class $\mathfrak{D}_l$ such that $D^\alpha u = 0$ on ∂G , $|\alpha| \leq 2m-1$. Then, every L -prepared matrix $W \in \mathfrak{D}_L$ such that $\Delta^j W$, $0 \leq j \leq m-1$, are nonsingular in $\bar{G}$ satisfies

$$(17) \quad \int_G (u^T l[u] - u^T L[W] W^{-1} u) \, dx = f[u] - F[u, W] + J[u, W] + Q[u, W].$$

Lemma 3.1 is a restatement of the identity (13). Lemma 3.2 is an immediate consequence of the Picone identity (Theorem 2.1).

On the basis of Lemma 3.1 we can extend to the case of systems the comparison theorems given in [39] for single elliptic differential inequalities.

THEOREM 3.1. Assume that

(i) W is an L -prepared matrix of class $\mathfrak{D}_L$ such that:

- (a) $(\Delta^j W^T) \sum_{p=j}^m \Delta^{p-j} (A_p \Delta^p W) \geq 0$, $0 \leq j \leq m-1$, in G ,
- (b) $(\Delta^{j-1} W^T) \sum_{p=j}^m \Delta^{p-j} (A_p \Delta^p W) \leq 0$, $2 \leq j \leq m$, in G ,
- (c) $W^T \sum_{p=1}^m \Delta^{p-1} (A_p \Delta^p W) < 0$ in G ,

(ii) there exists a nontrivial $u \in C^{2m}(\bar{G})$ with range in H such that $D^\alpha u = 0$ on ∂G , $|\alpha| \leq 2m-2$, and $F[u, W] \leq 0$.

Then, at least one of the matrices $W, \Delta W, \dots, \Delta^{m-1} W$ must be singular at some point of $\bar{G}$.

PROOF. Suppose to the contrary that $\Delta^j W$, $0 \leq j \leq m-1$, are nonsingular in $\bar{G}$. Then, Lemma 3.1 implies that the equality (16) holds. By the hypothesis

(i) (a) both $\int_G u^T L[W] W^{-1} u \, dx$ and $Q[u, W]$ are nonnegative. Since u is nontrivial and W is nonsingular in $\bar{G}$, it is easy to see that $D_i u - (D_i W) W^{-1} u = W D_i (W^{-1} u) \neq 0$, $1 \leq i \leq n$, in G . In view of this fact and the hypothesis (i) (b)–(c) we see that $J[u, W] > 0$. It follows that $F[u, W] > 0$, which contradicts the inequality $F[u, W] \leq 0$.

The following comparison theorem is "strong" in the sense that the conclusion applies to G rather than $\bar{G}$.

THEOREM 3.2 (Strong Comparison Theorem). *Assume that $\partial G \in C^{2m}$ and that*

(i) *W is an L -prepared matrix of class $\mathfrak{D}_L$ such that:*

$$(a) \quad (\Delta^j W^T) \sum_{p=j}^m \Delta^{p-j} (A_p \Delta^p W) \geq 0, \quad 0 \leq j \leq m-1, \quad \text{in } G,$$

$$(b) \quad (\Delta^{j-1} W^T) \sum_{p=j}^m \Delta^{p-j} (A_p \Delta^p W) \leq 0, \quad 2 \leq j \leq m, \quad \text{in } G,$$

$$(c) \quad W^T \sum_{p=1}^m \Delta^{p-1} (A_p \Delta^p W) < 0 \quad \text{in } G,$$

$$(d) \quad \Delta^j W, \quad 1 \leq j \leq m-1, \quad \text{are nonsingular in } \bar{G},$$

(ii) *there exists a nontrivial $u \in \mathfrak{D}_L$ such that:*

$$D^\alpha u = 0 \quad \text{on } \partial G, \quad |\alpha| \leq 2m-1,$$

$$\int_G u^T l[u] dx \leq 0,$$

$$f[u] \geq F[u, W].$$

Then, W must be singular at some point of G , unless $u = W\gamma$, where γ is a nonzero constant vector.

PROOF. The limit argument due to Swanson [34] will be adapted. Suppose to the contrary that W is nonsingular in G . Then, there exists a unique $v \in C^{2m}(G)$ such that $u = Wv$ in G . Since $\partial G \in C^{2m}$, $u \in \mathfrak{D}_L$ and $D^\alpha u = 0$ on ∂G , $|\alpha| \leq 2m-1$, it follows that u belongs to the Sobolev space $\dot{H}_{2m}(G)$ which is the closure in the norm

$$(18) \quad \|u\|_{2m} = \left(\int_G \sum_{|\alpha| \leq 2m} |D^\alpha u|^2 dx \right)^{1/2}$$

of the class $C_0^\infty(G)$ of infinitely differentiable vector functions with compact support in G . (See e.g. Agmon [1].)

Let $\{u_\nu\}$ be a sequence of functions in $C_0^\infty(G)$ converging to u in the norm (18). From Lemma 3.1 we see that

$$(19) \quad F[u_\nu, W] \geq J[u_\nu, W] \geq 0.$$

Since A_k , $0 \leq k \leq m$, are uniformly bounded in G , there exists a positive constant K_1 such that

$$|F[u_\nu, W] - F[u, W]| \leq K_1 (\|u_\nu\|_{2m} + \|u\|_{2m}) \|u_\nu - u\|_{2m}.$$

Since $\|u_\nu - u\|_{2m} \rightarrow 0$ as $\nu \rightarrow \infty$, it follows from (19) that

$$f[u] \geq F[u, W] \geq 0.$$

On the other hand, the hypothesis (ii) implies that $f[u] \leq 0$. Thus we have a contradiction unless $F[u, W] = 0$.

If $F[u, W] = 0$, then let B be a ball with $\bar{B} \subset G$ and define

$$\begin{aligned} \tilde{J}_B[u_\nu, W] \equiv & -2 \int_B \sum_{i=1}^n (D_i u_\nu - (D_i W) v_\nu)^T \left(\sum_{p=1}^m \Delta^{p-1} (A_p \Delta^p W) W^{-1} \right) \\ & \times (D_i u_\nu - (D_i W) v_\nu) dx, \end{aligned}$$

where $v_\nu = W^{-1}u_\nu$. Applying the Schwarz inequality, we obtain the following estimate

$$\begin{aligned} |\tilde{J}_B[u_\nu, W] - \tilde{J}_B[u, W]| \leq & K_2 (\|u_\nu - u\|_{2m, B} + \|v_\nu - v\|_{2m, B}) \\ & \times (\|u_\nu\|_{2m, B} + \|v_\nu\|_{2m, B} + \|u\|_{2m, B} + \|v\|_{2m, B}), \end{aligned}$$

where K_2 is a positive constant and the subscript B indicates the integrals involved in the norm (18) are evaluated over B instead of G . It is easy to see that $\|v_\nu - v\|_{2m, B} \rightarrow 0$ as $\|u_\nu - u\|_{2m, B} \rightarrow 0$. Hence, $\tilde{J}_B[u_\nu, W] \rightarrow \tilde{J}_B[u, W]$ as $\nu \rightarrow \infty$. From Lemma 3.1 we have

$$F[u_\nu, W] \geq \tilde{J}_B[u_\nu, W] \geq 0.$$

Since $F[u_\nu, W] \rightarrow F[u, W] = 0$ as $\nu \rightarrow \infty$, it follows that $\tilde{J}_B[u, W] = 0$, and consequently, $D_i(W^{-1}u) \equiv 0$ in B , $1 \leq i \leq n$. Since B is arbitrary, there exists a nonzero constant vector γ such that $W^{-1}u = \gamma$ in G and hence in $\bar{G}$ by continuity. This proves the theorem.

We conclude by observing that the Picone identity (Lemma 3.2) can be applied to obtain a lower bound for the first eigenvalue of the linear eigenvalue problem

$$(20) \quad \begin{aligned} l[u] &= \lambda u && \text{in } G, \\ D^\alpha u &= 0 && \text{on } \partial G, |\alpha| \leq 2m-1. \end{aligned}$$

THEOREM 3.3. *Let λ_1 be the first eigenvalue of the problem (20) and let u_1 be an associated eigenfunction. Suppose there exist a linear operator L of the form (2) and an L -prepared matrix $W \in \mathfrak{D}_L$ such that:*

$$\begin{aligned} & \Delta^j W, 0 \leq j \leq m-1, \text{ are nonsingular in } \bar{G}, \\ & (\Delta^j W^T) \sum_{p=j}^m \Delta^{p-j} (A_p \Delta^p W) \geq 0, 1 \leq j \leq m-1, \text{ in } G, \\ & (\Delta^{j-1} W^T) \sum_{p=j}^m \Delta^{p-j} (A_p \Delta^p W) \leq 0, 1 \leq j \leq m, \text{ in } G. \end{aligned}$$

If

$$V[u_1] \equiv \int_G \sum_{k=0}^m (\Delta^k u_1^T)(a_k - A_k) \Delta^k u_1 \, dx \geq 0,$$

then

$$\lambda_1 \geq \inf_{\|u\|_0=1} \int_G u^T L[W] W^{-1} u \, dx.$$

REMARK. For comparison theorems for elliptic equations and related results we refer the reader to the papers [2–12, 14, 17–22, 24, 25, 27, 30, 31, 33–36, 39].

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DEPARTMENT OF MATHEMATICS
HIROSHIMA UNIVERSITY
730 HIROSHIMA-SHI