On the seventh Lie dimension subgroups

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§1. Introduction

Let G be a group with the lower central series $G = G_1 \supset G_2 \supset \cdots \supset G_n \supset G_{n+1} \supset \cdots$. Let $\mathbf{Z}G$ be the group ring of G over $\mathbf{Z}$, the ring of rational integers, and $\Delta(G)$ the augmentation ideal of $\mathbf{Z}G$. Define the Lie powers of $\Delta(G)$ as $\Delta^{(1)}(G) = \Delta(G)$ and, inductively, $\Delta^{(n)}(G) = (\Delta^{(n-1)}(G), \Delta(G))\mathbf{Z}G$, where $(\Delta^{(n-1)}(G), \Delta(G))$ is the Lie bracket defined by

$$(\alpha, \beta) = \alpha\beta - \beta\alpha, \quad \alpha \in \Delta^{(n-1)}(G), \quad \beta \in \Delta(G).$$

The Lie powers $\Delta^{(n)}(G)$ of $\Delta(G)$ define the n -th Lie dimension subgroup $D_{(n)}(G)$ of G :

$$D_{(n)}(G) = G \cap (1 + \Delta^{(n)}(G)).$$

Sandling [4] showed $D_{(n)}(G) = G_n$ for any n with $1 \leq n \leq 6$, and Hurley and Sehgal [2] constructed a group G such that $D_{(n)}(G) \neq G_n$ for all n with $n \geq 9$ by modifying the Gupta group [1], and hence the question remains open for $n = 7$ and 8.

In this paper we show that the exponent of $D_{(7)}(G)/G_7$ divides 2 for any finite group G .

§2. Notation and preliminary results

Let c, d be natural integers. We denote by (c, d) the greatest common divisor of c and d , and put

$$\binom{d}{c} = d(d-1) \cdots (d-c+1)/c(c-1) \cdots 1.$$

Let A be an abelian group. Then $\mathrm{Sp}^n(A) = A^{\otimes n}/J$, where $A^{\otimes n}$ is the n -th tensor product of A , and J is the subgroup of $A^{\otimes n}$ generated by all elements $x_1 \otimes x_2 \otimes \cdots \otimes x_n - x_{\sigma(1)} \otimes x_{\sigma(2)} \otimes \cdots \otimes x_{\sigma(n)}$ for $x_1, x_2, \dots, x_n \in A$, and $\sigma \in S_n$, the symmetric group of degree n . Denote by $x_1 \vee x_2 \vee \cdots \vee x_n$ the class of $\mathrm{Sp}^n(A)$

to which $x_1 \otimes x_2 \otimes \cdots \otimes x_n$ belongs, and by $\bigvee^n x$ the class $x \vee x \vee \cdots \vee x$ (n times). In particular, put $\text{Sp}^0(A) = \mathbf{Z}$.

Following Losey [3], we introduce a uniqueness basis for a finite group. Let G be a finite group with the lower central series $G = G_1 \supset G_2 \supset \cdots \supset G_c \supset G_{c+1} = 1$. The lower central series induces a new weight function w^* on $G' = G_2$: for any element g of G' ,

$$w^*(g) = \begin{cases} k-1 & (g \in G_k - G_{k+1}) \\ \infty & (g = 1). \end{cases}$$

We note $w^*(g) \geq 1$ for any $g \in G'$. If $g \neq 1$ is an element of G' , define $o^*(g)$ to be the order of the coset $\bar{g} = gG_{w^*(g)+2}$ in $G_{w^*(g)+1}/G_{w^*(g)+2}$. Each quotient group G_i/G_{i+1} is a finite abelian group whose operation is denoted additively, and hence there exist elements $x_{i1}, x_{i2}, \dots, x_{i\lambda(i)}$ with $\bar{x}_{ij} = x_{ij}G_{i+1}$ such that any element $\bar{g} \in G_i/G_{i+1}$ can be uniquely written in the form

$$\bar{g} = u(1)\bar{x}_{i1} + u(2)\bar{x}_{i2} + \cdots + u(\lambda(i))\bar{x}_{i\lambda(i)},$$

where $0 \leq u(j) < o^*(x_{ij})$ for all j . Moreover, we choose x_{ij} so that $o^*(x_{ij})$ divides $o^*(x_{ij+1})$ for all j . Set

$$\Phi = \{x_{ij} \mid i = 2, 3, \dots, c; j = 1, 2, \dots, \lambda(i)\}.$$

Order Φ by setting $x_{ij} < x_{kl}$ if $i < k$ or $i = k$ and $j < l$. Then every element $g \in G'$ can be uniquely written in the form

$$(\#) \quad g = \prod_{i=2}^c x_{i1}^{\alpha_{i1}} x_{i2}^{\alpha_{i2}} \cdots x_{i\lambda(i)}^{\alpha_{i\lambda(i)}}$$

for some integers α_{ij} with $0 \leq \alpha_{ij} < o^*(x_{ij})$, $2 \leq i \leq c$, $1 \leq j \leq \lambda(i)$, where $\prod$ runs in the order of increasing i from left to right. Put $m = \sum_{i=2}^c \lambda(i)$, and consider an m -sequence $\alpha = (\alpha_{ij}) = (\alpha_{21}, \dots, \alpha_{2\lambda(2)}, \alpha_{31}, \dots, \alpha_{3\lambda(3)}, \dots, \alpha_{c1}, \dots, \alpha_{c\lambda(c)})$ for non-negative integers α_{ij} . The set of all m -sequences is naturally well ordered, that is, we define $\alpha = (\alpha_{ij}) < \beta = (\beta_{ij})$ if there exist k and l with $2 \leq k \leq c$ and $1 \leq j \leq \lambda(k)$ such that $\alpha_{ij} = \beta_{ij}$ ($2 \leq i \leq k-1$, $1 \leq j \leq \lambda(i)$), $\alpha_{kj} = \beta_{kj}$ ($1 \leq j \leq l-1$), and $\alpha_{kl} < \beta_{kl}$. An m -sequence $\alpha = (\alpha_{ij})$ is *basic* if $0 \leq \alpha_{ij} < o^*(x_{ij})$ for all i and j . It follows from the uniqueness of the expression (#) that there is a bijection between the elements of G' and the basic m -sequences. The weight W^* of an m -sequence $\alpha = (\alpha_{ij})$ is defined to be

$$W^*(\alpha) = \sum_{\substack{2 \leq i \leq c \\ 1 \leq j \leq \lambda(i)}} w^*(x_{ij}) \alpha_{ij} = \sum_{2 \leq i \leq c} (i-1) \left(\sum_{1 \leq j \leq \lambda(i)} \alpha_{ij} \right).$$

Given an m -sequence $\alpha = (\alpha_{ij})$, we define the proper product $P(\alpha) \in \mathbf{Z}G$ by

$$P(\alpha) = \prod_{i=2}^c (x_{i1} - 1)^{\alpha_{i1}} (x_{i2} - 1)^{\alpha_{i2}} \cdots (x_{i\lambda(i)} - 1)^{\alpha_{i\lambda(i)}},$$

where $\prod$ runs in the order of increasing i from left to right. If α is a basic m -sequence, then $P(\alpha)$ is called a *basic product*. We introduce notation for cutting an element g of G' modulo the i -th subgroup G_i . Let $P_i(g)$ be the product of weight $i - 1$ in the expression ($\#$):

$$P_i(g) = x_{i1}^{\alpha_{i1}} x_{i2}^{\alpha_{i2}} \cdots x_{i\lambda(i)}^{\alpha_{i\lambda(i)}},$$

and $\bar{P}_i(g)$ the coset of $P_i(g)$ modulo G_{i+1} :

$$\bar{P}_i(g) = \sum_{j=1}^{\lambda(i)} \alpha_{ij} \bar{x}_{ij} \in G_i/G_{i+1}.$$

On the other hand, we put

$$W_{(m)} = W_{(m)}(G) = \sum_{i=1}^m \bigotimes \mathrm{Sp}^{a_i}(G_i/G_{i+1}),$$

where $\sum$ runs over all non-negative integers $a_1, a_2, \dots, a_m$ such that $\sum_{i=2}^m (i-1)a_i = m - 1$. In particular,

$$W_{(4)} = \mathrm{Sp}^3(G_2/G_3) \oplus (G_2/G_3 \otimes G_3/G_4) \oplus G_4/G_5$$

$$W_{(5)} = \mathrm{Sp}^4(G_2/G_3) \oplus (\mathrm{Sp}^2(G_2/G_3) \otimes G_3/G_4) \oplus (G_2/G_3 \otimes G_4/G_5) \\ \oplus \mathrm{Sp}^2(G_3/G_4) \oplus G_5/G_6$$

$$W_{(6)} = \mathrm{Sp}^5(G_2/G_3) \oplus (\mathrm{Sp}^3(G_2/G_3) \otimes G_3/G_4) \oplus (\mathrm{Sp}^2(G_2/G_3) \otimes G_4/G_5) \\ \oplus (G_2/G_3 \otimes \mathrm{Sp}^2(G_3/G_4)) \oplus (G_2/G_3 \otimes G_5/G_6) \\ \oplus (G_3/G_4 \otimes G_4/G_5) \oplus G_6/G_7.$$

We fix the set of representatives t of G with respect to $G' = G_2$, namely

$$G = \bigcup_{t \in T} tG_2.$$

Denote $a(ij) = o^*(x_{ij})$ ($2 \leq i \leq 6, 1 \leq j \leq \lambda(i)$). Moreover put

$$x_{ij}^{a(ij)} \equiv \prod_{k=1}^{\lambda(i+1)} x_{i+1k}^{a(ijk)} \cdot \prod_{l=1}^{\lambda(i+2)} x_{i+2l}^{a'(ijl)} \pmod{G_{i+3}}, \quad 2 \leq i \leq 5, \quad 1 \leq j \leq \lambda(i)$$

$$[x_{2i}^{a(2i)}, x_{2j}] \equiv \prod_{k=1}^{\lambda(5)} x_{5k}^{\alpha_k(ij)} \cdot \prod_{l=1}^{\lambda(6)} x_{6l}^{\alpha_l(ij)'} \pmod{G_7}, \quad 1 \leq i < j \leq \lambda(2)$$

$$[x_{3i}, x_{3j}] \equiv \prod_{k=1}^{\lambda(6)} x_{6k}^{\delta_k^{(ij)}} \pmod{G_7}, \quad 1 \leq i < j \leq \lambda(3).$$

§3. Structure of $\Delta^{(n)}(G)$ for $n = 4, 5, 6$ and 7

Sandling [4] showed that, for $n \geq 2$,

$$\Delta^{(n)}(G) = \sum_j \prod \Delta(G_{n_j}) \mathbf{Z}G,$$

where $\sum$ runs over all n_j , $n \geq n_j \geq 2$ with $\sum_j (n_j - 1) = n - 1$.

On the other hand, we can show the following:

LEMMA 3.1. *Let G be a group with G_2/G_3 having finite exponent. Then we have for any $n \geq 2$,*

$$\text{rank}_{\mathbf{Z}} \Delta^{(n)}(G) = \text{rank}_{\mathbf{Z}} \Delta^{(2)}(G).$$

In particular, if G is a finite group, then for any $n \geq 2$,

$$\text{rank}_{\mathbf{Z}} \Delta^{(n)}(G) = |G| - |G/G_2| = |G/G_2|(|G_2| - 1).$$

PROOF. We can easily prove the lemma in three steps by a method similar to that in Losey [3]. The first step is to show that $e\Delta^{(n)}(G)$ is contained in $\Delta^{(n+1)}(G)$ for any $n \geq 2$ if the exponent e of G_2/G_3 is finite, and the second one is to show that $e\Delta^{(m)}(G)$ is contained in $\Delta^{(m+2)}(G)$ for any $m \geq 0$. The final step is to show $\text{rank}_{\mathbf{Z}} \Delta^{(n+2)}(G) = \text{rank}_{\mathbf{Z}} \Delta^{(2)}(G)$ by using the fact that $\Delta^{(2)}(G)$ is isomorphic to $e\Delta^{(2)}(G)$. Q.E.D.

Using the above lemma, we have the following Lemmas 3.2, 3.3 by the same methods as in the proofs of [3, Lemma 7], [5, Lemma 4] and [6, Lemma 3.5].

LEMMA 3.2. $\Delta^{(4)}(G)$ has a $\mathbf{Z}$ -free basis consisting of

- (1) $t(x_{ij} - 1)^{a(ij)}, \quad 2 \leq i \leq 3, \quad (i-1)a(ij) \geq 3, \quad 1 \leq j \leq \lambda(i)$
- (2) $ta(2i)(x_{2i} - 1)(x_{2j} - 1), \quad 1 \leq i \leq j \leq \lambda(2)$
- (3) $tP(\alpha), \quad \alpha : \text{basic}, \quad W^*(\alpha) \geq 3,$

for all $t \in T$.

LEMMA 3.3. For $u = 5, 6$ and 7 , $\Delta^{(u)}(G)$ has a system of $\mathbf{Z}$ -generators consisting of

- (1) $t(x_{ij} - 1)^{a(ij)}, \quad 2 \leq i \leq u - 1, \quad (i - 1)a(ij) \geq u - 1, \quad 1 \leq j \leq \lambda(i)$
- (2) $t(x_{ij} - 1)^{a(ij)}(x_{kl} - 1), \quad 2 \leq i \leq k, \quad i + k < u, \quad (i - 1)a(ij) + k \geq u,$
 $1 \leq j \leq \lambda(i), \quad 1 \leq l \leq \lambda(k)$
- (3) $t(x_{ij} - 1)(x_{kl} - 1)^{a(kl)}, \quad 2 \leq i \leq k, \quad i + k < u, \quad (k - 1)a(kl) + i \geq u,$
 $1 \leq j \leq \lambda(i), \quad 1 \leq l \leq \lambda(k)$
- (4) $t(x_{ij} - 1)^{a(ij)}(x_{kl} - 1)(x_{mn} - 1), \quad 2 \leq i \leq k \leq m, \quad i + k + m - 1 < u,$
 $(i - 1)a(ij) + k + m - 1 \geq u, \quad 1 \leq j \leq \lambda(i),$
 $1 \leq l \leq \lambda(k), \quad 1 \leq n \leq \lambda(m)$
- (5) $t(x_{ij} - 1)(x_{kl} - 1)^{a(kl)}(x_{mn} - 1), \quad 2 \leq i \leq k \leq m, \quad i + k + m - 1 < u,$
 $(k - 1)a(kl) + i + m - 1 \geq u, \quad 1 \leq j \leq \lambda(i),$
 $1 \leq l \leq \lambda(k), \quad 1 \leq n \leq \lambda(m)$
- (6) $t(x_{ij} - 1)(x_{kl} - 1)(x_{mn} - 1)^{a(mn)}, \quad 2 \leq i \leq k \leq m, \quad i + k + m - 1 < u,$
 $(m - 1)a(mn) + i + k - 1 \geq u, \quad 1 \leq j \leq \lambda(i),$
 $1 \leq l \leq \lambda(k), \quad 1 \leq n \leq \lambda(m)$
- (7) $t(x_{ij} - 1)^{a(ij)}(x_{kl} - 1)(x_{mn} - 1)(x_{pq} - 1), \quad 2 \leq i \leq k \leq m \leq p,$
 $i + k + m + p - 2 < u, \quad (i - 1)a(ij) + k + m + p - 2 \geq u,$
 $1 \leq j \leq \lambda(i), \quad 1 \leq l \leq \lambda(k), \quad 1 \leq n \leq \lambda(m), \quad 1 \leq q \leq \lambda(p)$
- (8) $t(x_{ij} - 1)(x_{kl} - 1)^{a(kl)}(x_{mn} - 1)(x_{pq} - 1), \quad 2 \leq i \leq k \leq m \leq p,$
 $i + k + m + p - 2 < u, \quad (k - 1)a(kl) + i + m + p - 2 \geq u,$
 $1 \leq j \leq \lambda(i), \quad 1 \leq l \leq \lambda(k), \quad 1 \leq n \leq \lambda(m), \quad 1 \leq q \leq \lambda(p)$
- (9) $t(x_{ij} - 1)(x_{kl} - 1)(x_{mn} - 1)^{a(mn)}(x_{pq} - 1), \quad 2 \leq i \leq k \leq m \leq p,$
 $i + k + m + p - 2 < u, \quad (m - 1)a(mn) + i + k + p - 2 \geq u,$
 $1 \leq j \leq \lambda(i), \quad 1 \leq l \leq \lambda(k), \quad 1 \leq n \leq \lambda(m), \quad 1 \leq q \leq \lambda(p)$
- (10) $t(x_{ij} - 1)(x_{kl} - 1)(x_{mn} - 1)(x_{pq} - 1)^{a(pq)}, \quad 2 \leq i \leq k \leq m \leq p,$
 $i + k + m + p - 2 < u, \quad (p - 1)a(pq) + i + k + m - 2 \geq u,$
 $1 \leq j \leq \lambda(i), \quad 1 \leq l \leq \lambda(k), \quad 1 \leq n \leq \lambda(m), \quad 1 \leq q \leq \lambda(p)$

- (11) $t(a(ij), a(kl))(x_{ij} - 1)(x_{kl} - 1), \quad 2 \leq i \leq k, \quad i + k = u,$
 $1 \leq j \leq \lambda(i), \quad 1 \leq l \leq \lambda(k)$
- (12) $t(a(ij), a(kl), a(mn))(x_{ij} - 1)(x_{kl} - 1)(x_{mn} - 1), \quad 2 \leq i \leq k \leq m,$
 $i + k + m - 1 = u, \quad 1 \leq j \leq \lambda(i), \quad 1 \leq l \leq \lambda(k), \quad 1 \leq n \leq \lambda(m)$
- (13) $t(a(ij), a(kl), a(mn), a(pq))(x_{ij} - 1)(x_{kl} - 1)(x_{mn} - 1)(x_{pq} - 1),$
 $2 \leq i \leq k \leq m \leq p, \quad i + k + m + p - 2 = u,$
 $1 \leq j \leq \lambda(i), \quad 1 \leq l \leq \lambda(k), \quad 1 \leq n \leq \lambda(m), \quad 1 \leq q \leq \lambda(p)$
- (14) $t(a(ij), a(kl), a(mn), a(pq), a(rs))(x_{ij} - 1)(x_{kl} - 1)(x_{mn} - 1)(x_{pq} - 1)$
 $\times (x_{rs} - 1),$
 $2 \leq i \leq k \leq m \leq p \leq r, \quad i + k + m + p + r - 3 = u,$
 $1 \leq j \leq \lambda(i), \quad 1 \leq l \leq \lambda(k), \quad 1 \leq n \leq \lambda(m),$
 $1 \leq q \leq \lambda(p), \quad 1 \leq s \leq \lambda(r)$
- (15) $tP(\alpha), \quad \alpha : \text{basic}, \quad W^*(\alpha) \geq u - 1,$
for all $t \in T$.

§4. Exponent of $D_{(\tau)}(G)/G_{\tau}$

We denote by $W_{(m)}^* = W_{(m)}^*(G)$ the free group generated by a basis of $W_{(m)} = W_{(m)}(G)$. Note that an element in such a basis will have finite order in $W_{(m)}$, but infinite order in $W_{(m)}^*$.

For simplicity in this section, put $a(i) = a(2i)$ ($1 \leq i \leq \lambda(2)$), $b(i) = a(3i)$ ($1 \leq i \leq \lambda(3)$), $c(i) = a(4i)$ ($1 \leq i \leq \lambda(4)$), $d(i) = a(5i)$ ($1 \leq i \leq \lambda(5)$), $a_{ik} = a(2ik)$ ($1 \leq i \leq \lambda(2), 1 \leq k \leq \lambda(3)$), $b_{ik} = a(3ik)$ ($1 \leq i \leq \lambda(3), 1 \leq k \leq \lambda(4)$), $c_{ik} = a(4ik)$ ($1 \leq i \leq \lambda(4), 1 \leq k \leq \lambda(5)$), $d_{ik} = a(5ik)$ ($1 \leq i \leq \lambda(5), 1 \leq k \leq \lambda(6)$).

Now, we define a homomorphism $\psi : \Delta^{(4)}(G) \longrightarrow W_{(4)}^* \oplus W_{(5)}^* \oplus W_{(6)}$ by

- 1) $\psi(t(x_{2i} - 1)^{a(i)}) = \begin{cases} \bigvee^{a(i)} \bar{x}_{2i}, & 3 \leq a(i) \leq 5, \\ 0, & a(i) \geq 6 \end{cases}, \quad 1 \leq i \leq \lambda(2)$
- 2) $\psi(t(x_{3i} - 1)^{b(i)}) = \begin{cases} \bigvee^{b(i)} x_{3i}, & b(i) = 2, \\ 0, & b(i) \geq 3 \end{cases}, \quad 1 \leq i \leq \lambda(3)$
- 3) $\psi(ta(i)(x_{2i} - 1)(x_{2j} - 1))$

- $$\begin{aligned}
&= - \sum_{h=2}^4 \binom{a(i)}{h} \left(\bigvee^h \bar{x}_{2i} \vee \bar{x}_{2j} \right) + (\bar{x}_{2j} \otimes \overline{x_{2i}^{a(i)}}) + (\bar{x}_{2j} \otimes \sum_{h=4}^5 \bar{P}_h(x_{2i}^{a(i)})) \\
&\quad + \left(\bar{x}_{2j} \otimes \sum_{k=1}^{\lambda(3)} \binom{a_{ik}}{2} \bigvee^2 \bar{x}_{3k} \right) + \left(\bar{x}_{2j} \otimes \sum_{1 \leq k < l \leq \lambda(3)} a_{ik} a_{il} (\bar{x}_{3k} \vee \bar{x}_{3l}) \right) \\
&\quad + [\overline{x_{2i}^{a(i)}}, x_{2j}] + \bar{P}_6([x_{2i}^{a(i)}, x_{2j}]) + (\bar{x}_{2j} \otimes [\overline{x_{2i}^{a(i)}}, x_{2j}]), \quad 1 \leq i \leq j \leq \lambda(2)
\end{aligned}$$
- 4) $\psi(t(x_{2i} - 1)(x_{2j} - 1)(x_{2k} - 1)) = \bar{x}_{2i} \vee \bar{x}_{2j} \vee \bar{x}_{2k},$
 $(x_{2i} - 1)(x_{2j} - 1)(x_{2k} - 1) : \text{basic}, \quad 1 \leq i \leq j \leq k \leq \lambda(2)$
- 5) $\psi(t(x_{2i} - 1)(x_{3j} - 1)) = \bar{x}_{2i} \otimes \bar{x}_{3j}, \quad 1 \leq i \leq \lambda(2), \quad 1 \leq j \leq \lambda(3)$
- 6) $\psi(t(x_{4i} - 1)) = \bar{x}_{4i}, \quad 1 \leq i \leq \lambda(4)$
- 7) $\psi(t(x_{2i} - 1)(x_{2j} - 1)(x_{2k} - 1)(x_{2l} - 1)) = \bar{x}_{2i} \vee \bar{x}_{2j} \vee \bar{x}_{2k} \vee \bar{x}_{2l},$
 $(x_{2i} - 1)(x_{2j} - 1)(x_{2k} - 1)(x_{2l} - 1) : \text{basic}, \quad 1 \leq i \leq j \leq k \leq l \leq \lambda(2)$
- 8) $\psi(t(x_{2i} - 1)(x_{2j} - 1)(x_{3k} - 1)) = (\bar{x}_{2i} \vee \bar{x}_{2j}) \otimes \bar{x}_{3k},$
 $(x_{2i} - 1)(x_{2j} - 1) : \text{basic}, \quad 1 \leq i \leq j \leq \lambda(2), \quad 1 \leq k \leq \lambda(3)$
- 9) $\psi(t(x_{2i} - 1)(x_{4j} - 1)) = \bar{x}_{2i} \otimes \bar{x}_{4j}, \quad 1 \leq i \leq \lambda(2), \quad 1 \leq j \leq \lambda(4)$
- 10) $\psi(t(x_{3i} - 1)(x_{3j} - 1)) = \bar{x}_{3i} \vee \bar{x}_{3j}, \quad (x_{3i} - 1)(x_{3j} - 1) : \text{basic},$
 $1 \leq i \leq j \leq \lambda(3)$
- 11) $\psi(t(x_{5i} - 1)) = \bar{x}_{5i}, \quad 1 \leq i \leq \lambda(5)$
- 12) $\psi(t(x_{2i} - 1)(x_{2j} - 1)(x_{2k} - 1)(x_{2l} - 1)(x_{2m} - 1))$
 $= \bar{x}_{2i} \vee \bar{x}_{2j} \vee \bar{x}_{2k} \vee \bar{x}_{2l} \vee \bar{x}_{2m},$
 $(x_{2i} - 1)(x_{2j} - 1)(x_{2k} - 1)(x_{2l} - 1)(x_{2m} - 1) : \text{basic},$
 $1 \leq i \leq j \leq k \leq l \leq m \leq \lambda(2)$
- 13) $\psi(t(x_{2i} - 1)(x_{2j} - 1)(x_{2k} - 1)(x_{3l} - 1)) = (\bar{x}_{2i} \vee \bar{x}_{2j} \vee \bar{x}_{2k}) \otimes \bar{x}_{3l},$
 $(x_{2i} - 1)(x_{2j} - 1)(x_{2k} - 1) : \text{basic}, \quad 1 \leq i \leq j \leq k \leq \lambda(2), \quad 1 \leq l \leq \lambda(3)$
- 14) $\psi(t(x_{2i} - 1)(x_{2j} - 1)(x_{4k} - 1)) = (\bar{x}_{2i} \vee \bar{x}_{2j}) \otimes \bar{x}_{4k},$
 $(x_{2i} - 1)(x_{2j} - 1) : \text{basic}, \quad 1 \leq i \leq j \leq \lambda(2), \quad 1 \leq k \leq \lambda(4)$
- 15) $\psi(t(x_{2i} - 1)(x_{3j} - 1)(x_{3k} - 1)) = \bar{x}_{2i} \otimes (\bar{x}_{3j} \vee \bar{x}_{3k}),$
 $(x_{3j} - 1)(x_{3k} - 1) : \text{basic}, \quad 1 \leq i \leq \lambda(2), \quad 1 \leq j \leq k \leq \lambda(3)$

$$16) \quad \psi(t(x_{2i} - 1)(x_{5j} - 1)) = \bar{x}_{2i} \otimes \bar{x}_{5j}, \quad 1 \leq i \leq \lambda(2), \quad 1 \leq j \leq \lambda(5)$$

$$17) \quad \psi(t(x_{3i} - 1)(x_{4j} - 1)) = \bar{x}_{3i} \otimes \bar{x}_{4j}, \quad 1 \leq i \leq \lambda(3), \quad 1 \leq j \leq \lambda(4)$$

$$18) \quad \psi(t(x_{6i} - 1)) = \bar{x}_{6i}, \quad 1 \leq i \leq \lambda(6)$$

$$19) \quad \psi(tP(\alpha)) = 0, \quad \alpha : \text{basic}, \quad W^*(\alpha) \geq 6.$$

We denote by R the subgroup of $W_{(4)}^* \oplus W_{(5)}^* \oplus W_{(6)}^*$ generated by the elements of the following types (I)~(IX):

$$(I) \quad \gamma_i = c(i)\bar{x}_{4i} - \overline{x_{4i}^{c(i)}} - \bar{P}_6(x_{4i}^{c(i)}), \quad 1 \leq i \leq \lambda(4)$$

$$(II) \quad \gamma'_i = d(i)\bar{x}_{5i} - \overline{x_{5i}^{d(i)}}, \quad 1 \leq i \leq \lambda(5)$$

$$(III) \quad \begin{aligned} \gamma_{ij} = & \sum_{h=2}^4 \left\{ \frac{a(j)}{a(i)} \binom{a(i)}{h} (\vee^h \bar{x}_{2i} \vee \bar{x}_{2j}) - \binom{a(j)}{h} (\bar{x}_{2i} \vee^h \bar{x}_{2j}) \right\} \\ & - \frac{a(j)}{a(i)} (\bar{x}_{2j} \otimes \overline{x_{2i}^{a(i)}}) - \frac{a(j)}{a(i)} \sum_{h=4}^5 (\bar{x}_{2j} \otimes \bar{P}_h(x_{2i}^{a(i)})) \\ & - \frac{a(j)}{a(i)} \left(\bar{x}_{2j} \otimes \sum_{k=1}^{\lambda(2)} \binom{a_{ik}}{2} \vee^2 \bar{x}_{3k} \right) \\ & - \frac{a(j)}{a(i)} \left(\bar{x}_{2j} \otimes \sum_{1 \leq k < l \leq \lambda(3)} a_{ik} a_{il} (\bar{x}_{3k} \vee \bar{x}_{3l}) \right) + (\bar{x}_{2i} \otimes \overline{x_{2j}^{a(j)}}) \\ & + \sum_{h=4}^5 (\bar{x}_{2i} \otimes \bar{P}_h(x_{2j}^{a(j)})) + \left(\bar{x}_{2i} \otimes \sum_{1 \leq k < l \leq \lambda(3)} a_{jk} a_{jl} (\bar{x}_{3k} \vee \bar{x}_{3l}) \right) \\ & + \left(\bar{x}_{2i} \otimes \sum_{k=1}^{\lambda(3)} \binom{a_{jk}}{2} \vee^2 \bar{x}_{3k} \right) - \frac{a(j)}{a(i)} \left\{ \overline{[x_{2i}^{a(i)}, x_{2j}]} \right. \\ & \left. + \bar{P}_6([x_{2i}^{a(i)}, x_{2j}]) + (\bar{x}_{2j} \otimes \overline{[x_{2i}^{a(i)}, x_{2j}]} \right\}, \quad 1 \leq i \leq j \leq \lambda(2) \end{aligned}$$

$$(IV) \quad \begin{aligned} \gamma_{ijk} = & \sum_{h=1}^3 \binom{a(i)}{h} (\vee^h \bar{x}_{2i} \vee \bar{x}_{2j} \vee \bar{x}_{2k}) - ((\bar{x}_{2j} \vee \bar{x}_{2k}) \otimes \overline{x_{2i}^{a(i)}}) \\ & - ((\bar{x}_{2j} \vee \bar{x}_{2k}) \otimes \bar{P}_4(x_{2i}^{a(i)})) - (\bar{x}_{2j} \otimes \overline{[x_{2i}^{a(i)}, x_{2k}]}) \\ & - (\bar{x}_{2k} \otimes \overline{[x_{2i}^{a(i)}, x_{2j}]}), \quad 1 \leq i \leq j \leq k \leq \lambda(2) \end{aligned}$$

$$\begin{aligned}
\tilde{\gamma}_{ijk} &= \sum_{h=1}^3 \binom{a(j)}{h} (\bar{x}_{2i} \overset{h}{\vee} \bar{x}_{2j} \vee \bar{x}_{2k}) - ((\bar{x}_{2i} \vee \bar{x}_{2k}) \otimes \overline{x_{2j}^{a(j)}}) \\
&\quad - ((\bar{x}_{2i} \vee \bar{x}_{2k}) \otimes \overline{P_4(x_{2j}^{a(j)})}) - (\bar{x}_{2i} \otimes \overline{[x_{2j}^{a(j)}, x_{2k}]}), \\
&\quad 1 \leq i < j \leq k \leq \lambda(2) \\
\tilde{\gamma}_{ijk} &= \sum_{h=1}^3 \binom{a(k)}{h} (\bar{x}_{2i} \vee \bar{x}_{2j} \overset{h}{\vee} \bar{x}_{2k}) - ((\bar{x}_{2i} \vee \bar{x}_{2j}) \otimes \overline{x_{2k}^{a(k)}}) \\
&\quad - ((\bar{x}_{2i} \vee \bar{x}_{2j}) \otimes \overline{P_4(x_{2k}^{a(k)})}), \quad 1 \leq i \leq j < k \leq \lambda(2) \\
(V) \quad \varepsilon_{ijkl} &= \sum_{h=1}^2 \binom{a(i)}{h} (\overset{h}{\vee} \bar{x}_{2i} \vee \bar{x}_{2j} \vee \bar{x}_{2k} \vee \bar{x}_{2l}) - ((\bar{x}_{2j} \vee \bar{x}_{2k} \vee \bar{x}_{2l}) \otimes \overline{x_{2i}^{a(i)}}), \\
&\quad 1 \leq i \leq j \leq k \leq l \leq \lambda(2) \\
\tilde{\varepsilon}_{ijkl} &= \sum_{h=1}^2 \binom{a(j)}{h} (\bar{x}_{2i} \overset{h}{\vee} \bar{x}_{2j} \vee \bar{x}_{2k} \vee \bar{x}_{2l}) - ((\bar{x}_{2i} \vee \bar{x}_{2k} \vee \bar{x}_{2l}) \otimes \overline{x_{2j}^{a(j)}}), \\
&\quad 1 \leq i < j \leq k \leq l \leq \lambda(2) \\
\tilde{\tilde{\varepsilon}}_{ijkl} &= \sum_{h=1}^2 \binom{a(k)}{h} (\bar{x}_{2i} \vee \bar{x}_{2j} \overset{h}{\vee} \bar{x}_{2k} \vee \bar{x}_{2l}) - ((\bar{x}_{2i} \vee \bar{x}_{2j} \vee \bar{x}_{2l}) \otimes \overline{x_{2k}^{a(k)}}), \\
&\quad 1 \leq i \leq j < k \leq l \leq \lambda(2) \\
\tilde{\tilde{\tilde{\varepsilon}}}_{ijkl} &= \sum_{h=1}^2 \binom{a(l)}{h} (\bar{x}_{2i} \vee \bar{x}_{2j} \vee \bar{x}_{2k} \overset{h}{\vee} \bar{x}_{2l}) - ((\bar{x}_{2i} \vee \bar{x}_{2j} \vee \bar{x}_{2k}) \otimes \overline{x_{2l}^{a(l)}}), \\
&\quad 1 \leq i \leq j \leq k < l \leq \lambda(2) \\
(VI) \quad \delta_{ij} &= \sum_{h=1}^3 \binom{a(i)}{h} (\overset{h}{\vee} \bar{x}_{2i} \otimes \bar{x}_{3j}) - (\overline{x_{2i}^{a(i)}} \vee \bar{x}_{3j}) - (\bar{x}_{3j} \otimes \overline{P_4(x_{2i}^{a(i)})}) \\
&\quad - \sum_{1 \leq j < h \leq \lambda(3)} a_{ih} [\overline{x_{3h}}, \overline{x_{3j}}], \quad 1 \leq i \leq \lambda(2), \quad 1 \leq j \leq \lambda(3) \\
\tilde{\delta}_{ij} &= \sum_{h=1}^2 \binom{b(j)}{h} (\bar{x}_{2i} \otimes \overset{h}{\vee} \bar{x}_{3j}) - (\bar{x}_{2i} \otimes \overline{x_{3j}^{b(j)}}) - (\bar{x}_{2i} \otimes \overline{P_5(x_{3j}^{b(j)})}), \\
&\quad 1 \leq i \leq \lambda(2), \quad 1 \leq j \leq \lambda(3)
\end{aligned}$$

$$\begin{aligned}
\text{(VII)} \quad \theta_{ijk} &= \sum_{h=1}^2 \binom{a(i)}{h} ((\bigvee^h \bar{x}_{2i} \vee \bar{x}_{2j}) \otimes \bar{x}_{3k}) - (\bar{x}_{2j} \otimes (\bar{x}_{3k} \vee \overline{x_{2i}^{a(i)}})), \\
&1 \leq i \leq j \leq \lambda(2), \quad 1 \leq k \leq \lambda(3) \\
\tilde{\theta}_{ijk} &= \sum_{h=1}^2 \binom{a(j)}{h} ((\bar{x}_{2i} \bigvee^h \bar{x}_{2j}) \otimes \bar{x}_{3k}) - (\bar{x}_{2i} \otimes (\bar{x}_{3k} \vee \overline{x_{2j}^{a(j)}})), \\
&1 \leq i < j \leq \lambda(2), \quad 1 \leq k \leq \lambda(3) \\
\tilde{\tilde{\theta}}_{ijk} &= b(k)((\bar{x}_{2i} \vee \bar{x}_{2j}) \otimes \bar{x}_{3k}) - ((\bar{x}_{2i} \vee \bar{x}_{2j}) \otimes \overline{x_{3k}^{b(k)}}), \\
&1 \leq i \leq j \leq \lambda(2), \quad 1 \leq k \leq \lambda(3) \\
\text{(VIII)} \quad \mu_{ij} &= \sum_{h=1}^2 \binom{a(i)}{h} (\bigvee^h \bar{x}_{2i} \otimes \bar{x}_{4j}) - (\overline{x_{2i}^{a(i)}} \otimes \bar{x}_{4j}), \\
&1 \leq i \leq \lambda(2), \quad 1 \leq j \leq \lambda(4) \\
\tilde{\mu}_{ij} &= c(j)(\bar{x}_{2i} \otimes \bar{x}_{4j}) - (\bar{x}_{2i} \otimes \overline{x_{4j}^{c(j)}}), \quad 1 \leq i \leq \lambda(2), \quad 1 \leq j \leq \lambda(4) \\
\text{(IX)} \quad \eta_{ij} &= b(i)(x_{3i} \vee x_{3j}) - (x_{3j} \otimes \overline{x_{3i}^{b(i)}}), \quad 1 \leq i \leq j \leq \lambda(3) \\
\tilde{\eta}_{ij} &= b(j)(\bar{x}_{3i} \vee \bar{x}_{3j}) - (\bar{x}_{3i} \otimes \overline{x_{3j}^{b(j)}}), \quad 1 \leq i < j \leq \lambda(3).
\end{aligned}$$

We would like to prove that the additive abelian group $\Delta^{(4)}(G)/\Delta^{(7)}(G)$ is isomorphic to $\{W_{(4)}^* \oplus W_{(5)}^* \oplus W_{(6)}\}/R$.

For simplicity, we denote by N the additive subgroup of $\Delta^{(4)}(G)$ generated by all basic products $P(\alpha)$ with $W^*(\alpha) \geq 6$.

Then we have the following series of lemmas for the map ψ . We omit the straightforward proofs.

LEMMA 4.1. *For $i = 4, 5$ and 6 , and any $t \in T$, we have*

$$\psi(t(x_{ij} - 1)^{a(ij)}) \equiv 0 \pmod{R}, \quad 1 \leq j \leq \lambda(i).$$

LEMMA 4.2. *For $k = 3, 4$ and 5 and any $t \in T$, we have*

$$\psi\left(t \prod_{j=1}^k (x_{2i_j} - 1)\right) \equiv \bar{x}_{2i_1} \vee \bar{x}_{2i_2} \vee \cdots \vee \bar{x}_{2i_k} \pmod{R}, \quad 1 \leq i_1 \leq i_2 \leq \cdots \leq i_k \leq \lambda(2).$$

LEMMA 4.3. *For any i and j with $1 \leq i \leq j \leq \lambda(3)$, we have*

$$\psi(t(x_{3i} - 1)(x_{3j} - 1)) \equiv \bar{x}_{3i} \vee \bar{x}_{3j} \pmod{R}, \quad t \in T.$$

LEMMA 4.4. For $k = 3$ and 4 , any i, j with $1 \leq i \leq j \leq \lambda(2)$, and any l with $1 \leq l \leq \lambda(k)$, we have

$$\psi(t(x_{2i} - 1)(x_{2j} - 1)(x_{kl} - 1)) \equiv (\bar{x}_{2i} \vee \bar{x}_{2j}) \otimes \bar{x}_{kl} \pmod{R}, \quad t \in T.$$

LEMMA 4.5. For any i with $1 \leq i \leq \lambda(2)$, and any j, k with $1 \leq j \leq k \leq \lambda(3)$, we have

$$\psi(t(x_{2i} - 1)(x_{3j} - 1)(x_{3k} - 1)) \equiv \bar{x}_{2i} \otimes (\bar{x}_{3j} \vee \bar{x}_{3k}) \pmod{R}, \quad t \in T.$$

LEMMA 4.6. For any i, j, k with $1 \leq i \leq j \leq k \leq \lambda(2)$, and any l with $1 \leq l \leq \lambda(3)$, we have

$$\begin{aligned} \psi(t(x_{2i} - 1)(x_{2j} - 1)(x_{2k} - 1)(x_{3l} - 1)) \\ \equiv (\bar{x}_{2i} \vee \bar{x}_{2j} \vee \bar{x}_{2k}) \otimes \bar{x}_{3l} \pmod{R}, \quad t \in T. \end{aligned}$$

By considering the map ψ on each generator of $\Delta^{(7)}(G)$ in Lemma 3.3 with $u = 7$ and by using Lemmas 4.1~4.6, we can easily show the following:

LEMMA 4.7. $\psi(\Delta^{(7)}(G)) = R$.

By Lemma 4.7, the homomorphism $\psi : \Delta^{(4)}(G) \longrightarrow W_{(4)}^* \oplus W_{(5)}^* \oplus W_{(6)}$ induces a homomorphism

$$\psi^* : \Delta^{(4)}(G)/\Delta^{(7)}(G) \longrightarrow \{W_{(4)}^* \oplus W_{(5)}^* \oplus W_{(6)}\}/R$$

by $\psi^*(u + \Delta^{(7)}(G)) = \psi(u) + R$, $u \in \Delta^{(4)}(G)$.

Conversely, we define a homomorphism $\phi^* : \{W_{(4)}^* \oplus W_{(5)}^* \oplus W_{(6)}\}/R \rightarrow \Delta^{(4)}(G)/\Delta^{(7)}(G)$ by defining an element of $\Delta^{(4)}(G)/\Delta^{(7)}(G)$ on each of $\mathbf{Z}$ -basis of $W_{(4)}^*$, $W_{(5)}^*$, $W_{(6)}$ such that $\psi^* \cdot \phi^*$ is the identity map on $\{W_{(4)}^* \oplus W_{(5)}^* \oplus W_{(6)}\}/R$, and $\phi^* \cdot \psi^*$ is one on $\Delta^{(4)}(G)/\Delta^{(7)}(G)$ by Lemmas 4.1~4.6. Hence we get the following:

PROPOSITION 4.8. The additive group $\Delta^{(4)}(G)/\Delta^{(7)}(G)$ is isomorphic to $\{W_{(4)}^* \oplus W_{(5)}^* \oplus W_{(6)}\}/R = \{W_{(4)}^*(G) \oplus W_{(5)}^*(G) \oplus W_{(6)}(G)\}/R$.

To calculate the exponent of $D_{(7)}(G)/G_7$, we may assume $G_7 = 1$, namely, G has the lower central series

$$G = G_1 \supset G_2 \supset G_3 \supset G_4 \supset G_5 \supset G_6 \supset G_7 = 1.$$

We define a homomorphism $\Phi : G_4 \rightarrow \Delta^{(4)}(G)/\Delta^{(7)}(G)$ by $\Phi(x) = x - 1 + \Delta^{(7)}(G)$, $x \in G_4$. Then

$$\text{Ker } \Phi = G_4 \cap (1 + \Delta^{(7)}(G)) = G \cap (1 + \Delta^{(7)}(G)) = D_{(7)}(G),$$

since $D_{(4)}(G) = G_4$. Here if we identify $\Delta^{(4)}(G)/\Delta^{(7)}(G)$ with $\{W_{(4)}^* \oplus W_{(5)}^* \oplus W_{(6)}^*\}/R$ then we have

$$\text{Ker } \Phi = \{x \in G_4 \mid \psi(x-1) \in R\}.$$

Since G_4 is abelian, we may write the operation of G_4 additively. Let $x = \sum_{1 \leq i \leq \lambda(4)} r_i x_{4i} + \sum_{1 \leq j \leq \lambda(5)} s_j x_{5j} + \sum_{1 \leq k \leq \lambda(6)} t_k x_{6k}$ be any element of $\text{Ker } \Phi$. Then it follows that

$$x-1 \equiv \sum_i r_i (x_{4i}-1) + \sum_j s_j (x_{5j}-1) + \sum_k t_k (x_{6k}-1) \pmod{N},$$

and hence

$$\psi(x-1) = \sum_i r_i \bar{x}_{4i} + \sum_j s_j \bar{x}_{5j} + \sum_k t_k \bar{x}_{6k}.$$

On the other hand, since $\psi(x-1)$ belongs to R , $\psi(x-1)$ is a $\mathbf{Z}$ -linear combination of the elements of the relations (I)~(IX). Denote the coefficients of the relations (I)~(IX) by $u_i (1 \leq i \leq \lambda(4))$, $u'_i (1 \leq i \leq \lambda(5))$, $u_{ij} (1 \leq i \leq j \leq \lambda(2))$, $u_{ijk} (1 \leq i \leq j \leq k \leq \lambda(2))$, $\tilde{u}_{ijk} (1 \leq i < j \leq k \leq \lambda(2))$, $\tilde{\tilde{u}}_{ijk} (1 \leq i \leq j < k \leq \lambda(2))$, $u_{ijkl} (1 \leq i \leq j \leq k \leq l \leq \lambda(2))$, $\tilde{u}_{ijkl} (1 \leq i < j \leq k \leq l \leq \lambda(2))$, $\tilde{\tilde{u}}_{ijkl} (1 \leq i \leq j < k \leq l \leq \lambda(2))$, $\tilde{\tilde{\tilde{u}}}_{ijkl} (1 \leq i \leq j \leq k < l \leq \lambda(2))$, $v_{ij} (1 \leq i \leq \lambda(2), 1 \leq j \leq \lambda(3))$, $\tilde{v}_{ij} (1 \leq i \leq \lambda(2), 1 \leq j \leq \lambda(3))$, $v_{ijk} (1 \leq i \leq j \leq \lambda(2), 1 \leq k \leq \lambda(3))$, $\tilde{v}_{ijk} (1 \leq i < j \leq \lambda(2), 1 \leq k \leq \lambda(3))$, $\tilde{\tilde{v}}_{ijk} (1 \leq i \leq j \leq \lambda(2), 1 \leq k \leq \lambda(3))$, $w_{ij} (1 \leq i \leq \lambda(2), 1 \leq j \leq \lambda(4))$, $\tilde{w}_{ij} (1 \leq i \leq \lambda(2), 1 \leq j \leq \lambda(4))$, $w'_{ij} (1 \leq i \leq j \leq \lambda(3))$, $\tilde{w}'_{ij} (1 \leq i < j \leq \lambda(3))$, respectively.

Then comparing the coefficients of $\bar{x}_{4i} (1 \leq i \leq \lambda(4))$, $\bar{x}_{5j} (1 \leq j \leq \lambda(5))$ and $\bar{x}_{6k} (1 \leq k \leq \lambda(6))$, we have

$$\begin{aligned} r_i &= u_i c(i), \quad 1 \leq i \leq \lambda(4), \\ s_j &= - \sum_{1 \leq i \leq \lambda(4)} u_i c_{ij} + u'_j d(j) - \sum_{1 \leq h < i \leq \lambda(2)} u_{hi} \frac{a(i)}{a(h)} \alpha_j^{(hi)}, \quad 1 \leq j \leq \lambda(5) \\ t_k &= - \sum_{1 \leq i \leq \lambda(4)} u_i c'_{ik} - \sum_{1 \leq i \leq \lambda(5)} u'_i d_{ik} - \sum_{1 \leq i < j \leq \lambda(2)} u_{ij} \frac{a(j)}{a(i)} \alpha_k^{(ij)'} \\ &\quad - \sum_{\substack{1 \leq i \leq \lambda(2) \\ 1 \leq j \leq \lambda(3)}} v_{ij} \left(\sum_{j < h} a_{ih} \delta_k^{(hj)} \right), \quad 1 \leq k \leq \lambda(6) \end{aligned}$$

and hence

$$x = - \sum_{1 \leq i < j \leq \lambda(2)} u_{ij} \frac{a(j)}{a(i)} [x_{2i}^{a(i)}, x_{2j}] - \sum_{\substack{1 \leq i \leq \lambda(2) \\ 1 \leq j \leq \lambda(3)}} v_{ij} \left(\sum_{j < k} a_{ik} [x_{3k}, x_{3j}] \right).$$

Further, comparing the coefficients of $\sqrt[2]{\bar{x}_{2i} \vee \bar{x}_{2j}}$ ($1 \leq i \leq j \leq \lambda(2)$), $\bar{x}_{2i} \sqrt[2]{\bar{x}_{2j}}$ ($1 \leq i < j \leq \lambda(2)$), $\bar{x}_{2i} \otimes \bar{x}_{3k}$ ($1 \leq i \leq \lambda(2), 1 \leq k \leq \lambda(3)$), $\bar{x}_{2i} \otimes \bar{x}_{4l}$ ($1 \leq i \leq \lambda(2), 1 \leq l \leq \lambda(4)$), $\bar{x}_{3j} \vee \bar{x}_{3k}$ ($1 \leq j < k \leq \lambda(3)$), we have

$$\begin{aligned} 2 \sum_{i < j} u_{ij} \frac{a(j)}{a(i)} [x_{2i}^{a(i)}, x_{2j}] &= \sum_{i,k} v_{ik} [x_{2i}^{a(i)}, x_{3k}] \\ \sum_{i,k} v_{ij} [x_{2i}^{a(i)}, x_{3k}] &= \sum_{j < k} \left(\sum_i v_{ik} a_{ij} - \sum_i v_{ij} a_{ik} \right) [x_{3j}, x_{3k}] \\ &= -2 \sum_{j < k} \left(\sum_i v_{ij} a_{ik} \right) [x_{3j}, x_{3k}] \\ &= -2 \sum_{i,j} v_{ij} \left(\sum_{j < k} a_{ik} [x_{3j}, x_{3k}] \right). \end{aligned}$$

Therefore we have

$$\begin{aligned} 2x &= -2 \sum_{i < j} u_{ij} \frac{a(j)}{a(i)} [x_{2i}^{a(i)}, x_{2j}] - 2 \sum_{i,j} v_{ij} \left(\sum_{j < k} a_{ik} [x_{3k}, x_{3j}] \right) \\ &= - \sum_{i,k} v_{ik} [x_{2i}^{a(i)}, x_{3k}] - 2 \sum_{i,j} v_{ij} \left(\sum_{j < k} a_{ik} [x_{3k}, x_{3j}] \right) = 0. \end{aligned}$$

Thus the exponent of $D_{(\tau)}(G)/G_7$ divides 2.

Thus we have the following:

THEOREM A. *If G is any finite group, then the exponent of $D_{(\tau)}(G)/G_7$ divides 2.*

COROLLARY. *If G is a finite p -group with $p \neq 2$, then $D_{(\tau)}(G) = G_7$.*

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