Zero-one law for a Markov chain

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1. Introduction

Let $\mathbf{X} = \{X_n\}_{n=0}^\infty$ be an $\mathbf{R}^d$ -valued random sequence, μ be the distribution of $\mathbf{X}$ on $(\mathbf{R}^d)^\infty$, the space of the sequences in $\mathbf{R}^d$, and let μ_{X_n} be the distribution of X_n . The symbol $\mathcal{B}$ denotes the Borel field of $(\mathbf{R}^d)^\infty$ and $\mathcal{B}_\mu$ the μ -completion of $\mathcal{B}$, and $(\mathbf{R}^d)_0^\infty$ denotes the set of sequences in $\mathbf{R}^d$ with only finitely many non-zero entries. Define

$$(\mathbf{R}^d)^{(n)} = \{(x_j)_{j=0}^\infty \in (\mathbf{R}^d)^\infty \mid x_j = 0 \text{ for } j > n\}, \quad n \geq 0.$$

We denote by $\mathcal{B}_\infty$ the tail σ -field of $\mathbf{X}$, that is, $\mathcal{B}_\infty = \bigcap_{n \geq 0} \mathcal{B}_n$, where $\mathcal{B}_n$ is the smallest σ -field that makes $X_n, X_{n+1}, \dots$ measurable, and we say $\mathcal{B}_\infty$ is *trivial* if $P(A) = 0$ or 1 for all $A \in \mathcal{B}_\infty$. It is well known that independent random sequences, random walks and random sequences with strong mixing conditions are tail trivial (see Kolmogorov zero-one law, Hewitt-Savage zero-one law [2] and [4]).

The purpose of this paper is to prove the following :

THEOREM. *Let $\mathbf{X} = \{X_n\}_{n=0}^\infty$ be an $\mathbf{R}^d$ -valued Markov chain and assume the following conditions (i), (ii) and (iii).*

- (i) *The tail σ -field of $\mathbf{X}$ is trivial.*
- (ii) *For every n , there exists a σ -finite Borel measure m_n on $\mathbf{R}^d$ such that $m_n(V) = 0$ for every hyperplane V of $\mathbf{R}^d$ and that $P(X_n \in dy \mid X_{n-1})$ is absolutely continuous with respect to m_n almost surely.*
- (iii) *$\mu_{X_0}(V) = 0$ for every hyperplane V of $\mathbf{R}^d$.*

Then we have

$$\mu(L + \mathbf{a}) = 0 \text{ or } 1 \quad \text{for every affine subspace } L + \mathbf{a} \text{ of } (\mathbf{R}^d)^\infty.$$

For an independent random sequence $\mathbf{X}$ with continuous marginal distributions, Hoffmann-Jørgensen [1, THEOREM 3.1] proved $\mu(L) = 0$ or 1 for every subspace L , and this paper extends his result to the case of a Markov chain.

We have proved in [3] that a tail trivial random sequence with absolutely continuous marginal distributions satisfies zero-one law for subgroups, where the

Lebesgue measure, the Haar measure on $\mathbf{R}^d$, plays an essential role. In this paper, by a method different from [3], we prove a zero-one law for affine subspaces of a tail trivial Markov chain when m_n is a continuous measure, which may not be the Lebesgue measure.

2. Proof and corollary

PROOF OF THEOREM. Let L be a subspace of $(\mathbf{R}^d)^\infty$, $\mathbf{a} \in (\mathbf{R}^d)^\infty$ and assume $L + \mathbf{a} \in \mathbf{B}_\mu$. Since μ is a Radon measure, $L + \mathbf{a}$ contains a σ -compact subset K such that $\mu((L + \mathbf{a}) \setminus K) = 0$. Let L' be the subspace generated by the σ -compact set $K - \mathbf{a}$. Then L' is σ -compact, so that $L' \in \mathcal{B}$, and we have

$$K = (K - \mathbf{a}) + \mathbf{a} \subset L' + \mathbf{a} \subset L + \mathbf{a} \quad \text{and} \quad \mu((L + \mathbf{a}) \setminus (L' + \mathbf{a})) = 0.$$

Therefore we may assume $L + \mathbf{a} \in \mathcal{B}$ without loss of generality. Moreover we may assume $\mathbf{a} = \mathbf{0}$, because $\mathbf{X} - \mathbf{a} = \{X_n - a_n\}_{n=0}^\infty$ satisfies the assumptions (i), (ii) and (iii).

Let G be the complementary subspace of $(\mathbf{R}^d)_0^\infty \cap L$ in $(\mathbf{R}^d)_0^\infty$. Since $G = \bigcup_n (G \cap (\mathbf{R}^d)^{(n)})$ and each $G \cap (\mathbf{R}^d)^{(n)}$ is σ -compact, we have

$$(\mathbf{R}^d)_0^\infty = ((\mathbf{R}^d)_0^\infty \cap L) \oplus G \quad (\text{algebraic direct sum}) \quad \text{and} \quad G \in \mathcal{B}.$$

Since $(\mathbf{x}, \mathbf{y}) \mapsto \mathbf{x} + \mathbf{y} : L \times G \rightarrow L \oplus G$ is bimeasurable by Theorem 3.9 of [5, page 21], we have $L \oplus G \in \mathcal{B}$, and hence $L \oplus G$ is a tail event, because $(\mathbf{R}^d)_0^\infty \subset L \oplus G$. Therefore $\mu(L \oplus G) = 0$ or 1 ; in particular, we have $\mu(L) = 0$ or 1 when $G = \{\mathbf{0}\}$. Therefore it is enough to prove $\mu(L) = 0$ in case $G \neq \{\mathbf{0}\}$.

Assuming $G \neq \{\mathbf{0}\}$, we define the projections

$$p : L \oplus G \rightarrow L : \mathbf{x} + \mathbf{y} \mapsto \mathbf{x}, \quad q : L \oplus G \rightarrow G : \mathbf{x} + \mathbf{y} \mapsto \mathbf{y}, \quad (\mathbf{x} \in L, \quad \mathbf{y} \in G),$$

and

$$\pi_j : (\mathbf{R}^d)^\infty \rightarrow \mathbf{R}^d : (x_n)_{n=0}^\infty \mapsto x_j, \quad j \geq 0.$$

Then p and q are measurable. Since $G \neq \{\mathbf{0}\}$, we have $q(\mathbf{x}) \neq \mathbf{0}$ for some $\mathbf{x} \in (\mathbf{R}^d)_0^\infty$. Define

$$k = \min\{n \geq 0 \mid q(\mathbf{x}) \neq \mathbf{0} \text{ for some } \mathbf{x} \in (\mathbf{R}^d)^{(n)}\},$$

and choose an $r \geq 0$ such that

$$\pi_r \circ q(\mathbf{x}) \neq \mathbf{0} \quad \text{for some } \mathbf{x} \in (\mathbf{R}^d)^{(k)}. \quad (1)$$

Fix such a pair $k, r \geq 0$. Define

$$\sigma_k : (x_j)_{j=0}^\infty \mapsto (x_0, \dots, x_k, 0, 0, \dots) : (\mathbf{R}^d)^\infty \rightarrow (\mathbf{R}^d)^{(k)},$$

and set $\tau_k(\mathbf{x}) = \mathbf{x} - \sigma_k(\mathbf{x})$. Since $x_r = \pi_r \circ p(\mathbf{x}) = \pi_r \circ p(\sigma_k(\mathbf{x}) + \tau_k(\mathbf{x}))$ for $\mathbf{x} \in L$, we have

$$\begin{aligned}\mu(L) &\leq P(X_r = \pi_r \circ p \circ \sigma_k(\mathbf{X}) + \pi_r \circ p \circ \tau_k(\mathbf{X})) \\ &= \mathbf{E}[P(X_r = \pi_r \circ p \circ \sigma_k(\mathbf{X}) + \pi_r \circ p \circ \tau_k(\mathbf{X}) \mid X_{k+1})].\end{aligned}$$

The Markov property implies

$$\begin{aligned}&P(X_r = \pi_r \circ p \circ \sigma_k(\mathbf{X}) + \pi_r \circ p \circ \tau_k(\mathbf{X}) \mid X_{k+1}) \\ &= \begin{cases} \int_{\mathbf{R}^d} P(\pi_r \circ p \circ \tau_k(\mathbf{X}) = z \mid X_{k+1}) P(X_r - \pi_r \circ p \circ \sigma_k(\mathbf{X}) \in dz \mid X_{k+1}) & \text{if } r \leq k, \\ \int_{\mathbf{R}^d} P(X_r - \pi_r \circ p \circ \tau_k(\mathbf{X}) = z \mid X_{k+1}) P(\pi_r \circ p \circ \sigma_k(\mathbf{X}) \in dz \mid X_{k+1}) & \text{if } r > k. \end{cases}\end{aligned}$$

Assume $r \leq k$ and define linear maps

$$\begin{aligned}\varphi : (x_0, \dots, x_k) &\mapsto \pi_r \circ q(x_0, \dots, x_k, 0, 0, \dots) : (\mathbf{R}^d)^{k+1} \rightarrow \mathbf{R}^d, \\ \psi : x_k &\mapsto \varphi(0, \dots, 0, x_k) : \mathbf{R}^d \rightarrow \mathbf{R}^d.\end{aligned}$$

By (1) and the choice of k , ψ is not identically zero, so that $\psi^{-1}(x_k - \varphi(x_0, \dots, x_{k-1}, 0))$ is contained in a hyperplane of $\mathbf{R}^d$, and thus we have

$$m_k\left(\psi^{-1}(z - \varphi(x_0, \dots, x_{k-1}, 0))\right) = 0 \quad \text{for all } x_0, \dots, x_{k-1}, z \in \mathbf{R}^d. \quad (2)$$

Denote by $p_n(x, dy)$ the Radon-Nikodym derivative of $P(X_n \in dy \mid X_{n-1} = x)$ with respect to m_n , and define $m_0 = \mu_{X_0}$,

$$g(y) = \int_{(\mathbf{R}^d)^{k+1}} p_1(x_0, x_1) \cdots p_{k+1}(x_k, y) dm_0(x_0) \cdots dm_k(x_k), \quad y \in \mathbf{R}^d,$$

and

$$h(y) = \begin{cases} g(y)^{-1} & \text{if } g(y) \neq 0, \\ 0 & \text{if } g(y) = 0. \end{cases}$$

Since $g(y)$ is the Radon-Nikodym derivative of $\mu_{X_{k+1}}$ with respect to m_{k+1} , we have almost surely,

$$\begin{aligned}&P((\pi_r - \pi_r \circ p) \circ \sigma_k(\mathbf{X}) = z \mid X_{k+1}) \\ &= P(\pi_r \circ q \circ \sigma_k(\mathbf{X}) = z \mid X_{k+1}) \\ &= P(\varphi(X_0, \dots, X_k) = z \mid X_{k+1}) \\ &= h(X_{k+1}) \cdot \int_{(\mathbf{R}^d)^{k+1}} p_1(x_0, x_1) \cdots p_{k+1}(x_k, X_{k+1}) \\ &\quad \times I_{\{z - \varphi(x_0, \dots, x_{k-1}, 0)\}}(\psi(x_k)) dm_0(x_0) \cdots dm_k(x_k) \quad \text{for all } z \in \mathbf{R}^d.\end{aligned}$$

Hence by (2), we have almost surely,

$$P(X_r - \pi_r \circ p \circ \sigma_k(\mathbf{X}) = z \mid X_{k+1}) = 0 \quad \text{for all } z \in \mathbf{R}^d.$$

On the other hand, there exists almost surely a sequence $\{z_j\}_{j=0}^\infty$ in $\mathbf{R}^d$ such that

$$P(\pi_r \circ p \circ \tau_k(\mathbf{X}) = z \mid X_{k+1}) = 0 \quad \text{if } z \notin \{z_j\}_{j=0}^\infty.$$

Hence we have almost surely,

$$\begin{aligned} & \int_{\mathbf{R}^d} P(\pi_r \circ p \circ \tau_k(\mathbf{X}) = z \mid X_{k+1}) P(X_r - \pi_r \circ p \circ \sigma_k(\mathbf{X}) \in dz \mid X_{k+1}) \\ &= \sum_{j=0}^\infty P(\pi_r \circ p \circ \tau_k(\mathbf{X}) = z_j \mid X_{k+1}) P(X_r - \pi_r \circ p \circ \sigma_k(\mathbf{X}) = z_j \mid X_{k+1}) = 0, \end{aligned}$$

so that

$$\mu(L) \leq \mathbf{E} \left[\int_{\mathbf{R}^d} P(\pi_r \circ p \circ \tau_k(\mathbf{X}) = z \mid X_{k+1}) P(X_r - \pi_r \circ p \circ \sigma_k(\mathbf{X}) \in dz \mid X_{k+1}) \right] = 0.$$

Next, assume $r > k$. Then by (1), we have $\pi_r \circ p(\mathbf{x}) \neq 0$ for some $\mathbf{x} \in (\mathbf{R}^d)^{(k)}$, and $\pi_r \circ p(\mathbf{x}) = 0$ for all $\mathbf{x} \in (\mathbf{R}^d)^{(k-1)}$. Define $\varphi(x_0, \dots, x_k) = \pi_r \circ p(x_0, \dots, x_k, 0, 0, \dots)$. Then the proof can proceed similarly as in the case $r \leq k$.

Therefore we have $\mu(L) = 0$ when $G \neq \{\mathbf{0}\}$. This completes the proof. $\square$

COROLLARY. Let $\mathbf{X} = \{X_n\}_{n=0}^\infty$ be an $\mathbf{R}^d$ -valued Markov chain as in Theorem, and set $\mu_{X_0} = \delta_b$ for any fixed $b \in \mathbf{R}^d$. Then

$$\mu(L + \mathbf{a}) = 0 \text{ or } 1 \text{ for every affine subspace } L + \mathbf{a} \text{ of } (\mathbf{R}^d)^\infty.$$

PROOF. As is shown in the proof of Theorem, we may assume $L \in \mathcal{B}$ without loss of generality.

If $(L + \mathbf{a}) \cap \pi_0^{-1}(\{b\}) = \emptyset$, then $\mu(L + \mathbf{a}) = 0$. Therefore we may assume that there exists a $\mathbf{v} = (v_n)_{n=0}^\infty \in L + \mathbf{a}$ with $v_0 = b$. Since $L + \mathbf{a} = L + \mathbf{v}$, we have

$$\begin{aligned} \{\mathbf{X} \in L + \mathbf{a}, X_0 = b\} &= \{(b, X_1, X_2, \dots) \in L + \mathbf{v}\} \\ &= \{(0, X_1, X_2, \dots) \in L + \mathbf{v} - (b, 0, 0, \dots)\} \\ &= \{(0, X_1, X_2, \dots) \in L + (0, v_1, v_2, \dots)\} \\ &= \{(X_1, X_2, \dots) \in L_0 + (v_1, v_2, \dots)\} \end{aligned}$$

where $L_0 = \{(x_j)_{j=1}^\infty \in (\mathbf{R}^d)^\infty \mid (0, x_1, x_2, \dots) \in L\}$. Then $L_0 \in \mathcal{B}$ and $(X_1, X_2, \dots)$ satisfies all the assumptions of Theorem. Hence we have $\mu(L + \mathbf{a}) = 0$ or 1. $\square$

EXAMPLE. Let $\mathbf{X} = \{X_n\}_{n=0}^{\infty}$ be a real Markov chain, and assume that the tail σ -field of $\mathbf{X}$ is trivial, every transition probability is absolutely continuous with respect to the Lebesgue measure and the initial distribution is either a continuous measure or a unit measure at a point. Then the distribution μ of $\mathbf{X}$ satisfies the zero-one law for affine subspaces.

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References

- [1] J. Hoffmann-Jørgensen, Integrability of seminorms, the 0-1 law and the affine kernel for product measures, *Studia Math.*, **61** (1977), 137-159.
- [2] M. Loève, *Probability Theory I*, 4th ed., Springer, New York Berlin Heidelberg, 1977.
- [3] H. Mizumachi and H. Sato, Continuity of quasi-invariant measures and zero-one laws on groups, *J. Functional Analysis*, **120**(1) (1994), 188-200.
- [4] S. Orey, *Limit theorems for Markov chain transition probabilities*, Van Nostrand, 1971.
- [5] K. R. Parthasarathy, *Probability measures on metric spaces*, Academic Press, New York San Francisco London, 1967.

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