Retarded resolvent operators for linear retarded functional differential equations in a Banach space

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(Received August 7, 1992)
(Revised August 25, 1993)

1. Introduction

Consider the linear retarded functional differential equation of the form

\[
\begin{aligned}
\frac{du(t)}{dt} &= Au(t) + \sum_{i=1}^{m} B_i u(t - h_i) + \int_{-h}^{0} C(s)u(s + t)ds + f(t) \\
\end{aligned}
\]

for a.a. t \in (0, \infty)  \quad (E)

\[
\begin{aligned}
u(0) &= x, \\ u(t) &= \varphi(t) \quad \text{for } t \in [-h, 0)
\end{aligned}
\]

in a Banach space \(X\), where the numbers \(0 < h_1 < h_2 < \cdots < h_m \leq h\) are fixed, \(x\) is a given initial value, \(\varphi\) and \(f\) are a given initial function and a forcing function, respectively. As a concrete model for such an equation one may take the following partial functional differential equation

\[
\begin{aligned}
\begin{cases}
    w_1(t, x) &= w_{xx}(t, x) + w_x(t, x) + \int_{-h}^{0} (a(s)w_x(s+t, x) + b(s)w_{xx}(s+t, x))ds + f(t, x), \\
    w(0, x) &= w(x), \\
    w(t, -\infty) &= w(t, \infty) = 0, \\
    t &> 0, -\infty < x < \infty,
\end{cases}
\end{aligned}
\]

\(1.1\)

In this paper we investigate the equation (E) by using the techniques developed for integrodifferential equations in [2, 3, 6–8, 16, 23, 25]. Our main purpose of this paper is: firstly, under fairly general conditions (H1) and (H2) stated below, to introduce the retarded resolvent operator (RRO) for (E), which is a family of strongly continuous bounded linear operators \(\{R(t)\}_{t \geq 0}\) in \(X\) satisfying the resolvent equations associated with (E) (see Definition 3.1 below); secondly, to prove that the existence of an RRO is equivalent to the wellposedness of the homogeneous equation of (E) in the sense of Krein [13]; thirdly, to give conditions to guarantee the existence of RROs. We also give a representation formula of the solution of an inhomogeneous equation by using the RRO.
Existence, uniqueness and representation of the solution for the scalar equation of the form (E) with \( C(t) = 0 \) and \( x = \varphi(0) \) have been given in [1, Theorems 3.1, 3.7]. Since then, in Banach spaces many authors have studied (E) for the case where \( A \) is a generator of \( C_0 \)-semigroup, \( B_i \) and \( C(t) \) are bounded operators (e.g. [9, 19, 20] and their references). Among others, Nakagiri [19] has constructed the fundamental solution and obtained a variation of constant formula of (mild) solution in the case where \( B_i \) and \( C(t) \) are bounded operators, in connection with his work on the control problems in infinite dimensional Banach spaces. There are a comparably few papers for unbounded ones (e.g. [4, 14, 15, 18, 24]). Concerning the case where \( B_i \) and \( C(t) \) are unbounded operators in Hilbert spaces, global existence, uniqueness and regularity of solutions have been investigated in [4]. For the case \( C(t) = 0 \) (null operator), the representation formula of mild solutions has been studied in [18] under the different conditions from ours. Under the condition that \( m = 1 \), \( A \) is the generator of an analytic semigroup, \( C(\cdot) = a(\cdot)B \) with \( a(\cdot) \) being a Hölder continuous real valued function and \( B_1, B \) are closed linear operators, Tanabe [24] has proved the existence, uniqueness and several properties of the fundamental solution and shown that the mild solution expressed by Nakagiri's formula ([20], [21]) is actually the strong solution of (E) provided that \( \varphi \) and \( f \) are Hölder continuous functions.

In connection with our standpoint in this paper, it is meaningful to notice that on \([0, h_1)\) the solution \( u \) of (E) with \( \varphi = 0 \) satisfies the linear integrodifferential equation of the form

\[
\begin{aligned}
\left\{
\begin{array}{l}
\frac{du(t)}{dt} = Au(t) + \int_0^t B(t - s)u(s) ds + f(t) \\
\quad u(0) = x.
\end{array}
\right.
\end{aligned}
\]  

(E1)

Existence of solutions for (E1) and wellposedness have been investigated by many authors under various hypotheses on \( A \) and \( B \) (e.g. [2, 7, 23, 25] and their references). Most of them have obtained wellposedness of (E1) by proving the existence of a resolvent operator for (E1) (cf. [7]), and by showing that any (strong) solution is represented by the variation of constant formula. The study of the converse problem originates with Miller [16]. Under the general setting on spaces and operators, equivalence of existence of resolvent operators and wellposedness of the homogeneous equation of (E1) have been established in [7]. On the existence of the resolvent operators there are two essentially different types of known results. For the so-called hyperbolic type, in [7] it is assumed that \( A \) generates a \( C_0 \)-semigroup on \( X \) and that \( B(\cdot)x \in W^{1,1}_{\text{loc}}(R^+; X) \) and \( B'(\cdot)x \in BV_{\text{loc}}(R^+; X) \) (the space of functions of locally bounded variation) hold for each \( x \in D(A) \), which is known to be almost best possible. On the other hand, for the parabolic type, as in [23], \( A \) is supposed to generate an analytic \( C_0 \)-semigroup but the regularity of \( B(\cdot) \) is not required contrary to the former case. The papers of Miller [16], Grimmer and Prüss
[7], and Tsuruta [25] for the homogeneous case of (E1) are closely related to our study.

We shall briefly outline the content of this paper. In §2, the notations and the assumptions on functions and operators used throughout the paper are given, and the definition of strong solution of (E) is given, too. In §3, we first give the definition of the RRO for (E) and study its fundamental properties. Next, by using them, we shall show that A can be characterized by $R(\cdot)$, more precisely $A = R'(0+)$, the strong right derivative of $R(t)$ at $t = 0$, and that A generates a $C_0$-semigroup $\{T(t)\}_{t \geq 0}$ under the additional conditions. The former result is already known in the case of integrodifferential equations. In §4, we shall establish the representation formula of the solution by means of the RRO. It is known as a variation of constant formula, which has been obtained by many authors using the fundamental solution under the more restrictive conditions. We shall also discuss the problem, for which classes of $\varphi$ and $f$ mild solutions become strong solutions. In this section equivalence of wellposedness of (E) and existence of an RRO is established, too. Under the suitable conditions, it will be given in Theorem 4.4 that RROs for (E) and fundamental solutions of (E) in the sense of Nakagiri [19, 20] are equivalent. In §5, we shall give necessary and sufficient conditions for the existence of an RRO $R(\cdot)$ having an exponential bound on the magnitude of $R(\cdot)$ by means of the Laplace transform. Theorem 5.2, corresponding to the Hille-Yosida theorem in differential equations, is one of the main results in this section. Similar theorems on integrodifferential equations have already been obtained in [3, 7]. These theorems are not as easy to apply as in the semigroup case. However, in the theoretical point of view, it is of great importance. In fact, our second example in §7 illustrates a circumstance where the equation involving $A$, which does not generate a $C_0$-semigroup, can have an RRO. Hence it is not always possible to solve (E) by perturbation methods. It is known that there is the same circumstance in integrodifferential equations [7]. In §6, by Tsuruta’s methods [25], for the so-called hyperbolic type we shall give another sufficient condition for the existence of the RRO having exponential order. As a consequence of our main theorem in §6, we give a representation formula which corresponds to the exponential formula in semigroup theory [22] if $B_i = 0$ and $C(t) \equiv 0$. We obtain also a result concerning the invariant sets under $T(t)$, i.e., closed sets $K$ satisfying $T(t)K \subset K$ for $t \geq 0$ [3]. An example, which tells us that $T(t)K \subset K$, $t \geq 0$, does not imply $R(t)K \subset K$, $t \geq 0$, is also given in §7. Finally in §7, as an example, we shall show that our results are applicable to (1.1).

2. Preliminaries

$\mathbb{N}$ will denote the set of all positive integers, and we let $\mathbb{N}_m = \{n \leq \mathbb{N} \mid n \leq m\}$ and $\mathbb{H}_m = \{h_i \mid i \in \mathbb{N}_m\}$. Let $\mathbb{C}$ be the set of all complex numbers and $\mathbb{R}$ be the
set of all real numbers. The symbol \( \mathbb{X} \) will denote a given complex Banach space with norm \( | \cdot | \). Let \( I_h \equiv [-h, 0] \) and \( J \) be an interval in \( \mathbb{R}^+ \equiv [0, \infty) \). For any Banach space \( \mathbb{Z} \) the set of all bounded linear operators from \( \mathbb{X} \) to \( \mathbb{Z} \) will be denoted by \( B(\mathbb{X}, \mathbb{Z}) \), \( B(\mathbb{X}, \mathbb{X}) = B(\mathbb{X}) \) and the norm of its element will be also denoted by \( | \cdot | \). The Banach space of all continuous mapping \( \varphi \) from \( J \) into \( \mathbb{X} \) will be denoted by \( C(J; \mathbb{Z}) \) with the sup norm \( | \cdot |_{\infty} \), and \( C^1(J; \mathbb{Z}) \) will denote the set of functions \( f \in C(J; \mathbb{Z}) \) such that \( f' \in C(J; \mathbb{Z}) \). Let \( L^1(J; \mathbb{Z}) \) denote the set of all integrable functions from \( J \) to \( \mathbb{X} \). Let \( G(J; \mathbb{Y}) \) denote the set of all regulated functions \( f \), i.e., \( f' \) has only discontinuity of the first kind (see [11, p.16]). For a locally integrable function \( u : \mathbb{R}^+ \rightarrow \mathbb{X} \), its Laplace transform will be denoted by \( \hat{u}(\lambda) \) whenever it exists. The function \( \chi_J \) denotes the characteristic function of the interval \( J \).

Throughout this paper we assume operators \( A, B_i \) \( (i \in \mathbb{N}_m) \) and \( C(s) \) \( (s \in I_h) \) are linear ones and moreover assume the following conditions (H0), (H1) and (H2).

(H0) \( A \) is a densely defined closed linear operator with domain \( D(A) \).

It is convenient to introduce a Banach space \( \mathbb{Y} \equiv D(A) \) endowed with the graph norm of \( A \), \( |x|_{\mathbb{Y}} \equiv |x| + |Ax| \) for \( x \in \mathbb{Y} \).

(H1) For each \( i \in \mathbb{N}_m \) \( D(A) \subset D(B_i) \) and \( B_i \) is (possibly not closed) relatively bounded with respect to \( A \) for \( i \in \mathbb{N}_m \), i.e., there exist nonnegative constants \( a \) and \( b \) such that for \( y \in D(A) \)

\[
|B_i y| \leq a|y| + b|Ay| \quad \text{(cf. [12]).}
\]

In other words (H1) means that \( B_i, i \in \mathbb{N}_m, \) are operators with domain \( D(B_i) \) such that \( D(A) \subset \bigcap_{i=1}^m D(B_i) \) and \( |B_i y| \leq b_i|y|_{\mathbb{Y}}, \ y \in \mathbb{Y} \) for some constants \( b_i \).

(H2) \( D(C(t)) \supset D(A) \) for all \( t \in I_h \), the function \( C(\cdot)x \) is strongly measurable for \( x \in D(A) \) and there is \( c \in L^1(I_h) \) such that

\[
|C(t)x| \leq c(t)|x|_{\mathbb{Y}} \quad \text{for a.a.} \ t \in I_h.
\]

We set

\[
B(t)y \equiv - \sum_{i=1}^m \chi_{(-\infty,-h_i]}(t)B_i y - \int_t^0 C(s)yds \quad \text{for} \ t \in I_h \quad \text{and} \ y \in \mathbb{Y}
\]

and

\[
b(t) = - \sum_{i=1}^m \chi_{(-\infty,-h_i]}(t)b_i - \int_t^0 c(s)ds.
\]

It is known ([12, Remark IV.1.4]) that the restriction \( B_i |_{\mathbb{Y}} \) of \( B_i \) to \( D(A) \) can be regarded as bounded operators, so \( B(t) |_{\mathbb{Y}} \) belongs to \( B(\mathbb{Y}, \mathbb{X}) \) for a.a. \( t \in I_h \). Thus \( B(\cdot) \) is of bounded semi-variation and the interior integral \( \int_{-h}^0 dB(r) \cdot u(r + t) \) in the sense of Hönig [11, p.7] is well defined as long as \( u \) belongs to \( G([-h, \infty); \mathbb{Y}) \).
Retarded resolvent operators

\[ \int_{-h}^{0} dB(s) \cdot u(s + t) = \sum_{i=1}^{m} B_i u(t - h_i) + \int_{-h}^{0} C(s) u(s + t) ds \quad \text{for} \quad t \geq 0. \quad (2.1) \]

Moreover for any \( u \in G([-h, \infty); \mathcal{Y}) \) we have

\[ \left| \int_{-h}^{0} dB(r) \cdot u(r + t) \right| \leq \int_{-h}^{0} db(r) \cdot |u(r + t)|_{\mathcal{Y}} \quad \text{for} \quad t \geq 0. \]

It is also known ([11, p.9, Theorem 1.2]) that if \( u \) is a continuous function, then the Stieltjes integral \( \int_{-h}^{0} dB(r) \cdot u(r + t) \) exists and

\[ \int_{-h}^{0} dB(r) \cdot u(r + t) = \int_{-h}^{0} dB(r) \cdot u(r + t) \quad \text{for} \quad t \geq 0. \]

We denote the resolvent operator of \( A \) by \( R(\lambda : A) \) and the resolvent set by \( \rho(A) \). If the operator

\[ \lambda - \Delta(\lambda) \equiv \lambda - A - \int_{-h}^{0} e^{\lambda r} \cdot dB(r) \]

has an inverse \( (\lambda - \Delta(\lambda))^{-1} \) in \( B(\mathcal{X}) \), we denote it by \( R(\lambda : \Delta) \equiv R(\lambda : \Delta(\lambda)) \). We call the set of all values \( \lambda \) in \( \mathbb{C} \), for which \( R(\lambda : \Delta) \) exists, the retarded resolvent set \( \rho(\Delta) \equiv \rho(\Delta(\lambda)) \) (cf. [20]). By \( g \in W^{1,1}(J; \mathcal{X}) \) we will mean that \( g \) is absolutely continuous and differentiable for a.a. \( t \) on \( J \) with \( g' \in L^{1}(J; \mathcal{X}) \) and that \( g(t) = g(0) + \int_{0}^{t} g'(s) ds \) holds on \( J \).

We study (E) in the form

\[ \frac{du(t)}{dt} = Au(t) - \int_{-h}^{0} dB(r) \cdot u(r + t) + f(t) \quad \text{for a.a.} \quad t > 0, \]

\[ u(0) = x, \quad u(t) = \varphi(t) \quad \text{for} \quad t \in [-h, 0). \]

**Definition 2.1.** By a solution of (E), we mean a function \( u : [-h, \infty) \rightarrow \mathcal{X} \) satisfying \( u(t) \in \mathcal{D}(A) \) for \( t \geq 0 \), \( u |_{\mathbb{R}+} \in C(\mathbb{R}+; \mathcal{Y}) \) and

\[ \begin{aligned}
& u(t) = x + \int_{0}^{t} \left( Au(s) + \int_{-h}^{0} dB(r) \cdot u(r + s) + f(s) \right) ds \quad \text{for} \quad t > 0, \\
& u(t) = \varphi(t) \quad \text{for} \quad t \in [-h, 0). 
\end{aligned} \]

Under the condition (H1) the restriction of \( B_i u(\cdot) \) to \( \mathbb{R}+ \) also belongs to \( C(\mathbb{R}+; \mathcal{X}) \) if \( u \) is a solution of (E). Further we remark that the solution \( u(t) \) is differentiable and satisfies (E) for \( t > 0 \) if \( x = \varphi(0) \), but if \( x \neq \varphi(0)u(t) \) is not
differentiable at $t = h_i, i \in \mathbb{N}_m$. At $t = h_i$ both the right-hand derivative $u'(t + 0)$ and the left-hand derivative $u'(t - 0)$ always exist and satisfy

$$u'(t + 0) - u'(t - 0) = B_i x - B_i \varphi(0).$$

3. Definition of a retarded resolvent operator and its properties

In this section we shall define a retarded resolvent operator $R(\cdot)$ and present some of its properties under the conditions (H0)-(H2).

DEFINITION 3.1. Let bounded linear operators $R_L(t)$ and $R_R(t)$ satisfy:

1. $R_L(\cdot)x$ and $R_R(\cdot)x$ are continuous on $\mathbb{R}^+$ for any $x \in X$;
2. $R_L(0) = R_R(0) = I$ and $R_L(t) = R_R(t) = O$ for $t \in [-h, 0)$;
3. $R_R(t)D(A) \subset D(A)$ on $\mathbb{R}^+$;
4. $AR_R(\cdot)y$ is continuous on $\mathbb{R}^+$ for $y \in D(A)$;
5. for $y \in D(A)$, $R_L(\cdot)y$ and $R_R(\cdot)y$ are differentiable on $(0, \infty)$ except at $t \in \mathbb{N}_m$;
6. for all $y \in D(A)$ and $t \in (0, \infty)$ except at $t \in \mathbb{N}_m$, the following resolvent equations hold:

\begin{equation}
(\frac{d}{dt})R_L(t)y = R_L(t)Ay + \int_{-h}^{0} R_L(t + r) \cdot dB(r)y;
\end{equation}

\begin{equation}
(\frac{d}{dt})R_R(t)y = AR_R(t)y + \int_{-h}^{0} dB(r) \cdot R_R(r + t)y,
\end{equation}

where in (3.1) let the integral $\int_{-h}^{0} R_L(t + r) \cdot dB(r)y$ conventionally mean

\begin{equation}
\sum_{i=1}^{m} R_L(t - h_i) B_i y + \int_{-h}^{0} R_L(t + s) C(s)yds.
\end{equation}

If both $R_L(t)$ and $R_R(t)$ exist and coincide on $D(A)$ for each $t \geq 0$, then we call it the retarded resolvent operator (RRRO) for (E) and denote it by $R(t)$ simply.

In order to prove our main results of this section we require some lemmas, which also will be used in the following sections.

LEMMA 3.1. (a) Let $u \in G([a, b]; Y)$. Then we have

$$\int_{a}^{b} B_i u(s)ds = B_i \int_{a}^{b} u(s)ds \quad \text{for} \quad i \in \mathbb{N}_m.$$

(b) Let $u \in C([0, \infty); Y)$ with $u(t) = 0$ for $t < 0$. Then for $t \geq 0$ we have

$$\int_{0}^{t} \left( \int_{-h}^{0} dB(r) \cdot u(r + s) \right) ds = \int_{-h}^{0} dB(r) \cdot \left( \int_{0}^{r+t} u(s)ds \right).$$
PROOF. (a) Since regulated functions are Darboux integrable [11, p.18] and Darboux integrable functions are Riemann integrable [11, p.14], the integral \( \int_a^b u(s)ds \) exists in \( Y \). As the restriction \( B_i|_Y \) of \( B_i \) to \( Y \) is an element of \( B(Y, X) \) for \( i \in \mathbb{N}_m \), it follows that
\[
\int_a^b B_i u(s)ds = \int_a^b B_i|_Y u(s)ds = B_i|_Y \int_a^b u(s)ds = B_i \int_a^b u(s)ds.
\]
(b) By (2.1), (a) and Fubini’s theorem we have
\[
\int_0^t \left( \int_{-h}^0 dB(r) \cdot u(r + s) \right) ds = \int_0^t \left( \sum_{i=1}^m B_i u(s - h_i) + \int_{-h}^0 C(r)u(r + s)dr \right) ds
\]
\[
= \sum_{i=1}^m B_i \int_0^{t-h_i} u(s)ds + \int_{-h}^0 \int_0^t C(r)u(r + s)ds dr.
\]
By the same reasoning as above we have
\[
\int_{-h}^0 \int_0^t C(r)u(r + s)ds dr = \int_{-h}^0 C(r) \int_r^{r+t} u(s)ds dr
\]
\[
= \int_{-h}^0 C(r) \int_0^{r+t} u(s)ds dr,
\]
since \( u(s) = 0 \) on \([r, 0]\) for \( r < 0 \). Hence, by (2.1) we have (3.4). \( \square \)

**Lemma 3.2.** Let \( J \equiv [0, a) \). Let \( M, \alpha \) be nonnegative numbers and \( z : J \rightarrow \mathbb{R}^+ \) be continuous. If
\[
z(t) \leq Me^{\alpha t} + \int_{-h}^0 dB(r) \cdot z(r + t), \quad t \in J,
\]
and \( z(t) = 0 \) for \( t \in [-h, 0) \), then
\[
z(t) \leq kM e^{(\alpha + \beta)t} \quad \text{for} \quad t \in J,
\]
where \( k \) and \( \beta \) are positive constants independent of \( a \).

**Proof.** Define \( c(t) = 0 \) for \( t < -h \) and set \( \tilde{c}(t) = c(-t) \) for \( t \geq 0 \). Let \( r_{\tilde{c}} \) be the solution of the equation ([8, pp.36–37])
\[
r_{\tilde{c}} = \tilde{c} + \tilde{c} * r_{\tilde{c}} = \tilde{c} + r_{\tilde{c}} * \tilde{c},
\]
where \( (\tilde{c} * r_{\tilde{c}})(t) = \int_0^t \tilde{c}(t-s)r_{\tilde{c}}(s)ds \) and etc. If \( a \leq h_1 \), (3.5) is obvious. Suppose \( a > h_1 \). Without loss of generality we may assume that \( \alpha \) is sufficiently large so that \( \int_0^{h_1} e^{-\alpha r} \tilde{c}(r)dr < 1 \). We set
\[
k = \left( 1 - \int_0^{h_1} e^{-\alpha r} \tilde{c}(s)ds \right)^{-1} \quad \text{and} \quad \beta = (1/h_1) \log \left( 1 + k \int_{-h}^0 e^{\alpha r} \cdot dB(r) \right).
\]
Then we have
\[ z(t) \leq M e^{\alpha t} + \int_0^t c(t-s)z(s)ds, \quad t \in [0, h_1), \]
so that the generalized Gronwall lemma ([8, p.257]) yields
\[ z(t) \leq M e^{\alpha t} + \int_0^t r_c(s)M e^{\alpha(t-s)}ds \leq M e^{\alpha t} \left( 1 + \int_0^{h_1} e^{-\alpha s} r_c(s)ds \right). \]
As (3.6) implies
\[
\int_0^{h_1} e^{-\alpha s} r_c(s)ds = \int_0^{h_1} e^{-\alpha s} \tilde{c}(s)ds + \int_0^{h_1} e^{-\alpha s} (\tilde{c} * r_c)(s)ds \\
\leq \int_0^{h_1} e^{-\alpha s} \tilde{c}(s)ds + \left( \int_0^{h_1} e^{-\alpha s} \tilde{c}(s)ds \right) \left( \int_0^{h_1} e^{-\alpha s} r_c(s)ds \right),
\]
we have
\[
1 + \int_0^{h_1} e^{-\alpha s} r_c(s)ds \\
\leq 1 + \left( 1 - \int_0^{h_1} e^{-\alpha s} \tilde{c}(s)ds \right)^{-1} \int_0^{h_1} e^{-\alpha s} \tilde{c}(s)ds = k.
\]
Hence we obtain \( z(t) \leq kMe^{\alpha t} \) for \([0, h_1)\). Suppose (3.5) holds for \( t \in [0, nh_1) \), \( n \geq 1 \). Then for \( t \in [nh_1, (n+1)h_1) \), setting \( \tau = t - nh_1 \), we have
\[
z(t) \leq M e^{\alpha t} + \sum_{i=1}^m b_i z(t - h_i) + \left( \int_{-h}^{-\tau} + \int_0^0 \right) c(s)z(s + t)ds \\
\leq M \left( e^{\alpha t} + \sum_{i=1}^m b_i k e^{(\alpha + \beta)(t-h_1)} + \int_{-h}^{-h_1} c(s) k e^{(\alpha + \beta)(t+s)}ds \right) \\
+ kM e^{(\alpha + \beta)t} \int_{-h}^{h_1} e^{-\alpha r} \tilde{c}(r)dr + \int_{-h}^{\tau} \tilde{c}(\tau - s) z(s + nh_1)ds \\
\leq M e^{(\alpha + \beta)t} \left( e^{-\beta h_1} \left( 1 + k \int_{-h}^{0} e^{\alpha r} \cdot db(r) \right) + k \int_{-h}^{h_1} e^{-\alpha r} \tilde{c}(r)dr \right) \\
+ \int_{-h}^{\tau} \tilde{c}(\tau - s) z(s + nh_1)ds \\
= M e^{(\alpha + \beta)t} \left( 1 + k \int_{-h}^{h_1} e^{-\alpha r} \tilde{c}(r)dr \right) + \int_{-h}^{\tau} \tilde{c}(\tau - s) z(s + nh_1)ds
\]
or
\[
y(\tau) \leq M e^{(\alpha + \beta)(\tau + nh_1)} \left( 1 + k \int_{-h}^{h_1} e^{-\alpha s} \tilde{c}(s)ds \right) + \int_{-h}^{\tau} \tilde{c}(\tau - s) y(s)ds,
\]
where we set \( y(\tau) \equiv z(\tau + nh_1) (= z(t)) \). Applying the generalized Gronwall lemma [8, p.257], it follows that

\[
\begin{align*}
\int_0^T e^{-\alpha r} c(s) ds &+ \int_0^T e^{-\alpha r} c(r) dr ds \\
\leq M e^{(\alpha + \beta)t} \left( 1 + k \int_0^{h_1} e^{-\alpha s} c(s) ds + \int_0^{h_1} e^{-\alpha s} c(r) dr ds \right) \\
+ &k \int_0^{h_1} e^{-\alpha s} c(s) ds.
\end{align*}
\]

Applying Fubini’s theorem and using (3.4) again, we have

\[
\begin{align*}
k \int_0^T e^{-\alpha s} c(s) ds &
= k \int_0^{h_1} e^{-\alpha s} c(s) ds \left( \int_0^T e^{-\alpha s} c(s) ds - \int_0^T e^{-\alpha s} c(r) dr ds \right) \\
&- k \int_0^{h_1} e^{-\alpha s} c(s) ds \left( \int_0^T e^{-\alpha s} c(r) dr ds - \int_0^T e^{-\alpha s} c(r) dr ds \right) \\
&= (k - 1) \int_0^T e^{-\alpha s} c(s) ds - k \int_0^T e^{-\alpha s} c(r) dr ds.
\end{align*}
\]

Substituting the relation into (3.7), for \( t \in [nh_1, (n+1)h_1] \) we have

\[
z(t) \leq M e^{(\alpha + \beta)t} \left( 1 + k \int_0^{h_1} e^{-\alpha s} c(r) dr \right) = k M e^{(\alpha + \beta)t}.
\]

This completes the proof (cf. Exercise 4 in [1, p.61]).

**Lemma 3.3.** Let \( R_R(\cdot) \) and \( R_L(\cdot) \) be the operator valued functions defined in Definition 3.1. For any \( x \in X \) let

\[
S_R(t)x \equiv \int_0^t R_R(s) x ds, \quad S_L(t)x \equiv \int_0^t R_L(s) x ds, \quad t \geq 0,
\]

and \( S_R(t) = S_L(t) = 0 \) for \( t < 0 \). Then we have
(1) $S_R(t) \subset D(A)$ for $t \geq 0$;
(2) $AS_R(t)x$ is continuous for $x \in X$ and $t \geq 0$;
(3) $R_R(t)x = x + AS_R(t)x + \int_{-h}^{0} dB(r) \cdot S_R(r + t)x$ for $x \in X$ and $t \geq 0$;
(4) for any $\tau > 0$ there exist constants $M(\tau)$ and $\beta$ such that
\[ |S_R(t)x|_Y \leq M(\tau)e^{\beta t}|x| \quad \text{for} \quad t \in J = [0, \tau] \quad \text{and} \quad x \in X; \]
(5) $R_L(t)y = y + S_L(t)Ay + \int_{-h}^{0} S_L(t + r) \cdot dB(r)y$ for $y \in D(A)$ and $t \geq 0$;
(6) if $(E)$ admits an RRO $R(\cdot)$, then $R(t)$ is a bounded linear operator from $Y$ to $Y$ for $t \geq 0$.

**Proof.** Let $y \in D(A)$. Since $S_L(t)$ is continuous in the operator norm, we see that $S_L(t)$ is of bounded semi-variation, owing to its definition [11, pp.21-22]. Thus by [11, p.26, Theorem 4.12] the integral $\int_{-h}^{0} dS_L(t + r) \cdot B(r)y$ exists and by [11, p.9, Theorem 1.2] the integral $\int_{-h}^{0} dS_L(t + r) \cdot B(r)y$ exists, too. Finally by [11, p.8, Theorem 1.3] the Stieltjes integral $\int_{-h}^{0} S_L(t + r) \cdot dB(r)y$ is well defined. It is also obvious that we have
\[
\int_{-h}^{0} S_L(t + r) \cdot dB(r)y = \sum_{i=1}^{m} S_L(t - h_i)B_iy + \int_{-h}^{0} S_L(t + s)C(s)yds.
\]
Recall that the integral $\int_{-h}^{0} R_L(t + r) \cdot dB(r)y$ means (3.3). Integrating (3.1) from $0$ to any $t$ we obtain
\[
R_L(t)y - y = \int_{0}^{t} \sum_{i=1}^{m} R_L(s - h_i)B_iyds + \int_{0}^{t} \int_{-h}^{0} R_L(s + r)C(r)y dr ds
\]
\[= \sum_{i=1}^{m} S_L(t - h_i)B_iy + \int_{-h}^{0} S_L(t + r)C(r)ydr = \int_{-h}^{0} S_L(t + r) \cdot dB(r)y,
\]
in which we used Fubini’s theorem and the fact that $R_L(t) = O$ for $t < 0$. (5) is thus proved. Integrating (3.2) and then applying (b) of Lemma 3.1, we have
\[
AS_R(t)y - (R_R(t)y - y) = - \int_{-h}^{0} dB(r) \cdot S_R(r + t)y. \quad (3.8)
\]
Hence by (3.8) we obtain
\[
|S_R(t)y|_Y = |S_R(t)y| + |AS_R(t)y| \leq |S_R(t)y| + |R_R(t)y - y| + \left| \int_{-h}^{0} dB(r) \cdot S_R(r + t)y \right|
\]
\[\leq M(\tau)|y| + \int_{-h}^{0} db(r) \cdot |S_R(r + t)y|_Y,
\]
where $M(\tau)$ is some constant satisfying
\[ |S_R(t)x| + |R_R(t)x - x| \leq M(\tau)|x| \quad \text{for } x \in X \text{ and } t \in J.\]

Here we note that $M(\tau) < \infty$ because of (1) in Definition 3.1 and the uniform boundedness theorem. Thus, by Lemma 3.2, (4) holds for $y \in D(A)$. Since $D(A)$ is dense in $X$, $AS_R(t)$ has a bounded extension to all of $X$, and this proves (1). Moreover since $A$ is closed, $S_R(t)X \subset D(A)$ for $t \geq 0$, and (4) holds. Also (3) follows from (3.8). Similarly we can obtain (2). Let $R(\cdot)$ be an RRO. Then taking (3.1), (3.2) and (1) and (4) of Definition 3.1 into account we have
\[
AR(t)y = R(t)Ay + \int_{-h}^{0} R(t+r) \cdot dB(r) y - \int_{-h}^{0} dB(r) \cdot R(r+t)y
\]
for $t \geq 0$ and $y \in D(A)$. Applying Lemma 3.2 we can find some constants $N(t)$ and $\gamma$ such that $|R(t)y|_Y \leq N(t)e^{\gamma t}|y|_Y$. This completes the proof.

Let $R(\cdot)$ be an RRO for $E$. We define $S(t)x \equiv \int_{0}^{t} R(s) xds$ for $x \in X$ and $t \geq 0$ and $S(t) \equiv 0$ for $t < 0$. Then we can see that $S(t)X \subset D(A)$ for $t \geq 0$, and by Lemma 3.3 $R(t)$ satisfies
\[
R(t)x - x = AS(t)x + \int_{-t}^{0} C(s)S(t+s)xds \quad \text{for } t \in [0, h_1].
\]

We now prove that the operator $A$ is characterized by the RRO (cf. [7, Theorem 6]).

**Theorem 3.1.** Suppose $(E)$ admits an RRO $R(\cdot)$. Then
\[
D(A) = \left\{ x \in X \mid \lim_{t \to 0^+} (1/t)(R(t)x - x) \text{ exists}\right\}
\]
and
\[
Ax = \lim_{t \to 0^+} (1/t)(R(t)x - x) \text{ for every } x \in D(A).
\]

**Proof.** Let
\[
D(\tilde{A}) = \left\{ x \in X \mid \tilde{A}x = \lim_{t \to 0^+} (1/t)(R(t)x - x) \text{ exists}\right\}.
\]

Then from (3.1) we know that $D(A) \subset D(\tilde{A})$. Now let $x \in D(\tilde{A})$. For any $\eta > 0$ we can find $\varepsilon \in (0, h_1)$ and $M_1 > 0$ such that $|S(s)| \leq sM_1$ and $|R(s)x - x - s\tilde{A}x| \leq s\eta$ for every $s \in [0, \varepsilon]$. Let $\tilde{c}$ and $\tau_{\varepsilon}$ be the functions defined in the proof of Lemma 3.2. Then from (3.9) we obtain for $t \in [0, h_1]$
\[
|S(t)x|_Y \leq |S(t)x| + |R(t)x - x| + \int_{0}^{t} \tilde{c}(t-s)|S(s)x|_Y ds,
\]
and
\[ |AS(t)x - (R(t)x - x)| \leq \int_0^t \overline{c}(t - s)|S(s)x|y ds. \tag{3.11} \]

In (3.10) using the generalized Gronwall lemma we have
\[ |S(t)x| \leq |S(t)x| + |R(t)x - x| + \int_0^t \overline{c}(t - s)(|S(s)x| + |R(s)x - x|) ds. \]

Therefore by Fubini’s theorem and (3.6) we obtain
\[ \int_0^t \overline{c}(t - s)|S(s)x|y ds \leq \int_0^t \overline{c}(t - s)(|S(s)x| + |R(s)x - x|) ds \]
\[ + \int_0^t \overline{c}(t - s) \int_0^s \overline{c}(s - r)(|S(r)x| + |R(r)x - x|) dr ds \]
\[ = \int_0^t (\overline{c}(t - s) + \overline{c} * \overline{c})(t - s)(|S(s)x| + |R(s)x - x|) ds \]
\[ = \int_0^t \overline{c}(t - s)(|S(s)x| + |R(s)x - x|) ds. \tag{3.12} \]

Combining (3.11) and (3.12) we obtain
\[ |AS(t)x - (R(t)x - x)| \leq \int_0^t \overline{c}(t - s)(|S(s)x| + |R(s)x - x|) ds \]
\[ \leq t(M_1|x| + \eta + |\overline{A}x|) \int_0^t \overline{c}(s) ds \quad \text{for} \quad t \in (0, \varepsilon]. \]

Hence \((1/t)(AS(t)x - (R(t)x - x)) \to 0\) as \(t \to 0^+\) or \(\lim_{t \to 0^+} (1/t)AS(t)x = \overline{A}x\).
Closedness of \(A\) shows that \(x \in D(A)\) and \(Ax = \overline{A}x\), since \((1/t)S(t)x \to x\) as \(t \to 0^+\). Thus we have the conclusion. \(\square\)

Now we can prove the following result.

**Theorem 3.2.** Suppose that (E) admits an RRO \(R(\cdot)\) and that there is a constant \(c > 0\) such that
\[ |\int_{-t}^0 C(s)R(s + t)y ds| \leq c|y| \quad \text{for} \quad y \in D(A) \text{ and } t \in [0, h_1]. \]

Then \(A\) is the infinitesimal generator of a \(C_0\)-semigroup.

**Proof.** Let
\[ G(t)y = \int_{-t}^0 C(s)R(s + t)y ds \quad \text{for} \quad y \in D(A) \text{ and } t \in [0, h_1]. \]
Then by our assumption $G(t)$ has a bounded extension for each $t \in [0, h_1]$ so that we denote the bounded extension of $G(t)$ by $\overline{G}(t)$. Let

$$\overline{G}_S(t)x \equiv \int_0^t \overline{G}(s)x\,ds$$

and $R_{(-\overline{G})}$ be the solution of

$$R_{(-\overline{G})}(t)x + \overline{G}(t)x = -\int_0^t \overline{G}(t-s)R_{(-\overline{G})}(s)x\,ds$$

$$= -\int_0^t R_{(-\overline{G})}(t-s)\overline{G}(s)x\,ds \quad (3.13)$$

for $t \in [0, h_1]$ and $x \in X$. Then by Fubini’s theorem we have

$$\overline{G}_S(t)x = \int_{-t}^0 C(s)S(s+t)x\,ds$$

and from (3.9) it follows that

$$R(t)x - x = AS(t)x + \overline{G}_S(t)x, \quad t \in [0, h_1]. \quad (3.14)$$

Moreover from (5) of Lemma 3.3 we obtain

$$\overline{G}(t)y = \overline{G}_S(t)y + \int_{-t}^0 \overline{G}_S(t+s)C(s)y\,ds + \int_{-t}^0 C(s)y\,ds \quad (3.15)$$

for $t \in [0, h_1]$ and $y \in \mathcal{D}(A)$.

If $A$ generates a $C_0$-semigroup $\{T(t)\}_{t \geq 0}$, since $R(t)y, y \in \mathcal{D}(A)$, satisfies the differential equation

$$\frac{d}{dt}R(t)y = AR(t)y + \overline{G}(t)y \quad \text{for} \quad t \in [0, h_1],$$

we have

$$R(t)y = T(t)y + \int_0^t T(t-s)\overline{G}(s)y\,ds$$

or

$$R(t)x = T(t)x + \int_0^t T(t-s)\overline{G}(s)x\,ds \quad \text{for} \quad x \in X,$$

since $\mathcal{D}(A)$ is dense in $X$. Hence using Fubini’s theorem we can compute to get

$$(R * R_{(-\overline{G})}x)(t) - (T * R_{(-\overline{G})}x)(t) = (T * \overline{G}) * R_{(-\overline{G})}x(t)$$

$$= (T * (\overline{G} * R_{(-\overline{G})}x))(t) = -(T * R_{(-\overline{G})}x)(t) - (T * \overline{G}x)(t)$$
or

\[
\int_0^t R(t-s)R_{(-\overline{G})}(s)xds = -\int_0^t T(t-s)\overline{G}(s)xds,
\]

from which we deduce that

\[
T(t)x = R(t)x + \int_0^t R(t-s)R_{(-\overline{G})}(s)xds \quad \text{for} \quad t \in [0, h_1] \text{ and } x \in X.
\]

Accordingly we inductively define the strongly continuous operator valued function \( T(\cdot) \) by

\[
\begin{cases}
T(t)x = R(t)x + \int_0^t R(t-s)R_{(-\overline{G})}(s)xds & \text{for } t \in [0, h_1] \text{ and } x \in X, \\
T(t+h_1) = T(t)T(h_1) & \text{for } t \in \mathbb{R}^+.
\end{cases}
\]

From the definition of \( T(\cdot) \) we have

\[
T(t)x = R(t)x - \int_0^t T(t-s)\overline{G}(s)xds \quad \text{for } x \in X
\]

and by integrating it we obtain

\[
T_S(t)x - S(t)x = \int_0^t S(t-s)R_{(-\overline{G})}(s)xds = -\int_0^t T(t-s)\overline{G}_S(s)xds, \quad (3.16)
\]

where \( T_S(t)x \equiv \int_0^t T(s)xds, \ x \in X \). Since \( S(t)X \subset D(A) \) and \( AS(t)x, \ x \in X, \) is continuous by Lemma 3.3, it follows that \( T_S(t)X \subset D(A) \) and by (3.13) and (3.14)

\[
AT_S(t)x = AS(t)x + \int_0^t AS(t-s)R_{(-\overline{G})}(s)xds \\
= (R(t)x - x - \overline{G}_S(t)x) + \int_0^t (R(t-s) - I - \overline{G}_S(t-s))R_{(-\overline{G})}(s)xds \\
= T(t)x - x, \quad t \in [0, h_1]. \quad (3.17)
\]

Now since there exist constants \( M \geq 1 \) and \( \omega_1 \) such that \( |T(t)| \leq Me^{\omega_1 t} \) for \( t \in [0, h_1] \) we obtain

\[
|T(t)| \leq Me^{\omega t} \quad \text{for} \quad t \geq 0, \quad (3.18)
\]

where \( \omega \equiv \omega_1 + \omega_2 \) and \( \omega_2 \equiv (1/h_1) \log M(\geq 0) \). In fact, if (3.18) holds on \([0, nh_1), \ n \geq 1\), then for \( t \in [nh_1, (n+1)h_1) \) we have

\[
|T(t)| = |T(t-h_1)T(h_1)| \leq Me^{\omega(t-h_1)}Me^{\omega_1 h_1} = Me^{\omega t}.
\]
For $\lambda > \omega$ and $x \in X$ let $\widehat{T}(\lambda)x \equiv \int_0^\infty e^{-\lambda t}T(t)xdt$. Since
\[ |e^{-\lambda h_1}T(h_1)| \leq e^{-\omega h_1}M e^{-\lambda h_1} = e^{-\omega h_1} < 1 \]
for $\lambda > \omega$, $P(\lambda) \equiv (I - e^{-\lambda h_1}T(h_1))^{-1}$ exists, and so we have
\[ \widehat{T}(\lambda)x = \int_0^{h_1} e^{-\lambda t}T(t)P(\lambda)xdt. \]
Furthermore integration by parts yields
\[ \widehat{T}(\lambda)x = e^{-\lambda h_1}T_S(h_1)P(\lambda)x + \lambda \int_0^{h_1} e^{-\lambda t}T_S(t)P(\lambda)xdt. \]
Hence $\widehat{T}(\lambda)x \in D(A)$ and by (3.17) we see
\[ A\widehat{T}(\lambda)x = e^{-\lambda h_1}(T(h_1) - I)P(\lambda)x + \lambda \int_0^{h_1} e^{-\lambda t}(T(t) - I)P(\lambda)xdt \]
so that $(\lambda - A)\widehat{T}(\lambda)x = x$ follows.

On the other hand by using (3.15), (3.16) and Fubini's theorem we get
\[ R(t)y - T(t)y = (T * \tilde{G}y)(t) \]
\[ = (T * \tilde{G}S_Ay)(t) + ((T * \tilde{G}S)(T \tilde{C}y))(t) + (T_S * \tilde{C}y)(t) \]
\[ = -T_S(t)Ay + S(t)Ax + (S \ast \tilde{C}y)(t) = -T_S(t)Ay + R(t)y - y \]
or
\[ T(t)y = y + T_S(t)Ay \quad \text{for} \quad t \in [0, h_1] \quad \text{and} \quad y \in D(A), \]
where $\tilde{C}(t)$ denotes $C(-t)$. From (3.17) we see that $T(t)D(A) \subset D(A)$ and $AT(t)y = Ay + AT_S(t)Ay = T(t)Ay$ for $t \in [0, h_1]$ and $y \in D(A)$. Furthermore it is not difficult to see that $P(\lambda)D(A) \subset D(A)$ and $AP(\lambda) = P(\lambda)A$ on $D(A)$ for $\lambda > \omega$. This yields $\widehat{T}(\lambda)(\lambda - A)y = y$ and so $R(\lambda : A) = \widehat{T}(\lambda)$ for all $\lambda > \omega$. Thus $\{T(t)\}_{t \geq 0}$ is a $C_0$-semigroup with the infinitesimal generator $A$ (see [5, Corollary VIII.1.16]).

4. Representations of the solution of the inhomogeneous equation

In this section we shall present a representation formula for the solution of (E) in terms of the RRO. A similar formula has been given in [19] and [24] under the conditions different from ours. We shall also give an equivalent condition for the
existence of the RRO of (E) by defining wellposedness and using the representation formula.

**Theorem 4.1.** Suppose (E) admits the operator valued function $R_L(\cdot)$ defined in Definition 3.1. Let $x \in X$, $\varphi \in L^1(I_h; Y)$ and $f \in L^1_{\text{loc}}(\mathbb{R}^+; X)$. If $u(t)$ is a solution of (E), then we have for $t \geq 0$

$$u(t) = R_L(t)x + \int_0^h R_L(t-s)(g\varphi)(s)ds + \int_0^t R_L(t-s)f(s)ds,$$

(4.1)

where

$$(g\varphi)(s) = \sum_{i=1}^m \chi_{[0,h_i]}(s)B_i\varphi(s-h_i) + \int_{-h}^s C(r)\varphi(r+s)dr, \ a.a. \ s \in [0,h].$$

**Remark 4.1.** In [21, p.36] Nakagiri has introduced the operator $F_1: L^p(I_h; X) \rightarrow L^p(I_h; X)$ defined by

$$(F_1\varphi)(s) = \sum_{i=1}^m \chi_{[-h_i,0]}(s)B_i\varphi(-s-h_i) + \int_{-h}^s C(r)\varphi(r-s)dr, \ a.a. \ s \in I_h,$$

under the condition that $B_i \in B(X)$ ($i \in \mathbb{N}_m$) and $C(\cdot) \in L^1(I_h; B(X))$. By using (H1), (H2) and the Hausdorff-Young inequality, we also see that the operator $g: L^1(I_h; Y) \rightarrow L^1([0,h]; X)$ is well defined, and hence, in (4.1) the integral $\int_0^h R_L(t-s)(g\varphi)(s)ds$ also is well defined.

**Proof of Theorem 4.1.** Since we have

$$u(t) = \begin{cases} 
  x + \int_0^t \left( Au(s) + \int_{-h}^0 dB(r) \cdot u(r+s) \right) ds + \int_0^t f(s)ds, & t \geq 0, \\
  \varphi(t), & t \in [-h,0),
\end{cases}$$

by (2.1), (5) of Lemma 3.3 and the fact that $R_L(t) = S_L(t) = O$ for $t \in [-h,0)$, we obtain

$$\int_0^t R_L(t-s) \left( u(s) - x - \int_0^s f(r)dr \right) ds$$

$$= \int_0^t R_L(t-s) \int_0^s \left( Au(r) + \int_{-h}^0 dB(p) \cdot u(p+r) \right) dr ds$$

$$= \int_0^t S_L(t-s)Au(s)ds + \sum_{i=1}^m \int_0^t S_L(t-s-h_i)B_iu(s)ds$$

$$+ \int_0^h S_L(t-s) \left( \sum_{i=1}^m \chi_{[0,h_i]}(s)B_i\varphi(s-h_i) + \int_{-h}^s C(r)\varphi(r+s)dr \right) ds$$
where we used Fubini's theorem. Therefore we have
and by differentiating it we obtain (4.1).

As a corollary we see that there can be at most one RRO.

**COROLLARY 4.1.** If \( R_L(t) \) and \( R_R(t) \) both exist on \( \mathbb{R}^+ \) then the RRO \( R(\cdot) \) of (E) exists and is unique.

**PROOF.** Since \( u(\cdot) = R_R(\cdot)y, \ y \in \mathcal{D}(A), \) is a solution of (E) with \( \varphi \equiv 0 \) and \( f \equiv 0 \) for \( y \in \mathcal{D}(A) \) by (3.2), we have \( RR(t)y = RL(t)y \) on \( \mathbb{R}^+ \) by Theorem 4.1. Thus \( RR(t) = RL(t) \) on \( \mathbb{R}^+ \), since \( \mathcal{D}(A) \) is dense. Uniqueness is now evident.

**DEFINITION 4.1.** Let \( R(\cdot) \) be an RRO of (E). Let \( x \in X, f \in L^1(\mathbb{R}^+; X) \) and \( \varphi \in L^1(I_{\infty}; Y) \). Then we shall call \( u, \) defined by (4.1) with \( RL \) replaced by \( R, \) the mild solution of (E) (cf. [21, Lemma 4.1]).

As a consequence we also have the following result.

**COROLLARY 4.2.** Suppose that (E) admits an RRO \( R(\cdot) \). Suppose \( x, x_n \in X, \varphi, \varphi_n \in L^1(I_{\infty}; Y) \); \( f, f_n \in L^1_{\text{loc}}(\mathbb{R}^+; X) \) and let \( x_n \to x \) in \( X, \varphi_n \to \varphi \) in \( L^1(I_{\infty}; Y) \) and \( f_n \to f \) in \( L^1([0, \tau]; X) \) for any \( \tau \in (0, \infty) \) as \( n \to \infty \). Then the mild solutions \( u_n \) of (E) with initial data \( (x_n, \varphi_n) \) and a forcing function \( f_n \) converge uniformly on \([0, \tau]\) to the mild solution of (E) with initial data \((x, \varphi)\) and \( f \).

**PROOF.** We see easily that \( R(t)x_n + \int_0^t R(t-s)f_n(s)ds \) converges uniformly on \([0, \tau]\) to \( R(t)x + \int_0^t R(t-s)f(s)ds \) as \( n \to \infty \). To verify that

\[
\int_0^h R(t-s)(g\varphi_n)(s)ds \to \int_0^h R(t-s)(g\varphi)(s)ds
\]
uniformly on $[0, \tau]$ as $n \to \infty$, consider
\[
|(g \varphi_n)(s) - (g \varphi)(s)| \leq \sum_{i=1}^{m} \chi_{[0,h_i]}(s) b_i |\varphi_n(s-h_i) - \varphi(s-h_i)|_{Y} + \int_{-h}^{-s} c(r) |\varphi_n(r+s) - \varphi(r+s)|_{Y} dr.
\]
Let $M$ be a constant such that $|R(t)| \leq M$ for $t \in [0, \tau]$. Then using Fubini’s theorem we have
\[
\left| \int_{0}^{h} R(t-s)(g \varphi_n)(s) - \int_{0}^{h} R(t-s)(g \varphi)(s) ds \right|
\]
\[
\leq \int_{0}^{h} \left| R(t-s) \left( \sum_{i=1}^{m} \chi_{[0,h_i]}(s) b_i |\varphi_n(s-h_i) - \varphi(s)|_{Y} + \int_{-h}^{-s} c(r) |\varphi_n(r+s) - \varphi(r+s)|_{Y} dr \right) \right| ds
\]
\[
\leq M \left( \sum_{i=1}^{m} b_i \int_{-h_i}^{0} |\varphi_n(s) - \varphi(s)|_{Y} ds + \int_{-h}^{0} c(r) \int_{r}^{0} |\varphi_n(s) - \varphi(s)|_{Y} ds dr \right)
\]
\[
\leq M \int_{-h}^{0} db(r) \cdot \int_{-h}^{0} |\varphi_n(s) - \varphi(s)|_{Y} ds \to 0 \quad \text{as} \quad n \to \infty. \quad \Box
\]

We prove now following two results, corresponding to [22, Theorem 4.2.4] in the case of differential equations, and they will be used in order to obtain a representation formula of the solution of (E).

**THEOREM 4.2.** Suppose $R_R(t)$ is the operator defined in Definition 3.1 and let $f \in C(\mathbb{R}^+; X)$. Define
\[
u(t) \equiv \int_{0}^{t} R_R(t-s)f(s)ds \quad \text{for} \quad t \geq 0 \quad \text{with} \quad \nu(t) \equiv 0 \quad \text{on} \quad [-h, 0).
\]
Then the following statements are equivalent:

1. $u \in C^1(\mathbb{R}^+; X)$ with $u'(0) = 0$
2. $u \in C(\mathbb{R}^+; Y)$
3. $u$ is a solution of (E) on $\mathbb{R}^+$ with $x = 0$ and $\varphi = 0$.

**PROOF.** Clearly (3) implies (1). To prove that (1) implies (2), we use the same technique as in [7]. Without loss of generality we assume $f(0) = 0$. For $\varepsilon > 0$ consider $C^\infty$-functions $\rho_\varepsilon(\cdot)$ such that $\rho_\varepsilon(t) \geq 0$, $\rho_\varepsilon(t) \equiv 0$ for $t \geq \varepsilon$ and $\int_{0}^{\infty} \rho_\varepsilon(t) dt = 1$. Define $f_\varepsilon \in C^1(\mathbb{R}^+; X)$ by
\[
f_\varepsilon(t) \equiv (\rho_\varepsilon \ast f)(t) = \int_{0}^{t} \rho_\varepsilon(t-s)f(s)ds.
\]
Then
\[ u_\varepsilon \equiv \rho_\varepsilon \ast u = \rho_\varepsilon \ast R_R \ast f = R_R \ast f_\varepsilon \text{ on } \mathbb{R}^+ \text{ with } u_\varepsilon \equiv 0 \text{ on } I_h \]
also belongs to \( C^1(\mathbb{R}^+; \mathbb{X}) \) and satisfies \( u_\varepsilon(t) = \int_0^t S_R(t-s)f_\varepsilon'(s)ds \). Thus \( Au_\varepsilon, B_iu_\varepsilon \) and \( \int_{-h}^0 C(s)u_\varepsilon(s+t)ds \) are well defined and continuous on \( \mathbb{R}^+ \). Since \( u_\varepsilon \equiv 0 \) on \( I_h, B_iu_\varepsilon(t-h_i) \) is also continuous on \( \mathbb{R}^+ \) for \( i \in \mathbb{N}_m \), and we have

\[
\begin{align*}
    u'_\varepsilon(t) - Au_\varepsilon(t) &= \int_{-h}^0 dB(r) \cdot u_\varepsilon(r+t) \\
    &= \int_0^t \left( R_R(t-s)f'_\varepsilon(s) - AS_R(t-s)f'_\varepsilon(s) - \int_{-h}^0 dB(r) \cdot S_R(r+t-s)f'_\varepsilon(s) \right)ds \\
    &= \int_0^t f'_\varepsilon(s)ds = f_\varepsilon(t).
\end{align*}
\]

Next we show that \( Au_\varepsilon \) converges uniformly on any finite intervals. From (E) it follows that

\[
|u_\varepsilon(t) - u_\eta(t)|_Y = |u_\varepsilon(t) - u_\eta(t)| + |Au_\varepsilon(t) - Au_\eta(t)|
\leq |u_\varepsilon(t) - u_\eta(t)| + |u'_\varepsilon(t) - u'_\eta(t)|
+ \left| \int_{-h}^0 dB(r) \cdot u_\varepsilon(r+t) - \int_{-h}^0 dB(r) \cdot u_\eta(r+t) \right| + |f_\varepsilon(t) - f_\eta(t)|
\leq C(\varepsilon, \eta) + \int_{-h}^0 db(r) \cdot |u_\varepsilon(r+t) - u_\eta(r+t)|_Y,
\]

where \( \varepsilon, \eta > 0 \) and \( C(\varepsilon, \eta) \equiv \|u_\varepsilon - u_\eta\|_\infty + \|u'_\varepsilon - u'_\eta\|_\infty + \|f_\varepsilon - f_\eta\|_\infty \). Thus for some constants \( \beta > 0, k > 0 \) and any \( \tau \in (0, \infty) \) we have

\[
|u_\varepsilon(t) - u_\eta(t)|_Y \leq kC(\varepsilon, \eta)e^{\beta t} \leq kC(\varepsilon, \eta)e^{\beta \tau} \quad \text{for } t \in J \equiv [0, \tau]
\]
by Lemma 3.2. Since \( f_\varepsilon \to f, \ u_\varepsilon \to u \) and \( u'_\varepsilon = \rho_\varepsilon \ast u' \to u' \) in \( C(J; \mathbb{X}) \) as \( \varepsilon \to 0 \), it follows that \( C(\varepsilon, \eta) \to 0 \) as \( \varepsilon \to 0, \eta \to 0 \). Therefore there exists a function \( u_0 \in C(\mathbb{R}^+; \mathbb{X}) \) such that \( Au_\varepsilon \to u_0 \) in \( C(J; \mathbb{X}) \). Consequently we have \( u \in C(\mathbb{R}^+; \mathbb{Y}) \) and \( u_0 = Au \) by the closedness of \( A \).

To prove that (2) implies (3) suppose \( u \in C(\mathbb{R}^+; \mathbb{Y}) \). Then we have

\[
\int_0^t Au(s)ds = A \int_0^t u(s)ds = A \int_0^t S_R(t-s)f(s)ds = \int_0^t AS_R(t-s)f(s)ds,
\]
and similarly by using Lemma 3.1 and Fubini's theorem we can prove

\[
\sum_{i=1}^m \int_0^{t-h} B_iu(s)ds = \int_0^t \sum_{i=1}^m B_iS_R(t-s-h_i)f(s)ds
\]
and
\[
\int_0^t \left( \int_{-h}^0 C(r)u(r+s)dr \right)ds = \int_0^t \left( \int_{-h}^0 C(r)S_R(r+t-s)f(s)dr \right)ds.
\]
Therefore, we obtain
\[
\int_0^t \left( Au(s) + \int_{-h}^0 dB(r) \cdot u(r+s) \right)ds
= \int_0^t \left( AS_R(t-s)f(s) + \sum_{i=1}^m B_i S_R(t-s-h_i)f(s) \right.
+ \int_{-h}^0 C(r)S_R(r+t-s)f(s)dr \bigg)ds
= \int_0^t \left( AS_R(t-s)f(s) + \int_{-h}^0 dB(r) \cdot S_R(r+t-s)f(s) \right)ds
= \int_0^t (R_R(t-s)f(s) - f(s))ds = u(t) - \int_0^t f(s)ds.
\]
Thus \( u \) is a solution of (E) with \( x = 0 \) and \( \varphi \equiv 0 \). \( \square \)

**THEOREM 4.3.** Suppose \( R_R(\cdot) \) is the function defined in Definition 3.1. Let \( \varphi \in C(I_h; Y) \) and \( u \in C(R^+; Y) \) with \( u = \varphi \) on \([-h, 0)\). Let
\[
u(t) \equiv \int_0^t R_R(t-s)g(s)ds, \quad t \geq 0,
\]
where \( g \) is defined by
\[
g(s) \equiv \begin{cases} (g\varphi)(s) + \sum_{i=1}^m \chi_{[h_i,h]}(s)B_i\varphi(0) & \text{for } s \in [0,h], \\ g(h) & \text{for } s \in (h,\infty). \end{cases}
\]
Then the following statements are equivalent:
1. \( v \in C^1(R^+; X) \),
2. \( u \in C(R^+; Y) \),
3. \( u \) is a solution of (E) on \( R^+ \) with \( x = 0 \) and \( f \equiv 0 \).

**PROOF.** Note that \( g \in C(R^+; X) \) and
\[
u(t) = -\sum_{i=1}^m S_R(t-h_i)B_i\varphi(0) + v(t) \quad \text{for } t \geq 0.
\]
If \( v \in C^{1}([\mathbb{R}+; X]) \), then \( v \in C([\mathbb{R}+; Y]) \) by Theorem 4.2 and so \( u \in C([\mathbb{R}+; Y]) \) since \( S_{R}(\cdot)x \) belongs to \( C([\mathbb{R}+; Y]) \), by Lemma 3.3, for any \( x \in X \).

Next suppose \( u \in C([\mathbb{R}+; Y]) \). Then we have

\[
\int_{0}^{t} Au(s)ds = \int_{0}^{h} A S_{R}(t-s)(g \varphi)(s)ds,
\]

\[
\int_{0}^{t} \sum_{i=1}^{m} B_{i} u(s - h_{i})ds = \sum_{i=1}^{m} \int_{0}^{t} \chi_{[i, h_{i}]}(s) B_{i} \varphi(s - h_{i})ds + \int_{0}^{h} \sum_{i=1}^{m} B_{i} S_{R}(t - h_{i} - s)(g \varphi)(s)ds
\]

for \( t \geq 0 \) and

\[
\int_{0}^{t} \left( \int_{-h}^{s} C(r)u(r + s)dr \right)ds = \int_{0}^{t} \left( \int_{-h}^{-s} \chi_{[-h, 0]}(r)C(r)\varphi(r + s)dr \right)ds + \int_{0}^{t} \left( \int_{-s}^{0} \chi_{[-h, 0]}(r)C(r)u(r + s)dr \right)ds
\]

\[
= \int_{0}^{t} \int_{-h}^{-s} \chi_{[-h, 0]}(r)C(r)\varphi(r + s)dr \right)ds + \int_{0}^{t} \left( \int_{-s}^{0} C(r)S_{R}(r + t - s)(g \varphi)(s)dr \right)ds \quad \text{for } t \leq h,
\]

\[
= \int_{0}^{t} \int_{-h}^{-s} C(r)S_{R}(r + t - s)(g \varphi)(s)dr \right)ds \quad \text{for } t > h.
\]

Hence we have

\[
\int_{0}^{t} \left( Au(s) + \int_{-h}^{0} dB(r) \cdot u(r + s) \right)ds
\]

\[
= \int_{0}^{t} \left( A S_{R}(s) + \sum_{i=1}^{m} B_{i} S_{R}(s - h_{i}) + I \right)(g \varphi)(t - s)ds + \int_{0}^{t} \int_{-s}^{0} C(r) S_{R}(r + s)(g \varphi)(t - s)drds \quad \text{for } t \leq h
\]

and

\[
= \int_{0}^{t} \left( A S_{R}(t - s) + \sum_{i=1}^{m} B_{i} S_{R}(t - h_{i} - s) + I \right)(g \varphi)(s)ds + \int_{0}^{h} \int_{-h}^{0} C(r) S_{R}(r + t - s)(g \varphi)(s)drds \quad \text{for } t > h.
\]
Consequently for $t \geq 0$ we have

$$\int_0^t \left( Au(s) + \int_{-h}^0 dB(r) \cdot u(r + s) \right) ds = \int_0^h R_R(t - s)(g\varphi)(s)ds = u(t).$$

This proves that $u$ is a solution of $(E)$ with $x = 0$ and $f \equiv 0$.

Next we assume (3) holds. Then $Av$ belongs to $C(\mathbb{R}^+; X)$ so that $v(t) \in D(A)$ for all $t \geq 0$ and

$$Av(t) = Au(t) + \sum_{i=1}^m AS_R(t - h_i)B_i\varphi(0).$$

Thus $v \in C(\mathbb{R}^+; Y)$, and hence Theorem 4.2 implies (1). \qed

We are now ready to state sufficient conditions in order that the function $w$ defined by

$$w(t) = R_R(t)y + \int_0^h R_R(t - s)(g\varphi)(s)ds + \int_0^t R_R(t - s)f(s)ds, \quad t \geq 0,$$

with $w(t) = \varphi(t)$ for $t \in [-h, 0)$, is a solution of $(E)$.

**COROLLARY 4.3.** Suppose $R_R(\cdot)$ in Definition 3.1 exists and $R_R(t)$ is a bounded linear operator from $Y$ to $Y$. Let $y \in D(A)$ and $\varphi \in C(I_h; Y)$. Moreover assume the following conditions hold:

$$\begin{align*}
(\varphi_1) \quad & (g\varphi) \in L^1([0, h]; Y), \\
(F1) \quad & f \in L^1_{\text{loc}}(\mathbb{R}^+; Y) \cap C(\mathbb{R}^+; X).
\end{align*}$$

Then $w$ is a solution of $(E)$.

**COROLLARY 4.4.** Suppose $R_R(\cdot)$ in Definition 3.1 exists. Let $y \in D(A)$, and $\varphi \in C(I_h; Y)$. Moreover assume that the following conditions hold:

$$\begin{align*}
(\varphi_2) \quad & g \in W^{1,1}([0, h]; X), \\
(F2) \quad & f \in W^1_{\text{loc}}(\mathbb{R}^+; X).
\end{align*}$$

Then $w$ is a solution of $(E)$.

We next consider the equation

$$\begin{cases}
\frac{du(t)}{dt} = Au(t) + \int_{-h}^0 dB(r) \cdot u(r + t) \quad \text{for a.a. } t > 0, \\
u(0) = y \in D(A) \quad \text{and } u(t) = 0 \quad \text{for } t \in [-h, 0)
\end{cases} \quad (4.2)$$

and give an equivalent condition for the existence of the RRO of $(E)$ by defining wellposedness.

**DEFINITION 4.2.** $(E)$ is called wellposed if for every $y \in D(A)$ there is a unique solution $u(\cdot; y)$ of (4.2) on $\mathbb{R}^+$ with $u(0; y) = y$ and $u(t; y) = 0$ on $[-h, 0)$ such that $y_n \to 0$ ($y_n \in D(A)$) implies $u(t; y_n) \to 0$ uniformly on bounded intervals (cf. [7]).
THEOREM 4.4. (E) admits an RRO $R(\cdot)$ if and only if (E) is wellposed.

PROOF. Let (E) be wellposed. Then the RRO $R(\cdot)$ for (E) has to be given by

$$R(t)y = u(t; y) \quad \text{for all } y \in \mathcal{D}(A) \quad \text{and} \quad t \geq -h.$$  

It is obvious that $R(t)$ is a linear operator and satisfies the conditions (2) through (5) of Definition 3.1 and the second resolvent equation (3.2). The assertion that $R(t)$ is a strongly continuous bounded linear operator can be verified by using exactly the same argument as in [7, Theorem 4]. In order to prove that (3.1) holds for this $R(\cdot)$, consider

$$\varphi(t) \equiv y \quad \text{for } t \in I_h \quad \text{and} \quad f(t) \equiv -Ay - \int_{-h}^{0} dB(s)y \quad \text{for } t \geq 0,$$

where $y \in \mathcal{D}(A)$. Then since these $f$ and $\varphi$ satisfy the conditions (F2) and (φ2) in Corollary 4.4 it follows that

$$u(t) = R(t)y - \int_{0}^{t} R(t-s) \left[Ay + \int_{-h}^{0} dB(r)y\right] ds + \int_{-h}^{h} R(t-s)(g\varphi)(s) ds$$

$$= R(t)y - S(t)Ay - \int_{-h}^{0} S(t+s) \cdot dB(s)y \quad \text{for } t \geq 0$$

is a solution of (E) with these $(y, \varphi)$ and $f$. On the other hand it is obvious that $v(t) = y$ for $t \in [-h, \infty)$ is also a solution of (E) for the same $(y, \varphi)$ and $f$ above. Hence by the uniqueness of the solution, for all $t \geq 0$ and $y \in \mathcal{D}(A)$ we obtain

$$y = v(t) = u(t) = R(t)y - S(t)Ay - \int_{-h}^{0} S(t+s) \cdot dB(s)y.$$

Then differentiating it we have (3.1). Hence, $R(\cdot)$ is an RRO of (E). The converse is obvious by Theorem 4.1 and Definition 3.1.

Next we refer to the relation between our RRO and the fundamental solution in the sense of Nakagiri [19].

DEFINITION 4.3. Let $A$ generate a $C_0$-semigroup $\{T(t)\}_{t \geq 0}$, $B_i \in B(X)$ for all $i \in \mathbb{N}_m$, $C(\cdot) \in L^1(I_h; B(X))$ and $G(\cdot)$ is a unique solution of

$$G(t) = \begin{cases} T(t) + \int_{0}^{t} T(t-s) \int_{-h}^{0} dB(r) \cdot G(r+s) ds, & t \geq 0, \\ 0, & t < 0 \end{cases}$$

in $B(X)$. Then a family of strongly continuous bounded linear operators $\{G(t)\}_{t \geq 0}$ is called the fundamental solution of (E).
**Theorem 4.5.** Under the condition of Definition 4.3 if (E) has an RRO \( R(\cdot) \), it is a fundamental solution of (E), and conversely.

**Proof.** Let \( R(\cdot) \) be an RRO. Then owing to Theorem 4.1 and Lemma 3.1, we see that \( u(\cdot) \equiv R(\cdot)y, y \in \mathcal{D}(A) \), is a unique solution of

\[
u(t) = T(t)y + \int_0^t T(t-s) \int_{-h}^0 \cdot dB(r) \cdot u(r+s)ds\]

for \( t \geq 0 \). Since \( \mathcal{D}(A) \) is dense in \( X \), \( G(\cdot) \) is an RRO in our sense. The converse is clear by [20, Corollary 2.1].

Finally we remark that if \( A \) generates a \( C_0 \)-semigroup \( \{T(t)\}_{t \geq 0} \) and (E) admits an RRO \( R(\cdot) \), then it further satisfies the relation

\[
R(t)y = T(t)y + \int_0^t R(t-s) \int_{-h}^0 \cdot dB(r) \cdot T(r+s)yds
\]

for \( t \geq 0 \) and \( y \in \mathcal{D}(A) \) provided that \( T(t) = O \) for \( t < 0 \).

5. Laplace transform of a retarded resolvent operator

It would be interesting to know whether RROs always have exponential bounds or not under the conditions (H0)–(H2). We are not able to prove this. However, in some cases (see Theorem 6.1 below) an RRO \( R(\cdot) \) satisfies the following growth condition.

(H3) There are constants \( M \geq 1 \) and \( \omega \in \mathbb{R} \) such that \( |R(t)| \leq Me^{\omega t} \) for all \( t \geq 0 \).

If (H3) is valid, as in [7], we have the relation

\[
R(\lambda : \Delta)x = \int_0^\infty e^{-\lambda t}R(t)xdt \quad \text{for} \quad \Re \lambda > \omega_0 \quad \text{and} \quad x \in X,
\]

where \( \omega_0 \) is some constant satisfying \( \omega_0 \geq \omega \). By (5) of Lemma 3.3 we know that \( R(\cdot)y \) is of bounded variation on any bounded interval of \( \mathbb{R}^+ \). Hence by the inversion formula ([10, Chap. VI]), if \( y \in \mathcal{D}(A) \), then

\[
R(t)y = (1/2\pi i) \int_{\gamma-i\infty}^{\gamma+i\infty} e^{\lambda t}R(\lambda : \Delta)y\lambda d\lambda \quad \text{for} \quad \gamma > \omega_0
\]

and for every \( \delta > 0 \), the integral converges uniformly in \( t \) on \([\delta, 1/\delta]\).

In order to prove our main result (Theorem 5.1 below) of this section we require some lemmas.

**Lemma 5.1.** Suppose there exists an \( \omega > 0 \) such that \( \lambda \in \rho(\Delta) \) for all \( \lambda > \omega \) and \( \lim \sup_{\lambda \to -\infty} |\lambda R(\lambda : \Delta)| < \infty \). Then we have
There exist \( \lambda > \omega \) such that \( \sup_{\lambda > \lambda} |AR(\lambda : \Delta)| < \infty \) and
\[
\int_{-h}^{0} e^{\lambda r} \cdot dB(r) R(\lambda : \Delta) \to 0 \quad \text{as} \quad \lambda \to \infty;
\]

(2) there exists \( \omega_1 > 0 \) such that \( \rho(A) \supset \{ \lambda \in \mathbb{R} \mid \lambda > \omega_1 \} \) and
\[
|R(\lambda : A)| = O(1/\lambda) \quad \text{as} \quad \lambda \to \infty;
\]

(3) \( \lambda^n R(\lambda : A)x \to x \) as \( \lambda \to \infty \) for all \( x \in \mathbb{K} \) and \( n \in \mathbb{N} \). In particular, \( \mathcal{D}(A^n) \) is dense in \( \mathbb{K} \) for each \( n \in \mathbb{N} \);

(4) \( \lambda R(\lambda : \Delta)x \to x \) as \( \lambda \to \infty \) for all \( x \in \mathbb{K} \);

(5) \( \lambda^2 R(\lambda : \Delta)y - \lambda y \to Ay \) as \( \lambda \to \infty \) for all \( y \in \mathcal{D}(A) \).

**Proof.** From the relation
\[
(\lambda - A - \int_{-h}^{0} e^{\lambda r} \cdot dB(r)) R(\lambda : \Delta)x = x \quad \text{for} \quad x \in \mathbb{K} \quad \text{and} \quad \lambda > \omega
\]
we obtain the estimate
\[
|AR(\lambda : \Delta)x| = \left| \left( \lambda - \int_{-h}^{0} e^{\lambda r} \cdot dB(r) \right) R(\lambda : \Delta)x - x \right|
\]
\[
\leq |\lambda R(\lambda : \Delta)x - x| + \left| \int_{-h}^{0} e^{\lambda r} \cdot dB(r) R(\lambda : \Delta)x \right|
\]
\[
\leq |\lambda R(\lambda : \Delta)x - x| + \int_{-h}^{0} e^{\lambda r} \cdot dB(r)(|R(\lambda : \Delta)x| + |AR(\lambda : \Delta)x|).
\]
Therefore we have by the assumption \( \limsup_{\lambda \to \infty} |\lambda R(\lambda : \Delta)| < \infty \) that
\[
|AR(\lambda : \Delta)| \leq \left( 1 - \int_{-h}^{0} e^{\lambda r} \cdot dB(r) \right)^{-1} \left( 1 + |\lambda R(\lambda : \Delta)| \left( 1 + \int_{-h}^{0} (e^{\lambda r}/\lambda) \cdot dB(r) \right) \right)
\]
for sufficiently large \( \lambda \) and
\[
\left| \int_{-h}^{0} e^{\lambda r} \cdot dB(r) R(\lambda : \Delta) \right| \leq \int_{-h}^{0} e^{\lambda r} \cdot dB(r)(|R(\lambda : \Delta)| + |AR(\lambda : \Delta)|) \to 0
\]
as \( \lambda \to \infty \). Hence (1) holds. Next note that
\[
\lambda - A = \left( I + \int_{-h}^{0} e^{\lambda r} \cdot dB(r) R(\lambda : \Delta) \right)(\lambda - \Delta(\lambda)),
\]
so \( R(\lambda : A) \) exists for sufficiently large \( \lambda \) and by the inequality
\[
\limsup_{\lambda \to \infty} |\lambda R(\lambda : A)| \leq \limsup_{\lambda \to \infty} \left( |\lambda R(\lambda : \Delta)| \left( 1 + \int_{-h}^{0} e^{\lambda r} \cdot dB(r) R(\lambda : \Delta) \right)^{-1} \right).
\]
the assertion (2) holds. For (3) and (4) see [7, Corollary 3]. Finally (5) follows directly from (1), (4) and the relation
\[
\lambda^2 R(\lambda : \Delta)y - \lambda y = \lambda R(\lambda : \Delta) \left( A y + \int_{-h}^{0} e^{\lambda r} \cdot dB(r) y \right).
\]
\[
\square
\]

Analogously we can prove the following Lemma.

**Lemma 5.2.** Suppose there exists an \( \omega > 0 \) such that \( \lambda \in \rho(A) \) for all \( \lambda > \omega \) and \( \lim \sup_{\lambda \to \infty} |\lambda R(\lambda : A)| < \infty \). Then

1. \( f_\infty \approx r . dB(r)R(A : A) \to 0 \) as \( \lambda \to \infty \);
2. there exists an \( \omega_1 > 0 \) such that \( \rho(\Delta) \supset \{ \lambda \in \mathbb{R} | \lambda > \omega_1 \} \) and
\[
|R(\lambda : \Delta)| = O(1/\lambda) \text{ as } \lambda \to \infty.
\]

**Lemma 5.3** (cf. [20]). The retarded resolvent set \( \rho(\Delta) \) is open in \( \mathbb{C} \) and the retarded resolvent \( R(\lambda : \Delta) \) is analytic in \( \lambda \) on \( \rho(\Delta) \).

**Proof.** Let \( \lambda \in \rho(\Delta) \) and \( \mu \in \mathbb{C} \). Since \( AR(\lambda : \Delta) \in B(\mathbb{X}) \) and \( R(\lambda : \Delta) \) satisfies, by (H1) and (H2),
\[
|B_i R(\lambda : \Delta)| \leq a|R(\lambda : \Delta)| + b|AR(\lambda : \Delta)|, \quad i \in \mathbb{N}_m,
\]
and \( C(t) \) satisfies
\[
|C(t)R(\lambda : \Delta)| \leq c(t)(|R(\lambda : \Delta)| + |AR(\lambda : \Delta)|),
\]
it is easily seen that
\[
\lim_{\mu \to \lambda} \int_{-h}^{0} (e^{\lambda s} - e^{\mu s}) \cdot dB(s)R(\lambda : \Delta) = O \text{ in } B(\mathbb{X}).
\]

Therefore the assertion of this lemma follows from the relation
\[
\mu - \Delta(\mu) = \left\{ I - \left( (\lambda - \mu)R(\lambda : \Delta) - \int_{-h}^{0} (e^{\lambda s} - e^{\mu s}) \cdot dB(s)R(\lambda : \Delta) \right) \right\}^{-1}(\mu - \Delta(\lambda)).
\]
\[
\square
\]

We now prove the main result of this section which has some interesting applications.

**Theorem 5.1.** The following statements are equivalent:
1. \( (E) \) has an \textbf{RRO} \( R(\cdot) \) satisfying (H3),

\[
\lambda^2 R(\lambda : \Delta)y - \lambda y = \lambda R(\lambda : \Delta) \left( A y + \int_{-h}^{0} e^{\lambda r} \cdot dB(r) y \right).
\]
(2) there exists an operator valued function \( R_L(\cdot) \), defined in Definition 3.1, satisfying (H3) and there exists a constant \( \alpha \geq \omega \) such that \( \lambda \in \rho(\Delta) \) for all \( \lambda > \alpha \) and \( \limsup_{\lambda \to \infty} |AR(\lambda : \Delta)| < \infty \).

**Proof.** By the consideration made preceding Lemma 5.1 and Definition 3.1, it is evident that (1) implies (2). Suppose (2) is valid. Then \( \rho(A) \) is nonempty by Lemma 5.1. Let

\[
F(t)x = \mu S_L(t)x - (R_L(t) - I)x \quad \text{for } x \in \mathcal{X},
\]

\[
K_i = B_i R(\mu : A) \quad \text{and} \quad L(t) = C(t) R(\mu : A),
\]

where \( \mu \in \rho(A) \). Then by (H1) \( K_i \) is a bounded linear operator for each \( i \in \mathbb{N}_m \), by (H2) \( L(t) \in B(\mathcal{X}) \) for \( t \in I_h \) with \( L(\cdot)x \in L^1(I_h; \mathcal{X}) \) for \( x \in \mathcal{X} \), and \( F(t) \) is also a strongly continuous bounded linear operator with \( F(0) = 0 \). By (H3) there are constants \( M_1 \geq M \) and \( \alpha \geq \omega \) such that \( |F(t)| \leq M_1 e^{\alpha t} \) for \( t \geq 0 \). Define

\[
K(t) = - \sum_{i=1}^{m} \chi_{(-\infty,-h_i]_i}(t) K_i - \int_t^0 L(s)ds, \quad t \in I_h;
\]

\[
k(t) = - \sum_{i=1}^{m} \chi_{(-\infty,-h_i]_i}(t) |K_i| - \int_t^0 |L(s)|ds, \quad t \in I_h.
\]

Then it is easy to see that \( K(\cdot) \) is of bounded semi-variation in the sense of Hönig [11, p.22].

Consider the integral equation of the form

\[
u(t) = F(t)x + \int_{-h}^0 dK(r) \cdot u(r + t) \quad \text{for } t \geq 0,
\]

with \( u(t) = 0 \) for \( t \in [-h,0) \). To prove the existence and uniqueness of solutions of (5.1), we will use the Banach fixed point theorem. Fix \( \tau > 0 \) and \( x \in \mathcal{X} \), and put \( C = \{ u \in C([-h,\tau]; \mathcal{X}) \mid u(t) = 0 \text{ for } t < 0 \} \). We define a mapping \( F : C \to C \) by

\[
(Fu)(t) = F(t)x + \int_{-h}^0 dK(r) \cdot u(r + t) \quad \text{for } t \in [0,\tau]
\]

with \( (Fu)(t) = 0 \) for \( t \in [-h,0) \). By [11, p.24, Theorem 4.6] the Stieltjes integral \( \int_{-h}^0 dK(r) \cdot u(r + t) \) exists and \( \int_{-h}^0 dK(r) \cdot u(r + t) = \int_{-h}^0 dK(r) \cdot u(r + t) \) holds ([11, p.7]), and we get

\[
|(Fu)(t) - (Fu)(t')| \leq |F(t)x - F(t')x|
\]

\[
+ \int_{-h}^0 dk(r) \cdot \sup_{\theta \in I_h} |u(t + \theta) - u(t' + \theta)| \quad \text{for } t,t' \in [0,\tau],
\]
since we have
\[
\int_{-h}^{0} dK(r) \cdot u(t+r) = \sum_{i=1}^{m} K_i u(t-h_i) + \int_{-h}^{0} L(s) u(s+t) ds
\]
and
\[
\left| \int_{-h}^{0} dK(r) \cdot u(t+r) \right| \leq \sum_{i=1}^{m} |K_i| |u(t-h_i)| + \int_{-h}^{0} |L(s)| |u(s+t)| ds
\]
\[
= \int_{-h}^{0} \cdot dk(r) \cdot |u(r+t)|.
\]
Hence, it is easy to see that the mapping \( F \) is well defined. Now we choose \( \lambda > 0 \) independent of \( \tau \) such that
\[
\int_{-h}^{0} dk(r) \cdot e^{\lambda r} = \sum_{i=1}^{m} |K_i| e^{-\lambda h_i} + \int_{-h}^{0} L(s) e^{\lambda s} ds < 1
\]
and introduce the norm \( \| \cdot \|_{\lambda} \) by \( \| u \|_{\lambda} = \sup \{|u(t)|e^{-\lambda t} | t \in [0, \tau]| \} \) for \( u \in C \). It is immediate that \( C \) is a Banach space endowed with \( \| \cdot \|_{\lambda} \) and with respect to this norm \( F \) is a contraction mapping:
\[
\| Fu - Fv \|_{\lambda} \leq \int_{-h}^{0} dk(r) \cdot e^{\lambda r} \| u - v \|_{\lambda} \quad \text{for} \quad u, v \in C.
\]
Hence, by the Banach fixed point theorem, (5.1) has a unique solution on \([0, \tau]\). Since \( \tau > 0 \) is arbitrary, (5.1) has a unique solution on \( \mathbb{R}^+ \) with \( u(t) = 0 \) for \( t < 0 \). Let \( u \) be the continuous solution of (5.1) with \( u(t) = 0 \) for \( t \in [-h, 0) \). Then we have
\[
|u(t)| \leq M_1 e^{\alpha t}|x| + \int_{-h}^{0} \cdot dk(r) \cdot |u(r+t)|.
\]
Therefore, by Lemma 3.2 we obtain
\[
|u(t)| \leq kM_1 e^{(\alpha+\beta)t}|x|,
\]
for some positive constants \( k \) and \( \beta \). Since we can choose \( \omega_2 \) sufficiently large so that
\[
Q(\lambda) = \left( I - \int_{-h}^{0} e^{\lambda s} \cdot dK(s) \right)^{-1}
\]
eexists in \( B(\mathcal{X}) \) for \( \text{Re} \lambda > \omega_2 \), \( \tilde{u}(\lambda) \) exists and satisfies \( \tilde{u}(\lambda) \equiv Q(\lambda)\tilde{F}(\lambda)x \) for \( \text{Re} \lambda > \alpha + \beta + \omega_2 \). Define \( v(t) \equiv R(\mu : A)u(t) \). Then we have
\[
\tilde{v}(\lambda) = R(\mu : A)\tilde{u}(\lambda) = R(\mu : A)Q(\lambda)\tilde{F}(\lambda)x
\]
\[
= (1/\lambda)R(\mu : A)Q(\lambda)(\mu R(\lambda : \Delta) - (\lambda R(\lambda : \Delta) - I))x
\]
On the other hand by (5) of Lemma 3.3 we have

\[ \lambda \tilde{S}_L(\lambda)y = \tilde{R}_L(\lambda)y = (1/\lambda)y + \tilde{S}_L(\lambda)Ay + \int_{-h}^{0} e^{\gamma r} \cdot dB(r)y \]

doing this for \( y = R(\lambda : \Delta) x \) and \( \Re \lambda > \alpha + \beta + \omega_2 \). Therefore we have \( \tilde{S}_L(\lambda)x = (1/\lambda)R(\lambda : \Delta)x \) for \( x \in X \). Hence, by the uniqueness of the Laplace transform, we have \( R(\mu : A)u(t) = \nu(t) = S_L(t)x \) for \( x \in X \). This shows that \( S_L(t)X \subset D(A) \), \( AS_L(\cdot)x \) is continuous and that \( S_L(t) \) satisfies

\[ (\mu - A)S_L(t)x = u(t) = F(t)x + \int_{-h}^{0} dK(r) \cdot u(r + t) \]
\[ = F(t)x + \int_{-h}^{0} dK(r) \cdot (\mu - A)S_L(r + t)x \]
\[ = \mu S_L(t)x - (R_L(t)x - x) + \int_{-h}^{0} dB(r) \cdot S_L(r + t)x, \]

or

\[ R_L(t)x = x + AS_L(t)x + \int_{-h}^{0} dB(r) \cdot S_L(r + t)x. \quad (5.2) \]

If \( y \in D(A) \), then \( R_L(t)y \in D(A) \) by (5) of Lemma 3.3 and we have

\[ AR_L(t)y = Ay + AS_L(t)Ay + \int_{-h}^{0} AS_L(t + r) \cdot dB(r)y \]

so \( R_L(\cdot)y \in C(\mathbb{R}^+; Y) \). Thus

\[ AS_L(t)y = \int_{0}^{t} AR_L(s)yds, \]
\[ B_i S_L(t - h_i)y = \int_{0}^{t - h_i} B_i R_L(s)yds = \int_{0}^{t} B_i R_L(s - h_i)yds \]

for each \( i \in \mathbb{N}_m \) and

\[ \int_{-h}^{0} C(s)S_L(s + t)yds = \int_{0}^{t} \left( \int_{-h}^{0} C(r)R_L(r + s)ydr \right)ds. \]

Combining these relations with (5.2), we have

\[ R_L(t)y = y + \int_{0}^{t} AR_L(s)yds + \int_{-h}^{0} dB(r) \cdot R_L(r + s)yds. \]
This proves that $R_L(\cdot)$ is an RRO for (E).

The following result is an interesting consequence from Theorem 5.1 and it is actually the Hille-Yosida theorem in semigroup theory if $B_i \equiv O$ for $i \in \mathbb{N}_m$ and $C(s) \equiv O$ for $s \in I_h$. Our proof is similar to that of [7, Theorem 8], so for details see that (cf. [2]).

**Theorem 5.2.** A necessary and sufficient condition for the existence of an RRO $R(\cdot)$ satisfying (H3) is that there are some constants $\omega$ and $M \geq 1$ such that $\lambda \in \rho(\Delta)$ for all $\Re \lambda > \omega$ and that $R(\lambda : \Delta)$ satisfies

$$|(1/n!)(d^n/d\lambda^n)R(\lambda : \Delta)| \leq M(\Re \lambda - \omega)^{-n-1}$$

for all $\Re \lambda > \omega$ and $n \in \mathbb{N} \cup \{0\}$.

**Proof.** Necessity is easy. To prove sufficiency we define

$$R_n(t) \equiv e^{-nt} \left( I + \sum_{k=0}^{\infty} (-1)^k \frac{(n^2 t)^{k+1} R^{(k)}(n : \Delta)}{k!(k+1)!} \right) \text{ for } t \geq 0,$$

with $R_n(t) \equiv O$ for $t \in [-h, 0)$. From our assumption it is easily checked that the series converges absolutely and uniformly on each finite interval and in addition, we have

$$|R_n(t)| \leq M \exp(n\omega/(n - \omega)) < Me^{2\omega t} \text{ for } n > 2\omega.$$

Hence $R_n(t)$ is well defined for $t \geq 0$.

We will show that for $x \in XR_n(t)x$ converges to $R_L(t)x$, where $R_L(t)$ is the operator valued function defined in Definition 3.1, as $n \to \infty$ uniformly for $t$ bounded. To this aim we further define

$$S_n(t) \equiv \int_0^t R_n(s)ds \text{ for } t \geq 0 \text{ and } S_n(t) = O \text{ for } t \in [-h, 0).$$

Then the Laplace transform of $R_n(t)$ is given by

$$\hat{R}_n(\lambda) = 1/(\lambda + n) + ((1/\lambda)\psi_n(\lambda))^2 R(\psi_n(\lambda) : \Delta)$$

for $\Re \lambda > n\omega/(n - \omega)$, where $\psi_n(\lambda) \equiv \lambda n/(\lambda + n)$. Hence $\hat{R}_n(\lambda) \to R(\lambda : \Delta)$ as $n \to \infty$ uniformly for bounded $\Re \lambda > \omega$. Also $S_n(t)$ has the Laplace transform $\hat{S}_n(\lambda)$, for which we can apply the inversion formula to get

$$S_n(t) = 1/(2\pi i) \int_{\gamma - i\infty}^{\gamma + i\infty} e^{\lambda t} \hat{S}_n(\lambda)d\lambda = 1/(2\pi i) \int_{\gamma - i\infty}^{\gamma + i\infty} e^{\lambda t} \hat{R}_n(\lambda)d\lambda/\lambda,$$
where $\gamma > \text{Max}(\omega, 0)$. Therefore from the relation

$$S_n(t)y = ty + 1/(2\pi i) \int^{\gamma + i\infty}_{\gamma - i\infty} e^{\lambda t} (n/(\lambda + n)) R(\psi_n(\lambda) : \Delta)$$

$$= ty + \int_{-h}^{0} e^{\psi_n(\lambda)} \cdot dB(r)y \, d\lambda/\lambda^2$$

for $y \in \mathcal{D}(A)$ we see that

$$S_n(t)y \to S(t)y$$

$$= ty + 1/(2\pi i) \int^{\gamma + i\infty}_{\gamma - i\infty} e^{\lambda t} R(\lambda : \Delta) \left( Ay + \int_{-h}^{0} e^{\lambda r} \cdot dB(r)y \right) d\lambda/\lambda^2$$

as $n \to \infty$ uniformly for $t$ bounded. Thus $\lim_{n \to \infty} S_n(t)x = S(t)x$ exists for all $x \in \mathcal{X}$ uniformly for $t$ bounded, since $S_n(t)$ is uniformly bounded with respect to $n$ for bounded $t \geq 0$, and we have $|S(t)| \leq \int_{0}^{t} M e^{2\omega s} ds$ and $\widetilde{S}(\lambda)x = \int_{0}^{\infty} e^{-\lambda t} S(t)x dt$.

Since

$$\widetilde{S}(\lambda)x = \lim_{n \to \infty} \widetilde{S}_n(\lambda)x = (1/\lambda) \lim_{n \to \infty} \widetilde{R}_n(\lambda)x = (1/\lambda) R(\lambda : \Delta)x$$

for $x \in \mathcal{X}$, we have

$$R(\lambda : \Delta)y = (1/\lambda)y + \widetilde{S}(\lambda)Ay + \widetilde{S}(\lambda) \int_{-h}^{0} e^{\lambda r} \cdot dB(r)y$$

for $y \in \mathcal{D}(A)$. By [10, Theorem 6.3.3] we obtain

$$R_n(t)y - y$$

$$= e^{-nt} \sum_{k=0}^{\infty} \frac{(-1)^k (n^2 t)^{k+1}}{k!(k+1)!} \frac{d^k/d\lambda^k}{d\lambda} \left( \widetilde{S}(\lambda)Ay + \widetilde{S}(\lambda) \int_{-h}^{0} e^{\lambda r} \cdot dB(r)y \right)_{\lambda=n}$$

$$= e^{-nt} \sum_{k=0}^{\infty} \frac{(n^2 t)^{k+1}}{k!(k+1)!} \int_{0}^{\infty} s^k e^{-ns} \left( S(s)Ay + \int_{-h}^{0} S(s+r) \cdot dB(r)y \right) ds$$

$$\to S(t)Ay + \int_{-h}^{0} S(t+r) \cdot dB(r)y \quad \text{as} \quad n \to \infty$$

uniformly for $t$ bounded. Hence for $x \in \mathcal{X}$, $\lim_{n \to \infty} R_n(t)x = R(t)x$ also exists uniformly for $t$ bounded and $R(t)$ satisfies $S(t)x = \int_{0}^{t} R(s)x ds$ for $x \in \mathcal{X}$. Therefore

$$R(t)y = y + S(t)Ay + \int_{-h}^{0} S(t+r) \cdot dB(r)y \quad \text{for} \quad y \in \mathcal{D}(A).$$
This shows that $R(t) = R_\ell(t)$ exists and the statement (2) of Theorem 5.1 holds. Thus applying Theorem 5.1 we have the conclusion.

The following result is a simple application of Theorem 5.1 (cf. [7, Theorem 7]).

**Corollary 5.1.** Let $R(\cdot)$ be an RRO for (E) satisfying (H3). Then $A$ commutes with $R(t)$ for all $t \geq 0$ if and only if there is $\mu \in \rho(A)$ such that $R(\mu : A)$ commutes with $B_0(t)$ for a.a $t \geq 0$, where

$$B_0(t)y = \sum_{i=1}^{m} \chi_{[0,\infty)}(t-h_i)B_iy + \int_{-h}^{0} \chi_{[0,\infty)}(t+s)C(s)yds, \quad y \in \mathcal{D}(A).$$

**Proof.** Let $\mu \in \rho(A)$. Then the following chain of equivalences is easily checked:

$$AR(t)y = R(t)Ay \quad \text{for all } y \in \mathcal{D}(A) \text{ and } t \geq 0$$

iff $R(\mu : A)R(t) = R(t)R(\mu : A)$ for all $t \geq 0$

iff $R(\mu : A)R(\lambda : \Delta) = R(\lambda : \Delta)R(\mu : A)$ for all $\lambda > \omega_0$, where $\omega_0$ is some constant satisfying $\omega_0 \geq \omega$

iff $\left( \lambda - A - \int_{-h}^{0} e^{\lambda r} \cdot dB(r) \right) R(\mu : A)y = R(\mu : A) \left( \lambda - A - \int_{-h}^{0} e^{\lambda r} \cdot dB(r) \right) y$

for all $y \in \mathcal{D}(A)$ and $\lambda > \omega_0$

iff $\int_{-h}^{0} e^{\lambda r} \cdot dB(r)R(\mu : A)y = R(\mu : A)\int_{-h}^{0} e^{\lambda r} \cdot dB(r)y$

for all $y \in \mathcal{D}(A)$ and $\lambda > \omega_0$

iff $B_0(\lambda)R(\mu : A)y = R(\mu : A)B_0(\lambda)y$ for all $y \in \mathcal{D}(A)$ and $\lambda > \omega_0$

iff $B_0(t)R(\mu : A)y = R(\mu : A)B_0(t)y$ for all $y \in \mathcal{D}(A)$ and a.a. $t$.

\[ \square \]

6. **Another existence result for $R(t)$**

The aim of this section is to give another existence theorem for an RRO satisfying (H3) under additional conditions by using Tsuruta’s method ([25]) for the most part. We assume (H2) holds and $A$ is the infinitesimal generator of a $C_0$-semigroup $\{T(t)\}_{t \geq 0}$ satisfying $|T(t)| \leq Me^{\omega t}$ for $t \geq 0$ with $\omega \in \mathbb{R}$. In what follows we also assume $T(t) = O$ for $t < 0$. Moreover, we suppose that the next conditions (H4) and (H5) hold.

(H4) For each $i \in \mathbb{N}_m \mathcal{D}(A) \subset \mathcal{D}(B_i)$, $B_iR(\mu : A) \in \mathcal{B}(X)$ for some $\mu > \omega$ and that there exist functions $p_i \in L^1_{loc}(\mathbb{R}^+) \text{ satisfying}$

$$|B_iT(t)y| \leq p_i(t)e^{\omega t}|y| \quad \text{for a.a. } t > 0 \text{ and } y \in \mathcal{D}(A).$$

(H5) There exists a constant $c$ such that
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\[ |\int_{-t}^{0} C(s)T(s + t)y ds| \leq ce^{\omega t}|y| \text{ for } y \in \mathcal{D}(A) \text{ and } t \geq 0, \]

where we set \( C(s) = 0 \) for \( s < -h \).

**Remark 6.1.** In the condition (H4) (cf. [17]), if we further assume that \( B_i, \ i \in \mathbb{N}_m \), are closed operators, then the condition \( B_i R(\mu : A) \subseteq B(\mathcal{X}) \) for some \( \mu > \omega \) is automatically satisfied (cf. [5, p.631]).

**Remark 6.2.** In practical applications of the theory of linear functional differential equations, deciding what conditions would enable us to construct the RRO is an important problem. We assumed the conditions (H4) and (H5) by means of a \( C_0 \)-semigroup \( \{T(t)\}_{t \geq 0} \), while in [24] the existence of the RRO in our sense has been proved for the case \( m = 1 \) and \( C(\cdot) = a(\cdot)B \) in (E) under the conditions that (i) \( A \) is a densely defined closed linear operator which generates an analytic semigroup \( \{T(t)\}_{t \geq 0} \) in \( \mathcal{X} \); (ii) \( B_1 \) and \( B \) are closed linear operators with domains \( \mathcal{D}(B_1) \) and \( \mathcal{D}(B) \) containing \( \mathcal{D}(A) \); (iii) \( a(\cdot) \) is a real valued Hölder continuous function defined in \( I_h \). It can be seen by (3.70) and (3.71) in [24] that our condition (H5) is satisfied under the conditions (i)–(iii). However (H4), in which we assumed \( p(t) \in L^1_{\text{loc}}(\mathbb{R}^+) \), is not satisfied under the conditions (i), (ii) in general.

**Remark 6.3.** Let (H4) be satisfied. Then it is known (cf. [5, 12, 17]) that for all \( i \in \mathbb{N}_m \) \( B_i R(\lambda : A) \subseteq B(\mathcal{X}) \) for all \( \Re \lambda > \omega \) and we have

\[ |B_i y| = |B_i R(\mu : A)(\mu - A)y| \leq |B_i R(\mu : A)|(1 + |\mu|)|y|, \quad i \in \mathbb{N}_m, \]

for \( y \in \mathcal{D}(A) \), so that (H1) is satisfied.

It is easily seen that for each \( i \in \mathbb{N}_m \) there exists a unique bounded extension \( \overline{B_i T}(t) \) of \( B_i T(t) \) for a.a. \( t > 0 \) and

\[ |\overline{B_i T}(t)| \leq p(t)e^{\omega t}, \quad \text{a.a. } t > 0, \]

holds, where we write \( B = \sum_{i=1}^{m} B_i \) and \( p(t) = \sum_{i=1}^{m} p_i(t) \).

First of all we set

\[ \mathcal{F}(t)y = \int_{-t}^{0} C(s)T(s + t)y ds \text{ for } y \in \mathcal{D}(A) \text{ and } t \geq 0. \]

Let \( \overline{\mathcal{F}}(t) \) be the bounded extension of \( \mathcal{F}(t) \), define \( R_{\overline{\mathcal{F}}}(t) \in B(\mathcal{X}) \) by

\[ R_{\overline{\mathcal{F}}}(t)x = \overline{\mathcal{F}}(t)x + \int_{0}^{t} \overline{\mathcal{F}}(t - s)R_{\overline{\mathcal{F}}}(s)x ds \]

\[ = \overline{\mathcal{F}}(t)x + \int_{0}^{t} R_{\overline{\mathcal{F}}}(t - s)\overline{\mathcal{F}}(s)x ds \text{ for } x \in \mathcal{X} \text{ and } t \geq 0, \quad (6.1) \]
whose existence is known ([25]). We set
\[ U_T(t)x = T(t)x + \int_0^t T(t-s)R_F(s)xds \quad \text{for } x \in X \text{ and } t \geq 0 \]  
with \( U_T(t) = O \) for \( t \in [-h, 0) \). It is also known ([25]) that we get
\[ U_T(t)x = T(t)x + UT(t-s)F(s)xds \quad \text{for } x \in X \text{ and } t \geq 0 \]  
and \( U_T(t)D(A) \subset D(A) \) for \( t \geq 0 \) with
\[ \int_{-t}^0 C(s)U_T(s+t)yds = R_F(t)y \quad \text{for } y \in D(A) \text{ and } t \geq 0. \]

Following [22, p.19] we introduce a new norm \( \|x\| \) satisfying
\[ \|T(t)\| \leq e^{\omega t} \quad \text{and} \quad |x| \leq \|x\| \leq M|x| \quad \text{for } x \in X. \]  
Then, by means of Gronwall's inequality, it follows easily from (6.1) and (6.2) that
\[ |R_F(t)| \leq ce^{(\omega+c)t} \quad \text{and} \quad \|R_F(t)\| \leq cMe^{(\omega+cM)t}, \quad t \geq 0; \]
\[ |U_T(t)| \leq Me^{(\omega+c)t} \quad \text{and} \quad \|U_T(t)\| \leq e^{(\omega+cM)t}, \quad t \geq 0. \]  

We are now in a position to establish our main result of this section, which asserts the existence of \( R(\cdot) \) satisfying the growth condition (H3) and some other properties.

**THEOREM 6.1.** There exists an \textbf{RRO} \( R(\cdot) \) satisfying
\[ \|R(t)\| \leq e^{\Omega t} \quad \text{for } t \geq 0, \]  
where
\[ \omega_1 \equiv (1/h_1) \log \left( 1 + Me^{-\omega_1} \int_0^{h_1} e^{-cMs}p(s)ds \right), \]  
and \( \Omega = \omega + cM + \omega_1. \) If \( \text{Re} \lambda > \omega_0, \) we have
\[ R(\lambda : \Delta)x = \int_0^\infty e^{-\lambda t}R(t)xdt \quad \text{for } x \in X, \]  
where \( \omega_0 \equiv \inf\{\alpha \mid |R(t)| \leq Me^{\alpha t}, \quad t \geq 0, \text{ for some } M > 0\}. \) Furthermore, there exists \( c_1 \in L^1_{\text{loc}}(\mathbb{R}^+) \) such that
\[ \left| \int_{-t}^0 dB(s) \cdot R(s+t)y \right| \leq c_1(t)y, \quad \text{for a.a. } t > 0 \text{ and } y \in D(A). \]
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PROOF. We may assume that \( \omega + c > 0 \) and set

\[
k = \frac{(e^{\omega_1 h_1} - 1)/(e^{(\omega_1 + cM)h_1} - 1)}.
\]  

(6.9)

In order to prove (6.6) we define the operators \( R(t) \) inductively by

\[
R(t)x = U_T(t)x + H(t)x + \int_0^t H(s)R_F(t-s)xds, \quad x \in X,
\]

(6.10)

where \( R(t) \equiv O \) for \( t < 0 \) and for \( t \geq 0 \) and \( x \in X \) we set

\[
H(t)x \equiv \int_0^t \sum_{i=1}^m R(t-s-h_i)B_iT(s)xds \quad \text{with} \quad H(t) \equiv O \quad \text{for} \quad t < 0.
\]

It is easy to see that

\[
H(t)x = H(t-h_1)T(h_1)x + \sum_{i=1}^m \int_0^{h_1} R(t-s-h_i)B_iT(s)xds, \quad x \in X.
\]

(6.11)

Since it is necessary not only to prove (6.6) but also to make use of Laplace transforms \( R(\lambda) \) and \( H(\lambda) \) in the following argument, we prove by induction

\[
\|R(t)\| \leq e^{\Omega t} \quad \text{and} \quad \|H(t)\| \leq k(e^{\Omega_1 t} - e^{\omega_1 t})
\]

(6.12)

for \( t \geq 0 \) simultaneously. For \( t \in [0, h_1) \) it is clear that \( H(t) = O \) and \( R(t) = U_T(t) \), and we have \( \|R(t)\| \leq e^{(\omega + cM)t} \). Next we assume (6.12) holds for \( t \in [0, nh_1), \ n \geq 1 \). Then, for \( t \in [nh_1, (n+1)h_1) \) we have by (6.11), (6.4) and the induction hypothesis

\[
\|H(t)x\| \leq k(e^{\Omega(t-h_1)} - e^{\omega(t-h_1)})e^{\omega_1 h_1}\|x\|
\]

\[
+ \sum_{i=1}^m \int_0^{h_1} e^{\Omega(t-s-h_i)} M|B_iT(s)x|ds \quad \text{for} \quad x \in X.
\]

(6.13)

By (H4), (6.4), (6.7) and \( \Omega = \omega + cM + \omega_1 \) it follows that

\[
\sum_{i=1}^m \int_0^{h_1} e^{\Omega(t-s-h_i)} M|B_iT(s)x|ds \leq e^{\Omega(t-h_1)} M\int_0^{h_1} e^{-(cM+\omega_1)p(s)}ds\|x\|
\]

\[
\leq e^{\Omega(t-h_1)} e^{\omega_1 h_1} \left( Me^{-\omega_1} \int_0^{h_1} e^{-cMsp(s)}ds \right)\|x\|
\]

\[
= e^{\Omega(t-h_1)} e^{\omega_1 h_1} (e^{\omega_1 h_1} - 1)\|x\|.
\]

(6.14)

Combining (6.13) and (6.14) we obtain

\[
\|H(t)x\| \leq (e^{\Omega(t-h_1)} e^{\omega_1 h_1} (k + (e^{\omega_1 h_1} - 1)) - ke^{\omega t})\|x\| \quad \text{for} \quad x \in X.
\]
Since \( k + (e^{\omega_1 h_1} - 1) = ke^{(\omega_1 + cM) h_1} \) by (6.9), we obtain
\[
\|H(t)\| \leq k(e^{\Omega t} - e^{\omega t}) \quad \text{for } t \in [nh_1, (n+1)h_1).
\] (6.15)

Moreover by (6.10), (6.5) and (6.15) we have
\[
\|R(t)\| \leq (e^{(\omega + cM)t} + k(e^{\Omega t} - e^{\omega t}))
+ \int_{h_1}^{t} k(e^{\Omega s} - e^{\omega s})cMe^{(\omega + cM)(t-s)}ds \equiv I + II,
\] (6.16)

where we used the fact \( H(t) = O \) for \( t \in [0, h_1) \). Since \( \Omega = \omega + cM + \omega_1 \) and \( kcM \leq (1-k)\omega_1 \), it follows that
\[
II \leq e^{(\omega + cM)t} \int_{h_1}^{t} (1-k)\omega_1 e^{\omega_1 s}ds - ke^{(\omega + cM)t} \int_{h_1}^{t} cMe^{-cM s}ds
= (1-k)e^{\Omega t} - (1-k)e^{(\omega + cM)t}e^{\omega_1 h_1} - ke^{(\omega + cM)t}e^{-cM h_1} + ke^{\omega t}.
\] (6.17)

Combining (6.16), (6.17) and (6.9) and noting that \( k(e^{\omega_1 h_1} - e^{-cM h_1}) = e^{-cM h_1}(e^{\omega_1 h_1} - 1) \), we have
\[
\|R(t)\| \leq e^{\Omega t} + e^{(\omega + cM)t}(1 - e^{\omega_1 h_1} + k(e^{\omega_1 h_1} - e^{-cM h_1}))
= e^{\Omega t} - e^{(\omega + cM)t}(1 - e^{-cM h_1})(e^{\omega_1 h_1} - 1) \leq e^{\Omega t}
\]
for \( t \in [nh_1, (n+1)h_1) \). Then it is clear that \( R(t) \) is a strongly continuous bounded operator for each \( t \geq 0 \) and (6.6) holds.

We now prove (6.8). From (6.10) and (6.3) it follows that
\[
(R \ast \hat{F}x)(t) = (U_T \ast \hat{F}x)(t) + (H \ast \hat{F}x)(t) + (H \ast (R_{\hat{F}} \ast \hat{F}x))(t)
= U_T(t)x - T(t)x + (H \ast R_{\hat{F}}x)(t).
\] (6.18)

Hence from (6.10) and (6.18) we have
\[
R(t)x = T(t)x + H(t)x + \int_{0}^{t} R(t-s)\hat{F}(s)xds
\] (6.19)

and if \( \text{Re } \lambda > \Omega \)
\[
\hat{F}(\lambda)x = \int_{-h}^{0} e^{\lambda s}C(s)R(\lambda : A)xds, \quad x \in X;
\]
\[
B_1R(\lambda : A)x = \int_{0}^{h_1} e^{-\lambda t}B_1T(t)(I - e^{-\lambda h_1}T(h_1))^{-1}xdt, \quad x \in X.
\]
From (6.11) we have also
\[ \hat{H}(\lambda)x = \hat{R}(\lambda)\left(\sum_{i=1}^{m} e^{-\lambda h_i} \int_{0}^{h_1} e^{-\lambda s} B_i T(s)(I - e^{-\lambda h_1} T(h_1))^{-1} x ds\right) \]
\[ = \hat{R}(\lambda)\left(\sum_{i=1}^{m} e^{-\lambda h_1} B_i R(\lambda : A)\right)x, \quad x \in X. \]

Hence if \(\text{Re} \, \lambda > \Omega\), owing to (6.19), we have
\[ \hat{R}(\lambda)x = R(\lambda : A)x + \hat{R}(\lambda) \int_{-h}^{0} e^{\lambda s} \cdot dB(s) R(\lambda : A)x, \quad x \in X, \]
and so, setting \(x = (\lambda - A)y\) for \(y \in D(A)\), we have
\[ \hat{R}(\lambda)\left(\lambda - A - \int_{-h}^{0} e^{\lambda s} \cdot dB(s)\right)y = \hat{R}(\lambda) \left(x - \int_{-h}^{0} e^{\lambda s} \cdot dB(s) R(\lambda : A)x ds\right) \]
\[ = R(\lambda : A)x = R(\lambda : A)(\lambda - A)y = y. \]

By the uniqueness of Laplace transform we obtain
\[ R(t)y - y = \int_{0}^{t} \left(R(s)Ay + \int_{-h}^{0} R(s + r) \cdot dB(r)y\right) ds \]
for \(t \geq 0\) and \(y \in D(A)\). Thus differentiating it we have (3.1). Moreover it is not difficult to see that there exists some constant \(\gamma \geq \omega\) such that
\[ \left|\int_{-h}^{0} e^{\lambda r} \cdot dB(r) R(\lambda : A)\right| < 1 \quad \text{for} \quad \text{Re} \, \lambda > \gamma. \]

This means
\[ \lambda - \Delta(\lambda) = \left(I - \int_{-h}^{0} e^{\lambda s} \cdot dB(s) R(\lambda : A)\right)(\lambda - A) \]
is invertible for \(\text{Re} \, \lambda > \gamma\). According to Theorem 5.1, \(R(\cdot)\) is an RRO, which satisfies (6.8) for \(\text{Re} \, \lambda > \Omega\). However, by Lemma 5.3, (6.8) is also true for \(\text{Re} \, \lambda > \omega_0\).

To prove the last assertion we set for \(j \in \mathbb{N}_m\)
\[ \phi_j(t) \equiv e^{\omega t} \left(p_j(t) + \int_{0}^{t} p_j(s)e^{\gamma(t-s)} ds\right), \quad t \geq 0; \]
\[ q(t) \equiv \sum_{i=1}^{m} p_i(t - h_i)e^{\omega(t-h_i)} + ce^{\omega t}, \quad t \geq 0; \]
\[ \psi_j(t) \equiv \phi_j(t) + \int_{0}^{t} r_q(t - s) \phi_j(s) ds, \quad t \geq 0. \]
Here we assume $\phi_j(t) = q(t) = \psi_j(t) = 0$ for $t < 0$. From (6.2) it follows that for $y \in \mathcal{D}(A)$

$$
\int_0^t U_T(s)yds = T_S(t)y + \int_0^t T_S(t-s)R_F(s)yds,
$$

where $T_S(t)x = \int_0^t T(s)x ds$, $x \in \mathcal{X}$, and for $j \in \mathbb{N}_m$

$$
\int_0^t B_j U_T(s)yds = B_j \int_0^t U_T(s)yds
$$

$$
= B_j T_S(t)y + \int_0^t B_j T_S(s)R_F(t-s)yds
$$

$$
= \int_0^t B_j \overline{T}(s)yds + \int_0^t \left( \int_0^s B_j \overline{T}(r)R_F(t-s)ydr \right) ds
$$

$$
= \int_0^t B_j \overline{T}(s)yds + \int_0^t B_j \overline{T}(s) \int_0^{t-s} R_F(r)ydr ds, \quad y \in \mathcal{D}(A).
$$

Hence differentiating it we have

$$
B_j U_T(t)y = \overline{B}_j \overline{T}(t)y + \int_0^t \overline{B}_j \overline{T}(t-s)R_F(s)yds, \quad t > 0,
$$

and so $|B_j U_T(t)y| \leq \phi_j(t)||y||$. This shows the existence of $\overline{B}_j U_T(t)$. If we define $H_S(t)x = \int_0^t H(s)x ds$, $x \in \mathcal{X}$, then by the same procedure as above we see that $H_S(t)x \in \mathcal{D}(A)$ for $t \geq 0$ and $H_S(\cdot)x \in C(\mathbb{R}^+; \mathcal{Y})$ for each $x \in \mathcal{X}$, since $H_S$ satisfies

$$
H_S(t)x = \int_0^t S(t-s) \sum_{i=1}^m \overline{B}_i \overline{T}(s-h_i)x ds, \quad x \in \mathcal{X}.
$$

For $t \in [0, h_1)$ we have $R(t) = U_T(t)$ and $H(t) = O$ so that $|\overline{B}_j R(t)| \leq \psi_j(t)$ and $|\overline{B}_j H(t)| = 0$ are satisfied. Suppose that $\overline{B}_j R(t)$ and $\overline{B}_j H(t)$ exist and satisfy $|\overline{B}_j R(t)| \leq \psi_j(t)$ and $|\overline{B}_j H(t)| \leq c_j^{(n)}(t)$, a.a. $t$ on $[0, nh_1)$, $n \geq 1$, for some $c_j^{(n)} \in L_1^{loc}(\mathbb{R}^+)$. Since it is easy to check that

$$
\overline{B}_j \overline{T}(t)y = \int_0^t \overline{B}_i \overline{T}(s)Ayds + B_i y \quad \text{for } y \in \mathcal{D}(A),
$$

Fubini’s theorem yields for $t \in [nh_1, (n+1)h_1)$

$$
H(t)y = \sum_{i=1}^m \left( \int_0^t S(t-s)\overline{B}_i \overline{T}(s-h_i)Ayds + S(t-h_i)B_i y \right).
$$

This implies $H(t)y \in \mathcal{D}(A)$ and

$$
B_j H(t)y = \sum_{i=1}^m \left( \int_0^t B_j S(t-s)\overline{B}_i \overline{T}(s-h_i)Ayds + B_j S(t-h_i)B_i y \right).$$
Since $B_j S(t)x = \int_0^t B_j \overline{R}(s) x ds$, $x \in X$, we have further

$$B_j H(t)y = \sum_{i=1}^m \left( \int_0^t \int_0^{t-s} B_j \overline{R}(r) B_i T(s-h_i) Ay dr ds + \int_0^{t-h_i} B_j \overline{R}(s) B_i y ds \right)$$

$$= \sum_{i=1}^m \int_0^t B_j \overline{R}(t-s-h_i) \left( \int_0^s B_i T(r) Ay dr + B_i y \right) ds$$

$$= \sum_{i=1}^m \int_0^t B_j \overline{R}(t-s-h_i) B_i T(s) y ds.$$

Hence

$$|B_j H(t)y| \leq \left( (\psi_j * q)(t) - \int_0^t \psi_j(t-s) ce^{\omega s} ds \right) |y|$$

$$= \left( (r_q * \phi_j)(t) - \int_0^t \psi_j(t-s) ce^{\omega s} \right) |y|, \quad y \in D(A),$$

and $\overline{B_j H}(t)$ exists. Clearly there is $c_j^{(n+1)} \in L^1_{\text{loc}}(\mathbb{R}^+)$ such that $|\overline{B_j H}(t)| \leq c_j^{(n+1)}(t)$ and we have

$$\left| \int_0^t \overline{B_j H}(t-s) R_F(s) y ds \right| \leq c \left( \int_0^t (r_q * \phi_j)(t-s) e^{(\omega+c)s} ds \right.$$

$$- \int_0^t \psi_j(t-s) (e^{(\omega+c)s} - e^{\omega s}) ds \right) |y| \quad \text{for } t \in [nh_1, (n+1)h_1).$$

Moreover we have

$$\int_0^t B_j R(s) y ds - \int_0^t B_j \overline{U_T}(s) y ds - \int_0^t B_j \overline{H}(s) y ds$$

$$= B_j \int_0^t H_S(t-s) R_F(s) y ds = \int_0^t B_j H_S(t-s) R_F(s) y ds$$

$$= \int_0^t \left( \int_0^s B_j \overline{H}(s-r) R_F(r) y dr \right) ds, \quad y \in D(A).$$

Differentiating it again we have

$$B_j R(t)y = \overline{B_j U_T}(t)y + \overline{B_j H}(t)y + \int_0^t \overline{B_j H}(t-s) R_F(s) y ds$$

and we can estimate

$$|B_j R(t)y| \leq (\phi_j(t) + (\phi_j * r_q)(t)) |y|$$

$$+ \int_0^t ((r_q * \phi_j)(t-s) - \psi_j(t-s)) ce^{(\omega+c)s} ds |y|$$
This shows the existence of $B_j R(t)$ and $|B_j R(t)| \leq \psi_j(t)$ for a.a. $t > 0$. We have gone through the details rather carefully so that we may henceforth merely refer to this type of argument. Finally to complete the proof we have to show the existence of $\psi \in L^1_{\text{loc}}(\mathbb{R}^+)$ satisfying

$$\left| \int_{-t}^{0} C(s) R(s + t) y ds \right| \leq \psi(t)|y| \quad \text{for a.a. } t > 0 \text{ and } y \in D(A).$$

Now for $y \in D(A)$ we set

$$\psi(t) \equiv ce^{\omega t} \left( 1 + \int_{0}^{t} r_q(s)e^{-\omega s} ds \right) \quad \text{for } t \geq 0;$$

$$V(t)y \equiv \int_{-t}^{0} C(s) R(s + t)y ds \quad \text{for } t \geq 0 \text{ with } V(t) = O \text{ for } t < 0;$$

$$W(t)y \equiv \int_{-t}^{0} C(s) H(s + t)y ds \quad \text{for } t \geq 0 \text{ with } W(t) = O \text{ for } t < 0.$$

If $V(\cdot)$ satisfies $|V(t)y| \leq \psi(t)|y|$ for $t \in [0, nh_1)$, $n \geq 1$, using the same technique as above, we can see that for $y \in D(A)$ and $t \geq 0$

$$W(t)y = \sum_{i=1}^{m} \int_{0}^{t} V(t - s - h_i) B_i T(s) y ds;$$

$$V(t)y = R_F(t)y + W(t)y + \int_{0}^{t} W(t - s) R_F(s)y ds;$$

$$|W(t)y| \leq \left( (\psi * q)(t) - \int_{0}^{t} \psi(t - s)ce^{\omega s} ds \right)|y|. $$

So, if $t \in [nh_1, (n + 1)h_1)$ we assure that $|V(t)y| \leq \psi(t)|y|$ as above. Hence the extension $V(t)$ exists and satisfies $|V(t)| \leq \psi(t)$ for all $t \geq 0$. This completes the proof.

Having established the existence of the RRO satisfying the growth condition (H3), let us now discuss some of its further properties.

**Corollary 6.1 (cf. [19]).** Let $B_i \in B(\mathbb{X})$, $i \in \mathbb{N}_m$. Suppose that the function $C(\cdot)x$ is strongly measurable for $x \in \mathbb{X}$ and there is a function $c_2 \in L^1(I_h)$ such that $|C(t)x| \leq c_2(t)|x|$ for a.a. $t$. Then

$$|R(t)| \leq Me^{(\omega + M\text{Var }|B|)t}, \quad t \geq 0,$$

where $\text{Var }|B| \equiv \sum_{i=1}^{m} |B_i| + \int_{-h}^{0} e^{\omega r} c_2(r)dr$. 

PROOF. Since \(|U(\cdot)|\) is measurable and bounded on each finite interval ([5; Chap. VIII. 1.3]), by (6.10) the inequality
\[
|R(t)| \leq Me^{\omega t} + \int_0^t \left( \sup_{0 \leq r \leq s} |U(r)| \right) M \left( \sum_{i=1}^m |B_i| \right) e^{\omega(t-s)} ds
\]
\[+ \int_0^t |R(s)| M \left( \int_{-h}^0 e^{\omega r} c_2(r) dr \right) e^{\omega(t-s)} ds, \quad t \geq 0,
\]
is well defined. Using Gronwall's inequality we have the conclusion.

**Corollary 6.2.** For all \(x \in X\) and \(t \geq 0\) we have
\[T(t)x = \lim_{\lambda \to \infty} R(t/n)^n x,
\]
the convergence being uniform on bounded intervals for fixed \(x\).

**Proof.** From (6.6) we have \(\|R(t)^k\| \leq e^{k\omega t}\) for \(k \in \mathbb{N}\). Thus by [22, Corollary 3.5.4], Theorems 3.1 and 3.2, we obtain the result (cf. [3]).

The following result is an obvious consequence of Corollary 6.2.

**Corollary 6.3.** Let \(K\) be a closed subset of \(X\) such that \(R(t)K \subseteq K\) for all \(t \geq 0\). Then we have also \(T(t)K \subseteq K\) for all \(t \geq 0\).

### 7. Examples

In this section we consider three examples. The equation in the first example is a typical one to which our results in §6 are applicable. The second one illustrates the fact that though \(A\) is not the infinitesimal generator of a \(C_0\)-semigroup, the equation involving \(A\) can have an RRO (cf. [7]). The final one shows that in Corollary 6.3 the converse is not true (cf. [7, Theorem 4]).

**Example 7.1** (cf. [5, Chap. VIII]). Let \(X = C[-\infty, \infty]\),
\[A = d^2/dx^2\]
with \(D(A) = \{g \in X \mid g', g'' \in X\}\),
\[B = d/dx\]
with \(D(B) = \{g \in X \mid g' \in X\}\)
and
\[C(t) = c_1(t)A + c_2(t)B, \quad \text{where} \quad c_1 \in C^1(I_h) \quad \text{and} \quad c_2 \in C(I_h).
\]
Then it follows that \(A\) generates a \(C_0\)-semigroup \(\{T(t)\}_{t \geq 0}\) satisfying \(|T(t)| \leq 1\) and \(B\) is a closed operator with \(D(A) \subseteq D(B)\). Moreover we have for \(g \in D(A)\) and \(t > 0\)
\[
|BT(t)g| \leq (\pi t)^{-1/2} |g|;
\]
\[
\left| \int_{-h}^0 C(s)T(s+t)g ds \right| \leq (2|c_1|_\infty + |c'_1|_\infty h + 2(h/\pi)^{1/2}|c_2|_\infty) |g|.
\]
Therefore conditions (H4) and (H5) are satisfied, and hence (E) with \( m = 1 \) and \( h_1 = h \) has an RRO.

**Example 7.2.** In \( X = \ell^2(\mathbb{N}) \) we consider (E) with \( m = 1 \), \( h_1 = h \) and \( f = \varphi \equiv 0 \). For \( n \in \mathbb{N} \) let

\[
(Ax)_n = \alpha_n x_n, \quad D(A) = \{(x_n) \in \ell^2(\mathbb{N}) \mid (n^2 x_n) \in \ell^2(\mathbb{N})\};
\]

\[
(Bx)_n = \beta_n x_n, \quad D(B) = X;\]

\[
(Cx)_n = \gamma_n x_n, \quad D(C) = D(A) \quad \text{and} \quad C(t) = b(t)C,
\]

where

\[
\lambda_n = 2n^2 i, \quad \alpha_n = \lambda_n + n(1 - i), \quad \beta_n = b(h)\gamma_n/\lambda_n,
\]

\[
\gamma_n = (\lambda_n/b_n)e^{\pi i/4}, \quad b(t) = (\pi |t|)^{-1/2}
\]

and

\[
b_n = 2^{1/2} n \int_0^h e^{-\lambda_n r} b(t) dt - ib(h) e^{-\lambda_n h/(2n)}.
\]

It is easily checked that \( \lambda = \lambda_n \) is a zero point of

\[
\lambda - \alpha_n - e^{-\lambda h} \beta_n - \gamma_n \int_0^h e^{-\lambda t} b(t) dt.
\]

Let \( r_n \) be the solution of

\[
r_n(t) = e^{\lambda_n t} + (\gamma_n/\lambda_n) \int_0^h (e^{\lambda_n s} c_n(s) - b(s)) r_n(t - s) ds, \quad t \geq 0,
\]

with \( r_n(t) = 0 \) for \( t \in [-h, 0) \), where

\[
c_n(t) \equiv \int_t^h e^{-\lambda_n r} b(r)/(2r) dr \quad \text{for} \quad t \in (0, h].
\]

Since \( c_n(t) \) is continuous, \( |c_n(t)| \leq b(t) \) for each \( t \in (0, h] \) and \( c_n \in L^1((0, h)) \), the existence and uniqueness of solutions of (7.1) is guaranteed by the usual continuation process (cf. [1, Chap. 3]). We can easily see that \( c_n \) satisfies the relation

\[
e^{\lambda_n t} c_n(t) - b(t) = -e^{\lambda_n (t-h)} b(h) - \lambda_n e^{\lambda_n t} \int_t^h e^{-\lambda_n \tau} b(\tau) d\tau.
\]

Hence integration by parts yields

\[
\lambda_n \int_0^t r_n(s) ds - (e^{\lambda_n t} - 1)
\]

\[
= - \int_0^h \lambda_n e^{\lambda_n s} \left( \beta_n e^{-\lambda_n h} + \gamma_n \int_s^h e^{-\lambda_n r} b(\tau) d\tau \right) \left( \int_0^{t-s} r_n(r) dr \right) ds.
\]
Therefore $r_n(t)$ satisfies

$$r_n(t) = 1 + \int_0^t \left( r_n(s) \lambda_n + r_n(s-h) \beta_n + \int_{-h}^0 r_n(s+r)b(r)\gamma_n dr \right) ds \quad (7.2)$$

for $t \geq 0$. Moreover it can be seen that $r_n(t)$ is uniformly bounded in $n$ for each finite intervals. In fact, we have from (7.1)

$$|r_n(t)| \leq 1 + 4 \int_0^t b(t-s) r_n(s) ds \quad \text{for} \quad t \geq 0,$$

and applying a comparison theorem we obtain

$$|r_n(t)| \leq s_{4b}(t), \quad t \geq 0, \quad (7.3)$$

where $s_{4b}$ is the solution of $s_{4b}(t) = 1 + 4 \int_0^t b(t-s) s_{4b}(s) ds$. Since $\text{Re} \alpha_n \to \infty$ as $n \to \infty$, the operator $A$ cannot generate a $C_0$-semigroup of bounded linear operators in $X$. However, we can see that (E) is wellposed. By (7.3) the operator $R(t)$ defined by $(R(t)x)_n = r_n(t)x_n$ for $t \geq -h$ is a bounded operator. It is not difficult to assure that $R(t)$ is strongly continuous for each $t \geq 0$ and $R(0) = I$. Further by (7.2) it is easily seen that $R(t)$ satisfies

$$R(t)y - y = \int_0^t \left( R(s)Ay + R(s-h)By + \int_{-h}^0 R(s+r)C(r)y dr \right) ds$$

$$= \int_0^t \left( AR(s)y + BR(s-h)y + \int_{-h}^0 C(r)R(r+s)y dr \right) ds$$

for $y \in D(A)$ and $t \geq 0$, thus $R(\cdot)$ is an RRO of (E).

**Example 7.3.** Let $X = C$, $K = \mathbb{R}^+$ and $h = \pi$. Consider the scalar equation

$$u'(t) = 2u(t) + (i-1)u(t-h) - \int_{-h}^0 C(s)u(s+t)ds, \quad t \geq 0,$$
with \( u(t) = 0 \) for \( t \in [-h, 0) \), where

\[
C(t) = \begin{cases} 2 & \text{for } t \in [-h, 0] \\ 0 & \text{otherwise} \end{cases}
\]

Then by denoting \((a)_+ = \max(a, 0)\), \( R(t) \) is given by

\[
R(t) = e^t (\cos t + \sin t) + \sum_{k=1}^{\infty} \left( (i - 1)^k / k! \right) (t - kh)_+ e^{(1-i)(t-kh)} \\
+ (1 + i) \int_{0}^{t} e^{(1+i)(t-s)} \sum_{k=1}^{\infty} \left( (i - 1)^k / k! \right) (s - kh)_+ e^{(1-i)(s-kh)} ds,
\]

with \( R(t) = 0 \) for \( t < 0 \). It is clear that \( T(t) = e^{2t} \) and \( T(t)K \subset K \) for all \( t \geq 0 \).

However, \( R(t)x, x \in K \), does not belong to \( K \) at least for \( t \in (3h/4, h) \) since \( R(t) = e^t (\cos t + \sin t) \) for \( t \in [0, h) \).

ACKNOWLEDGEMENTS. The authors wish to express their appreciation to Professor Shohei Sugiyama for his valuable suggestions and encouragements. They also express their gratitude to the referee for his careful reading of this paper and his useful comments.

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