Two-phase free boundary problem for viscous incompressible thermo-capillary convection

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(Received July 5, 1993)

1. Introduction

Let $\Omega^{(1)}$ and $\Omega^{(2)}$ be two bounded domains in $\mathbb{R}^3$ which are filled with fluids (1) and (2), respectively, at the initial moment. We assume that $\partial \Omega^{(1)} \equiv \Sigma^{(1)} \cup \Gamma$, $\partial \Omega^{(2)} \equiv \Sigma^{(2)} \cup \Gamma$, $\Sigma^{(1)} \cap \Gamma = \phi$, $\Sigma^{(2)} \cap \Gamma = \phi$, $\Sigma^{(1)} \cap \Sigma^{(2)} = \phi$ ($\Gamma$ is the initial interface between fluids (1) and (2), $\Sigma^{(1)}$, $\Sigma^{(2)}$ are fixed). Then our problem consists in determining the domain $\Omega^{(j)}(t)$, occupied by the fluid $j$ $(j = 1, 2)$ at the moment $t > 0$ together with the velocity vector field $v^{(j)}(x, t) = (v_1^{(j)}, v_2^{(j)}, v_3^{(j)})(x, t)$, the pressure $p^{(j)}(x, t)$ and with the absolute temperature $\theta^{(j)}(x, t)$ of the fluid $j$, satisfying the system of Navier-Stokes equations:

$$\frac{\partial}{\partial t} v^{(j)} + v^{(j)} \cdot \nabla v^{(j)} = \nabla \cdot (\kappa^{(j)} \nabla \theta^{(j)}) \quad (x \in \Omega^{(j)}(t), \ t > 0), $$

where $\nabla = (\nabla_1, \nabla_2, \nabla_3)$, $\nabla_i = \frac{\partial}{\partial x_i}$ $(i = 1, 2, 3)$, $P^{(j)} = P^{(j)}(v^{(j)}, p^{(j)}) = -p^{(j)} I + 2\mu^{(j)} D(v^{(j)})$ is the stress tensor, $I$ is the $3 \times 3$ unit matrix, $D(v)$ is the velocity deformation tensor with $(i, k)$ elements $(D(v))_{ik} \equiv \frac{1}{2} \left( \frac{\partial v_i}{\partial x_k} + \frac{\partial v_k}{\partial x_i} \right)$ $(i, k = 1, 2, 3)$, $f^{(j)} = (f_1^{(j)}, f_2^{(j)}, f_3^{(j)})$ is a given vector field of external mass forces. $\rho^{(j)}$, $\mu^{(j)}$ and $\kappa^{(j)}$ are, respectively, the density of the fluid, the coefficient of viscosity and the coefficient of heat conductivity, which are all assumed to be positive constants. Here and in what follows we shall use the well-known notation of vector analysis and the summation convention.
The initial condition is

\begin{align*}
\frac{\partial(v^{(1)}, \theta^{(1)})}{\partial t} \big|_{t=0} &= (v_0^{(1)}, \theta_0^{(1)})(x) \quad (x \in \Omega^{(1)}(0) \equiv \Omega^{(1)}), \\
\frac{\partial(v^{(2)}, \theta^{(2)})}{\partial t} \big|_{t=0} &= (v_0^{(2)}, \theta_0^{(2)})(x) \quad (x \in \Omega^{(2)}(0) \equiv \Omega^{(2)}). 
\end{align*}

The boundary condition on the free surface \( \Gamma(t) \) (i.e., the interface between fluids (1) and (2) at the moment \( t > 0 \)) is (see, for example [12, 16])

\begin{align*}
v^{(1)} &= v^{(2)}, \quad P^{(1)} n - P^{(2)} n = \sigma H n + \nabla^{(s)} \sigma, \\
\theta^{(1)} &= \theta^{(2)}, \quad \kappa^{(1)} \nabla \theta^{(1)} \cdot n = \kappa^{(2)} \nabla \theta^{(2)} \cdot n \quad (x \in \Gamma(t), \ t > 0), 
\end{align*}

while on the fixed boundary,

\begin{align*}
v^{(1)} &= 0, \quad \theta^{(1)} = \theta^{(2)} \quad (x \in \Sigma^{(1)}, \ t > 0), \\
v^{(2)} &= 0, \quad \theta^{(2)} = \theta^{(2)} \quad (x \in \Sigma^{(2)}, \ t > 0),
\end{align*}

where \( n = n(x, t) \) is the unit normal vector pointing to \( \Omega^{(2)}(t) \) at \( x \in \Gamma(t), H(x, t) \) is the twice mean curvature of \( \Gamma(t) \), \( \nabla^{(s)} = \nabla - n \cdot \nabla \) is the surface gradient and \( \sigma = \sigma(\theta^{(s)}) \left( \theta^{(s)} = \frac{1}{2}(\theta^{(1)} + \theta^{(2)}) \big|_{\Gamma(t)} \right) \) is the coefficient of surface tension at the interface between fluids (1) and (2). The signature of \( H \) is chosen in such a way that \( H n = \Delta(t) x \), where \( \Delta(t) \) is the Laplace-Beltrami operator on \( \Gamma(t) \). If \( \Gamma(t) \) is given by the equation \( x = x(s_1, s_2; t) \ (s_1, s_2) \in \mathbb{R}^2 \), then

\begin{align*}
\Delta(t) &= \frac{1}{\sqrt{g}} \frac{\partial}{\partial s_\alpha} \left( g^{\alpha \beta} \sqrt{g} \frac{\partial}{\partial s_\beta} \right), \\
g &= \det(g_{\alpha \beta}), \quad (g^{\alpha \beta}) = (g_{\alpha \beta})^{-1} \ (\alpha, \beta = 1, 2).
\end{align*}

We decompose the second condition of (1.3) into the tangential and the normal components.

\begin{align*}
2 \Pi \left( \mu^{(1)} D(v^{(1)}) n - \mu^{(2)} D(v^{(2)}) n \right) &= \sigma' \Pi \nabla \theta^{(s)}, \\
n \cdot (P^{(1)} n - P^{(2)} n) &= \sigma n \cdot \Delta(t) x,
\end{align*}

where \( \Pi \psi = \psi - (\psi \cdot n)n \) is the projection operator onto the tangent plane to \( \Gamma(t) \) and \( \sigma' = \frac{\partial \sigma}{\partial \theta^{(s)}} \).

In addition to the dynamical boundary conditions written above, we shall impose the kinematic condition on \( \Gamma(t) \):

\begin{equation}
\left[ \frac{D}{Dt} \right]^{(j)} F(x, t) = 0 \quad (x \in \Gamma(t), \ t > 0),
\end{equation}
if $\Gamma(t)$ is given by $F(x, t) = 0$ ($j = 1$ or $2$).

To avoid the difficulties connected with the presence of an unknown surface $\Gamma(t)$, it is convenient to choose $\xi \equiv X(0; x, t) \in \Omega^{(j)}$ as new independent variables, where $X(\tau; x, t)$ is the solution of following system of ordinary differential equations:

$$\frac{d}{d\tau} X(\tau; x, t) = v^{(j)}(X(\tau; x, t), \tau), \quad X(t; x, t) = x, \quad 0 \leq \tau \leq t. \quad (1.6)$$

If $v^{(j)}$ has a suitable smoothness, then the fundamental existence theorem of ordinary differential equations yields that $(1.6)$ has a unique solution curve $X(\tau; x, t), \ x \in \Omega^{(j)}(t), \ 0 \leq \tau \leq t$. Hence this gives the relationship between the so-called Eulerian coordinates $\{x\}$ and the Lagrangian coordinates $\{\xi\}$:

$$x = X(t; \xi, 0) = \xi + \int_0^t u^{(j)}(\xi, \tau) d\tau \equiv X_{u^{(j)}}(\xi, t), \quad (1.7)$$

where $u^{(j)}(\xi, t) \equiv v^{(j)}(X(t; \xi, 0), t) = v^{(j)}(x, t)$.

In the Lagrangian coordinates, the problem $(1.1)^{(j)}-(1.5)$ takes the form

$$\left\{ \begin{array}{l}
\rho^{(j)} \frac{\partial u^{(j)}}{\partial t} = \nabla_{u^{(j)}} \cdot P_{u^{(j)}} + \rho^{(j)} f^{(j)}(X_{u^{(j)}}, t), \quad \nabla_{u^{(j)}} \cdot u^{(j)} = 0, \\
\frac{\partial \theta^{(j)}}{\partial t} = \nabla_{u^{(j)}} \cdot (\kappa^{(j)} \nabla_{u^{(j)}} \theta^{(j)}) \quad \text{in} \quad Q_{T^{(j)}} \equiv \Omega^{(j)} \times (0, T), \\
(u^{(j)}, \theta^{(j)}) \mid_{t=0} = (v_0^{(j)}, \theta_0^{(j)})(\xi) \quad \text{on} \quad \Omega^{(j)},
\end{array} \right. \quad (1.8)^{(j)}$$

$$\left\{ \begin{array}{l}
u^{(1)} - u^{(2)} = 0, \\
2\mu^{(1)} \prod_{u^{(1)}} D_{u^{(1)}} (u^{(1)})^{-1} n_{u^{(1)}} - 2\mu^{(2)} \prod_{u^{(2)}} D_{u^{(2)}} (u^{(2)})^{-1} n_{u^{(2)}} = \frac{\sigma(\theta^{(s)})}{2} \sum_{j=1}^2 n_{u^{(j)}} \cdot \Delta_{u^{(j)}}(t) X_{u^{(j)}}, \\
\theta^{(1)} - \theta^{(2)} = 0, \quad \kappa^{(1)} \nabla_{u^{(1)}} \theta^{(1)} \cdot n_{u^{(1)}} = \kappa^{(2)} \nabla_{u^{(2)}} \theta^{(2)} \cdot n_{u^{(2)}} = 0 \\
on \Gamma_T \equiv \Gamma \times (0, T), \quad \Gamma \equiv \Gamma(0),
\end{array} \right. \quad (1.10)^{(j)}$$

where $\theta^{(j)}(\xi, t) \equiv \bar{\theta}^{(j)}(X_{u^{(j)}}(\xi, t), t)$, $\nabla_{u^{(j)}} = t \left( \frac{\partial X_{u^{(j)}}}{\partial \xi} \right)^{-1} \nabla$, $\nabla = (\nabla_1, \nabla_2, \nabla_3)$, $\nabla_i = \frac{\partial}{\partial \xi_i}$ ($i = 1, 2, 3$). From Liouville's theorem we have $\det \left( \frac{\partial X_{u^{(j)}}}{\partial \xi} \right) = 1$ so that we can write $\nabla_{u^{(j)}}$ as $A_{u^{(j)}} \nabla$, $A_{u^{(j)}} = \left( A_{ik}^{(j)} \right)$, with $A_{ik}^{(j)}$ being the $(i, k)$-cofactor.
of the Jacobian matrix of the transformation (1.7). \( P_{u(j)}(u(j), q(j)) = -q(j)I + 2\mu(j)D_{u(j)}(u(j)), \) 

\[ q(j)(\xi, t) = p(j)(X_{u(j)}(\xi, t), t), \]

\[ (D_{u(j)}(w(j)))_{ik} = \frac{1}{2} \left( A_{kl}(j) \frac{\partial w_{l}(j)}{\partial \xi_k} + A_{km}(j) \frac{\partial w_{k}(j)}{\partial \xi_m} \right). \]

Using \( F(X_{u(j)}(\xi, t), t) = F(X_{u(j)}(\xi, 0), 0) = F(\xi, 0) \equiv F_{0}(\xi), \) we can write \( n_{u(j)}(\xi, t) = n(X_{u(j)}(\xi, t), t) \) as \( \frac{A_{u(j)}(\xi)}{A_{u(j)}(\xi) n_{0}(\xi)}, \) where \( n_{0}(\xi) = n(\xi, 0), \xi \in \Gamma. \) \( \prod_{u(j)} \psi = \psi - (\psi \cdot n_{u(j)}) n_{u(j)}. \) \( \Delta_{u(j)}(t) \) is the Laplace-Beltrami operator on \( \Gamma \) parametrized by (1.7).

It is convenient to rewrite the boundary condition (1.10) in the form

\[ n_{u(1)} \cdot P_{u(1)} n_{u(1)} - n_{u(2)} \cdot P_{u(2)} n_{u(2)} = \frac{\sigma(\theta(s))}{2} \sum_{j=1}^{2} n_{u(j)} \cdot \Delta_{u(j)}(t) \int_{0}^{t} u(j) \, dt \]

\[ = (\sigma_{0} H_{0}(\xi)) + \frac{1}{2} \sum_{j=1}^{2} (\sigma(\theta(s)) n_{u(j)} \cdot \Delta_{u(j)}(t) \xi - \sigma(\theta(0)) n_{0} \cdot \Delta(0) \xi), \]

where \( \sigma_{0}(\xi) = \sigma(\theta(0)(\xi)), \) \( \theta(0) = \frac{1}{2}(\theta_{(1)} + \theta_{(2)}) \big|_{\Gamma}, \) and \( \frac{1}{2} H_{0}(\xi) \) is the mean curvature of \( \Gamma. \)

Before stating our main theorem, we introduce some notation to describe compatibility conditions. From (1.8)-(1.11), we have

\[ \frac{\partial u(j)}{\partial t} \bigg|_{t=0} = \frac{1}{\rho(j)} \nabla \cdot P(v_{0}(j), q_{0}(j)) + f_{0}(j) \equiv v_{0t}(j), \]

where \( f_{0}(j) = f(j)(X_{u(j)}, t) \bigg|_{t=0}. \) And let \( (q_{0(1)}, q_{0(2)}) \) be a solution of the problem

\[ \begin{cases}
\nabla^{2} q_{0(j)} = \rho(j) \nabla \cdot f_{0(j)} & \text{in } \Omega(j), \\
q_{0(1)} - q_{0(2)} = 2n_{0} \cdot (\mu(1) D(v_{0(1)}) n_{0} - \mu(2) D(v_{0(2)}) n_{0) - \sigma_{0} H_{0}, \\
\frac{1}{\rho(1)} \frac{\partial q_{0(1)}}{\partial n} - \frac{1}{\rho(2)} \frac{\partial q_{0(2)}}{\partial n} = \frac{\mu(1)}{\rho(1)} \nabla^{2} v_{0(1)} - \frac{\mu(2)}{\rho(2)} \nabla^{2} v_{0(2)} + (f_{0(1)} - f_{0(2)}) \cdot n_{0} & \text{on } \Gamma, \\
\frac{1}{\rho(j)} \frac{\partial q_{0(j)}}{\partial n} = \frac{\mu(j)}{\rho(j)} \nabla^{2} v_{0(j)} + f_{0(j)} \cdot n(j) & \text{on } \Sigma(j)
\end{cases} \]

\( (j = 1, 2), \) where \( n^{(j)} \) is the unit outward normal vector to \( \Sigma^{(j)}. \) Similarly, let us put \( \frac{\partial \theta(j)}{\partial t} \bigg|_{t=0} = \nabla \cdot (\kappa(j) \nabla \theta_{0(j)}) = \theta_{0t}(j). \)

The following is the main result of this paper (as for function spaces, see §2).

**Theorem 1.1.** Suppose that \( \frac{1}{2} < l < 1, T > 0, \) and that for \( j = 1, 2, \)
(i) $\Omega(j)$ are two bounded domains prescribed above with the boundaries $\Gamma, \Gamma(j) \in W^{2+1/2}_{2}(\Omega(j))$, 
(ii) $\rho(j), \mu(j), \kappa(j)$ are positive constants and $\sigma \in W^{5+1}_{2}(R_{+})$, 
$R_{+} = \{x \in R \mid x > 0\}, \quad \sigma > 0,$
(iii) $(v_{0}(j), \theta_{0}(j)) \in W^{2+1}_{2}(\Omega(j))$, \quad $\theta_{0}(j) > 0,$
(iv) $f(j) \in W^{5+1,5/2+1/2}_{2}(R^{3}_{T}), \quad R^{3}_{T} = \{(x, t) \mid x \in R^{3}, t \in (0, T)\},$
$\theta_{e}(j) \in W^{5/2+1,5/4+1/2}_{2}(\Sigma_{T}(j))$,
(v) (compatibility conditions)

\[ \nabla \cdot v_{0}(j) = 0 \text{ in } \Omega(j), \]

\[
\begin{aligned}
\{ v_{0}(1) = v_{0}(2), \quad v_{0t}(1) = v_{0t}(2), \\
2 \prod_{0}^{1}(\mu(1)D(v_{0}(1))n_{0} - \mu(2)D(v_{0}(2))n_{0}) = \sigma'(\theta_{0}(s)) \prod_{0}^{1} \nabla \theta_{0}(s), \\
\theta_{0}(1) = \theta_{0}(2), \quad \theta_{0t}(1) = \theta_{0t}(2), \quad \kappa(1) \nabla \theta_{0}(1) \cdot n_{0} = \kappa(2) \nabla \theta_{0}(2) \cdot n_{0} \text{ on } \Gamma \\
\prod_{0}^{1} \psi = \psi - (\psi \cdot n_{0})n_{0}, \\
v_{0}(j) = v_{0t}(j) = 0, \quad \theta_{0}(j) = \theta_{0}(j)|_{t=0}, \quad \theta_{0t}(j) = \frac{\partial \theta_{e}(j)}{\partial t} |_{t=0} \text{ on } \Sigma(j).
\end{aligned}
\]

Then there exists a unique solution $(u^{(1)}, q^{(1)}, \theta^{(1)}, u^{(2)}, q^{(2)}, \theta^{(2)})$ to the problem (1.8)(j)–(1.11), such that

\[ (u^{(j)}, \theta^{(j)}) \in W^{2+1,3/2+1/2}_{2}(Q_{T_{1}}(j)), \quad \nabla q^{(j)} \in W^{1+1,1/2+1/2}_{2}(Q_{T_{1}}(j)), \]

\[ q^{(1)} - q^{(2)} |_{\Gamma \in W^{2+1,3/4+1/2}_{2}(\Gamma_{T_{1}})} \text{ for some } T_{1} \in (0, T]. \]

Moreover, the inequality holds:

\[ \|v(j), q(j), \theta(j)\|_{W(Q_{T_{1}}(j))} \]

\[ = 2 \sum_{j=1}^{2} \left( \|u(j), \theta(j)\|_{W^{2+1,3/2+1/2}_{2}(Q_{T_{1}}(j))} + \|\nabla q(j)\|_{W^{1+1,1/2+1/2}_{2}(Q_{T_{1}}(j))} \right) \\
+ \|q^{(1)} - q^{(2)}\|_{W^{2+1,3/4+1/2}_{2}(\Gamma_{T_{1}})} \\
\leq c_{1} \sum_{j=1}^{2} \left( \|v_{0}(j), \theta_{0}(j)\|_{W^{2+1}_{2}(\Omega(j))} + \|f(j)\|_{W^{5+1,5/2+1/2}_{2}(R^{3}_{T})} \\
+ \|\theta_{e}(j)\|_{W^{5/2+1,5/4+1/2}_{2}(\Sigma_{T}(j))} \right) + \|\sigma_{0} H_{0}\|_{W^{2+1}_{2}(\Gamma)} \equiv c_{1} N. \]

The magnitude of $T_{1} < \infty$ increases unboundedly as $N$ tends to zero.
REMARK 1.1. Of course, the uniqueness of \((q^{(1)}, q^{(2)})\) in Theorem 1 means that \((q^{(1)}, q^{(2)})\) is unique up to an arbitrary function of \(t\).

REMARK 1.2. The condition \(l > \frac{1}{2}\) is necessary for using various imbedding theorems, while \(l < 1\) is used for minimizing the number of compatibility conditions. Therefore, throughout this paper, we assume \(\frac{1}{2} < l < 1\).

The proof of Theorem 1.1 consists in solving the linearized system and the method of successive approximations.

We consider the linearized system

\[
\begin{cases}
\frac{\partial u^{(j)}}{\partial t} - \frac{\mu^{(j)}}{\rho^{(j)}} \nabla^2 w^{(j)} u^{(j)} + \frac{1}{\rho^{(j)}} \nabla w^{(j)} q^{(j)} = \Phi_1^{(j)}, & \nabla w^{(j)} \cdot u^{(j)} = \Phi_2^{(j)}, \\
\frac{\partial \theta^{(j)}}{\partial t} - \kappa^{(j)} \nabla^2 w^{(j)} \theta^{(j)} = \Phi_3^{(j)} & \text{in } Q_T^{(j)}, \\
(u^{(j)}, \theta^{(j)}) \big|_{t=0} = (v_0^{(j)}, \theta_0^{(j)})(\xi) & \text{on } \Omega^{(j)},
\end{cases}
\]

where \(w^{(1)}, w^{(2)}, \tilde{\theta}^{(s)}\) are given functions.

\[u^{(1)} - u^{(2)} = 0,\]
\[2 \left( \mu^{(1)} \prod_{w^{(1)}} D_{w^{(1)}} (u^{(1)}) n_{w^{(1)}} - \mu^{(2)} \prod_{w^{(2)}} D_{w^{(2)}} (u^{(2)}) n_{w^{(2)}} \right) \]
\[-\frac{1}{2} \sigma'(\tilde{\theta}^{(s)}) \sum_{j=1}^{2} \prod_{w^{(j)}} \nabla w^{(j)} \theta^{(j)} = \Phi_4,\]
\[-q^{(1)} + 2\mu^{(1)} n_{w^{(1)}} \cdot D_{w^{(1)}} (u^{(1)}) n_{w^{(1)}} \]
\[+ q^{(2)} - 2\mu^{(2)} n_{w^{(2)}} \cdot D_{w^{(2)}} (u^{(2)}) n_{w^{(2)}} \]
\[-\frac{1}{2} \sigma(\tilde{\theta}^{(s)}) \sum_{j=1}^{2} n_{w^{(j)}} \cdot \Delta w^{(j)} (t) \int_0^t u^{(j)} d\tau = \Phi_5 + \int_0^t \Phi_6 d\tau,\]
\[
\theta^{(1)} - \theta^{(2)} = 0, \\
\kappa^{(1)} \nabla w^{(1)} \theta^{(1)} \cdot n_{w^{(1)}} - \kappa^{(2)} \nabla w^{(2)} \theta^{(2)} \cdot n_{w^{(2)}} = \Phi_7 & \text{on } \Gamma_T, \\
u^{(j)} = 0, & \theta^{(j)} = \theta_e^{(j)} \text{ on } \Sigma_T^{(j)},
\]

where \(w^{(1)}, w^{(2)}, \tilde{\theta}^{(s)}\) are given functions.

**Theorem 1.2.** Suppose that \(\frac{1}{2} < l < 1\), \(T > 0\) and that for \(j = 1, 2\),

(i) \(\Gamma, \Sigma^{(j)} \in W_2^{5/2+l},\)
(ii) \(\rho^{(j)}, \mu^{(j)}, \kappa^{(j)}\) are positive constants and \(\sigma \in W_2^{4+l}(R_+), \quad \sigma > 0,\)
(iii) \((v_0^{(j)}, \theta_0^{(j)}) \in W_2^{2+l}(\Omega^{(j)}), \quad \theta_0^{(j)} > 0,\)
(iv) \((\Phi_1^{(j)}, \Phi_3^{(j)}) \in W_2^{1+1/2+l/2}(Q_T^{(j)}), \quad \Phi_2^{(j)} \in W_2^{2+l,4+l/2}(Q_T^{(j)}),\)
\[\Phi_2^{(j)} = \nabla \cdot \Phi_2^{(j)}, \quad \Phi_2^{(j)} \in W_2^{3/2+l/2}(Q_T^{(j)}),\]
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\((\Phi_2 \gamma^{(1)}(1) - \Phi_2 \gamma^{(2)}(2)) \cdot n_0 |_{\Gamma} = 0, \quad \Phi_2 \gamma^{(j)}(j) \cdot n_0 |_{\Sigma_{\gamma}^{(j)}} = 0,\)

\((\Phi_4, \Phi_5, \Phi_7) \in W^{3/2+1,3/4+1/2}_2(\Gamma_T), \quad \Phi_6 \in W^{1/2+1,1/4+1/2}_2(\Gamma_T),\)

\(\theta_\gamma^{(j)} \in W^{5/2+1,5/4+1/2}_2(\Sigma_{\gamma}^{(j)}),\)

and the compatibility conditions analogous to Theorem 1.1 hold. Furthermore we assume that

\(w^{(j)} \in W^{3+1,3/2+1/2}_2(Q_{\gamma}^{(j)}), \quad \tilde{\theta}^{(s)} \in W^{5/2+1,5/4+1/2}_2(\Gamma_T)\)

are given functions satisfying \(\tilde{\theta}^{(s)}|_{t=0} = \theta_0^{(s)}\) and the inequality

\[c_2(T) \left( \sum_{j=1}^{2} \|w^{(j)}\|_{W^{3+1,3/2+1/2}_2(Q_{\gamma}^{(j)})} + \|\tilde{\theta}^{(s)}\|_{W^{5/2+1,5/4+1/2}_2(\Gamma_T)} \right) \leq \delta \]  

(1.17)

with some power function \(c_2(T)\) of \(T\) such that \(c_2(T) \to 0\) as \(T \to 0\) and a sufficiently small number \(\delta > 0\). Then the problem (1.13)\((j)\)–(1.16) has a unique solution \((u^{(1)}, q^{(1)}, \theta^{(1)}, u^{(2)}, q^{(2)}, \theta^{(2)})\) such that

\[(u^{(j)}, q^{(j)}) \in W^{3+1,3/2+1/2}_2(Q_{\gamma}^{(j)}), \quad \nabla q^{(j)} \in W^{1+1,1/2+1/2}_2(Q_{\gamma}^{(j)}), \quad \]  

(1.18)

\[q^{(1)} - q^{(2)} |_{\Gamma} \in W^{3/2+1,3/4+1/2}_2(\Gamma_T)\]

\[\|((u^{(j)}, q^{(j)}, \theta^{(j)}))\|_{W(Q_{\gamma}^{(j)})} \leq c_3 \left[ \sum_{j=1}^{2} \|((v_0^{(j)}, \theta_0^{(j)}))\|_{W^{2+1}(\Omega^{(j)})} \right.\]

\[+ \|((\Phi_1^{(j)}, \Phi_3^{(j)}))\|_{W^{2+1,1/2+1/2}_2(Q_{\gamma}^{(j)})} + \|\Phi_2^{(j)}\|_{W^{2+1,1/2}_2(Q_{\gamma}^{(j)})} \]

\[+ \|\Phi_4^{(j)}\|_{W^{2,3+2+1/2}_2(Q_{\gamma}^{(j)})} + \|((\Phi_4, \Phi_5, \Phi_7))\|_{W^{2+1,3/4+1/2}_2(\Gamma_T)} \]

\[+ \|\Phi_6\|_{W^{2,3+2+1/2}_2(\Gamma_T)} + \sum_{j=1}^{2} \|\theta^{(j)}\|_{W^{5/2+1,5/4+1/2}_2(\Sigma_{\gamma}^{(j)})} \right] \]

\[\equiv c_3\|((v_0^{(j)}, \theta_0^{(j)}), \Phi, \theta^{(j)} G(Q_{\gamma}^{(j)})), \]

where \(\Phi = ((\Phi_1^{(j)}, \Phi_2^{(j)}, \Phi_3^{(j)}), (\Phi_4, \Phi_5, \Phi_6, \Phi_7))\) and \(c_3\) depends on \(T\) nondecreasingly.

The essential part of the proof of Theorem 1.2 is to solve the problem linearized around \(t = 0\):

\[
\begin{cases}
\frac{\partial u^{(j)}}{\partial t} - \frac{\mu^{(j)}}{\rho^{(j)}} \nabla^2 u^{(j)} + \frac{1}{\rho^{(j)}} \nabla q^{(j)} = \Phi_1^{(j)}, \quad \nabla \cdot u^{(j)} = \Phi_2^{(j)}, \\
\frac{\partial \theta^{(j)}}{\partial t} - \kappa^{(j)} \nabla^2 \theta^{(j)} = \Phi_3^{(j)} \quad \text{in } Q_{T}^{(j)},
\end{cases}
\]  

(1.20)\((j)\)
\begin{equation}
\begin{aligned}
(u^{(j)}, \theta^{(j)}) \mid_{t=0} = (v_0^{(j)}, \theta_0^{(j)})(\xi) \quad \text{on } \Omega^{(j)},
\end{aligned}
\end{equation}

\begin{equation}
\begin{cases}
\begin{aligned}
& \left. u^{(1)} - u^{(2)} \right|_{t=0} = 0, \\
& 2 \prod_0^1 (\mu^{(1)} D(u^{(1)}) n_0 - \mu^{(2)} D(u^{(1)}) n_0) \\
& \quad - \frac{1}{2} \sigma'(\theta_0^{(s)}) \prod_0^1 (\nabla \theta^{(1)} + \nabla \theta^{(2)}) = \Phi_4, \\
& - q^{(1)} + 2 \mu^{(1)} n_0 \cdot D(u^{(1)}) n_0 + q^{(2)} - 2 \mu^{(2)} n_0 \cdot D(u^{(2)}) n_0 \\
& \quad - \frac{1}{2} \sigma(\theta_0^{(s)}) n_0 \cdot \Delta(0) \int_0^t (u^{(1)} + u^{(2)}) d\tau = \Phi_5 + \int_0^t \Phi_6 d\tau, \\
& \theta^{(1)} - \theta^{(2)} = 0, \quad \kappa^{(1)} \nabla \theta^{(1)} \cdot n_0 - \kappa^{(2)} \nabla \theta^{(2)} \cdot n_0 = \Phi_7 \quad \text{on } \Gamma_T,
\end{aligned}
\end{cases}
\end{equation}

\begin{equation}
\begin{aligned}
u^{(j)} = 0, \quad \theta^{(j)} = \theta_0^{(j)} \quad \text{on } \Sigma_T^{(j)}.
\end{aligned}
\end{equation}

**Theorem 1.3.** Suppose that the same assumptions as for Theorem 1.2 except the condition (1.17) are true. Then there exists a unique solution \((u^{(1)}, q^{(1)}, \theta^{(1)}, u^{(2)}, q^{(2)}, \theta^{(2)})\) to the problem (1.20)-(1.23) satisfying (1.18) and (1.19).

Hereafter, we shall use the letter \((x, t)\), instead of \((\xi, t)\), to write the independent variables of our problem in the Lagrangian form because of the notational convenience.

In conclusion to §1, we mention the related works published up to now.

There are some papers dealing with the free boundary problem for one phase incompressible viscous fluid motion. V.A. Solonnikov has studied the fluid motion of an isolated finite volume bounded only by a free surface. He showed that there exists a unique solution in a finite time interval ([26] for \(\sigma = 0\) and [27] for \(\sigma > 0\)) and this solution can be continued for all \(t > 0\) if the data is sufficiently close to some equilibrium state ([28, 31, 32] for \(\sigma > 0\) and [30] for \(\sigma = 0\)). Such a solution global in time was also obtained by A. Tani under more general situations ([41] for \(\sigma > 0\)). The same problem in the case of the fluid domain which is like an ocean of infinite extent and finite depth with a free surface on top was investigated by J.T. Beale ([3] for \(\sigma = 0\) and [4] for \(\sigma > 0\)), J.T. Beale and T. Nishida ([5] for \(\sigma > 0\)), G. Allain ([2] for \(\sigma > 0\)), G. Sylvester ([35] for \(\sigma = 0\)) and recently by A. Tani ([43] for \(\sigma > 0\)). Note that all these results mentioned above are concerned with the case of a constant coefficient of surface tension. The theory of thermo-capillary convection, \(i.e.,\) the free boundary problem with variable surface tension was discussed by V.V. Pukhnachov ([18, 19]). Recently, Russian mathematicians have studied the problems similar to ours. Namely, M.V. Lagunova and V.A. Solonnikov established the temporarily local existence theorem for one phase thermo-capillary convection problem in Hölder space ([11]) and I.V. Denisova has studied the linear theory of two phase incompressible viscous fluid motion ([7]).
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For compressible viscous heat-conducting fluid, (say, general fluid), it was proved by A. Tani that there exists a unique solution for the two phase problem in a finite time interval, which as far as the present author knows, is the only result giving an exact, time-dependent two phase flow, for both incompressible and compressible viscous fluid ([39, 40] for \( \sigma = 0 \) and [37] for \( \sigma > 0 \) but constant). Also A. Tani ([38] for \( \sigma = 0 \)), P. Secchi and A. Valli ([20] for \( \sigma = 0 \)) and V.A. Solonnikov and A. Tani ([34] for \( \sigma > 0 \) but constant) solved the one phase problem local in time.

For other free boundary problems of fluid dynamics such as water waves or gaseous star, we should refer to a survey article by T. Nishida ([17]) and the references therein for them.

In the forthcoming paper, we shall discuss the temporarily global solution of our problem ([36]).

2. Function spaces

Throughout this paper we use the Sobolev-Slobodetskii spaces defined as follows.

Let \( l > 0 \) be a non-integer, \( T > 0 \), \( \Omega \) be a domain in \( \mathbb{R}^3 \) and set \( \partial \Omega = \Gamma \), \( Q_T = \Omega \times (0, T) \) and \( \Gamma_T = \Gamma \times (0, T) \). We define

\[
W^{l,1/2}_2(Q_T) = \{ u(x, t) \mid \| u \|_{W^{l,1/2}_2(Q_T)} < \infty \}, \quad \text{where}
\]

\[
\| u \|_{W^{l,1/2}_2(Q_T)} = \left( \| u \|_{W^{l,0}_2(Q_T)}^2 + \| u \|_{W^{l,1/2}_2(Q_T)}^2 \right)^{1/2},
\]

\[
\| u \|_{W^{l,0}_2(Q_T)}^2 = \int_0^T \| u \|_{W^{l,0}_2(\Omega)}^2 dt,
\]

\[
\| u \|_{W^{l,1/2}_2(Q_T)}^2 = \int_\Omega \| u \|_{W^{l,1/2}_2(0, T)}^2 dx,
\]

\[
\| u \|_{W^{l,0}_2(\Omega)}^2 = \sum_{|k| < l} \| D_x^k u \|_{L^2(\Omega)}^2 + \| u \|_{W^{l,0}_2(\Omega)}^2,
\]

\[
\| u \|_{W^{l,1/2}_2(\Omega)}^2 = \sum_{|k| = l} \int_\Omega \int_\Omega \frac{|D_x^k u(x, t) - D_x^k u(y, t)|^2}{|x - y|^{3+2(l-[l])}} dx dy,
\]

\[
\| u \|_{W^{l,1/2}_2(0, T)}^2 = \sum_{j=0}^{[l/2]} \| D_t^j u \|_{L^2(0, T)}^2
\]

\[
+ \int_0^T \int_0^T \frac{|D_t^{[l/2]} u(x, t) - D_t^{[l/2]} u(x, \tau)|^2}{|t - \tau|^{1+2(l-[l/2])}} dt d\tau.
\]

We also define the space \( W^{l,1/2}_2(\Gamma_T) \) as \( L_2((0, T); W^{l,1/2}_2(\Gamma)) \cap L_2(\Gamma; W^{l,1/2}_2(0, T)) \).
Furthermore we introduce the Sobolev-Slobodetskii spaces with the exponential weight $e^{-2ht}$ ($h > 0$).

$$H_{h}^{l,1/2}(Q_T) = \{ u(x,t) \mid \|u\|_{H_{h}^{l,1/2}(Q_T)} < \infty \},$$

$$\|u\|_{H_{h}^{l,1/2}(Q_T)} = (\|u\|^2_{H_{h}^{l,0}(Q_T)} + \|u\|^2_{H_{h}^{l,1/2}(Q_T)})^{1/2},$$

$$\|u\|^2_{H_{h}^{l,0}(Q_T)} = \int_0^T e^{-2ht} \|u\|^2_{L_2(\Omega)} dt,$$

$$\|u\|^2_{H_{h}^{l,1/2}(Q_T)} = \int_0^T e^{-2ht} \|u\|^2_{L_2(\Omega)} dt + \int_{-\infty}^T e^{-2ht} dt \int_0^\infty \|D_t^{[l/2]}u_0(\cdot,t) - D_t^{[l/2]}u_0(\cdot,t-\tau)\|^2_{L_2(\Omega)} \tau^{-1-2(l/2-[l/2])} d\tau,$$

where $u_0(x,t) = u(x,t)(t > 0) = 0 (t < 0)$.

The same notation will be used for the spaces of vector fields, the norms of a vector supposed to be equal to the sum of the norms of all its components.

Further, we shall use the following simplified notations:

$$(v,u) \in W_{2}^{l,1/2}(Q_T) \text{ implies } v \in W_{2}^{l,1/2}(Q_T), \ u \in W_{2}^{l,1/2}(Q_T) \text{ and}$$

$$\|(v,u)\|_{W_{2}^{l,1/2}(Q_T)} = \|v\|_{W_{2}^{l,1/2}(Q_T)} + \|u\|_{W_{2}^{l,1/2}(Q_T)}.$$

**Remark 2.1.** For $T < \infty$, the space $H_{h}^{l,1/2}(Q_T)$ can be identified with the subspace of $W_{2}^{l,1/2}(Q_T)$ consisting of functions $u(x,t)$ that can be extended by zero into the domain $t < 0$ without loss of smoothness. In the case of $l > 1$, this implies $\frac{\partial^i u}{\partial t^i} \big|_{t=0} = 0, \ i = 0,1,2, \ldots, \left[\frac{l-1}{2}\right].$

Hence in §3, we shall discuss the problem (1.20)-(1.23) in the space $H_{h}^{l,1/2}(Q_T)$ instead of $W_{2}^{l,1/2}(Q_T)$ (See [33] for the detailed discussion of the relation between the spaces $H_{h}^{l,1/2}(Q_T)$ and $W_{2}^{l,1/2}(Q_T)$).

3. Linearized problem 1: Proof of Theorem 1.3

3.1. Model problem

First of all, we solve (1.20)-(1.23) in the case of $\Phi_1(j) = \Phi_2(j) = \Phi_3(j) \equiv 0$, $v_0(j) = \theta_0(j) \equiv 0, \sigma, \sigma' \equiv \text{const}, \sigma > 0$ and $\Omega^{(1)} = R^3 \setminus \{ x \in R^3 \mid x_3 > 0 \}$, $\Omega^{(2)} = R^3 \setminus \{ x \in R^3 \mid x_3 < 0 \}$, $\Gamma = \{ x_3 = 0 \}$, $n_0 = (0,0,-1)$, $\Sigma^{(1)} = \Sigma^{(2)} = \phi$:

$$\begin{align*}
\frac{\partial u^{(j)}}{\partial t} - \frac{\mu^{(j)}}{\rho^{(j)}} \nabla^2 u^{(j)} + \frac{1}{\rho^{(j)}} \nabla q^{(j)} &= 0, \quad \nabla \cdot u^{(j)} = 0, \\
\frac{\partial \theta^{(j)}}{\partial t} - \kappa^{(j)} \nabla^2 \theta^{(j)} &= 0 \quad \text{in } D_{\infty}^{(j)} \equiv R^3 \times (0, \infty).
\end{align*}$$

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\[(u^{(j)}, \theta^{(j)}) \big|_{t=0} = (0, 0) \quad \text{on} \quad \mathbb{R}^{3(j)}, \quad (3.2)\]

\[
\begin{align*}
    |u^{(1)} - u^{(2)}|_{x_3 = 0} &= (b_1, b_2, 0), \\
    - \mu^{(1)} \left( \frac{\partial u^{(1)}}{\partial x_k} + \frac{\partial u^{(1)}}{\partial x_3} \right) + \mu^{(2)} \left( \frac{\partial u^{(2)}}{\partial x_k} + \frac{\partial u^{(2)}}{\partial x_3} \right) \\
    &- \frac{\sigma'}{2} \left( \frac{\partial \theta^{(1)}}{\partial x_k} + \frac{\partial \theta^{(2)}}{\partial x_k} \right) \bigg|_{x_3 = 0} = b_{3+k} \quad (k = 1, 2), \\
    - q^{(1)} + 2\mu^{(1)} \frac{\partial u^{(1)}}{\partial x_3} + q^{(2)} - 2\mu^{(2)} \frac{\partial u^{(2)}}{\partial x_3} \\
    &+ \frac{\sigma}{2} \int_0^t \nabla^2 \left( u^{(1)} + u^{(2)} \right) d\tau \bigg|_{x_3 = 0} = b_{61} + \int_0^t b_{62} d\tau, \\
    \theta^{(1)} - \theta^{(2)} \bigg|_{x_3 = 0} &= b_7, \\
    - \kappa^{(1)} \frac{\partial \theta^{(1)}}{\partial x_3} + \kappa^{(2)} \frac{\partial \theta^{(2)}}{\partial x_3} \bigg|_{x_3 = 0} &= b_8 \quad \text{on} \quad D_\infty = \mathbb{R}^2 \times (0, \infty), \quad (3.3)
\end{align*}
\]

where \((b_k, b_{3+k}, b_{6k}, b_7, b_8) \quad (k = 1, 2)\) are given functions on \(D_\infty\) and \(\nabla^2 = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}\). Extend \((u^{(1)}, q^{(1)}, \theta^{(1)}, u^{(2)}, q^{(2)}, \theta^{(2)})\), \((b_k, b_{3+k}, b_{6k}, b_7, b_8)\) to the half space \(t < 0\) by \(0\) and apply the Fourier transformation with respect to \(x' = (x_1, x_2)\) and the Laplace transformation with respect to \(t\):

\[
F[f] = \hat{f}(\xi', x_3, s) = \int_0^\infty e^{-st} dt \int_{\mathbb{R}^2} e^{-i\xi' \cdot \xi'} f(x, t) dx'. \quad (3.4)
\]

Then we have the following system of ordinary differential equations:

\[
\begin{align*}
    \frac{\mu^{(j)}}{\rho^{(j)}} \left( r^{(j)^2} - \frac{d^2}{dx_3^2} \right) \hat{u}^{(j)} + \frac{1}{\rho^{(j)}} i\xi_k \hat{q}^{(j)} &= 0 \quad (k = 1, 2), \\
    \frac{\mu^{(j)}}{\rho^{(j)}} \left( r^{(j)^2} - \frac{d^2}{dx_3^2} \right) \hat{u}^{(j)} + \frac{1}{\rho^{(j)}} \frac{d}{dx_3} \hat{q}^{(j)} &= 0, \quad (3.5) \\
    i\xi_1 \hat{u}_1^{(j)} + i\xi_2 \hat{u}_2^{(j)} + \frac{d}{dx_3} \hat{u}_3^{(j)} &= 0, \\
    \kappa^{(j)} \left( r^{(j)^2} - \frac{d^2}{dx_3^2} \right) \hat{\theta}^{(j)} &= 0, \quad (3.6)
\end{align*}
\]

\[
\left( \hat{u}^{(j)}, \hat{\theta}^{(j)} \right) \bigg|_{t=0} = (0, 0),
\]
We shall seek the solution of (3.5)(j) of the form ([23, 24, 27]):

\[
\begin{align*}
\hat{u}_k^{(1)}(t) - \hat{u}_k^{(2)}(t) \bigg|_{x_3=0} &= \hat{b}_k \quad (k = 1, 2), \quad \hat{u}_3^{(1)}(t) - \hat{u}_3^{(2)}(t) \bigg|_{x_3=0} = 0, \\
-\mu^{(1)} \left( i\xi_k \hat{u}_3^{(1)} + \frac{d\hat{u}_k^{(1)}}{dx_3} \right) + \mu^{(2)} \left( i\xi_k \hat{u}_3^{(2)} + \frac{d\hat{u}_k^{(2)}}{dx_3} \right) \\
-\frac{\sigma'}{2} i\xi_k (\hat{\theta}^{(1)} + \hat{\theta}^{(2)}) \bigg|_{x_3=0} &= \hat{b}_{3+k} \quad (k = 1, 2), \\
-\hat{\theta}^{(1)} + 2\mu^{(1)} \frac{d\hat{u}_3^{(1)}}{dx_3} + \hat{\theta}^{(2)} - 2\mu^{(2)} \frac{d\hat{u}_3^{(2)}}{dx_3} \\
-\frac{\sigma}{2s} \xi'^2 (\hat{u}^{(1)} + \hat{u}^{(2)}) \bigg|_{x_3=0} &= \hat{b}_{61} + \frac{1}{s}\hat{b}_{62}, \\
\hat{\theta}^{(1)} - \hat{\theta}^{(2)} \bigg|_{x_3=0} &= \hat{b}_{71}, \quad -\kappa^{(1)} \frac{d\hat{\theta}^{(1)}}{dx_3} + \kappa^{(2)} \frac{d\hat{\theta}^{(2)}}{dx_3} \bigg|_{x_3=0} = \hat{b}_{8}, \\
(\hat{u}^{(1)}, \hat{\theta}^{(1)}, \hat{\theta}^{(1)}) &\to 0 \text{ as } x_3 \to +\infty, \quad (\hat{u}^{(2)}, \hat{\theta}^{(2)}, \hat{\theta}^{(2)}) \to 0 \text{ as } x_3 \to -\infty, \\
r^{(j)} &= \frac{\rho^{(j)}}{\mu^{(j)}} s + \xi'^2, \quad \bar{r}^{(j)} = \frac{s}{\kappa^{(j)}} + \xi'^2, \quad \xi'^2 = \xi_1^2 + \xi_2^2, \\
\arg r^{(j)}, \arg \bar{r}^{(j)} &\in \left(-\frac{\pi}{4}, \frac{\pi}{4}\right).
\end{align*}
\]

We shall seek the solution of (3.5)(j) of the form ([23, 24, 27]):

\[
\begin{align*}
\begin{pmatrix}
\hat{u}_1^{(1)}(t) \\
\hat{u}_2^{(1)}(t) \\
\hat{u}_3^{(1)}(t)
\end{pmatrix} &= c_1^{(1)} \begin{pmatrix}
r^{(1)} \\
0 \\
i\xi_1
\end{pmatrix} e^{-r^{(1)}x_3} + c_2^{(1)} \begin{pmatrix}
0 \\
r^{(1)} \\
i\xi_2
\end{pmatrix} e^{-r^{(1)}x_3} \\
&+ c_3^{(1)} \begin{pmatrix}
i\xi_1 \\
i\xi_2 \\
-|\xi'|
\end{pmatrix} e^{-|\xi'|x_3}, \\
\hat{\theta}^{(1)} &= -c_3^{(1)} \rho^{(1)} s e^{-|\xi'|x_3}, \quad \bar{\theta}^{(1)} = c_4^{(1)} e^{-\bar{r}^{(1)}x_3}, \\
\begin{pmatrix}
\hat{u}_1^{(2)}(t) \\
\hat{u}_2^{(2)}(t) \\
\hat{u}_3^{(2)}(t)
\end{pmatrix} &= c_1^{(2)} \begin{pmatrix}
r^{(2)} \\
0 \\
i\xi_1
\end{pmatrix} e^{r^{(2)}x_3} + c_2^{(2)} \begin{pmatrix}
0 \\
r^{(2)} \\
i\xi_2
\end{pmatrix} e^{r^{(2)}x_3} \\
&+ c_3^{(2)} \begin{pmatrix}
i\xi_1 \\
i\xi_2 \\
|\xi'|
\end{pmatrix} e^{\xi'^{2}x_3}, \\
\hat{\theta}^{(2)} &= -c_3^{(2)} \rho^{(2)} s e^{\xi'^{2}x_3}, \quad \bar{\theta}^{(2)} = c_4^{(2)} e^{\bar{r}^{(2)}x_3},
\end{align*}
\]

The constants \((c_1^{(1)}, c_2^{(1)}, c_3^{(1)}, c_4^{(1)}, c_1^{(2)}, c_2^{(2)}, c_3^{(2)}, c_4^{(2)})\) can be determined by substituting the formulas (3.8)(j) into the boundary conditions (3.7). After some
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Calculations, we obtain

\( \left( r^{(1)} - |\xi'| \right) c_{3}^{(1)} = \frac{\left( r^{(1)} - |\xi'| \right)}{\Psi} \left( M_{12} i \xi_{k} \hat{b}_{k} + M_{32} i \xi_{k} \hat{b}_{3+k} + M_{42} \hat{b}_{6} \right), \tag{3.9} \)

\( \left( r^{(2)} - |\xi'| \right) c_{3}^{(2)} = \frac{\left( r^{(2)} - |\xi'| \right)}{\Psi} \left( M_{14} i \xi_{k} \hat{b}_{k} + M_{34} i \xi_{k} \hat{b}_{3+k} + M_{44} \hat{b}_{6} \right), \tag{3.10} \)

\( c^{(1)} \left( 1 - |\xi'| \right) c_{3}^{(1)} = c^{(2)} \left( 1 + |\xi'| \right) c_{3}^{(2)} = \frac{1}{\Psi} \left( (M_{11} - |\xi'| M_{12}) i \xi_{k} \hat{b}_{k} \right. \)

\( \left. + (M_{31} - |\xi'| M_{32}) i \xi_{k} \hat{b}_{3+k} + (M_{41} - |\xi'| M_{42}) i \xi_{k} \hat{b}_{6} \right), \tag{3.11} \)

\( c_{k}^{(1)} r^{(1)} + i \xi_{k} c_{3}^{(1)} \)

\( = \frac{i \xi_{k}}{G} \left[ \mu^{(1)} \left( c^{(1)} - |\xi'| c_{3}^{(1)} + (r^{(1)} - |\xi'| c_{3}^{(1)}) \right) \right. \)

\( - \mu^{(2)} \left( c^{(2)} - |\xi'| c_{3}^{(2)} + (r^{(2)} - |\xi'| c_{3}^{(2)}) \right) + \frac{\mu^{(2)} r^{(2)}}{G} \hat{b}_{k} + \frac{1}{G} \hat{b}_{3+k}, \tag{3.12} \)

\( - c_{k}^{(2)} r^{(2)} + i \xi_{k} c_{3}^{(2)} \)

\( = \frac{i \xi_{k}}{G} \left[ \mu^{(1)} \left( c^{(1)} - |\xi'| c_{3}^{(1)} + (r^{(1)} - |\xi'| c_{3}^{(1)}) \right) \right. \)

\( - \mu^{(2)} \left( c^{(2)} - |\xi'| c_{3}^{(2)} + (r^{(2)} - |\xi'| c_{3}^{(2)}) \right) - \frac{\mu^{(2)} r^{(2)}}{G} \hat{b}_{k} + \frac{1}{G} \hat{b}_{3+k}, \tag{3.13} \)

Where \( c^{(j)} = i \xi_{1} c_{1}^{(j)} + i \xi_{2} c_{2}^{(j)}, \hat{b}_{3+k} = \frac{\sigma}{2} i \xi_{k} (c_{4}^{(1)} + c_{4}^{(2)}) + \hat{b}_{3+k}, \)

\( \left( \begin{array}{c}
  c_{4}^{(1)} \\
  c_{4}^{(2)}
\end{array} \right) = \frac{1}{G} \left( \begin{array}{c}
  \kappa^{(2)} r^{(2)} \hat{b}_{7} + \hat{b}_{8} \\
  -\kappa^{(1)} r^{(1)} \hat{b}_{7} + \hat{b}_{8}
\end{array} \right), \quad G = \mu^{(1)} r^{(1)} + \mu^{(2)} r^{(2)}, \quad \overline{G} = \kappa^{(1)} r^{(1)} + \kappa^{(2)} r^{(2)}, \)

\( (r^{(1)} - |\xi'|) M_{12} = -\frac{\rho^{(1)} \rho^{(2)}}{\mu^{(1)}} \frac{s}{r^{(1)} + |\xi'|} \)

\( \times \left[ \Psi^{(2)} - \rho^{(1)} s^{2} \frac{r^{(1)} - |\xi'|}{r^{(1)} + |\xi'|} - 4 \mu^{(1)} s \xi^{(2)} \frac{1}{r^{(1)} + |\xi'|} \right], \)

\( (r^{(1)} - |\xi'|) M_{32} = -\frac{\rho^{(1)} \rho^{(2)}}{s^{2}} \frac{s^{2}}{r^{(1)} + |\xi'|} \)

\( \times \left[ r^{(1)} \frac{r^{(2)} - |\xi'|}{r^{(2)} + |\xi'|} + r^{(2)} + \left( 2 \mu^{(1)} r^{(1)} + \frac{\sigma}{s} \xi^{(2)} \right) \frac{1}{r^{(2)} + |\xi'|} \right], \)

\( (r^{(1)} - |\xi'|) M_{42} = -\frac{\rho^{(1)} \rho^{(2)}}{s^{2}} \frac{s^{2}}{r^{(1)} + |\xi'|} \)

\( \times \left[ |\xi'| (r^{(1)} + r^{(2)}) - \frac{1}{\mu^{(2)}} \frac{|\xi'|}{r^{(2)} + |\xi'|} \right] \left( \mu^{(2)} (r^{(2)} + \xi^{(2)}) - \mu^{(1)} (r^{(1)} + \xi^{(2)}) \right) \right) \)
The following lemma plays a crucial role in our investigation.

**Lemma 3.1.** Let \( \text{Re } s > 0, \xi' \in \mathbb{R}^2 \). Then the estimate

\[
|s^{1/2} \sum_{j=1}^{2} |\Psi^{(j)}| + |s^{5/2} + |\xi'| |s|^2 + \sigma |\xi'|^2 |s| \leq c(h)|\Psi'|
\]

is valid, where \( c(h) \) is a nonincreasing function of \( h \).
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PROOF. First of all, we have $|\Psi'| \equiv |s^2| \geq |s|^2 \geq c_4 |s|^2 |\xi'|$ with $c_4 = 2 \inf \left( \frac{r^{(1)} - |\xi'|}{r^{(1)} + |\xi'|}, \frac{r^{(2)} - |\xi'|}{r^{(2)} + |\xi'|} \right) > 0$. For the case $|s| \leq |\xi'|^2$, it is obvious that $|s|^{5/2} = |s|^2 \cdot |s|^{1/2} \leq |s|^2 |\xi'| \leq c_4^{-1} |\Psi'|$. And we can derive

$$
\left| \sigma |s|^{\xi'} \sum_{j=1}^{2} \frac{1}{\mu^{(j)}} \frac{|\xi'|}{r^{(j)} + |\xi'|} \right|
\leq |\Psi'| + c_5 |s|^2 (|s|^{1/2} + |\xi'|) + c_6 \sum_{j=1}^{2} r^{(j)} + |\xi'|
+ c_7 |s|^2 |\xi'| \sum_{j=1}^{2} \frac{|\xi'|}{r^{(1)} + |\xi'|} r^{(j)} + |\xi'| \left| r^{(1)} - |\xi'| r^{(2)} - |\xi'| \right|
\leq |\Psi'| + c_8 |s|^{5/2} + c_9 |s|^2 |\xi'| \leq c_10 |\Psi'|,
$$

so that

$$
\sigma |\xi'| |s| \leq c_{11} |\Psi'| \quad \text{with} \quad c_{11} = c_{12}^{-1} c_{10}, \quad c_{12} = \inf \sum_{j=1}^{2} \frac{1}{\mu^{(j)}} \frac{|\xi'|}{r^{(j)} + |\xi'|} > 0,
$$

from the fact $|\xi'| \geq |s|^{1/2} \geq \sqrt{\kappa} > 0$. While for $|s| \geq |\xi'|^2$, the inequality $\sigma |\xi'| |s| \leq \sigma |s|^{3/2} |\xi'| \leq \frac{\sigma}{\sqrt{\kappa}} |s|^2 |\xi'| \leq \frac{\sigma}{\sqrt{\kappa}} |\Psi'|$ is valid. Since

$$
\left| s^2 (r^{(1)} + r^{(2)}) + \frac{\rho^{(2)}}{\mu^{(1)}} s^3 \frac{1}{r^{(1)} + |\xi'|} + \frac{\rho^{(1)}}{\mu^{(2)}} s^3 \frac{1}{r^{(2)} + |\xi'|} \right|
\leq |\Psi'| + c_7 |s|^2 |\xi'| \sum_{j=1}^{2} \frac{|\xi'|}{r^{(1)} + |\xi'|} \left| r^{(j)} \right|
+ \sigma |\xi'| |s| \left( \sum_{j=1}^{2} \frac{1}{\mu^{(j)}} \frac{|\xi'|}{r^{(j)} + |\xi'|} \right) + 2 |s|^2 |\xi'| \left( \frac{r^{(1)} - |\xi'|}{r^{(1)} + |\xi'|} + \frac{r^{(2)} - |\xi'|}{r^{(2)} + |\xi'|} \right)
$$

holds, we have $|s|^{5/2} \leq c_{13} \left( 1 + \frac{1}{\sqrt{\kappa}} \right) |\Psi'|$. Finally, we obtain

$$
\left| \frac{1}{\mu^{(1)}} \frac{s^2 \Psi^{(2)}}{r^{(1)} + |\xi'|} + \frac{1}{\mu^{(2)}} \frac{s^2 \Psi^{(1)}}{r^{(2)} + |\xi'|} \right|
\leq |\Psi'| + |s|^2 (|r^{(1)}| + |r^{(2)}|) + 2 |s|^2 |\xi'| \left( \frac{r^{(1)} - |\xi'|}{r^{(1)} + |\xi'|} + \frac{r^{(2)} - |\xi'|}{r^{(2)} + |\xi'|} \right)
\leq c(h) |\Psi'|.
$$

The proof is completed.

Next we introduce the new norms

$$
\| \cdot \|_{l,h,D^{(1)}} \quad \| \cdot \|_{l,h,D^{(2)}} \quad \text{and} \quad \| \cdot \|_{l,h,D^{\infty}}.
$$
The norm \(\| \cdot \|_{l,h,D_\infty^{(2)}}\) is defined in a similar way to the norm \(\| \cdot \|_{l,h,D_\infty^{(1)}}\) with \(R_+\) replaced by \(R_- = \{x_3 < 0\}\).

The next two lemmas are due to Solonnikov ([33]).

**Lemma 3.2.** The norm \(\| \cdot \|_{l,h,D_\infty^{(2)}}\) (resp. \(\| \cdot \|_{l,h,D_\infty^{(1)}}\)) is equivalent to the norm of \(H_0^{1,1/2}(D'_{\infty})\) (resp. \(H_0^{1,1/2}(D_{\infty}^{(1)})\)).

Let us put \(e_1^{(1)}(x_3) = e^{-r(1)x_3}\), \(e_2^{(1)}(x_3) = e^{-r(1)x_3 - |\xi'|x_3}\).

**Lemma 3.3.** Let \(h > 0\), \(s = h + i\xi_0\), \(k \geq 0\) be an integer and \(\alpha \in (0, 1)\). Then we have for any \(\xi' \in R^2\)

\[
\begin{align*}
(i) & \quad \int_0^\infty \left( \frac{d}{dx_3} \right)^k e_1^{(1)}(x_3)^2 dx_3 \leq c|r|^{2k-1}, \\
(ii) & \quad \int_0^\infty \int_0^\infty \left( \frac{d}{dx_3} \right)^k e_1^{(1)}(x_3 + z) - \left( \frac{d}{dx_3} \right)^k e_1^{(1)}(x_3)^2 \frac{dx_3 dz}{z^{1+2\alpha}} \leq c|r|^{2(k+\alpha)-1}, \\
(iii) & \quad \int_0^\infty \left( \frac{d}{dx_3} \right)^k e_2^{(1)}(x_3)^2 dx_3 \leq c\frac{|r|^{2k-1} + |\xi'|^{2k-1}}{|r|^2}, \\
(iv) & \quad \int_0^\infty \int_0^\infty \left( \frac{d}{dx_3} \right)^k e_2^{(1)}(x_3 + z) - \left( \frac{d}{dx_3} \right)^k e_2^{(1)}(x_3)^2 \frac{dx_3 dz}{z^{1+2\alpha}} \leq c\frac{|r|^{2(k+\alpha)-1} + |\xi'|^{2(k+\alpha)-1}}{|r|^2},
\end{align*}
\]

where \(c\) is a positive constant independent of \(r\) and \(|\xi'|\).

Of course, similar estimates hold for

\[
\begin{align*}
e_1^{(2)}(x_3) = e^{r(2)x_3}, & \quad e_2^{(2)}(x_3) = e^{r(2)x_3 - |\xi'|x_3} - \frac{r(2) - |\xi'|}{r(2)} x_3, 
\end{align*}
\]

and the estimates analogous to (i), (ii) are true for

\[
\begin{align*}
e^{r|x'|x_3}, & \quad e^{-r|x'|x_3}, e^{r|x'|x_3}, \quad e^{-r|x'|x_3}.
\end{align*}
\]
THEOREM 3.4. Let $h > 0$. Suppose

$$(b_1, b_2, b_7) \in H_h^{5/2+1,5/4+1/2}(D'_\infty), \quad (b_4, b_5, b_{61}, b_8) \in H_h^{3/2+1,3/4+1/2}(D'_\infty),$$

$$b_{02} \in H_h^{1/2+1,1/4+1/2}(D'_\infty).$$

Then for the solution $(u^{(1)}, q^{(1)}, \theta^{(1)}, u^{(2)}, q^{(2)}, \theta^{(2)})$ of (3.1)–(3.3), we have the estimate

$$
\|(u^{(j)}, q^{(j)}, \theta^{(j)})\|_{H(D_\infty^{(j)})} = 2 \left( \|(u^{(j)}, \theta^{(j)})\|_{3+1,h, D_\infty^{(j)}} + \|\nabla q^{(j)}\|_{1+1,h, D_\infty^{(j)}} \right) + \|q^{(1)} - q^{(2)}\|_{3/2+1,h, D'_\infty} \leq c_{14}(h) \left( \|(b_1, b_2, b_7)\|_{5/2+1,h, D'_\infty} + \|(b_4, b_5, b_{61}, b_8)\|_{3/2+1,h, D'_\infty} + \|b_{02}\|_{1/2+1,h, D'_\infty} \right) = c_{14}(h) \|b\|_{H(D'_\infty)},
$$

where $c(h)$ is a nonincreasing function of $h$.

PROOF. From Lemma 3.1, we have

$$
\|(a^{(1)} - |\xi'|)c^{(1)}_3, (a^{(2)} - |\xi'|)c^{(2)}_3, |c^{(1)} - |\xi'|c^{(1)}_3 | \leq c(h) \left[ \hat{b}_k + \frac{1}{|s|^{1/2} + |\xi'|} \left( \left| \hat{b}_{3+k} \right| + \hat{b}_{61} \right) + \frac{1}{|s| + |\xi'|^2} \hat{b}_{02} \right].
$$

Using this inequality, (3.8)–(3.12), definition of the norm $\|\cdot\|_{3+1,h, D_\infty^{(j)}}$ and Lemma 3.3, we obtain the required estimate (3.15). \qed

3.2. Nonhomogeneous system

We next consider the nonhomogeneous system in $D_T^{(j)} \equiv R^3 \times (0, T)$.

THEOREM 3.5. Let $b = (b_k, b_{3+k}, b_{6k}, b_7, b_8)$ ($k = 1, 2$) be as in Theorem 3.4 ($D'_\infty$ should be replaced by $D'_T \equiv R^2 \times (0, T)$). Assume also

$$
(\Phi_1^{(j)}, \Phi_3^{(j)}) \in H_h^{1+1,1/2+1/2}(D_T^{(j)}), \quad \Phi_2^{(j)} \in H_h^{2+1,1/2}(D_T^{(j)}), \quad \Phi_2^{(j)} = \nabla \cdot \Phi_2^{(j)}, \quad \Phi_2^{(j)} \in H_h^{0,3/2+1/2}(D_T^{(j)}), \quad (\Phi_2^{(j)} - \Phi_2^{(j)})_3 \big|_{x_3=0} = 0.
$$

Then there exists a unique solution $(u^{(1)}, q^{(1)}, \theta^{(1)}, u^{(2)}, q^{(2)}, \theta^{(2)})$ to the problem (1.20)–(3.2) such that

$$
(u^{(j)}, \theta^{(j)}) \in H_h^{3+1,3/2+1/2}(D_T^{(j)}), \quad \nabla q^{(j)} \in H_h^{1+1,1/2+1/2}(D_T^{(j)}),
$$
PROOF. Extending \((\Phi_1(j), \Phi_2(j), \Phi_2'(j), \Phi_3(j))(j=1,2), (b_k, b_{3+k}, b_{6+k}, b_7, b_8)(k=1,2)\) appropriately, we may assume \(T = \infty\) (see [33]). The solution \((u(j), q(j), \theta(j))(j=1,2)\) of (1.20), (3.2), (3.3) is given as follows:

\[
\begin{align*}
\begin{cases}
    u(j) &= u'(j) + \nabla \varphi(j) + \nabla \varphi'(j) + u''(j), \\
    q(j) &= \mu(j)\psi(j) - \rho(j) \left( \frac{\partial \varphi(j)}{\partial t} + \frac{\partial \varphi'(j)}{\partial t} \right) + q''(j), \\
    \theta(j) &= \theta'(j) + \theta''(j).
\end{cases}
\end{align*}
\]

Here \((u'(j), u''(2))\) and \((\theta'(1), \theta''(2))\) are the solutions of the heat equations with zero Dirichlet conditions:

\[
\begin{align*}
\left\{ \begin{aligned}
    \frac{\partial u'(j)}{\partial t} - \frac{\mu(j)}{\rho(j)} \nabla^2 u'(j) &= \Phi_1(j) \quad \text{in } D_\infty(j), \\
    u'(j) \big|_{t=0} &= 0 \quad \text{on } \mathbb{R}^3(j), \\
    u'(j) \big|_{x_3=0} &= 0 \quad \text{on } D'_\infty,
\end{aligned} \right. 
(3.17)(j)
\]

\[
\left\{ \begin{aligned}
    \frac{\partial \theta'(j)}{\partial t} - \kappa(j) \nabla^2 \theta'(j) &= \Phi_3(j) \quad \text{in } D_\infty(j), \\
    \theta'(j) \big|_{t=0} &= 0 \quad \text{on } \mathbb{R}^3(j), \\
    \theta'(j) \big|_{x_3=0} &= 0 \quad \text{on } D'_\infty;
\end{aligned} \right. 
(3.18)(j)
\]

\((\varphi'(1), \varphi'(2))\) satisfies the Laplace equation with Dirichlet boundary condition:

\[
\left\{ \begin{aligned}
    \nabla^2 \varphi(j) &= \Phi_2(j) - \nabla \cdot u'(j) = \Psi(j) - \nabla \cdot \Phi_2''(j) \quad \text{in } D_\infty(j), \\
    \varphi(j) \big|_{x_3=0} &= 0 \quad \text{on } D'_\infty;
\end{aligned} \right. 
(3.19)(j)
\]

\((\varphi'(j), \varphi'(2))\) satisfies the following diffraction problem:

\[
\left\{ \begin{aligned}
    \nabla^2 \varphi(j) &= 0 \quad \text{in } D_\infty(j), \\
    \rho^{(1)} \varphi'(1) - \rho^{(2)} \varphi'(2) \big|_{x_3=0} &= 0, \\
    \frac{\partial \varphi'(1)}{\partial x_3} - \frac{\partial \varphi'(2)}{\partial x_3} \big|_{x_3=0} &= -\left( \frac{\partial \varphi^{(1)}}{\partial x_3} - \frac{\partial \varphi^{(2)}}{\partial x_3} \right) \big|_{x_3=0} = A \quad \text{on } D'_\infty.
\end{aligned} \right. 
(3.20)
\]
Finally we define \((u''(j), q''(j), \theta''(j))\) as the solution of the problem of the form (3.1\((j))-(3.3)).

It can be easily verified that \((u''(j), \theta''(j))\) satisfy ([10])

\[
\sum_{j=1}^{2} \|(u''(j), \theta''(j))\|_{H_h^{3+1/2}((D_{\infty}(j)))} \leq c_{16} \sum_{j=1}^{2} \|\Phi_1^{(j)}, \Phi_3^{(j)}\|_{H_h^{3+1/2}((D_{\infty}(j)))},
\]

It is well known that \(\nabla \varphi^{(1)}\) satisfies ([1])

\[
\|\nabla \varphi^{(1)}\|_{W_2^{3/2}((R^3(1)))} \leq c_{17} \|\Psi^{(1)}\|_{W_2^{3/2}((R^3(1)))}.
\]

Since from Green's formula

\[
\varphi^{(1)} = \int_{R^3(1)} G(x, y) \Psi^{(1)}(y, t) dy = - \int_{R^3(1)} \nabla_y G(x, y) \cdot \Phi_2''(y, t) dy
\]

holds with the Green function \(G(x, y)\) in a half space, we find

\[
\|\nabla \varphi^{(1)}\|_{H_h^{3/2+1/2}((D_{\infty}(1)))} \leq c_{18} \|\Psi^{(1)}''\|_{H_h^{3/2+1/2}((D_{\infty}(1)))}
\]

by virtue of the Calderón-Zygmunt theorem for the singular integral operators ([6, 8]). Of course, the similar estimate is true for \(\nabla \varphi^{(2)}\).

The problem (3.20) can be solved in the same way as in §3.1. Indeed, after the Fourier-Laplace transformation (3.4), we have

\[
\varphi^{(1)}(x_3) = -\frac{\rho^{(2)}}{\rho} \frac{A e^{-|\xi'|x_3}}{|\xi'|} (x_3 > 0), \quad \varphi^{(2)}(x_3) = -\frac{\rho^{(1)}}{\rho} \frac{A e^{\xi'|x_3}}{|\xi'|} (x_3 < 0),
\]

where \(\rho = \rho^{(1)} + \rho^{(2)}\). The relation \(F^{-1}\left[\frac{e^{-|\xi'|x_3}}{|\xi'|}\right] = \frac{\delta(t)}{2\pi|x|} (x_3 > 0)\) and the convolution theorem imply

\[
\begin{align*}
\varphi^{(1)} &= -\frac{1}{2\pi} \int_{R^2} \frac{1}{|x-y'|} A(y', t) dy' (x_3 > 0), \\
\varphi^{(2)} &= \frac{1}{2\pi} \int_{R^2} \frac{1}{|x-y'|} A(y', t) dy' (x_3 < 0).
\end{align*}
\]
Hence, in the same way as for the problem (3.19)\(^{(j)}\), we arrive at the inequality

\[
\sum_{j=1}^{2} \| \nabla \varphi^{(j)} \|_{H^{3+1.3/2+i/2}(D_{\infty}(j))} \leq c_{19} \left( \sum_{j=1}^{2} \| \nabla \varphi^{(j)} \|_{H^{3+1.3/2+i/2}(D_{\infty}(j))} + \| \Phi_{2}^{\prime(j)} \|_{H^{0,3/2+i/2}(D_{\infty}(j))} \right)
\]

by integration by parts.

From (3.21)–(3.24), Theorem 3.4, Lemma 3.2 and the trace theorem ([33]), we derive (3.16). The uniqueness of the solution follows from the standard energy method ([33]). This completes the proof of Theorem 3.5.

**Remark 3.1.** Similar results hold for the initial and the first initial-boundary value problems in a half-space (see [21, 23, 24, 33]).

### 3.3. Proof of Theorem 1.3

We first prove Theorem 1.3 in the spaces \( H^{1.3/2}_{h}(Q_T) \). Namely, we shall show:

**Theorem 3.6.** Let us assume the conditions (i), (ii) of Theorem 1.2 and

(iii)' \( v_0^{(j)} = \theta_0^{(j)} \equiv 0 \).

(iv)' \( (\Phi, \theta^{(j)}) \) satisfy the condition (iv) of Theorem 1.2 in the spaces \( H^{1.3/2}_{h}(Q_T) \).

If \( h > 0 \) is sufficiently large, then there exists a unique solution \( (u^{(1)}, q^{(1)}, \pi^{(1)}, u^{(2)}, q^{(2)}, \pi^{(2)}) \) to (1.20)\(^{(j)}\)–(1.23) satisfying (1.18) and (1.19) in the spaces \( H^{1.3/2}_{h}(Q_T) \).

**Proof.** We decompose the solution of (1.20)\(^{(j)}\)–(1.23)\(^{(j)}\) with \( v_0^{(j)} = \theta_0^{(j)} \equiv 0 \) as \( (u^{(j)}, q^{(j)}, \pi^{(j)}) = (w^{(j)}, \pi^{(j)}) + (u^{(j)}, q^{(j)}, \theta^{(j)}) \), where \( (w^{(j)}, \pi^{(j)}) \), \( (u^{(j)}, q^{(j)}, \theta^{(j)}) \) \( (j = 1, 2) \) satisfy the following systems of equations, respectively:

\[
\begin{aligned}
\frac{\partial w^{(j)}}{\partial t} - \frac{\mu^{(j)}}{\rho^{(j)}} \nabla^2 w^{(j)} + \frac{1}{\rho^{(j)}} \nabla \pi^{(j)} &= \Phi_1^{(j)}, \quad \nabla \cdot w^{(j)} = \Phi_2^{(j)} \quad \text{in} \quad Q_T^{(j)}, \\
n(w^{(j)}) |_{t=0} &= 0 \quad \text{on} \quad \Omega^{(j)}, \\
w^{(1)} - w^{(2)} &= 0, \\
2 \prod_0^{(j)} (\mu^{(1)} D^{(w^{(1)})} n_0 - \mu^{(2)} D^{(w^{(2)})} n_0) &= \Phi_4, \\
- (\pi^{(1)} - \pi^{(2)}) + 2n_0 \cdot (\mu^{(1)} D^{(w^{(1)})} n_0 - \mu^{(2)} D^{(w^{(2)})} n_0) &= \Phi_5 \quad \text{on} \quad \Gamma_T, \\
w^{(j)} &= 0 \quad \text{on} \quad \Sigma_T^{(j)},
\end{aligned}
\]
We shall first prove the solvability of the problem (3.26).

We introduce the coverings of $S_i = \sum_{z(1)} U z(2)$ and associated smooth cut-off functions ([22, 37]).

Let $A$ be an arbitrary small number. We can construct two systems $\{w(k)\}$ and $\{H(k)\}$ as follows:

(i) $w(k) \subset 5z(k) \subset S$, $k < c = k(2)$;

(ii) for any $x \in \Sigma$, there exists $w(k)$ such that $x \in w(k)$ and $\text{dist}(x, -w(')) > \frac{3}{2}A$ for some $\delta > 0$;

(iii) there exists a number $N_0$ independent of $A$ such that $\sum_{j=1}^{\infty} \Delta(0) \sum_{j=1}^{\infty} \int_{0}^{t} w(j) d\tau = \frac{\sigma(\theta(s))}{2} n_0 \cdot \Delta(0) \sum_{j=1}^{\infty} \int_{0}^{t} w(j) d\tau = \frac{\Phi(0)}{2} n_0 \cdot \Delta(0) \sum_{j=1}^{\infty} \int_{0}^{t} w(j) d\tau$

$$= \int_{0}^{t} \Phi d\tau \div \frac{\sigma(\theta(s))}{2} n_0 \cdot \Delta(0) \sum_{j=1}^{\infty} \int_{0}^{t} w(j) d\tau \equiv \int_{0}^{t} \Phi d\tau,$$

$$\theta(1) - \theta(2) = 0, \quad \kappa(1) \nabla \theta(1) \cdot n_0 - \kappa(2) \nabla \theta(2) \cdot n_0 = \Phi(7) \quad \text{on } \Gamma_T,$$

$$u(1) = 0, \quad \theta(1) = \theta_e(1) \quad \text{on } \Sigma_T(1).$$

We shall first prove the solvability of the problem (3.26).

We introduce the coverings of $\Omega \equiv \Omega(1) \cup \Omega(2)$ and associated smooth cut-off functions ([22, 37]).

Let $\lambda$ be an arbitrary small number. We can construct two systems $\{\omega(k)\}$ and $\{\Omega(k)\}$ as follows:

(i) $\omega(k) \subset \Omega(k) \subset \Omega$; $U_k \omega(k) = U_k \Omega(k) = \Omega$;

(ii) for any $x \in \Omega$, there exists $\omega(k)$ such that $x \in \omega(k)$ and $\text{dist}(x, \Omega - \omega(k)) \geq \beta_1 \lambda$ for some $\beta_1 > 0$;

(iii) there exists a number $N_0$ independent of $\lambda$ such that $\bigcup_{k=1}^{N_0+1} \Omega(k) = \phi$;

(iv-1) if $\Omega(1) \cap \Gamma = \phi$, $\Omega(1) \cap \Sigma(j) = \phi$ (in this case, we shall denote $k = k'$), then $\omega(k')$ and $\Omega(k')$ are the cubes with the same center and with the length of their edges, in a direction parallel to axes, equal to $\lambda/2$ and $\lambda$, respectively;

(iv-2) if $\Omega(1) \cap \Gamma \neq \phi$, $\Omega(1) \cap \Sigma(j) \neq \phi$, (we shall denote $k = k''$), then we construct $\omega(k'')$ and $\Omega(k'')$ by means of the local rectangular coordinate system $\{y\}$ with the origin at some point $\xi(k'') \in \Gamma$, i.e., we take the inner normal to $\Omega(1)$ at $\xi(k'') \in \Gamma$ as the $y_3$-axis and place the $y_1$- and $y_2$-axes in the tangential plane at $\xi(k'')$. The local coordinates $\{y\}$ are connected with the original coordinates $\{x\}$ by the relation $y = L(k'')(x - \xi(k''))$ with the orthogonal matrix $L(k'')$; hence we may assume without loss of generality that $\{x\}$ coincides with $\{y\}$. We define $\omega(k'')$, 

\[
\begin{aligned}
\frac{\partial u^{(j)}}{\partial t} - \frac{\mu^{(j)}}{\rho^{(j)}} \nabla^2 u^{(j)} + \frac{1}{\rho^{(j)}} \nabla q^{(j)} = 0, \quad \nabla \cdot u^{(j)} = 0,
\frac{\partial \theta^{(j)}}{\partial t} - \kappa^{(j)} \nabla^2 \theta^{(j)} = \Phi_3^{(j)} \quad \text{in } Q_T^{(j)},
(u^{(j)}, \theta^{(j)}) \mid_{t=0} = (0, 0) \quad \text{on } \Omega^{(j)},
u^{(1)} - u^{(2)} = 0,
2 \prod_{0}^{(1)} (\mu^{(1)} D(u^{(1)}) n_0 - \mu^{(2)} D(u^{(2)}) n_0) - \sum_{j=1}^{2} \Pi_{0}^{(1)} \nabla \theta^{(j)} = 0,
-(q^{(1)} - q^{(2)}) + 2n_0 \cdot (\mu^{(1)} D(u^{(1)}) n_0 - \mu^{(2)} D(u^{(2)}) n_0)
- \frac{\sigma(\theta(s))}{2} n_0 \cdot \Delta(0) \sum_{j=1}^{t} \int_{0}^{t} w(j) d\tau
= \int_{0}^{t} \Phi d\tau \div \frac{\sigma(\theta(s))}{2} n_0 \cdot \Delta(0) \sum_{j=1}^{t} \int_{0}^{t} w(j) d\tau \equiv \int_{0}^{t} \Phi d\tau,
\theta^{(1)} - \theta^{(2)} = 0, \quad \kappa^{(1)} \nabla \theta^{(1)} \cdot n_0 - \kappa^{(2)} \nabla \theta^{(2)} \cdot n_0 = \Phi_7 \quad \text{on } \Gamma_T,
u^{(1)} = 0, \quad \theta^{(1)} = \theta_e^{(1)} \quad \text{on } \Sigma_T^{(j)}.
\end{aligned}
\]
where the equation $y_3 = F_0(y'; \xi^{(k'')})$ represents the boundary $\Gamma$ in the neighborhood $U$ of the point $\xi^{(k'')}$, and $\beta_2$ is a positive constant independent of $\lambda$. The function $F_0 \in W^{5/2+1}_2(U)$ can be extended into $\mathbb{R}^3$ in such a way that $F_0 \in W^{3+1}_2(\mathbb{R}^3)$, and $F_0(0) = \nabla F_0(0) = 0$ yields the inequality

$$\sup_U |D_x^\alpha F_0| \leq c\lambda^{2-|\alpha|} \quad (|\alpha| = 0, 1) \quad (3.27)$$

by virtue of the imbedding theorem (see [33]). Furthermore, the change of the variables $z_i = y_i (i = 1, 2)$, $z_3 = y_3 + F_0(z)$ takes $\Gamma^{(1)(k'')}$, $\Gamma^{(2)(k'')}$ into the cube

$$K^{(1)(k'')} = \{|z_i| \leq \beta_2 \lambda, \ (i = 1, 2), \ 0 \leq z_3 \leq 2\beta_2 \lambda\},$$

$$K^{(2)(k'')} = \{|z_i| \leq \beta_2 \lambda, \ (i = 1, 2), \ -2\beta_2 \lambda \leq z_3 \leq 0\},$$

and the boundary in $\Gamma^{(k'')}$, which we shall denote by $\Gamma^{(k'')}$, into

$$K^{(k'')} = \{|z_i| \leq \beta_2 \lambda, \ (i = 1, 2), \ z_3 = 0\};$$

(iv-3) if $\Omega^{(k)} \cap \Sigma^{(j)} \neq \phi, \Omega^{(k)} \cap \Gamma = \phi$, (in this case let us denote $k = k''$), we define $\omega^{(k'')}$ and $\Omega^{(k'')}$ in the same way as $\omega^{(1)(k'')}$ and $\Omega^{(1)(k'')}$ (for simplicity, we shall use the same letter $F_0$ describing the boundary $\Sigma^{(j)}$ in the neighbourhood $U$ of $\xi^{(k'')} \in \Sigma^{(j)}$, which will not cause any confusion).

Now we introduce two families of smooth functions $\{\zeta^{(k)}(x)\}$ and $\{\eta^{(k)}(x)\}$ associated with the coverings $\{\omega^{(k)}\}$ and $\{\Omega^{(k)}\}$:

$$\zeta^{(k)}(x) = \begin{cases} 1 & \text{if } x \in \omega^{(k)}, \\ 0 & \text{if } x \in \overline{\Omega} - \Omega^{(k)}, \end{cases} \quad 0 \leq \zeta^{(k)}(x) \leq 1, \quad |D_x^\alpha \zeta^{(k)}(x)| \leq c\lambda^{-|\alpha|},$$

and put $\eta^{(k)}(x) \equiv \frac{\zeta^{(k)}(x)}{\sum_k \zeta^{(k)}(x)^2}$.

Clearly, $\{\eta^{(k)}(x)\}$ has the following properties:

$$\eta^{(k)}(x) = 0 \text{ if } x \in \overline{\Omega} - \Omega^{(k)}, \quad \sum_k \zeta^{(k)}(x)\eta^{(k)}(x) = 1,$$

$$|D_x^\alpha \eta^{(k)}(x)| \leq c\lambda^{-|\alpha|}. \quad (3.28)$$
The solvability of (3.26) will be proved by the construction of the regularizer and the contraction argument.

The regularizer \( R \) is defined by the formula

\[
RG = \sum_k \eta^{(k)}(x)\left(u^{(1)(k)}, q^{(1)(k)}, \theta^{(1)(k)}, u^{(2)(k)}, q^{(2)(k)}, \theta^{(2)(k)}\right),
\]

where

\[
G = \left(0, 0, (0, 0, \Phi_3(j))\right), \left(0, 0, \int_0^t \Phi_6' d\tau, 0, \Phi_7\right), \theta_e(j)
\]

and

\[
\left(u^{(1)(k)}, q^{(1)(k)}, \theta^{(1)(k)}, u^{(2)(k)}, q^{(2)(k)}, \theta^{(2)(k)}\right)(x, t) \quad (k = k', k'', k''')
\]

are the unique solutions of the problems discussed in §3.2 corresponding to the cases \( k = k', k'', k''' \), respectively:

\[
\left(u^{(j)(k')}, q^{(j)(k')}, \theta^{(j)(k')}\right)(x, t) \quad (j = 1, 2)
\]

is the solution of the Cauchy problem (1.20)(j), (3.2)(j) with \(((0, 0), (\Phi_1(j), \Phi_2(j), \Phi_3(j)))\) replaced by \(((0, 0), (0, 0, \zeta(k') \Phi_3(j)))\).

\[
\left(u^{(1)(k'''), q^{(1)(k'''), \theta^{(1)(k'''), u^{(2)(k'''), q^{(2)(k'''), \theta^{(2)(k'''), (x, t)}}
\]

where \( \Pi_x^z \) denotes the coordinate transformation from \( \{z\} \) to \( \{x\} \) and

\[
(\bar{u}^{(1)(k'''), \bar{q}^{(1)(k'''), \bar{\theta}^{(1)(k'''), \bar{u}^{(2)(k'''), \bar{q}^{(2)(k'''), \bar{\theta}^{(2)(k'''), (z, t)}}
\]

is the solution of the problem (1.20)(j), (3.2)(j), (3.3) with

\[
\left((0, 0), (b_k, b_{3+k}, b_{61} + \int_0^t b_{62} d\tau, b_7, b_8)\right)
\]

replaced by

\[
\left((0, 0), (0, 0, \Phi_3(j)(k'''), \left(0, 0, \int_0^t \Phi_6' d\tau, 0, \Phi_7(k''')\right)\right), \sigma = \sigma(\theta_0'(\zeta(k'''))), \quad \sigma' = \sigma'(\theta_0'(\zeta(k'''))),
\]

and

\[
(\Phi_3(j)(k'''), \Phi_6'(k'''), \Phi_7'(k'''))(z, t) = \Pi_x^z \zeta^{(k''')}(x)(\Phi_3(j), \Phi_6', \Phi_7)(x, t).
\]

Similarly,

\[
(\bar{u}(j)(k'''), \bar{q}(j)(k'''), \bar{\theta}(j)(k'''))(x, t) = \Pi_x^z (\bar{u}(j)(k'''), \bar{q}(j)(k'''), \bar{\theta}(j)(k'''))(z, t),
\]
where \((\eta^{(j)}(k'), \theta^{(j)}(k'))(z, t)\) is the solution of the initial-boundary value problem (1.20)(j), (3.2)(j), (1.23)(j) in the half space with \(((0, 0), (\Phi_1^{(j)}, \Phi_2^{(j)}, 
abla^{(j)}), \sigma^{(j)}(k'))\) replaced by \(((0, 0), (0, 0, \Phi_3^{(j)}(k'), \theta^{(j)}(k'))\) for \((\Phi_3^{(j)}(k'), \theta^{(j)}(k'))\) defined in the same way.

After some calculations, one can show that \(\mathcal{R}G = (u''(1), q''(1), \theta''(1), u''(2), q''(2), \theta''(2))\) satisfies the problem (3.26) with \(G\) replaced by \(((0, 0), (\mathcal{M}_1^{(j)}G, -\mathcal{M}_2^{(j)}G, \mathcal{M}_3^{(j)}G), (-\mathcal{M}_4G, -\mathcal{M}_5G, \int_0^t \Phi_6' - \int_0^t \Phi_7' G, \Phi_7 - \mathcal{M}_7G), \sigma^{(j)}(k'))\). Here the operator \(\mathcal{M} = ((0, 0), (\mathcal{M}_1^{(j)}, \mathcal{M}_2^{(j)}, \mathcal{M}_3^{(j)}), (\mathcal{M}_4, \mathcal{M}_5, \mathcal{M}_6, \mathcal{M}_7), 0)\) is defined on the space equipped with the norm in the right hand side of (1.19) in the spaces \(H_0^{1,1/2}(Q_T)\) (see below). Applying \(\mathcal{R}\) repeatedly, we can obtain the desired solution in the form \((u'(1), q'(1), \theta'(1), u'(2), q'(2), \theta'(2)) = \mathcal{R}(I + \mathcal{M} + \cdots)G = \mathcal{R}(I - \mathcal{M})^{-1}G\), so that the solvability of (3.26) follows from the contraction argument.

The operator \(\mathcal{MG}\) can be represented explicitly:

\[
\begin{align*}
\mathcal{M}_1^{(j)}G &= \sum_{k''} \frac{\mu^{(j)}}{\rho^{(j)}} (\nabla^2 (\eta^{(k)}(u^{(j)}(k)) - \eta^{(k)} \nabla^2 u^{(j)}(k) + \\
&\quad - \sum_{k''} \frac{1}{\rho^{(j)}} (\nabla (\eta^{(k)} q^{(j)}(k)) - \eta^{(k)} \nabla q^{(j)}(k) + \\
&\quad + \sum_{k''} \eta^{(k)} \Pi^z_x \frac{1}{\rho^{(j)}} \left[ \mu^{(j)} (\nabla^2 - \nabla^2) u^{(j)}(k) - (\nabla - \nabla) u^{(j)}(k) \right] = \sum_{i=1}^3 \mathcal{M}_{1i}^{(j)}G, \\
\mathcal{M}_2^{(j)}G &= \sum_{k''} (\eta^{(k)} (\nabla \cdot u^{(j)}(k)) - \nabla \cdot (\eta^{(k)} u^{(j)}(k))) + \\
&\quad + \sum_{k''} \eta^{(k)} \Pi^z_x (\nabla - \nabla) u^{(j)}(k) = \sum_{i=1}^2 \mathcal{M}_{2i}^{(j)}G, \\
\mathcal{M}_3^{(j)}G &= \sum_{k''} \theta^{(j)} (\nabla^2 (\eta^{(k)} \theta^{(j)}(k)) - \eta^{(k)} \nabla^2 \theta^{(j)}(k)) + \\
&\quad + \sum_{k''} \eta^{(k)} \Pi^z_x \kappa^{(j)} (\nabla^2 - \nabla^2) \theta^{(j)}(k) = \sum_{i=1}^2 \mathcal{M}_{3i}^{(j)}G, \\
\mathcal{M}_4G &= \sum_{k''} 2 \Pi^0_x (\eta^{(k)} (\mu^{(1)}D(u^{(1)}(k))) n_0 - \mu^{(2)}D(u^{(2)}(k))) n_0 + \\
&\quad - (\mu^{(1)}D(\eta^{(k)} u^{(1)}(k))) n_0 - \mu^{(2)}D(\eta^{(k)} u^{(2)}(k))) n_0) + \\
&\quad + \sum_{k''} 2 \eta^{(k)} (\Pi'_0 (\mu^{(1)}D(u^{(1)}(k))) n_0 (\xi^{(k)})) - \mu^{(2)}D(u^{(2)}(k))) n_0 (\xi^{(k)})) + \\
&\quad - \Pi'_0 (\mu^{(1)}D(u^{(1)}(k))) n_0 (x) - \mu^{(2)}D(u^{(2)}(k))) n_0 (x)) + \\
&\quad + \sum_{k''} \eta^{(k)} \Pi^z_x \Pi'_0 (\mu^{(1)}(\overline{D}(u^{(1)}(k))) - \overline{D}(u^{(1)}(k)))
\end{align*}
\]
Two-phase free boundary problem

$$- \mu^{(2)}(\mathbf{D}(\eta^{(2)}(x)) - \mathbf{D}(\bar{u}^{(2)}(x))) \n_0(\xi^{(k)})$$

$$+ \sum_{k''} \frac{\sigma'((\theta_0)^{(s)}(x))}{2} \Pi_0 \sum_{j=1}^{2} \left( \n(\eta^{(k)} \theta^{(j)}(k)) - \eta^{(k)} \n \theta^{(j)(k)} \right)$$

$$+ \sum_{k''} \eta^{(k)} \left( \sigma'((\theta_0)^{(s)}(x)) - \sigma'((\theta_0)^{(s)}(\xi^{(k)})) \right) \Pi_0 \sum_{j=1}^{2} \n \theta^{(j)(k)}$$

$$+ \sum_{k''} \frac{\eta^{(k)}}{2} \sigma'((\theta_0)^{(s)}(\xi^{(k)})) \sum_{j=1}^{2} \left( \Pi_0 \n \theta^{(j)(k)} - \Pi_0' \n \theta^{(j)(k)} \right)$$

$$+ \sum_{k''} \frac{\eta^{(k)}}{2} \sigma'((\theta_0)^{(s)}(\xi^{(k)})) \Pi_0 \sum_{j=1}^{2} \left( \n - \n' \theta^{(j)(k)} \right),$$

$$\mathcal{M}_5G = \sum_{k''} 2\n_0 \cdot \left( \eta^{(k)} \left( \mu^{(1)} \mathbf{D}(u^{(1)(k)}) \n_0 - \mu^{(2)} \mathbf{D}(u^{(2)(k)}) \n_0 \right)$$

$$- (\mu^{(1)} \mathbf{D}(\eta^{(k)} u^{(1)(k)}) \n_0 - \mu^{(2)} \mathbf{D}(\eta^{(k)} u^{(2)(k)}) \n_0) \right)$$

$$+ \sum_{k''} \sum_{j=1}^{2} \left( \n_0(\xi^{(k)}) \cdot (\mu^{(1)} \mathbf{D}(u^{(1)(k)}) \n_0(\xi^{(k)}) - \mu^{(2)} \mathbf{D}(u^{(2)(k)}) \n_0(\xi^{(k))))$$

$$- \n_0(\xi^{(k)}) \cdot (\mu^{(1)} \mathbf{D}(u^{(1)(k)}) \n_0(\xi^{(k)}) - \mu^{(2)} \mathbf{D}(u^{(2)(k)}) \n_0(\xi^{(k)})) \right)$$

$$+ \sum_{k''} \eta^{(k)} \Pi_0 \sum_{j=1}^{2} \left( \mu^{(1)}(\mathbf{D}(u^{(1)(k)}) - \mathbf{D}(\bar{u}^{(1)(k)}))$$

$$- \mu^{(2)}(\mathbf{D}(u^{(2)(k)}) - \mathbf{D}(\bar{u}^{(2)(k)})) \right) \n_0(\xi^{(k)}),$$

$$\mathcal{M}_6G = \sum_{k''} \frac{\sigma((\theta_0)^{(s)}(x))}{2} \n_0 \cdot \sum_{j=1}^{2} \int_0^t \left( \Delta(0)(\eta^{(k)} u^{(j)(k)}) - \eta^{(k)} \Delta(0) u^{(j)(k)} \right) d\tau$$

$$+ \sum_{k''} \frac{\eta^{(k)}}{2} \left( \sigma((\theta_0)^{(s)}(x)) - \sigma((\theta_0)^{(s)}(\xi^{(k)})) \right) \n_0(\xi^{(k)}) \Delta(0) \sum_{j=1}^{2} \int_0^t u^{(j)(k)} d\tau$$

$$+ \sum_{k''} \frac{\eta^{(k)}}{2} \sigma((\theta_0)^{(s)}(\xi^{(k)})) \n_0(\xi^{(k)}) \cdot \Delta(0) \sum_{j=1}^{2} \int_0^t u^{(j)(k)} d\tau$$

$$+ \sum_{k''} \frac{\eta^{(k)}}{2} \sigma((\theta_0)^{(s)}(\xi^{(k)})) \Pi_0 \sum_{j=1}^{2} \int_0^t \left( \n^2 - \Delta(0) \bar{u}^{(j)(k)} \right) d\tau,$$

$$\mathcal{M}_7G = \sum_{k''} \left( \eta^{(k)}(\kappa^{(1)} \n \theta^{(1)(k)} - \kappa^{(2)} \n \theta^{(2)(k)} \cdot \n_0$$

$$- (\kappa^{(1)} \n (\eta^{(k)} \theta^{(1)(k)} \cdot \n_0 - \kappa^{(2)} \n (\eta^{(k)} \theta^{(2)(k)} \cdot \n_0)) \right)$$
where

\[ \nabla = g^{(k)} \nabla_z, \]

\[ g^{(k)} = (g_{id}^{(k)}) = \left( \frac{\partial g}{\partial z} \right)^{-1} = \frac{1}{1 + \nabla_3 F_0} \begin{pmatrix} 1 + \nabla_3 F_0 & 0 & -\nabla_1 F_0 \\ 0 & 1 + \nabla_3 F_0 & -\nabla_2 F_0 \\ 0 & 0 & 1 \end{pmatrix}, \]

\[ \prod_0' \psi = \psi - (\psi \cdot n_0(\xi(k)))n_0(\xi(k)), \quad \overline{D} = \Pi_\xi D, \quad \overline{\Delta}(0) = \Pi_\xi \Delta(0). \]

We shall estimate each term of \( \mathcal{MG} \). It is convenient to denote by \( c(\lambda, h) \) various positive constants of the form \( \frac{c_{20}}{\lambda^m h^2} + c_{21}(\lambda) + \frac{\lambda}{c_{22}(h)} \) where \( c_{20}, \alpha \) and \( \beta \) are positive constants independent of \( \lambda \) and \( h \) while \( c_{21}(\cdot) \) and \( c_{22}(\cdot) \) are nondecreasing functions of their respective argument and \( c_{21}(0) = c_{22}(0) = 0 \). We shall frequently use the well-known inequalities

\[ \|u\|_{H^{-m,m/2}_h(Q_T)} \leq c_{23} h^{-(l-m)/2} \|u\|_{H^{l,m/2}_h(Q_T)} \quad (0 < m < l), \quad (3.29) \]

\[ \|fg\|_{L_2(\Omega)} \leq \|f\|_{L_2(\Omega)} \|g\|_{L_2(\Omega)} \|g\|_{W^{1,2}_2(\Omega)} \quad (3.30) \]

\[ \|fg\|_{L_3(\Omega)} \leq \|f\|_{L_3(\Omega)} \|g\|_{L_3(\Omega)} \|g\|_{W^{1,2}_2(\Omega)}. \]

Let us proceed to estimate \( \|\mathcal{M}_{11} G\|_{H^{1+1/2}_h(Q_T^{(k)})} \). Here and in what follows, we omit the indices \( (j) \) : \( Q^{(k)}_T = \Omega^{(k)} \times (0, T) \). (Correctly speaking, for \( k = k'' \), we should replace \( \Omega^{(k)} \) by \( \Omega^{(1)(k)} \) or \( \Omega^{(2)(k)} \), but for both cases we simply write \( \Omega^{(k)} \).) Since

\[ \nabla^2 (\eta^{(k)} u^{(j)(k)}) = \eta^{(k)} \nabla^2 u^{(j)(k)} = 2(\nabla \eta^{(k)} \cdot \nabla) u^{(j)(k)} + \nabla^2 \eta^{(k)} u^{(j)(k)}, \]

it is clear that the term \( \mathcal{M}_{11} G \) is of lower order. In fact, for example,

\[ \sum_{k''} \frac{1}{\rho_0} (\nabla \eta^{(k)} \cdot \nabla) u^{(k)} \|_{H^{1+1/2}_h(Q_T^{(k)})} \]

\[ \leq c_{25} \lambda^{-(2+l)} \sum_{k''} \|\nabla u^{(k)}\|_{H^{1+1/2}_h(Q_T^{(k)})} \]

\[ \leq c_{26} \lambda^{-(2+l)} h^{-l/2} \sum_{k''} \|u^{(k)}\|_{H^{3+1/2}_h(Q_T^{(k)})}. \]
holds. Here we used the property (iii) of the covering \{Ω(k)\}, (3.28), (3.29) and the boundedness of the transformation Π^z_x. In the same way as above, we have

\[
\|(M_{11}G, M_{31}G)\|_{H^{1+1/2+1/2}_h(Q_T)} + \|M_{12}G\|_{H^{1+0}_h(Q_T)}
\]

\[
+ \|M_{21}G\|_{H^{2+1/2+1/2}_h(Q_T)} \leq c(\lambda, h) \sum_k \|((u(k), q(k), \theta(k))\|_{H(Q_T(k))}.
\]

On the other hand, since

\[
(\nabla^2 - \nabla^2)\overline{u}^{(k)} = ((g^{(k)})^2 - I)\nabla^2\overline{u}^{(k)} + g^{(k)}(\nabla \cdot g^{(k)})\nabla\overline{u}^{(k)},
\]

we have from (3.27) and (3.30) that

\[
\|\eta^{(k)}(x)\Pi^z_x(\nabla^2 - \nabla^2)\overline{u}^{(k)}\|_{H^{1+1/2+1/2}_h(Q_T(k))} \leq c(\lambda, h)\|\overline{u}^{(k)}\|_{H^{3+1/2+1/2}_h(D_T^{(k)})},
\]

where \(D_T^{(k)} = K^{(1)(k)} \times (0, T)\) or \(K^{(2)(k)} \times (0, T)\). The same is true for \(M_{13}G, M_{22}G\) and \(M_{32}G\):

\[
\|(M_{13}G, M_{32}G)\|_{H^{1+1/2+1/2}_h(Q_T)} + \|M_{22}G\|_{H^{2+1/2+1/2}_h(Q_T)}
\]

\[
\leq c(\lambda, h) \sum_{k^{'}, k''} \|((\overline{u}^{(k'), q^{(k')}}, \overline{q}^{(k)}, \overline{u}^{(k)})\|_{H(D_T^{(k')})}.
\]

Next we estimate \(\|\nabla\eta^{(k)} q^{(k)}\|_{H^{1+1/2+1/2}_h(Q_T^{(k)})}\). It is obvious that \((u^{(k')}, q^{(k')}) = (u^{(k'''}), q^{(k'''}) \equiv 0\), hence we only have to consider the case \(k = k''\). We see that \((\overline{q}^{(1)}, q^{(2)})\) satisfies

\[
\begin{align*}
\frac{1}{\rho^{(j)}} \nabla^2 \overline{q}^{(j)}(k) &= 0 \quad \text{in } D_T^{(j)(k)}, \\
\overline{q}^{(1)}(k) - \overline{q}^{(2)}(k) \big|_{z_3=0} &= 2\mu^{(1)} \frac{\partial \overline{u}^{(1)}(k)}{\partial z_3} - 2\mu^{(2)} \frac{\partial \overline{u}^{(2)}(k)}{\partial z_3} \\
+ \frac{\sigma}{2} \int_0^t \nabla^2 (\overline{u}^{(1)}(k) + \overline{u}^{(2)}(k)) dT \big|_{z_3=0} - \int_0^t \Phi_6(r^{(k)}) dT \equiv A,
\end{align*}
\]

\[
\begin{align*}
\frac{1}{\rho^{(1)}} \frac{\partial \overline{q}^{(1)}(k)}{\partial z_3} - \frac{1}{\rho^{(2)}} \frac{\partial \overline{q}^{(2)}(k)}{\partial z_3} \big|_{z_3=0} &= \mu^{(1)} \frac{\nabla^2 \overline{u}^{(1)}(k)}{\rho^{(1)}} - \mu^{(2)} \frac{\nabla^2 \overline{u}^{(2)}(k)}{\rho^{(2)}} \big|_{z_3=0} \equiv B \quad \text{on } D_T^{(k)} = K^{(r)(k)} \times (0, T).
\end{align*}
\]

After extending the functions into \(t > T\) with preservation of their classes, we apply the Fourier–Laplace transformation (3.4). The transformed problem can be solved
Applying the inverse Fourier transformation, we obtain

\[
\tilde{q}^{(1)}(k) = \frac{\rho^{(1)}(1)}{2\pi \rho} \left[ \int_{R^2} \left( \frac{1}{\rho^{(2)}} \frac{\partial}{\partial z_3} \frac{1}{|z-y'|} \tilde{A}(y', s) - \tilde{B}(y', s) \right) dy' \right],
\]

\[
\tilde{q}^{(2)}(k) = \frac{\rho^{(1)}(1)\rho^{(2)}(1)}{2\pi} \left[ \int_{R^2} \left( \frac{1}{\rho^{(1)}} \frac{\partial}{\partial z_3} \frac{1}{|z-y'|} \tilde{A}(y', s) - \tilde{B}(y', s) \right) dy' \right],
\]

where, \(\tilde{\cdot}\) denotes the Laplace transformation. From these formulas, we have (we again omit the indices \(j\))

\[
\|M_{12}G\|_{H^{(0,1/2+1/2)}_h(Q_T)} \leq c_{27}\lambda^{-1} \sum_{k''} \|\tilde{q}^{(k)}\|_{H^{(0,3/2+1/2)}_h(D_T^{(k)})} \tag{3.33}
\]

\[
\leq c_{28}\lambda^{-1} \sum_{k''} \left( \|\nabla_3 \bar{u}_3^{(k)}\|_{H^{(0,1/2+1/2)}_h(D_T^{(k)})} + h^{-1/2}\|\nabla^2 \bar{u}_3^{(k)}\|_{H^{(0,1/2)}_h(D_T^{(k)})} + h^{-3/2}\|\Phi_6^{(k)}\|_{H^{(0,1/4+1/2)}_h(D_T^{(k)})} \right)
\]

\[
\leq c(\lambda, h) \sum_{k''} \|\bar{u}^{(k)}\|_{H^{(3/2+1/2+1/2)}_h(D_T^{(k)})}.
\]

The term \(\Phi_2'\) can be evaluated similarly. Indeed, since \(\Pi_x(\nabla - \nabla) \cdot \bar{u}^{(k)} = (g^{(k)})^{-1}\nabla - \nabla) \cdot u^{(k)}\), the term \(M_2G\) is represented in the form \(M_2G = \Sigma \nabla \cdot \Phi_2'\) with

\[
\Phi_2'(k) = -\int_{R^3} \nabla_x G(x, y)(u^{(k)} \cdot \nabla)\eta^{(k)}(y, t)dy + \eta^{(k)}(g^{(k)})^{-1} - I)u^{(k)}
\]

\[-\int_{R^3} \nabla_x G(x, y)(\nabla \cdot (\eta^{(k)}(g^{(k)})^{-1} - I))u^{(k)}(y, t)dy = I + II + III.
\]

Using (1.20)\(_{1j}\), we can estimate the term I in the same way as (3.33) to obtain

\[
\|I\|_{H^{(0,3/2+1/2)}_h(Q_T^{(k)})} \leq c_{29} \left( \|\nabla \bar{u}^{(k)}\|_{H^{(0,1/2+1/2)}_h(D_T^{(k)})} + \|\tilde{q}^{(k)}\|_{H^{(0,0,1/2+1/2)}_h(D_T^{(k)})} \right)
\]

\[
\leq c(\lambda, h) \|(\bar{u}^{(k)}, \tilde{q}^{(k)})\|_{H(D_T^{(k)})}.
\]
While from (3.27), we have

$$\|II + III\|_{H^{0,3/2+1/2}(Q_T^{(k)})} \leq c(\lambda)\|(u^{(k)}, q^{(k)}, \theta^{(k)})\|_{H(Q_T^{(k)})}. \quad (3.35)$$

Next, we extend the functions on the boundary into $\Omega^{(1)(k)}$ or $\Omega^{(2)(k)}$ so as to estimate $M_4G, M_5G, M_6G$ and $M_7G$. We also extend $\theta_0(s) = \frac{1}{2}(\theta_0^{(1)} + \theta_0^{(2)}) |_T$ into $\Omega^{(k)} = \Omega^{(1)(k)} \cup \Omega^{(2)(k)}$ appropriately, which we simply write as $\theta_0$.

On the surface $\Gamma$, the normal vector $n_0(x)$ and the Laplace–Beltrami operator $\Delta(0)$ have the form

$$n_0 = \frac{1}{\sqrt{g(0)}}(\nabla F_0, \nabla F_0, -1),$$

$$\Delta(0) = \frac{1}{\sqrt{g(0)}} \frac{\partial}{\partial x_\alpha} \left( g^{\alpha\beta}(0) \sqrt{g(0)} \frac{\partial}{\partial x_\beta} \right) = \frac{1}{\sqrt{g(0)}} \frac{\partial}{\partial x_\alpha} \left( g_{\alpha\beta}(0) \frac{\partial}{\partial x_\beta} \right),$$

where

$$g_{\alpha\beta}(0) = \sum_{i=1}^2 \delta_{i\alpha} \delta_{i\beta} + \nabla_\alpha F_0 \nabla_\beta F_0 = \delta_{\alpha\beta} + \nabla_\alpha F_0 \nabla_\beta F_0, \quad (3.36)$$

$$g(0) = 1 + |\nabla F_0|^2.$$

Using the above formulas and the inequality

$$\sup_{x \in \Omega^{(k)}} |\sigma'((\theta_0)(x)) - \sigma'((\zeta^{(k)}(x)))| \leq c_30 \lambda,$$

we can estimate the terms $(M_4G, M_5G, M_6G, M_7G)$ in the same way as above:

$$\|(M_4G, M_5G, M_7G)\|_{H^{0,1}+1/2(Q_T)} + \|M_6G\|_{H^{0,1+1/2}(Q_T)} \quad (3.37)$$

$$\leq c(\lambda, h) \sum_{k'} \|(u^{(k)}, q^{(k)}, \theta^{(k)})\|_{H(D_{T^{(k)}})}.$$

From (3.31)–(3.35), (3.37), we see that the operator $\mathcal{M}$ is a contraction for sufficiently small $\lambda$ and large $h$. Thus the solvability of the problem (3.26) is proved.

Next, we solve the problem (3.25). Of course, the idea of the proof is the same as that of the previous one, hence we only sketch the proof, pointing out some differences between them. In this case, we shall use the covering $\{\omega^{(k)}\}$ instead of $\{\Omega^{(k)}\}$ to construct the solution. Note that $\zeta^{(k)} \equiv 1$ on $\omega^{(k)}$.

We first consider the equation

$$\left\{ \begin{array}{l}
\nabla^2 \varphi^{(j)} = \Phi_2^{(j)} \quad \text{in} \quad Q_T^{(j)}, \\
\varphi^{(1)} - \varphi^{(2)} \big|_{T=0} = 0, \quad \frac{\partial \varphi^{(1)}}{\partial n} - \frac{\partial \varphi^{(2)}}{\partial n} \big|_T = 0, \quad \frac{\partial \varphi^{(j)}}{\partial n} \big|_{\Sigma^{(j)}} = 0. \end{array} \right. \quad (3.38)$$
It can be shown that \((\varphi^{(1)}, \varphi^{(2)})\) satisfies
\[
\sum_{j=1}^{2} \|\varphi^{(j)}\|_{H^{3+1,3/2+1/2}_{h}(Q_{T}(j))} \leq c_{31} \sum_{j=1}^{2} \left( \|\Phi_{2}^{(j)}\|_{H^{3+1,0}_{h}(Q_{T}(j))} + \|\Phi_{2}^{\prime}(j)\|_{H^{0,3/2+1/2}_{h}(Q_{T}(j))} \right).
\] (3.39)

We next construct the divergence free vectors \((w'(1), w'(2))\) satisfying
\[
w'(1) - w'(2) \big|_{\Gamma} = -(\nabla \varphi^{(1)} - \nabla \varphi^{(2)}) \big|_{\Gamma}, \quad w'(j) \big|_{\Sigma_{j}} = -\nabla \varphi^{(j)} \big|_{\Sigma_{j}}
\] (3.40)
in the same way as in [9, 3]. Then we have
\[
\sum_{j=1}^{2} \|w'(j)\|_{H^{3+1,3/2+1/2}_{h}(Q_{T}(j))} \leq c_{32} \sum_{j=1}^{2} \|\nabla \varphi^{(j)}\|_{H^{5/2+1,5/4+1/2}_{h}(\Gamma^{\cup} \Sigma_{T}(j))}.
\] (3.41)

Finally, by virtue of the well-known decomposition of the vector \(\Phi_{1}'(j) = \Phi_{1}(j) - \frac{\partial}{\partial t}(\nabla \varphi^{(j)} + w'(j)) + \frac{\mu_{2}(j)}{\rho_{1}(j)} \nabla^{2}(\nabla \varphi^{(j)} + w'(j))\) into \(\Phi_{1}'(j) = \Phi_{1}''(j) + \nabla \varphi^{(j)} \in J_{2}(Q_{T}(j)) + G_{2}(Q_{T}(j))\), where \(J_{2}(Q_{T}(j)) = \{\nabla \cdot \Phi_{1}''(j) = 0 \text{ in } \Omega_{j}, \Phi_{1}''(j) \cdot n_{0} = 0 \text{ on } \partial \Omega_{j} \}\) and \(G_{2}(Q_{T}(j)) = \{\nabla \varphi^{(j)} \in L_{2}(Q_{T}(j))\}\) (see [9, 23]) the problem (3.25) is reduced to the same one for \(w''(j) = w'(j) - \nabla \varphi^{(j)} - w'(j), \pi''(j) = \pi'(j) - \rho(j)(\varphi^{(j)} + \varphi''(j))\) with \((\Phi_{1}(j), \Phi_{2}(j), \Phi_{4}, \Phi_{5})\) replaced by \(G(\Phi_{1}''(j) + \nabla \varphi''(j), 0, \Phi_{4}', \Phi_{5}')\). Here \(\varphi''(j)\) satisfy \(\nabla^{2} \varphi''(j) = 0 \text{ in } \Omega_{j}, \varphi''(j) = \varphi^{(j)} \text{ on } \Gamma,\)
\[
\Phi_{4}' = \Phi_{4} - 2 \sum_{l=0}^{1}(\mu^{(1)}D(\nabla \varphi^{(1)} + w'(1))n_{0} - \mu^{(2)}D(\nabla \varphi^{(2)} + w'(2))n_{0}) \big|_{\Gamma},
\]
\[
\Phi_{5}' = \Phi_{5} - 2n_{0} \cdot (\mu^{(1)}D(\nabla \varphi^{(1)} + w'(1))n_{0} - \mu^{(2)}D(\nabla \varphi^{(2)} + w'(2))n_{0}) \big|_{\Gamma}.
\]

For the reduced system, we can define the regularizer \(R\) in the same way as that of the previous problem:
\[
RG = \sum_{k} \eta^{(k)}(x)(w''(1)(k), \pi'(1)(k), w''(2)(k), \pi''(2)(k)) \big|_{\Gamma}.
\]

For \(k = k', (w''(j)(k), \pi''(j)(k))(x, t)\) is a solution to the problem
\[
\begin{cases}
\frac{\partial w''(j)(k)}{\partial t} - \frac{\mu(j)}{\rho(j)} \nabla^{2} w''(j)(k) + \frac{1}{\rho(j)} \nabla \pi''(j)(k) = \Phi_{1}''(j) + \nabla \varphi''(j), \\
\nabla \cdot w''(j)(k) = 0, \quad x \in \omega(k), \quad t > 0, \quad w''(j)(k) \big|_{t=0} = 0 \quad (j = 1, 2).
\end{cases}
\]
For \( k = k'' \), we define \((w''(j)(k), \pi''(j)(k)) (x, t)\) as \( \Pi^z_k (\bar{w}''(j)(k), \bar{\pi}''(j)(k))(z, t) \) where \((\bar{w}''(j)(k), \bar{\pi}''(j)(k))(z, t)\) satisfies

\[
\begin{align*}
\frac{\partial \bar{w}''(j)(k)}{\partial t} - \frac{\mu(j)}{\rho(j)} \nabla^2 \bar{w}''(j)(k) + \nabla \bar{\pi}''(j)(k) &= t^{(k)}(\Phi_1''(j) + \Pi^z \nabla \varphi''(j)), \\
\nabla \cdot \bar{w}''(j)(k) &= 0, \quad z \in \Pi^z \omega(k), \quad t > 0, \quad \bar{w}''(j)(k) \big|_{t=0} = 0, \\
\bar{w}''(1)(k) - \bar{w}''(2)(k) \big|_{z_3=0} &= 0, \\
-
\mu^{(1)} \left( \frac{\partial \bar{w}_3''(1)(k)}{\partial z_i} + \frac{\partial \bar{w}_1''(1)(k)}{\partial z_3} \right) + \mu^{(2)} \left( \frac{\partial \bar{w}_3''(2)(k)}{\partial z_i} + \frac{\partial \bar{w}_1''(2)(k)}{\partial z_3} \right) \big|_{z_3=0} &= (\Phi_4''), \quad i = 1, 2, \\
- \pi''(1)(k) + 2\mu^{(1)} \frac{\partial \bar{w}_3''(1)(k)}{\partial z_3} + \pi''(2)(k) - 2\mu^{(2)} \frac{\partial \bar{w}_3''(2)(k)}{\partial z_3} \big|_{z_3=0} &= \Phi_5',
\end{align*}
\]

with \((\Phi_1''(j), \Phi_4'', \Phi_5'')(z, t) = \Pi^z_k (\Phi_1''(j), \Phi_4'', \Phi_5')(x, t)\) (\( j = 1, 2 \)). For \( k = k''' \), we define \((w'''(j)(k), \pi'''(j)(k))(x, t)\) in the same way as above with the boundary condition of \( w''(j)(k) \) replaced by \( w''(j)(k) = 0 \) on \( z_3 = 0 \). Note that \( \nabla \cdot (\Phi_1'''(j) + \nabla \varphi'''(j)) = 0 \) and \( \nabla \cdot t^{(k)}(\Phi_1'''(j) + \Pi^z \nabla \varphi'''(j)) = 0 \) hold.

The operator corresponding to \( MG \) is also defined analogously. The only difference is that we must add the term

\[
\sum_{k''', k'''} \eta^{(k)} \Pi^z_k (I - t^{(k)}(\Phi_1'''(j) + \Pi^z \nabla \varphi'''(j)) \to M_1^{(j)} G.
\]

In the same way as in the case of the previous problem, we can show that the operator \( M \) is a contraction for sufficiently small \( \lambda \) and large \( h \), hence the solvability of (3.25).

For the proof of Theorem 1.3, we construct \((u''(j), \theta''(j))\) as solutions of the heat equations with the initial-boundary conditions given by (1.21)-(1.23). Theorem 3.6 is applicable to the difference \((u(j) - u''(j), q(j), \theta(j) - \theta''(j))\) which implies Theorem 1.3.

4. Linearized problem II: Proof of Theorem 1.2

We rewrite (1.13)-(1.16) as follows:
\[
\begin{aligned}
\left\{ \begin{array}{l}
\frac{\partial u^{(j)}}{\partial t} - \frac{u^{(j)}}{\rho^{(j)}} \nabla^2 u^{(j)} + \frac{1}{\rho^{(j)}} \nabla q^{(j)} = \Phi_1^{(j)} - \left( \frac{\mu^{(j)}}{\rho^{(j)}} (\nabla^2 - \nabla^2 w^{(j)}) u^{(j)} - \frac{1}{\rho^{(j)}} (\nabla - \nabla w^{(j)}) q^{(j)} \right) \\
\nabla \cdot u^{(j)} = \Phi_2^{(j)} - (\nabla w^{(j)} - \nabla) \cdot u^{(j)} \equiv \Phi_2^{(j)} - k_2^{(j)} ,
\end{array} \right. \\
\frac{\partial \theta^{(j)}}{\partial t} - \kappa^{(j)} \nabla^2 \theta^{(j)} = \Phi_3^{(j)} - \kappa^{(j)} (\nabla^2 - \nabla^2 w^{(j)}) \theta^{(j)} \\
\equiv \Phi_3^{(j)} - k_3^{(j)} \text{ in } Q_T^{(j)},
\end{aligned}
\]
\[
\begin{aligned}
(u^{(j)}, \theta^{(j)}) \big|_{t=0} = (v_0^{(j)}, \theta_0^{(j)})(x) \text{ on } \Omega^{(j)},
\end{aligned}
\]
\[
\begin{aligned}
&\left\{ \begin{array}{l}
2 \prod_0 \mu^{(1)} D(u^{(1)}) n_0 - \mu^{(2)} D(u^{(1)}) n_0 = \frac{\sigma'(\theta_0^{(s)})}{2} \sum_{j=1}^2 \prod_0 \nabla \theta^{(j)} \\
= \Phi_4 - \left(2 \mu^{(1)} (\prod_{w^{(1)}} D(u^{(1)}) n_{w^{(1)}}) - \prod_0 D(u^{(1)}) n_0 \right) \\
- 2 \mu^{(2)} (\prod_{w^{(2)}} D(u^{(2)}) n_{w^{(2)}}) - \prod_0 D(u^{(2)}) n_0 \right) \\
- \frac{1}{2} \sum_{j=1}^2 \left( \sigma'(\theta_0^{(s)}) \prod_{w^{(j)}} \nabla w^{(j)} \theta^{(j)} - \sigma'(\theta_0^{(s)}) \prod_0 \nabla \theta^{(j)} \right) = \Phi_4 - k_4 ,
\end{array} \right.
\end{aligned}
\]
\[
\begin{aligned}
-q^{(1)} + 2 \mu^{(1)} n_0 \cdot D(u^{(1)}) n_0 + q^{(2)} - 2 \mu^{(2)} n_0 \cdot D(u^{(2)}) n_0 \\
- \frac{\sigma(\theta_0^{(s)})}{2} n_0 \cdot \Delta(0) \sum_{j=1}^2 \int_0^t u^{(j)} \, d\tau
\end{aligned}
\]
\[
\begin{aligned}
= \Phi_5 + \int_0^t \Phi_6 \, d\tau - \left(2 \mu^{(1)} (n_{w^{(1)}} \cdot D(u^{(1)}) n_{w^{(1)}}) - n_0 \cdot D(u^{(1)}) n_0 \right) \\
- 2 \mu^{(2)} (n_{w^{(2)}} \cdot D(u^{(2)}) n_{w^{(2)})} - n_0 \cdot D(u^{(2)}) n_0 \right) \\
- \frac{1}{2} \sum_{j=1}^2 \left( \sigma(\theta_0^{(s)}) n_{w^{(j)}} \cdot \Delta_{w^{(j)}}(t) \int_0^t u^{(j)} \, d\tau - \sigma(\theta_0^{(s)}) n_0 \cdot \Delta(0) \int_0^t u^{(j)} \, d\tau \right) \\
= \Phi_5 + \int_0^t \Phi_6 \, d\tau - (k_5 + \int_0^t k_6 \, d\tau),
\end{aligned}
\]
\[
\begin{aligned}
\theta^{(1)} - \theta^{(2)} = 0 ,
\end{aligned}
\]
\[
\begin{aligned}
\kappa^{(1)} \nabla \theta^{(1)} \cdot n_0 - \kappa^{(2)} \nabla \theta^{(2)} \cdot n_0 \\
= \Phi_7 - \left( \kappa^{(1)} (\nabla w^{(1)} \theta^{(1)} \cdot n_{w^{(1)}} - \nabla \theta^{(1)} \cdot n_0) \\
- \kappa^{(2)} (\nabla w^{(2)} \theta^{(2)} \cdot n_{w^{(2)}} - \nabla \theta^{(2)} \cdot n_0) \right) \equiv \Phi_7 - k_7 \text{ on } \Gamma_T ,
\end{aligned}
\]
Two-phase free boundary problem

\[ u^{(j)} = 0, \quad \theta^{(j)} = \theta_{e}^{(j)} \text{ on } \Sigma_{T}^{(j)}. \quad (4.4)^{(j)} \]

Since \( \nabla \cdot A_{w}^{(j)} = 0 \), we can write \( k_{2}^{(j)} = (\nabla w^{(j)} - \nabla) \cdot u^{(j)} = -\nabla \cdot (I - tA_{w}^{(j)}))u^{(j)} = \nabla \cdot k_{2}^{(j)} \). Applying Theorem 1.3 to the problem (4.1)(j)-(4.4)(j), we obtain

\[
\| (u^{(j)}, q^{(j)}, \theta^{(j)}) \|_{W^{2+1,3/2+1/2}(Q_{T})} \leq c_{33}\| (\Phi, k, \theta_{e}, \theta_{0}) \|_{C(Q_{T})},
\]

where \( k = ((k_{1}(j), k_{2}(j), k_{3}(j)), (k_{4}, k_{5}, k_{6}, k_{7})) \).

Before calculating the norm of \( k \), we prepare auxilliary estimates. For brevity, we again omit the indices \( (j) \).

**Lemma 4.1.** Let \( w \in W_{2}^{3+1,3/2+1/2}(Q_{T}) \) be given, then for \( B_{w} = A_{w} - I \), we have the estimates

\[
\sup_{t} \| B_{w} \|_{W^{2+1,3/2+1/2}(Q_{T})} \leq c_{34} \delta, \quad (4.6)
\]

\[
\| B_{w} \|_{W^{2+1,3/2+1/2}(Q_{T})} \leq c_{35} \delta. \quad (4.7)
\]

**Lemma 4.2.** Under the same assumption as in Lemma 4.1, the estimates

\[
\| n_{w} - n_{0} \|_{W^{2+1,3/2+1/2}(Q_{T})} \leq c_{36} \delta, \quad (4.8)
\]

\[
\| \sigma'(\tilde{\theta}) - \sigma'(\theta_{0}) \|_{W^{2+1,3/2+1/2}(Q_{T})} \leq c_{37} \delta \quad (4.9)
\]

are valid.

**Proof.** The formulas

\[
n_{w} = \frac{(I + B_{w})}{\sqrt{1 + b}} n_{0}, \quad b = 2n_{0} \cdot B_{w} n_{0} + \| B_{w} n_{0} \|^{2},
\]

\[
n_{w} - n_{0} = \frac{1}{\sqrt{1 + b}} \left( B_{w} + \frac{b}{\sqrt{1 + b}} I \right) n_{0},
\]

\[
\frac{\partial n_{w}}{\partial t} = \frac{1}{2} \frac{1}{1 + b} \frac{\partial b}{\partial t} n_{w} + \frac{1}{\sqrt{1 + b}} \frac{\partial B_{w}}{\partial t} n_{0},
\]

and the condition \( \tilde{\theta} \big|_{t=0} = \theta_{0} \) yield the estimates (4.8) and (4.9).

**Lemma 4.3.** Under the same assumption as in Lemmas 4.1–4.2, the following estimates are true.

\[
\| \Delta_{w}(t) u \|_{W^{2+1,3/2+1/2}(Q_{T})} \leq c_{38} \| u \|_{W^{2+1,3/2+1/2}(Q_{T})}, \quad (4.10)
\]
Here $\Delta_w(t)$ is an operator obtained from $\Delta_w(t)$ by differentiating its coefficients with respect to $t$.

**PROOF.** On the surface $\Gamma$, $\Delta_w(t)$ is represented as

$$
\Delta_w(t) = \frac{1}{\sqrt{g(t)}} \frac{\partial}{\partial x_\alpha} \left( g^{\alpha \beta}(t) \sqrt{g(t)} \frac{\partial}{\partial x_\beta} \right) \equiv \frac{1}{\sqrt{g(t)}} \frac{\partial}{\partial x_\alpha} \left( \tilde{g}_{\alpha \beta}(t) \frac{\partial}{\partial x_\beta} \right),
$$

where

$$
g_{\alpha \beta}(t) = \sum_{i=1}^{2} (\delta_{i \alpha} + h_{i \alpha})(\delta_{i \beta} + h_{i \beta}) + (\nabla F_{0 \alpha} + h_{3 \alpha})(\nabla F_{0 \beta} + h_{3 \beta})
= g_{\alpha \beta}(0) + \varphi_{\alpha \beta}(h_{\alpha \beta}),
$$
given by (3.36), and

$$
h_{i \alpha} = \int_0^t \left( \frac{\partial w_i}{\partial x_\alpha} + \nabla F_0 \frac{\partial w_i}{\partial x_3} \right) d\tau, \quad i = 1, 2, 3, \quad \alpha = 1, 2.
$$

Therefore, the estimates (4.10)–(4.12) follow from the formulas

$$
\Delta(0) - \Delta_w(t) = \left( \frac{1}{\sqrt{g(0)}} - \frac{1}{\sqrt{g(t)}} \right) \frac{\partial}{\partial x_\alpha} \left( \tilde{g}_{\alpha \beta}(0) \frac{\partial}{\partial x_\beta} \right)
+ \frac{1}{\sqrt{g(t)}} \frac{\partial}{\partial x_\alpha} \left( \tilde{g}_{\alpha \beta}(0) - \tilde{g}_{\alpha \beta}(t) \right) \frac{\partial}{\partial x_\beta},
$$

$$
\dot{\Delta}_w(t) = -\frac{1}{2} \frac{1}{g(t)} \frac{\partial g(t)}{\partial t} \Delta_w(t) + \frac{1}{\sqrt{g(t)}} \frac{\partial}{\partial x_\alpha} \left( \tilde{g}_{\alpha \beta}(t) \frac{\partial}{\partial x_\beta} \right).
$$

From Lemmas 4.1–4.3, since $(\Pi_w - \Pi_0)\psi = ((n_w - n_0) \cdot \psi)n_0 + (n_w \cdot \psi)(n_0 - n_w)$, we have

$$
\| (k_1, k_3) \|_{W_2^{1+1/2,1+1/2}(Q_T)} + \| k_2 \|_{W_2^{2+1,1+1/2}(Q_T)}
+ \| k_2' \|_{W_2^{1+1/2}(Q_T)} \leq c_{41} \delta \| (u, \theta, q) \|_{W(Q_T)},
$$

$$
\| (k_4, k_5, k_7) \|_{W_2^{1+1/2,1+1/2}(Q_T)} \leq c_{42} \delta \| (u, q, \theta) \|_{W(Q_T)},
$$

$$
\| k_6 \|_{W_2^{1+1/2,1+1/2}(Q_T)} \leq c_{43} \delta \| (u, q, \theta) \|_{W(Q_T)}.
$$

(4.13)

(4.14)

(4.15)
From (4.13)-(4.15), it is easily seen that $k$ is a contraction operator provided that $\delta$ is small enough. Hence the proof of Theorem 1.2 is completed.

5. Nonlinear problem: Proof of Theorem 1.1

We shall solve (1.8)-(1.11) by the method of successive approximations. Let $(u^{(j)}(0), q^{(j)}(0), \theta^{(j)}(0)) = (0, 0, \theta_0)$ and $(u^{(j)}(m+1), q^{(j)}(m+1), \theta^{(j)}(m+1)) \ (m = 1, 2, \cdots)$ be a solution of the following linear problem:

$$
\left\{ \begin{align*}
\frac{\partial u^{(j)}(m+1)}{\partial t} - \frac{\mu^{(j)}}{\rho^{(j)}} \nabla^2 u^{(j)}(m+1) + \frac{1}{\rho^{(j)}} \nabla u^{(j)}(m+1) q^{(j)}(m+1) &= f^{(j)}(m), \\
\nabla u^{(j)}(m) \cdot u^{(j)}(m+1) &= 0, \\
\frac{\partial \theta^{(j)}(m+1)}{\partial t} - \kappa^{(j)} \nabla^2 \theta^{(j)}(m+1) &= 0 \text{ in } Q_T^{(j)}, \\
(u^{(j)}(m+1), \theta^{(j)}(m+1)) \big|_{t=0} &= (v_0^{(j)}, \theta_0^{(j)})(x) \text{ on } \Omega^{(j)},
\end{align*} \right. 
\right. 
$$

$$
\left\{ \begin{align*}
2\mu^{(1)} \prod_{u^{(1)}(m)} D_{u^{(1)}(m)} (u^{(1)}(m+1)) n_u^{(1)}(m) \\
- 2\mu^{(2)} \prod_{u^{(2)}(m)} D_{u^{(2)}(m)} (u^{(2)}(m+1)) n_u^{(2)}(m) \\
- \frac{\sigma'(\theta(s)(m))}{2} \sum_{j=1}^2 \prod_{u^{(j)}(m)} \nabla u^{(j)}(m) \theta^{(j)}(m+1) &= 0, \\
-q^{(1)}(m+1) + 2\mu^{(1)} n_u^{(1)}(m) \cdot D_{u^{(1)}(m)} (u^{(1)}(m+1)) n_u^{(1)}(m) \\
+ q^{(2)}(m+1) - 2\mu^{(2)} n_u^{(2)}(m) \cdot D_{u^{(2)}(m)} (u^{(2)}(m+1)) n_u^{(2)}(m) \\
- \frac{\sigma(\theta(s)(m))}{2} \sum_{j=1}^2 n_{u^{(j)}(m)} \Delta u^{(j)}(m)(t) \int_0^t u^{(j)}(m+1) d\tau &= (\sigma_0 H_0)(x) - \frac{1}{2} \sum_{j=1}^2 (\sigma(\theta(s)(m)) n_{\delta^2} \Delta(0)x \\
- \frac{\sigma'(\theta(s)(m))}{2} \sum_{j=1}^2 n_{u^{(j)}(m)} \Delta u^{(j)}(m)(t) \equiv (\sigma_0 H_0)(x) + \int_0^t J^{(m)} d\tau, \\
\theta^{(1)}(m+1) - \theta^{(2)}(m+1) &= 0, \\
\kappa^{(1)} \nabla u^{(1)}(m) \theta^{(1)}(m+1) n_u^{(1)}(m) - \kappa^{(2)} \nabla u^{(2)}(m) \theta^{(2)}(m+1) n_u^{(2)}(m) &= 0 \text{ on } \Gamma_T, \\
u^{(j)}(m+1) = 0, \quad \theta^{(j)}(m+1) = \theta_e^{(j)}(x, t) \text{ on } \Sigma_T^{(j)},
\end{align*} \right. 
\right. 
$$
where
\[ f^{(j)}(m) = f^{(j)}(X_{u^{(j)}(m)}, t), \quad \theta^{(s)}(m) = \frac{1}{2} (\theta^{(1)}(m) + \theta^{(2)}(m)) \bigg|_{\Gamma}, \]
and \((u^{(j)}(m), \theta^{(j)}(m))\) are given functions such that \((u^{(j)}(m), \theta^{(j)}(m)) \in W^{3+1/2+1/2}_r(Q^{(j)}_T)\) and \((u^{(j)}(m), \theta^{(j)}(m)) \big|_{t=0} = (v_0^{(j)}(x), \theta_0^{(j)}(x))\). According to Theorem 1.2, we have

\[ z^{(m+1)}(T) \equiv \sum_{j=1}^{2} \left( \left\| (u^{(j)}(m+1), q^{(j)}(m+1), \theta^{(j)}(m+1)) \right\|_{W^{1/2+1/2}_r(Q_T^{(j)})} \right. \]

\[ \leq c_{44} \left[ \sum_{j=1}^{2} \left( \left\| (v_0^{(j)}, \theta_0^{(j)}) \right\|_{W^{1+1/2+3/2}_r(Q^{(j)}_T)} + \left\| f^{(j)}(m) \right\|_{W^{1+1/2+3/2}_r(Q^{(j)}_T)} + \left\| \sigma_0 H_0 \right\|_{W^{1/2+1/4+1/4}_r(\Gamma_T)} + \sum_{j=1}^{2} \left\| \theta^{(j)}_0 \right\|_{W^{1/2+1/4+1/4}_r(\Sigma^{(j)}_T)} \right] \]

provided that \(T\) is sufficiently small so that \((u^{(j)}(m), \theta^{(j)}(m))\) satisfies the condition (1.17).

In the same way as in \(\S 4\), we have the following estimates for the terms in the right-hand side of (5.5).

**Lemma 5.1.** The estimates
\[ \left\| f^{(j)}(m) \right\|_{W^{1+1/2+3/2}_r(Q^{(j)}_T)} \leq c_{45} \left\| f^{(j)}(m) \right\|_{W^{1+1/2+3/2}_r(\mathbb{R}^{3+1})} + c_{46}(T) \left\| u^{(j)}(m) \right\|_{W^{3+1/2+3/2}_r(Q^{(j)}_T)}, \]
\[ \left\| j^{(m)} \right\|_{W^{1/2+3/2}_r(\Gamma_T)} \leq c_{47}(T, \zeta^{(m)}(T)) \]
is valid, here and below, where the constants \(c_{46}(\cdot)\) and \(c_{47}(\cdot, \cdot)\) are monotonically increasing functions of their respective argument and \(c_{46}(0) = c_{47}(0, \cdot) = c_{47}(\cdot, 0) = 0\).

Substituting (5.6), (5.7) into (5.5), we obtain
\[ z^{(m+1)}(T) \leq c_{48} N + c_{49}(T, z^{(m)}(T)). \]

Now let us take a positive constant \(M\) such that \(c_{48} N < M\) and, if necessary, choose \(T' \in (0, T]\) sufficiently small so as to satisfies \(c_{49}(T', M) < M - c_{48} N\). Under this situation, if \((u^{(j)}(m), q^{(j)}(m), \theta^{(j)}(m))\) are defined and satisfy \(z^{(m)}(T') < M\), then \(z^{(m+1)}(T') \leq c_{48} N + c_{49}(T', z^{(m)}(T')) < c_{48} N + c_{49}(T', M) < M\). Thus our sequence is well defined and satisfies
\[ z^{(m)}(T') < M, \quad (m = 0, 1, 2, \ldots). \]

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Henceforth, we take $T = T'$.

Next, we shall show the convergence of our sequence. Subtracting from (5.1)(j)-(5.4)(j) with similar relations for $(u^{(j)(m)}, q^{(j)(m)}, \theta^{(j)(m)})$ and setting $U^{(j)(m+1)} = u^{(j)(m+1)} - u^{(j)(m)}$, $Q^{(j)(m+1)} = q^{(j)(m+1)} - q^{(j)(m)}$ and $\Theta^{(j)(m+1)} = \theta^{(j)(m+1)} - \theta^{(j)(m)}$, we have

$$\begin{align*}
\frac{\partial U^{(j)(m+1)}}{\partial t} & = f^{(j)(m)} - f^{(j)(m-1)} + K_1^{(j)(m)} , \\
\nabla u^{(j)(m)} \cdot U^{(j)(m+1)} & = K_2^{(j)(m)} \equiv \nabla \cdot K_2^{(j)(m)} , \\
\frac{\partial \Theta^{(j)(m+1)}}{\partial t} & = -K_3^{(j)(m)} \Theta^{(j)(m+1)} = K_3^{(j)(m)} \quad \text{in } \Omega_T^{(j)} , \\
(U^{(j)(m+1)}, \Theta^{(j)(m+1)}) \big|_{t=0} & = (0,0) \quad \text{on } \Omega^{(j)} , \\
2\mu^{(1)} & \prod_{u^{(1)(m)}} D_{u^{(1)(m)}} (U^{(1)(m+1)}) n_{u^{(1)(m)}} , \\
-2\mu^{(2)} & \prod_{u^{(2)(m)}} D_{u^{(2)(m)}} (U^{(2)(m+1)}) n_{u^{(2)(m)}} , \\
-\sigma' (\theta^{(s)(m)}) & \sum_{j=1}^{2} \Pi_{u^{(j)(m)}} \nabla u^{(j)(m)} \Theta^{(j)(m+1)} = K_4^{(m)} , \\
-Q^{(1)(m+1)} & + 2\mu^{(1)} n_{u^{(1)(m)}} \cdot D_{u^{(1)(m)}} (U^{(j)(m+1)}) n_{u^{(1)(m)}} , \\
+ Q^{(2)(m+1)} & - 2\mu^{(2)} n_{u^{(2)(m)}} \cdot D_{u^{(2)(m)}} (U^{(2)(m+1)}) n_{u^{(2)(m)}} , \\
-\sigma (\theta^{(s)(m)}) & \sum_{j=1}^{2} n_{u^{(j)(m)}} \Delta u^{(j)(m)} (t) \int_0^t U^{(j)(m+1)} dt ,
\end{align*}$$

$$\begin{align*}
&= K_6^{(m)} + \int_0^t (K_6^{(m)} + J^{(m)}) d\tau ,
&= K_7^{(m)} \quad \text{on } \Gamma_T ,
&= K_7^{(m)} \quad \text{on } \Gamma_T ,
&= K_7^{(m)} \quad \text{on } \Sigma_T^{(j)} ,
\end{align*}$$

Here

$$K^{(m)} (u^{(j)(m)}, q^{(j)(m)}, \theta^{(j)(m)})$$

$$= k^{(m)} (u^{(j)(m)}, q^{(j)(m)}, \theta^{(j)(m)}) - k^{(m-1)} (u^{(j)(m)}, q^{(j)(m)}, \theta^{(j)(m)}),$$
\( k^{(m)} \) is defined by \( k \) in the right-hand side of (4.1)(j)-(4.3) with \((w(j), \tilde{\theta}(s))\) replaced by \((u(j)(m), \tilde{\theta}(s)(m))\) and \( J^{(m)} = j^{(m)} - j^{(m-1)} \).

Applying Theorem 1.2, under the condition (1.17), we have

\[
Z^{(m+1)}(T) = \sum_{j=1}^{2} \left\| \left( U^{(j)}(m+1), Q^{(j)}(m+1), \Theta^{(j)}(m+1) \right) \right\|_{W(Q_{T}^{(j)})}
\]

\[
\leq c_{50} \left( \sum_{j=1}^{2} \left\| f^{(j)}(m) - f^{(j)}(m-1) \right\|_{W_{2}^{1,1/2+1/2}(Q_{T}^{(j)})} + \left\| (0, 0), K^{(m)}, 0 \right\|_{G(Q_{T}^{(j)})} + \| f^{(m)} \|_{W_{2}^{1/2+1/4+1/2}(G_{T})} \right).
\]  (5.13)

The following estimates correspond to Lemmas 4.1-4.3 and 5.1. (For simplicity, we omit the indices \((j)\) henceforth.)

**Lemma 5.2.** Set \( B_{(m)} = B_{u(m)}, n_{(m)} = n_{u(m)}, \) and \( \Delta_{(m)} = \Delta_{u(m)} \). Then

\[
\sup_{t \in (0, T)} \left\| B_{(m)} - B_{(m-1)} \right\|_{W_{2}^{2+i,1/2}(Q)} \leq c_{51}(T)Z^{(m)}(T),
\]

\[
\left\| n_{(m)} - n_{(m-1)} \right\|_{W_{2}^{2+i,1/2}(Q_{T})} \leq c_{52}(T)Z^{(m)}(T),
\]

\[
\left\| \sigma'(\theta^{(m)}) - \sigma'(\theta^{(m-1)}) \right\|_{W_{2}^{2+i,1/2}(Q_{T})} \leq c_{53}(T)Z^{(m)}(T),
\]

\[
\left\| (\Delta_{(m)} - \Delta_{(m-1)})u^{(m)} \right\|_{W_{2}^{2+i,1/2+1/2}(Q_{T})} \leq c_{54}(T)Z^{(m)}(T),
\]

\[
\left\| \left( \Delta_{(m)} - \Delta_{(m-1)} \right) \right\|_{W_{2}^{2+i,1/2}(Q_{T})} \leq c_{55}(T)Z^{(m)}(T),
\]

\[
\left\| f^{(m)} - f^{(m-1)} \right\|_{W_{2}^{1+i,1/2+1/2}(Q_{T})} \leq c_{56}(T)Z^{(m)}(T),
\]

\[
\left\| J^{(m)} \right\|_{W_{2}^{1/2+i,1/4+1/2}(G_{T})} \leq c_{57}(T)Z^{(m)}(T).
\]

The constants \( c_{i}(T) \) \((i = 51, \cdots)\) depend on \( Z^{(m)}(T) + Z^{(m-1)}(T) \) nondecreasingly.

From (5.13) and Lemma 5.2, we derive the inequality

\[
Z^{(m+1)}(T) \leq c_{59}(T)Z^{(m)}(T)
\]  (5.14)

with \( c_{59}(T) \to 0 \) as \( T \to 0 \). Therefore, the sequence \( \{(u^{(j)}(m), q^{(j)}(m), \theta^{(j)}(m))\} \) is uniformly convergent to \( (u^{(j)}, q^{(j)}, \theta^{(j)}) \) as \( m \to \infty \) if \( T'' \in (0, T'] \) is taken so small as to satisfy

\[
0 < c_{59}(T'') < 1.
\]  (5.15)
It is obvious that this limit function \((u^{(j)}(m), \theta^{(j)}(m), \psi^{(j)}(m))\) is our desired solution of \((1.8)^{(j)}\)–\((1.11)^{(j)}\) satisfying

\[
(u^{(j)}, \theta^{(j)}) \in W^{3,1/2/2}_2(Q_T''), \quad \nabla q^{(j)} \in W^{1+1/2/2}_2(Q_T''),
\]

and \(q^{(1)}(\Gamma) - q^{(2)}(\Gamma) \in W^{3/2/2+1/2}_2(\Gamma_T'').\)

The uniqueness of this solution can be shown by making use of the inequality analogous to (5.14). Indeed, if two solutions (say, \((u^{(j)}, q^{(j)}, \psi^{(j)}), (u'^{(j)}, q'^{(j)}, \psi'^{(j)})\)) exist, then the differences \(U^{(j)} = u^{(j)} - u'^{(j)}, \tilde{Q}^{(j)} = q^{(j)} - q'^{(j)}, \tilde{\psi}^{(j)} = \psi^{(j)} - \psi'^{(j)}\) must satisfy the system of equations similar to (5.9)–(5.12). In exactly the same manner as above, we have

\[
Z(T'') \equiv \sum_{j=1}^{2} \| (U^{(j)}, Q^{(j)}, \psi^{(j)}) \|_{W(Q_T^{''}(\Gamma))} \leq c_{59}(T'') Z(T''),
\]

from which it follows, according to the condition (5.15), that \(Z(T'') = 0\), i.e., \(u^{(j)} = u'^{(j)}, \nabla q^{(j)} = \nabla q'^{(j)}, \theta^{(j)} = \theta'^{(j)}\) in \(Q_T^{''}(\Gamma)\), and \(q^{(1)}(\Gamma) - q^{(2)}(\Gamma) = q'^{(1)}(\Gamma) - q'^{(2)}(\Gamma)\) on \(\Gamma_T''\). Thus, the proof of our main Theorem is completed.

Acknowledgement. The author should like to express his sincere gratitude to Professor Atusi Tani for his kind advices. He also wishes to thank Professor Takao Kakita for his constant encouragement.

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Two-phase free boundary problem


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