The fluctuation results for the single point range of random walks in low dimensions

Dedicated to Professor Yuji Ito on his 60th birthday

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1. Introduction

Let $\{S_n\}_{n=0}^\infty$ be a random walk in the d dimensional integer lattice $\mathbb{Z}^d$, that is,

$$S_0 = 0, \quad S_n = \sum_{k=1}^n X_k,$$

where $\{X_n\}_{n=1}^\infty$ is a sequence of independent identically distributed random variables with values in $\mathbb{Z}^d$ defined on some probability space $(\Omega, \mathfrak{B}, P)$. Throughout this paper, we assume that group G generated by $\{z \in \mathbb{Z}^d; P(X_1 = z) > 0\}$ is isomorphic to $\mathbb{Z}^d$. If $G = \mathbb{Z}^d$, the random walk is called *adapted*. In terms of the characteristic function of X_1 , denoted by $\varphi(\xi)$ ($\xi \in (-\pi, \pi]^d$), adapted means exactly that the equation $\varphi(\xi) = 1$ has the unique solution $\xi = 0$. However, there may exist non-zero values of ξ satisfying $|\varphi(\xi)| = 1$. If $|\varphi(\xi)| < 1$ for all ξ except 0, the random walk is called *aperiodic*. This is equivalent to the property that for each $x \in \mathbb{Z}^d$, the smallest subgroup of $\mathbb{Z}^d$ which contains the set $\{z \in \mathbb{Z}^d; z = x + y, \text{ where } P(X_1 = y) > 0\}$ coincides with $\mathbb{Z}^d$. If $P(X_1 = e) = 1/2d$ for each unit vector $e \in \mathbb{Z}^d$, we call the random walk *simple*. The simple random walk is adapted but not aperiodic, which is called *periodic*.

The random walk is called *transient* if $q > 0$ and *recurrent* otherwise, where $q = P(S_n \neq 0, n = 1, 2, \dots)$. An equivalent criterion for transience is $\sum_{n=0}^\infty P(S_n = 0) < \infty$. When $\sum_{n=0}^\infty \sum_{k=n}^\infty P(S_k = 0)$ converges, the random walk is called *strongly transient*. If $d \geq 3$, the random walk is always transient and if $d \geq 5$, strongly transient.

The range of a random walk, denoted by R_n , means the number of distinct sites visited by the random walk in the first n steps. In 1951, Dvoretzky and Erdős [2] established the strong law of large numbers for the range of the simple random walk in $\mathbb{Z}^d$ for $d \geq 2$. Employing the ergodic theorem, Kesten, Spitzer, and Whitman (cf. [17]) proved that for all random walks R_n/n converges to q almost surely. For

transient random walks $ER_n \sim qn$, so that their results include the strong law of large numbers for the range of all transient random walks. The first work on the central limit theorem for R_n is by Jain and Orey [6] in 1968. They showed that if the random walk is strongly transient and $q < 1$, $\text{Var } R_n \sim \sigma^2 n$ for some positive constant σ^2 and $(R_n - qn)/\sigma\sqrt{n}$ converges toward the normal variable with mean 0 and variance 1 in the distribution sense. If $q = 1$, it is not interesting because $R_n = n + 1$ almost surely. In 1971, Jain and Pruitt [8] extended their result to the case which $d \geq 4$ and $q < 1$ by using a different technique from that of Jain and Orey. In 1974, they showed that if $d = 3$ and $q < 1$, there is a non-decreasing slowly varying function $\phi(n)$ such that $\text{Var } R_n \sim n\phi(n)$ and R_n obeys the central limit theorem [10].

In the two dimensional case, Jain and Pruitt [9] proved that R_n/ER_n is asymptotically equal to 1 almost surely for all recurrent random walks and $\text{Var } R_n \sim Kn^2/(\log n)^4$ for an adapted random walk with mean 0 and finite variance. However, they were not able to establish the central limit theorem. In 1986, Le Gall [12] finally answered their question and proved the limit distribution of $(R_n - ER_n)/\sqrt{\text{Var } R_n}$ is the distribution of the self-intersection local time of the two dimensional Brownian motion.

Our results concern the single point range of the random walk. Let Q_n be the number of distinct points entered once and only once by a random walk in the first n steps, which is called the single point range of the random walk. Erdős and Taylor [3] proved the strong law of large numbers for the single point range of the simple random walk when $d \geq 2$. In 1974, Pitt [14] showed that $EQ_n \sim q^2 n$ and $\lim_{n \rightarrow \infty} Q_n/n = q^2$ a.s. for all transient random walks. In 1992, Hamana [4] proved that if $d \geq 4$ and $q < 1$, $\text{Var } Q_n \sim \lambda^2 n$ for some positive constant λ^2 and the distribution of $(Q_n - q^2 n)/\lambda\sqrt{n}$ is asymptotically equal to the standard normal.

We shall consider these problems in the case of two or three dimensions. In Section 3, we will prove that if $d = 3$ and $q < 1$, there is a non-decreasing slowly varying function $\psi(n)$ such that $\text{Var } Q_n \sim n\psi(n)$, and in Section 4 we will give the proof that the distribution of $(Q_n - q^2 n)/\sqrt{n\psi(n)}$ converges to the normal with mean 0 and variance 1. Section 5 and Section 6 are devoted to the study in the two dimensional case. Although the asymptotic behavior of Q_n for $d \geq 3$ is quite similar to that of R_n , it is different for $d = 2$. We will show that $\text{Var } Q_n \sim Ln^2/(\log n)^6$ for some positive constant L and $(Q_n - EQ_n)/\sqrt{\text{Var } Q_n}$ converges to the self-intersection local time of the planar Brownian motion in the distribution sense for an aperiodic random walk with mean 0 and finite variance; however, we can not prove this in the general periodic case, but there is no problem with the simple random walk case.

2. Notation and preliminaries

In this section we will give some notation and two lemmas. For $x \in \mathbb{Z}^d$, the notation $P_x(\cdot)$ will be used to denote the probability measures of events related to the random walk starting at x . When $x = 0$, we will simply use $P(\cdot)$ instead of $P_0(\cdot)$. For $n \geq 0$ and $x, y \in \mathbb{Z}^d$, the notation $p^n(x, y)$ means $P_x(S_n = y)$ and note that $p^n(x, y) = p^n(0, y - x)$. For $x \in \mathbb{Z}^d$, τ_x will denote the first hitting time of x , *i.e.*

$$\tau_x = \inf\{n \geq 1; S_n = x\} \quad (\inf \emptyset = \infty).$$

The taboo probabilities are defined by

$$p_z^n(x, y) = P_x(S_n = y, \tau_z \geq n), \quad p_{zw}^n(x, y) = P(S_n = y, \tau_z, \tau_w \geq n).$$

The following lemma is very important.

LEMMA 2.1 (Jain - Pruitt [8], Spitzer [17]). *If $q < 1$, there is a positive constant A such that*

$$p^n(0, x) \leq An^{-\frac{1}{2}d}$$

for all $x \in \mathbb{Z}^d$ and $n \geq 1$.

This is the best estimate since $p^n(0, 0) = 2(d/4n\pi)^{(1/2)d} + o\{n^{-(1/2)d}\}$ when the random walk is simple. Another standard result is that for $n \geq 1$ and $x \in \mathbb{Z}^d$,

$$p_0^n(0, x) = p_x^n(0, x).$$

This can be checked easily by considering the reversed random walk.

We will use r_n for $P(\tau_0 > n)$, $\bar{r}_n$ for $P(n < \tau_0 < \infty)$ and $f_n = p_0^n(0, 0)$. For any random walk, the equality $r_n = q + \bar{r}_n$ holds. In particular, $r_n = \bar{r}_n$ when the random walk is recurrent. By Lemma 2.1, if $d \geq 3$

$$P_x(n < \tau_y < \infty) = \sum_{k=n+1}^{\infty} p_y^k(x, y) \leq Cn^{1-\frac{1}{2}d}$$

for all $x, y \in \mathbb{Z}^d$; especially, $\bar{r}_n \leq Cn^{1-(1/2)d}$. When $d = 2$, Kesten and Spitzer [11] proved that r_n is slowly varying. Here the meaning of slowly varying is that for any positive real number c , $r_{[cn]}/r_n \rightarrow 1$ as n tends to infinity, and for a real number x , $[x]$ will denote the integer part of x . If the two dimensional random walk is transient, the result of Kesten and Spitzer is trivial since $r_n = q + o(1)$; however, their result has important meaning in the recurrent case. We will need the simple observation about slowly varying functions in order to obtain the property of r_n .

LEMMA 2.2 (Jain - Pruitt [9]). *Let $\{\alpha(n)\}_{n=1}^{\infty}$ be a decreasing slowly varying sequence. Then there exists a positive constant c such that for all $j \leq n$, $n \geq 1$*

$$j\alpha(j) \leq c n \alpha(n).$$

This lemma implies that given a positive integer k , there exists a constant c_k such that for $j \leq n$

$$jr_j^k \leq c_k n r_n^k$$

for any two dimensional random walk.

If $\{a_n\}$ and $\{b_n\}$ ($b_n > 0$) are sequences of real numbers, then $a_n = o(b_n)$ means $a_n b_n^{-1} \rightarrow 0$; $a_n = O(b_n)$ means $a_n b_n^{-1}$ remains bounded; $a_n \sim b_n$ means $a_n b_n^{-1} \rightarrow 1$, as $n \rightarrow \infty$.

3. The variance of Q_n in the three dimensional case

We will consider the single point range of three dimensional random walks. Firstly we will give several lemmas which will be the main tools to estimate various probabilities. For $n \geq 0$ and $x, y \in \mathbb{Z}^d$, let

$$G_n(x, y) = \sum_{k=0}^n p^k(x, y).$$

For transient random walks, $\lim_{n \rightarrow \infty} G_n(x, y) < \infty$, which is denoted by $G(x, y)$ and is bounded by $G(0, 0) < \infty$ (cf. [17]). By Lemma 2.1, we obtain useful bounds.

LEMMA 3.1 (Jain-Pruitt [8]). *If $d = 3$ and $q < 1$, then*

$$\begin{aligned} \sum_{x \in \mathbb{Z}^3} p^n(0, x) \{G(u, x) + G(x, u)\} &\leq C n^{-\frac{1}{2}} \\ \sum_{x \in \mathbb{Z}^3} p^n(0, x) G(u, x) G(x, v) &\leq C n^{-1} \\ \sum_{x \in \mathbb{Z}^3} p^n(0, x) G(x, u) G(x, v) G(v, x) &\leq C n^{-\frac{3}{2}} \log n \end{aligned}$$

uniformly for $u, v \in \mathbb{Z}^3$.

By this lemma, it is clear that $\sum_x G_n(0, x) G(x, 0) = O(n^{1/2})$. However, this estimate can be improved.

LEMMA 3.2 (Jain-Pruitt [10]). *If $d = 3$, $q < 1$, and $\sum_x G(0, x)^2 G(x, 0) < \infty$, then*

$$\sum_{x \in \mathbb{Z}^3} G_n(0, x) G(x, 0) = o(n^{\frac{1}{2}}).$$

Simple calculations give the following three lemmas.

LEMMA 3.3 (Jain-Pruitt [8]). *If $d = 3$ and $q < 1$, then*

$$P_x(n < \tau_x < \infty, \tau_0 < \infty) \leq C n^{-\frac{1}{2}} \{G(0, x) + G(x, 0)\}$$

for $x \in \mathbb{Z}^3$.

LEMMA 3.4 (Jain-Pruitt [8]). For $x \neq 0$, let

$$b(x) = P_x(\tau_0 < \infty)P_x(\tau_x = \infty) - P_x(\tau_0 < \infty, \tau_x = \infty).$$

Then

$$b(x) = P_x(\tau_0 < \tau_x < \infty)P_x(\tau_0 = \infty).$$

In particular, $b(x) \leq G(0, x)G(x, 0)$.

LEMMA 3.5 (Jain-Pruitt [8]). For a transient random walk, if $x \neq 0$, then

$$\begin{aligned} \sum_{k=1}^m p_{0x}^k(0, x) - q \sum_{k=1}^m p_x^k(0, x) \\ = \sum_{k=1}^m p_x^k(0, x) \{ \bar{\tau}_{m-k} + P_0(\tau_x < \tau_0 \leq m - k) \}. \end{aligned}$$

We will introduce some indicator random variables. For $0 \leq i < j$, let

$$\begin{aligned} Z_i^j &= \begin{cases} 1 & \text{if } S_i \neq S_\alpha \text{ for } i < \alpha \leq j, \\ 0 & \text{otherwise,} \end{cases} \\ Y_j^i &= \begin{cases} 1 & \text{if } S_j \neq S_\alpha \text{ for } i \leq \alpha < j, \\ 0 & \text{otherwise,} \end{cases} \end{aligned}$$

and $Z_n^n = Y_n^n = 1$ for $n \geq 0$. Then we have $R_n = \sum_{j=0}^n Z_j^n = \sum_{j=0}^n Y_j^0$ and $Q_n = \sum_{j=0}^n Y_j^0 Z_j^n$. To approximate R_n and Q_n by a partial sum of stationary sequences, we need to define other indicator random variables. Let X_0, X_{-n} ($n = 1, 2, \dots$) be independent copies of X_1 and for $l < m$, define

$$X(l, m) = \sum_{k=l+1}^m X_k.$$

Using this notation, we can express

$$\begin{aligned} Z_i^j &= \begin{cases} 1 & \text{if } X(i, \alpha) \neq 0 \text{ for } i < \alpha \leq j, \\ 0 & \text{otherwise,} \end{cases} \\ Y_j^i &= \begin{cases} 1 & \text{if } X(\alpha, j) \neq 0 \text{ for } i \leq \alpha < j, \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

For each integer j , let

$$Z_j = \begin{cases} 1 & \text{if } X(j, \alpha) \neq 0 \text{ for } \alpha > j, \\ 0 & \text{otherwise,} \end{cases}$$

$$Y_j = \begin{cases} 1 & \text{if } X(\alpha, j) \neq 0 \text{ for } \alpha < j, \\ 0 & \text{otherwise,} \end{cases}$$

and for $i < j$, $W_i^j = Z_i^j - Z_i$ and $V_j^i = Y_j^i - Y_j$. Note that W_i^j and V_j^i are also indicator random variables. The following lemma is useful for the approximation of R_n and Q_n .

LEMMA 3.6 (Jain-Pruitt [10]). *If $d = 3$ and $q < 1$, then*

$$E \left| \sum_{j=0}^n W_j^n \right|^2, E \left| \sum_{j=0}^n V_j^0 \right|^2 = O(n),$$

If, in addition, $\sum G(0, x)^2 G(x, 0) < \infty$, then

$$E \left| \sum_{j=0}^n W_j^n \right|^2, E \left| \sum_{j=0}^n V_j^0 \right|^2 = o(n).$$

The assertion of W part was proved in [10]. By relabeling the indices of X , the estimate of V part can be shown in the same fashion as that of W part. Recall that Jain and Pruitt [10] proved that

$$\text{Var } R_n \sim n\phi(n),$$

where ϕ is a non-decreasing slowly varying function such that

$$\lim_{n \rightarrow \infty} \phi(n) = \begin{cases} \infty & \text{if } \sum_x G(0, x)^2 G(x, 0) = \infty \\ \rho^2 & \text{if } \sum_x G(0, x)^2 G(x, 0) < \infty \end{cases}$$

for some positive constant ρ^2 . From now on, we will consider two cases separately:

$$\begin{cases} \text{Case 1} & \sum_x G(0, x)^2 G(x, 0) = \infty, \\ \text{Case 2} & \sum_x G(0, x)^2 G(x, 0) < \infty. \end{cases}$$

LEMMA 3.7. *If $d = 3$ and $q < 1$, then*

$$\begin{aligned} \text{Var } R_n &= \sum_{j=1}^n \sum_{i=0}^{j-1} \left[\sum_{x \neq 0} p_{0x}^{j-i}(0, x) P_x(\tau_0 = \infty)^2 - q^2 \right] \\ &\quad + \begin{cases} O(n\sqrt{\phi(n)}) & \text{Case 1} \\ o(n) & \text{Case 2.} \end{cases} \end{aligned}$$

PROOF. We can prove it along the same line as Lemma 5.2 in [4]. Let

$$\Gamma_n^+ = \sum_{j=0}^n Z_j, \quad \Gamma_n^- = \sum_{j=0}^n Y_j.$$

Lemma 3.6 asserts that

$$E|R_n - \Gamma_n^+|^2, E|R_n - \Gamma_n^-|^2 = \begin{cases} O(n) & \text{Case 1} \\ o(n) & \text{Case 2} \end{cases}$$

and then we obtain

$$\begin{aligned} \text{Var } \Gamma_n^+ &= \text{Var } R_n + \begin{cases} O(n\sqrt{\phi(n)}) & \text{Case 1} \\ o(n) & \text{Case 2} \end{cases} \\ &\sim n\phi(n). \end{aligned}$$

Hence

$$\begin{aligned} \text{Var } R_n &= \text{Cov}(\Gamma_n^+, \Gamma_n^-) + \text{Cov}(R_n - \Gamma_n^+, R_n) + \text{Cov}(\Gamma_n^+, R_n - \Gamma_n^-) \\ &= \text{Cov}(\Gamma_n^+, \Gamma_n^-) + \begin{cases} O(n\sqrt{\phi(n)}) & \text{Case 1} \\ o(n) & \text{Case 2.} \end{cases} \end{aligned}$$

If $i \leq j$, Y_i and Z_j are independent, and therefore

$$\text{Var } R_n = \sum_{j=1}^n \sum_{i=0}^{j-1} \text{Cov}(Z_i, Y_j) + \begin{cases} O(n\sqrt{\phi(n)}) & \text{Case 1} \\ o(n) & \text{Case 2} \end{cases}$$

holds. For $0 \leq i < j \leq n$,

$$\begin{aligned} EZ_i Y_j &= \sum_{x \neq 0} p_{0x}^{j-i}(0, x) P_x(\tau_0 = \infty)^2, \\ EZ_i EY_j &= q^2. \end{aligned}$$

Thus we can conclude the assertion. $\blacksquare$

The theorem in this section is

THEOREM 3.8. *If $d = 3$ and $q < 1$, then there is a non-decreasing slowly varying function $\psi(n)$ such that*

$$\text{Var } Q_n \sim n\psi(n).$$

Moreover,

$$\psi(n) \sim \begin{cases} 4q^2\phi(n) & \text{Case 1} \\ \mu^2 & \text{Case 2} \end{cases}$$

for some positive constant μ^2 which is larger than $2q^2\rho^2$.

PROOF. Let

$$\Lambda_n = \sum_{j=0}^n Y_j Z_j.$$

Firstly we will estimate the variance of Λ_n . For $i < j$,

$$\begin{aligned} \text{Cov}(Y_i Z_i, Y_j Z_j) &= E(Y_i Z_i Y_j Z_j) - (E Y_i Z_i)(E Y_j Z_j) \\ &= \sum_{x \neq 0} p_{0x}^{j-i}(0, x) P_x(\tau_0, \tau_x = \infty)^2 - q^4 \\ &= \sum_{x \neq 0} p_{0x}^{j-i}(0, x) b(x)^2 \\ &\quad + 2q \sum_{x \neq 0} p_{0x}^{j-i}(0, x) b(x) P_x(\tau_0 = \infty) \\ &\quad + q^2 \sum_{x \neq 0} p_{0x}^{j-i}(0, x) P_x(\tau_0 = \infty)^2 - q^4. \end{aligned}$$

Therefore it holds that

$$\begin{aligned} (3.1) \quad \text{Var } \Lambda_n &= \sum_{j=0}^n \text{Var } Y_j Z_j + 2 \sum_{j=1}^n \sum_{i=0}^{j-1} \text{Cov}(Y_i Z_i, Y_j Z_j) \\ &= q^2(1 - q^2)n + 2 \sum_{j=1}^n \sum_{i=1}^j \sum_{x \neq 0} p_{0x}^i(0, x) b(x)^2 \\ &\quad + 4q \sum_{j=1}^n \sum_{i=1}^j \sum_{x \neq 0} p_{0x}^i(0, x) b(x) P_x(\tau_0 = \infty) + 2q^2 \text{Var } R_n \\ &\quad + \begin{cases} O(n\sqrt{\phi(n)}) & \text{Case 1} \\ o(n) & \text{Case 2} \end{cases} \end{aligned}$$

by Lemma 3.7.

(Case 1) $\sum_x G(0, x)^2 G(x, 0) = \infty$

By Lemma 3.1 and Lemma 3.4,

$$\sum_{i=1}^j \sum_{x \neq 0} p_{0x}^i(0, x) b(x)^2, \quad \sum_{i=1}^j \sum_{x \neq 0} p_{0x}^i(0, x) b(x) P_x(\tau_0 < \infty) = O(1).$$

Moreover,

$$\sum_{j=1}^n \sum_{i=1}^j \bar{r}_{j-i} \sum_{x \neq 0} p_x^i(0, x) b(x) = O(n^{\frac{1}{2}} \log n),$$

$$\sum_{j=1}^n \sum_{i=1}^j \sum_{x \neq 0} p_x^i(0, x) b(x) P_0(\tau_x < \tau_0 \leq j - i) = O(n).$$

Hence, applying Lemma 3.5,

$$\sum_{j=1}^n \sum_{i=1}^j \sum_{x \neq 0} p_{0x}^i(0, x) b(x) = q \sum_{j=1}^n \sum_{i=1}^j \sum_{x \neq 0} p_x^i(0, x) b(x) + O(n).$$

Combining these estimates, we have that

$$\text{Var } A_n = 4q^2 \sum_{j=1}^n \sum_{i=1}^j \sum_{x \neq 0} p_x^i(0, x) b(x) + 2q^2 \text{Var } R_n + O(n\sqrt{\phi(n)}).$$

Remember that Jain and Pruitt [10] showed

$$\text{Var } R_n = 2 \sum_{j=1}^n \sum_{i=1}^j \sum_{x \neq 0} p_x^i(0, x) b(x) + O(n\sqrt{\phi(n)}) \sim n\phi(n).$$

Then we obtain

$$\text{Var } A_n \sim 4q^2 \text{Var } R_n \sim 4q^2 n\phi(n).$$

(Case 2) $\sum_x G(0, x)^2 G(x, 0) < \infty$

Jain and Pruitt [10] proved $\text{Var } R_n \sim \rho^2 n$ for some positive constant ρ^2 in this case. Then we need to estimate the second term and the third term of the right hand side of (3.1). Let

$$a_j = \sum_{i=1}^j \sum_{x \neq 0} p_{0x}^i(0, x) b(x)^2.$$

Since $\{a_j\}_{j=1}^\infty$ is a bounded, non-decreasing, and non-negative sequence of real numbers, there is some non-negative constant a satisfying that $\lim_{j \rightarrow \infty} a_j = a$. Therefore we have

$$\sum_{j=1}^n \sum_{i=1}^j \sum_{x \neq 0} p_{0x}^i(0, x) b(x)^2 \sim an.$$

The assumption $\sum G(0, x)^2 G(x, 0) < \infty$ implies that

$$\sum_{i=1}^j \sum_{x \neq 0} p_{0x}^i(0, x) b(x) P_x(\tau_0 = \infty) \leq \sum_{x \neq 0} G_j(0, x) b(x) = O(1).$$

By a similar argument we can conclude that there is a non-negative constant a' such that

$$\sum_{j=1}^n \sum_{i=1}^j \sum_{x \neq 0} p_{0x}^i(0, x) b(x) P_x(\tau_0 = \infty) \sim a'n.$$

Then $\text{Var } \Lambda_n \sim \mu^2 n$, where $\mu^2 = q^2(1 - q^2) + 2(a + 2a'q + \rho^2 q^2)$.

Accordingly, we obtain

$$\text{Var } \Lambda_n \sim \begin{cases} 4q^2 n \phi(n) & \text{Case 1} \\ \mu^2 n & \text{Case 2.} \end{cases}$$

To complete the proof, we need the bound of $|\text{Var } Q_n - \text{Var } \Lambda_n|$. Since

$$\begin{aligned} |\text{Var } Q_n - \text{Var } \Lambda_n| &\leq 2|\text{Cov}(Q_n - \Lambda_n, \Lambda_n)| + \text{Var}(Q_n - \Lambda_n) \\ &\leq 2\sqrt{E|Q_n - \Lambda_n|^2 \cdot \text{Var } \Lambda_n} + E|Q_n - \Lambda_n|^2, \end{aligned}$$

it is sufficient to estimate $E|Q_n - \Lambda_n|^2$. By the definition of indicator random variables introduced in this section, we have

$$Q_n - \Lambda_n = \sum_{j=0}^n Y_j^0 W_j^n + \sum_{j=0}^n V_j^0 Z_j.$$

Lemma 3.6 shows that

$$E\left|\sum_{j=0}^n Y_j^0 W_j^n\right|^2, \quad E\left|\sum_{j=0}^n V_j^0 Z_j\right|^2 = \begin{cases} O(n) & \text{Case 1} \\ o(n) & \text{Case 2} \end{cases}$$

and then $E|Q_n - \Lambda_n|^2$ has the same estimate. This completes the proof of Theorem 3.8. $\blacksquare$

REMARK 3.9. Applying the above theorem, for the three dimensional adapted random walk with mean 0 and finite variance, we can express the asymptotic behavior of $\text{Var } Q_n$ exactly:

$$\text{Var } Q_n \sim 8q^6(2\pi)^{-2}(\det \Xi)^{-2}n \log n,$$

where Ξ^2 is the covariance matrix for X_1 by adopting the explicit formula for $\phi(n)$ proved by Jain and Pruitt [8] in Theorem 2.

4. The central limit theorem for Q_n when $d = 3$

To prove the central limit theorem for Q_n , we will define four more indicator random variables. For $n \geq 0$ and $0 \leq j < l$, let

$$Z_{j,l}^n = \begin{cases} 1 & \text{if } S_j \neq S_\alpha \text{ for } j < \alpha < l, \quad l < \alpha \leq n, \\ & \text{and } S_j = S_l, \\ 0 & \text{otherwise,} \end{cases} \quad \text{if } l < n,$$

$$Z_{j,n}^n = \begin{cases} 1 & \text{if } S_j \neq S_\alpha \text{ for } j < \alpha < n, \text{ and } S_j = S_l, \\ 0 & \text{otherwise,} \end{cases} \quad \text{if } l = n,$$

$$Z_{j,l} = \begin{cases} 1 & \text{if } S_j \neq S_\alpha \text{ for } j < \alpha < l, \alpha > l, \text{ and } S_j = S_l, \\ 0 & \text{otherwise,} \end{cases}$$

$$W_{j,l}^n = Z_{j,l}^n - Z_{j,l}.$$

The random variable $\sum_{j=0}^{n-1} \sum_{l=j+1}^n Z_{j,l}^n$, denoted by $R_n^{(2)}$, is equal to the number of distinct points visited at least twice by $\{S_n\}$ up to time n . It is clear that $Q_n = R_n - R_n^{(2)}$. In Section 3, the estimate

$$(4.1) \quad E|R_n - \Gamma_n^+|^2 = \begin{cases} O(n) & \text{Case 1} \\ o(n) & \text{Case 2} \end{cases}$$

was established. We will now obtain a similar assertion about $R_n^{(2)}$. The following two lemmas assure that $R_n^{(2)}$ is approximately a partial sum of a stationary sequence.

LEMMA 4.1. *If $d = 3$ and $q < 1$, then*

$$E \left| \sum_{j=0}^{n-1} \sum_{l=j+1}^n W_{j,l}^n \right|^2 = \begin{cases} O(n) & \text{Case 1} \\ o(n) & \text{Case 2.} \end{cases}$$

PROOF. We can estimate this analogously to Lemma 3.1 in [4], by means of which the assertion of Case 1 can be derived easily. In calculating this in the other case, we will use the same method as Lemma 3.1 in [10].

$$\begin{aligned} E \left| \sum_{j=0}^{n-1} \sum_{l=j+1}^n W_{j,l}^n \right|^2 &= \sum_{j=0}^{n-1} \sum_{l=j+1}^n E W_{j,l}^n \\ &\quad + 2 \sum_{j=1}^{n-2} \sum_{i=0}^{j-1} \sum_{l=j+1}^{n-1} \sum_{h=l+1}^n E W_{j,l}^n W_{i,h}^n \\ &\quad + 2 \sum_{j=1}^{n-2} \sum_{i=0}^{j-1} \sum_{l=j+2}^n \sum_{h=j+1}^{l-1} E W_{j,l}^n W_{i,h}^n \\ &\quad + 2 \sum_{j=2}^{n-1} \sum_{i=0}^{j-2} \sum_{l=j+1}^n \sum_{h=i+1}^{j-1} E W_{j,l}^n W_{i,h}^n \\ &= \text{I} + \text{II} + \text{III} + \text{IV}. \end{aligned}$$

For $0 \leq j < l \leq n$,

$$E W_{j,l}^n = f_{l-j} \bar{r}_{n-l} \leq C(l-j)^{-\frac{3}{2}} (n-l+1)^{-\frac{1}{2}}.$$

Hence we have $I = O(n^{1/2})$. Since, for $0 \leq i < j < l < h \leq n$,

$$EW_{j,l}^n W_{i,h}^n \leq \sum_x p^{j-i}(0, x) p^{l-j}(x, x) p^{h-l}(x, 0) P_0(n - h < \tau_0, \tau_x < \infty),$$

we obtain by neglecting the event $\{n - h < \tau_x < \infty\}$,

$$\begin{aligned} \text{II} &\leq C \sum_{l=2}^{n-1} \sum_{j=1}^{l-1} \sum_{h=l+1}^n (l-j)^{-\frac{3}{2}} (n-h+1)^{-\frac{1}{2}} \sum_x p^{h-l}(x, 0) G(0, x) \\ &\leq C \sum_{h=3}^n (n-h+1)^{-\frac{1}{2}} \sum_x G_h(0, x) G(x, 0) = \begin{cases} O(n) & \text{Case 1} \\ o(n) & \text{Case 2} \end{cases} \end{aligned}$$

by Lemma 2.1 and Lemma 3.1. For $0 \leq i < j < h < l \leq n$,

$$EW_{j,l}^n W_{i,h}^n \leq \sum_x p^{j-i}(0, x) p^{h-j}(x, 0) p^{l-h}(0, x) P_x(n - l < \tau_0, \tau_x < \infty).$$

Summing on i at first and next changing the order of the summation on j , l , and h , we have

$$\text{III} \leq C \sum_{l=3}^n \sum_{h=2}^{l-1} \sum_x p^{l-h}(0, x) G(x, 0) G(0, x) \bar{r}_{n-l} = O(n^{\frac{1}{2}} \log n).$$

The remainder term is complicated. We will estimate it in each case one by one. For $0 \leq i < h < j < l \leq n$,

$$\begin{aligned} EW_{j,l}^n W_{i,h}^n &\leq \sum_{x \neq 0} p^{h-i}(0, 0) p^{j-h}(x, 0) p^{l-j}(x, x) P_x(n - l < \tau_0, \tau_x < \infty) \\ &\leq C(h-i)^{-\frac{3}{2}} (l-j)^{-\frac{3}{2}} \sum_{x \neq 0} p^{j-h}(0, x) P_x(n - l < \tau_0, \tau_x < \infty). \end{aligned}$$

(Case 1) $\sum_x G(0, x)^2 G(x, 0) = \infty$

By Lemma 3.1 and Lemma 3.3,

$$\text{IV} \leq C \sum_{j=2}^{n-1} \sum_{i=0}^{j-2} \sum_{l=j+1}^n \sum_{h=i+1}^{j-1} (h-i)^{-\frac{3}{2}} (l-j)^{-\frac{3}{2}} (j-h)^{-\frac{1}{2}} (n-l+1)^{-\frac{1}{2}} = O(n).$$

(Case 2) $\sum_x G(0, x)^2 G(x, 0) < \infty$

By the inequality

$$P_x(n - l < \tau_0, \tau_x < \infty) \leq P_x(n - l < \tau_0 < \tau_x < \infty) + \bar{r}_{n-l} P_x(\tau_0 < \infty),$$

we have

$$\begin{aligned}
\text{IV} &\leq C \sum_{j=2}^{n-1} \sum_{h=1}^{j-1} \sum_{l=j+1}^n (l-j)^{-\frac{3}{2}} \sum_{x \neq 0} p^{j-h}(0, x) P_x(n-l < \tau_0 < \tau_x < \infty) \\
&\quad + C \sum_{j=2}^{n-1} \sum_{h=1}^{j-1} \sum_{l=j+1}^n (l-j)^{-\frac{3}{2}} (n-l+1)^{-\frac{1}{2}} \sum_{x \neq 0} p^{j-h}(0, x) P_x(\tau_0 < \infty) \\
&\leq C \sum_{j=2}^{n-1} \sum_{l=j+1}^n (l-j)^{-\frac{3}{2}} \sum_{x \neq 0} G(0, x) P_x(n-l < \tau_0 < \tau_x < \infty) \\
&\quad + C \sum_{j=2}^{n-1} \sum_{l=j+1}^n (l-j)^{-\frac{3}{2}} (n-l+1)^{-\frac{1}{2}} \sum_x G_j(0, x) G(x, 0) \\
&\leq C \sum_{l=3}^n \sum_x G(0, x) \{G(x, 0) - G_l(x, 0)\} G(x, 0) \\
&\quad + C \sum_{j=2}^{n-1} (n-j)^{-\frac{1}{2}} \sum_x G_j(0, x) G(x, 0).
\end{aligned}$$

Since $\sum_x G(0, x)^2 G(x, 0) < \infty$ and $G_l(x, 0) \rightarrow G(x, 0)$ as $l \rightarrow \infty$,

$$\lim_{l \rightarrow \infty} \sum_x G(0, x) \{G(x, 0) - G_l(x, 0)\} G(x, 0) = 0$$

by applying the dominated convergence theorem. Hence the first part is equal to $o(n)$. Using Lemma 3.2, we can easily check that the second part is also $o(n)$. ■

LEMMA 4.2. If $d = 3$ and $q < 1$, then

$$E \left| \sum_{j=0}^{n-1} \sum_{l=n+1}^{\infty} Z_{j,l} \right|^2 = \begin{cases} O(n) & \text{Case 1} \\ o(n) & \text{Case 2.} \end{cases}$$

PROOF. We can prove it along the same line as the proof of Lemma 3.2 in [4],

$$\begin{aligned}
E \left| \sum_{j=0}^{n-1} \sum_{l=n+1}^{\infty} Z_{j,l} \right|^2 &= \sum_{j=0}^{n-1} \sum_{l=n+1}^{\infty} E Z_{j,l} \\
&\quad + 2 \sum_{j=1}^{n-1} \sum_{i=0}^{j-1} \sum_{l=n+1}^{\infty} \sum_{h=l+1}^{\infty} E Z_{j,l} Z_{i,h}
\end{aligned}$$

$$\begin{aligned}
& + 2 \sum_{j=1}^{n-1} \sum_{i=0}^{j-1} \sum_{h=n+1}^{\infty} \sum_{l=h+1}^{\infty} EZ_{j,l} Z_{i,h} \\
& = \text{I} + \text{II} + \text{III}.
\end{aligned}$$

The first term can be computed easily, and we have that $\text{I} = O(n^{1/2})$. For $0 \leq i < j \leq n < l < h < \infty$,

$$\begin{aligned}
EZ_{j,l} Z_{i,h} & \leq \sum_x p^{j-i}(0, x) p^{l-j}(x, x) p^{h-l}(x, 0) \\
& \leq A(l-j)^{-\frac{3}{2}} \sum_x p^{j-i}(0, x) p^{h-l}(x, 0).
\end{aligned}$$

Hence

$$\begin{aligned}
\text{II} & \leq C \sum_{j=1}^{n-1} \sum_{i=0}^{j-1} (n-j)^{-\frac{1}{2}} \sum_x p^{j-i}(0, x) G(x, 0) \\
& \leq C n^{\frac{1}{2}} \sum_x G_n(0, x) G(x, 0) \\
& = \begin{cases} O(n) & \text{Case 1} \\ o(n) & \text{Case 2.} \end{cases}
\end{aligned}$$

The term III can be estimated similarly to the term IV in Lemma 4.1. For $0 \leq i < j \leq n < h < l < \infty$,

$$EZ_{j,l} Z_{i,h} \leq \sum_x p^{j-i}(0, x) p^{h-j}(x, 0) p^{l-h}(0, x).$$

(Case 1)

By Lemma 2.1 and Lemma 3.1,

$$\begin{aligned}
\text{III} & \leq C \sum_{j=1}^{n-1} \sum_{i=0}^{j-1} \sum_{h=n+1}^{\infty} (h-j)^{-\frac{3}{2}} \sum_x p^{j-i}(0, x) G(0, x) \\
& \leq C \sum_{j=1}^{n-1} \sum_{i=0}^{j-1} (n-j)^{-\frac{1}{2}} (j-i)^{-\frac{1}{2}} = O(n).
\end{aligned}$$

(Case 2)

After summing $EZ_{j,l} Z_{i,h}$ on l and i , we calculate it over all h in (n, ∞) . Then we obtain

$$\text{III} \leq C \sum_{j=1}^{n-1} \sum_x G(0, x) \{G(x, 0) - G_{n-j}(x, 0)\} G(0, x) = o(n).$$

Therefore we can conclude the assertion. $\blacksquare$

For $n \geq 0$, define $\Gamma_n^{(2)} = \sum_{j=0}^{n-1} \sum_{l=j+1}^{\infty} Z_{j,l}$. Combining Lemma 4.1 and Lemma 4.2, we have

$$(4.2) \quad E|R_n^{(2)} - \Gamma_n^{(2)}|^2 = \begin{cases} O(n) & \text{Case 1} \\ o(n) & \text{Case 2.} \end{cases}$$

In order to check the Lindeberg-Feller condition, we need the centered fourth moment of Q_n . In the case of R_n , Jain and Pruitt [10] showed that if $d = 3$ and $q < 1$, $E|R_n - ER_n|^4 = O(\{n\phi(n)\}^2)$.

LEMMA 4.3. *If $d = 3$ and $q < 1$, then*

$$E|Q_n - EQ_n|^4 = O(\{n\psi(n)\}^2).$$

PROOF. We have that

$$\begin{aligned} Q_{2n} &= \sum_{j=0}^n Y_j^0 Z_j^n + \sum_{j=n}^{2n} Y_j^n Z_j^{2n} \\ &\quad - \sum_{j=0}^{n-1} (Z_j^n - Z_j^{2n}) Y_j^0 - \sum_{j=n}^{2n} (Y_j^n - Y_j^0) Z_j^{2n} + Y_n^0 Z_n^n. \end{aligned}$$

Jain and Pruitt showed in their paper [10] that

$$E \left| \sum_{j=0}^{n-1} (Z_j^n - Z_j^{2n}) \right|^4 = O(\{n\phi(n)\}^2).$$

This implies that

$$E \left| \sum_{j=0}^{n-1} (Z_j^n - Z_j^{2n}) Y_j^0 \right|^4 = O(\{n\phi(n)\}^2) = O(\{n\psi(n)\}^2)$$

since $Z_j^n - Z_j^{2n} \geq 0$ and $0 \leq Y_j^0 \leq 1$ for $0 \leq j \leq n$. Considering the reversed random walk, we can obtain

$$E \left| \sum_{j=n}^{2n} (Y_j^n - Y_j^0) Z_j^{2n} \right|^4 = O(\{n\psi(n)\}^2).$$

Also

$$E \left| \sum_{j=0}^n (Y_j^0 Z_j^n - EY_j^0 Z_j^n) + \sum_{j=n}^{2n} (Y_j^n Z_j^{2n} - EY_j^n Z_j^{2n}) \right|^4$$

$$= 2E|Q_n - EQ_n|^4 + 6(\text{Var } Q_n)^2$$

since the two random variables are independent and have the same distribution as $Q_n - EQ_n$. Therefore

$$\{E|Q_{2n} - EQ_{2n}|^4\}^{\frac{1}{4}} \leq \{2E|Q_n - EQ_n|^4 + 6(\text{Var } Q_n)^2\}^{\frac{1}{4}} + O(\{n\psi(n)\}^{\frac{1}{2}}).$$

The remainder of the proof is the same as Lemma 4.1 in [10]. $\blacksquare$

We are now ready to show the central limit theorem. The main tool is the Lindeberg theorem for triangular arrays.

THEOREM 4.4. *If $d = 3$ and $q < 1$, then the $(Q_n - q^2n)/\sqrt{n\psi(n)}$ converges to the normal variable with mean 0 and variance 1 in the distribution sense.*

PROOF. Let $\Psi_n = \Gamma_n^+ - \Gamma_n^{(2)}$. We conclude that

$$E|Q_n - \Psi_n|^2 = \begin{cases} O(n) & \text{Case 1} \\ o(n) & \text{Case 2} \end{cases}$$

by (4.1) and (4.2). Then for any $\varepsilon > 0$,

$$P(|Q_n - \Psi_n| \geq \varepsilon \sqrt{n\psi(n)}) \leq \varepsilon^{-2} \{n\psi(n)\}^{-1} E|Q_n - \Psi_n|^2 \rightarrow 0$$

as n tends to infinity. Thus $(Q_n - \Psi_n)/\sqrt{n\psi(n)} \rightarrow 0$ in probability, so that it suffices to consider $(\Psi_n - q^2n)/\sqrt{n\psi(n)}$.

For $N, m \geq 0$ and $i = 1, 2, \dots, N$, let

$$\begin{aligned} A_i &= \sum_{j=(i-1)m}^{im-1} Z_j^{im}, & A_i^{(2)} &= \sum_{j=(i-1)m}^{im-1} \sum_{l=j+1}^{im} Z_{j,l}^{im}, \\ U_i &= \sum_{j=(i-1)m}^{im-1} W_j^{im}, & U_i^{(2)} &= \sum_{j=(i-1)m}^{im-1} \sum_{l=j+1}^{im} W_{j,l}^{im}, \\ V_i^{(2)} &= \sum_{j=(i-1)m}^{im-1} \sum_{l=im+1}^{\infty} Z_{j,l}. \end{aligned}$$

Then we have

$$\begin{aligned} \Gamma_{Nm}^+ - qNm &= \sum_{i=1}^N (A_i - EA_i) - \sum_{i=1}^N (U_i - EU_i) + (Z_{Nm} - q), \\ \Gamma_{Nm}^{(2)} - q(1-q)Nm &= \sum_{i=1}^N (A_i^{(2)} - EA_i^{(2)}) \end{aligned}$$

$$-\sum_{i=1}^N (U_i^{(2)} - EU_i^{(2)}) + \sum_{i=1}^N (V_i^{(2)} - EV_i^{(2)}).$$

Firstly, we will show that both A part and $A^{(2)}$ part are dominant in comparison with the other parts. Jain and Pruitt [10] derived that

$$(4.3) \quad \text{Var} \left(\sum_{i=1}^N U_i \right) = \begin{cases} O(N^{\frac{3}{2}}m) & \text{Case 1} \\ N \times o(m) & \text{Case 2.} \end{cases}$$

Then we will estimate $\text{Var}(\sum U_i^{(2)})$ and $\text{Var}(\sum V_i^{(2)})$.

$$\begin{aligned} & \text{Var} \left(\sum_{i=1}^N U_i^{(2)} \right) \\ &= \sum_{i=1}^N \sum_{j=(i-1)m}^{im-1} \sum_{s=(i-1)m}^{im-1} \sum_{l=j+1}^{im} \sum_{h=s+1}^{im} \text{Cov}(W_{j,l}^{im}, W_{s,h}^{im}) \\ & \quad + 2 \sum_{j=2}^N \sum_{i=1}^{j-1} \sum_{r=(j-1)m}^{jm-1} \sum_{s=(i-1)m}^{im-1} \sum_{l=r+1}^{jm} \sum_{h=s+1}^{im} \text{Cov}(W_{r,l}^{jm}, W_{s,h}^{im}) \\ &= \text{I} + \text{II}. \end{aligned}$$

Generally if $X, Y \geq 0$, then $\text{Cov}(X, Y) \leq EXY$. Replacing each term in I with $EW_{j,l}^{im}W_{s,h}^{im}$, we can estimate I in the same way as Lemma 4.1, which yields that $\text{I} \leq CNm$ in Case 1 and $\text{I} \leq \varepsilon Nm$ for given $\varepsilon > 0$ and sufficiently large m in Case 2.

Now we estimate the term II. For $0 \leq a < b \leq c < d$, we define $W_{a,b}^{c,d} = Z_{a,b}^c - Z_{a,b}^d$, which is a non-negative indicator random variable. Since, for $0 \leq (i-1)m < s < h \leq im \leq (j-1)m \leq r < l < jm$, $W_{s,h}^{im} = W_{s,h}^{im,r} + W_{s,h}^r$, and $W_{r,l}^{jm}$ is independent of $W_{s,h}^{im,r}$, it holds that

$$\begin{aligned} \text{Cov}(W_{r,l}^{jm}, W_{s,h}^{im}) &= \text{Cov}(W_{r,l}^{jm}, W_{s,h}^r) \\ &= EW_{r,l}^{jm}W_{s,h}^{r,l} + (EW_{r,l}^{jm}W_{s,h}^l - EW_{r,l}^{jm}EW_{s,h}^r). \end{aligned}$$

Noting $\sum_x p^a(0, x)p^b(x, 0) = p^{a+b}(0, 0)$ for $a, b \geq 0$, we have

$$\begin{aligned} EW_{r,l}^{jm}W_{s,h}^{r,l} &\leq \sum_{k=1}^{l-r} \sum_x p^{h-s}(0, 0)p^{r-h}(0, x)p^k(x, 0)p^{l-r-k}(0, x)\bar{r}_{jm-l} \\ &\leq C(h-s)^{-\frac{3}{2}}(r-h)^{-\frac{3}{2}}(jm-l+1)^{-\frac{1}{2}} \sum_{k=1}^{l-r} \sum_x p^{l-r-k}(0, x)p^k(x, 0) \\ &\leq C(h-s)^{-\frac{3}{2}}(r-h)^{-\frac{3}{2}}(jm-l+1)^{-\frac{1}{2}}(l-r)^{-\frac{1}{2}}. \end{aligned}$$

Changing the order of summing over h and s ,

$$\begin{aligned}
& \sum_{j=2}^N \sum_{i=1}^{j-1} \sum_{r=(j-1)m}^{jm-1} \sum_{h=(i-1)m+1}^{im} \sum_{l=r+1}^{jm} \sum_{s=(i-1)m}^{h-1} EW_{r,l}^{jm} W_{s,h}^{r,l} \\
& \leq C \sum_{j=2}^N \sum_{r=(j-1)m}^{jm-1} \sum_{l=r+1}^{jm} \sum_{h=1}^{(j-1)m} (r-h)^{-\frac{3}{2}} (l-r)^{-\frac{1}{2}} (jm-l+1)^{-\frac{1}{2}} \\
& \leq CNm^{\frac{1}{2}}.
\end{aligned}$$

On the other hand, for $0 \leq (i-1)m < s < h \leq im \leq (j-1)m \leq r < l < jm$, we have

$$\begin{aligned}
EW_{r,l}^{jm} W_{s,h}^l &= \sum_{x \neq 0} p_0^{h-s}(0,0) p_0^{r-h}(0,x) p_{0x}^{l-r}(x,x) P_x(\tau_0 < \infty, jm-l < \tau_x < \infty) \\
&\leq f_{h-s} f_{l-r} \sum_{x \neq 0} p_0^{r-h}(0,x) P_x(\tau_0 < \tau_x, jm-l < \tau_x < \infty) \\
&\quad + f_{h-s} f_{l-r} \sum_{x \neq 0} p_0^{r-h}(0,x) P_x(\tau_0 < \infty) \bar{r}_{jm-l}, \\
EW_{r,l}^{jm} EW_{s,h}^r &= f_{h-s} f_{l-r} \bar{r}_{jm-l} \sum_{x \neq 0} p_0^{r-h}(0,x) P_x(\tau_0 < \infty).
\end{aligned}$$

(Case 1)

By Lemma 3.1 and Lemma 3.3, the second term of $\mathbb{II}$ has the following upper bound;

$$\begin{aligned}
& C \sum_{j=2}^N \sum_{l=(j-1)m+1}^{jm} \sum_{r=(j-1)m}^{l-1} \sum_{h=1}^{(j-1)m} f_{l-r} \sum_{x \neq 0} p^{r-h}(0,x) P_x(\tau_0 < \tau_x, jm-l < \tau_x < \infty) \\
& \leq C \sum_{j=2}^N \sum_{l=(j-1)m+1}^{jm} (jm-l)^{-\frac{1}{2}} (jm)^{\frac{1}{2}} \\
& \leq CN^{\frac{3}{2}} m.
\end{aligned}$$

(Case 2)

By the same argument as in Lemma 4.1, we have

$$\begin{aligned}
& \sum_{j=2}^N \sum_{i=1}^{j-1} \sum_{r=(j-1)m}^{jm-1} \sum_{h=(i-1)m}^{im-1} \sum_{l=r+1}^{jm} \sum_{s=(i-1)m}^{h-1} (EW_{r,l}^{jm} W_{s,h}^l - EW_{r,l}^{jm} EW_{s,h}^r) \\
& \leq C \sum_{j=2}^N \sum_{l=(j-1)m+1}^{jm} \sum_{r=(j-1)m}^{l-1} f_{l-r} \sum_x G(0,x) P_x(\tau_0 < \tau_x, jm-l < \tau_x < \infty)
\end{aligned}$$

$$\begin{aligned}
&\leq C \sum_{j=2}^N \sum_{l=0}^{m-1} \sum_x G(0, x) P_x(l/2 < \tau_0 < \infty) G(0, x) \\
&\quad + C \sum_{j=2}^N \sum_{l=0}^{m-1} \sum_x G(0, x) G(x, 0) P_0(l/2 < \tau_x < \infty) \\
&= N \times o(m),
\end{aligned}$$

where the bound for $P_x(\tau_0 < \tau_x, l < \tau_x < \infty)$ is derived by observing that it must be the case that either the transition from x to 0 or the one from 0 to x after arriving at 0 takes at least $j/2$ steps while the random walk reaches x starting at x . Hence

$$\mathbb{I} \leq \begin{cases} CN^{\frac{3}{2}}m & \text{Case 1} \\ \varepsilon Nm & \text{Case 2} \end{cases}$$

for some positive constant C in Case 1 and given $\varepsilon > 0$ and sufficiently large m in Case 2. Then we have

$$\text{Var} \left(\sum_{i=1}^N U_i^{(2)} \right) = \begin{cases} O(N^{\frac{3}{2}}m) & \text{Case 1} \\ N \times o(m) & \text{Case 2.} \end{cases}$$

Next we will show the $V^{(2)}$ part has the same estimate.

$$\begin{aligned}
&\text{Var} \left(\sum_{i=1}^N V_i^{(2)} \right) \\
&= \sum_{i=1}^N \sum_{j=(i-1)m}^{im-1} \sum_{s=(i-1)m}^{im-1} \sum_{l=im+1}^{\infty} \sum_{h=im+1}^{\infty} \text{Cov}(Z_{j,l}, Z_{s,h}) \\
&\quad + 2 \sum_{j=2}^N \sum_{i=1}^{j-1} \sum_{r=(j-1)m}^{jm-1} \sum_{s=(i-1)m}^{im-1} \sum_{l=jm+1}^{\infty} \text{Cov}(Z_{r,l}, Z_{s,r}) \\
&\quad + 2 \sum_{j=2}^N \sum_{i=1}^{j-1} \sum_{r=(j-1)m}^{jm-1} \sum_{s=(i-1)m}^{im-1} \sum_{l=jm+1}^{\infty} \text{Cov}(Z_{r,l}, Z_{s,l}) \\
&\quad + 2 \sum_{j=2}^N \sum_{i=1}^{j-1} \sum_{r=(j-1)m}^{jm-1} \sum_{s=(i-1)m}^{im-1} \sum_{l=jm+1}^{\infty} \sum_{h=im}^{r-1} \text{Cov}(Z_{r,l}, Z_{s,h}) \\
&\quad + 2 \sum_{j=2}^N \sum_{i=1}^{j-1} \sum_{r=(j-1)m}^{jm-1} \sum_{s=(i-1)m}^{im-1} \sum_{l=jm+1}^{\infty} \sum_{h=r+1}^{\infty} \text{Cov}(Z_{r,l}, Z_{s,h}) \\
&= \text{I} + \mathbb{I} + \text{III} + \text{IV} + \text{V}.
\end{aligned}$$

The first three terms can be estimated in the same manner as the $U^{(2)}$ part by choosing the order of the summation carefully and noting that each term is bounded by CNm for some suitable constant C in Case 1 and εNm for given $\varepsilon > 0$ and sufficiently large m in Case 2. The remaining two terms are different from the others.

The method of estimating the fourth term is quite similar to Lemma 3.5. in [4]. For $s < h < r < l$,

$$\begin{aligned} EZ_{r,l}Z_{s,h} &= \sum_{x \neq 0} p_0^{h-s}(0,0)p_0^{r-h}(0,x)p_{0x}^{l-r}(x,x)P_x(\tau_0, \tau_x = \infty) \\ &\leq f_{h-s}f_{l-r} \sum_{x \neq 0} p_0^{r-h}(0,x)P_x(\tau_0, \tau_x = \infty) \\ EZ_{r,l}EZ_{s,h} &= q^2 f_{h-s}f_{l-r} \\ &= f_{h-s}f_{l-r} \sum_{x \neq 0} p_0^{r-h}(0,x)P_x(\tau_0 = \infty)P_x(\tau_x = \infty) \end{aligned}$$

since $P_0(\tau_0 = \infty) = P_x(\tau_x = \infty) = \sum_{x \neq 0} p_0^k(0,x)P_x(\tau_0 = \infty)$ for any $k \geq 1$. By Lemma 3.4,

$$\text{Cov}(Z_{r,l}, Z_{s,h}) \leq f_{h-s}f_{l-r} \sum_{x \neq 0} p_0^{r-h}(0,x)b(x).$$

(Case 1)

By Lemma 3.1,

$$\begin{aligned} \text{IV} &\leq C \sum_{j=2}^N \sum_{i=1}^{j-1} \sum_{r=(j-1)m}^{jm-1} \sum_{s=(i-1)m}^{im-1} \sum_{h=s+1}^{r-1} (h-s)^{-\frac{3}{2}} (jm-r)^{-\frac{1}{2}} (r-h)^{-1} \\ &\leq C \sum_{j=2}^N \sum_{r=(j-1)m}^{jm-1} \sum_{s=0}^{(j-1)m-1} (r-s)^{-1} (jm-r)^{-\frac{1}{2}} \\ &\leq CNm^{\frac{1}{2}} \log(Nm). \end{aligned}$$

(Case 2)

For fixed s , we sum over h in (s, r) and take the summation on i . Then

$$\begin{aligned} \text{IV} &\leq C \sum_{j=2}^N \sum_{r=(j-1)m}^{jm-1} \sum_{s=0}^{(j-1)m-1} \sum_{h=s+1}^{r-1} (h-s)^{-\frac{3}{2}} (jm-r)^{-\frac{1}{2}} \\ &\quad \times \sum_x p^{r-h}(0,x)G(x,0)G(0,x). \end{aligned}$$

Change the order of summing over h and s after widening the summation on s . Hence we have

$$IV \leq C \sum_{j=2}^N \sum_{r=(j-1)m}^{jm-1} (jm-r)^{-\frac{1}{2}} \leq CNm^{\frac{1}{2}}.$$

The fifth term also requires Lemma 3.4 in Section 3; however, it is more complicated to estimate than the fourth term. For $s < r < l$, we have

$$\sum_{h=r+1}^{\infty} EZ_{r,l}Z_{s,h} = \sum_{h=r+1}^{l-1} EZ_{r,l}Z_{s,h} + \sum_{h=l+1}^{\infty} EZ_{r,l}Z_{s,h}$$

and since for $s < r < l < h$,

$$EZ_{r,l}Z_{s,h} = \sum_{x \neq 0} p_0^{r-s}(0, x) p_{0x}^{l-r}(x, x) p_{0x}^{h-l}(x, 0) P_x(\tau_0, \tau_x = \infty),$$

it holds that

$$\begin{aligned} \sum_{h=l+1}^{\infty} EZ_{r,l}Z_{s,h} &\leq qf_{l-r} \sum_{x \neq 0} p_0^{r-s}(0, x) P_x(\tau_0 < \infty, \tau_0 < \tau_x) \\ &= qf_{l-r} \sum_{x \neq 0} p_0^{r-s}(0, x) P_x(\tau_0 < \tau_x < \infty) \\ &\quad + qf_{l-r} \sum_{x \neq 0} p_0^{r-s}(0, x) P_x(\tau_0 < \infty, \tau_x = \infty). \end{aligned}$$

On the other hand,

$$\sum_{h=r+1}^{\infty} EZ_{r,l}EZ_{s,h} = q^2 f_{l-r} \sum_{x \neq 0} p_0^{r-s}(0, x) P_x(\tau_0 < \infty).$$

Hence we obtain

$$\begin{aligned} &\sum_{h=r+1}^{\infty} \text{Cov}(Z_{r,l}, Z_{s,h}) \\ &\leq \sum_{h=r+1}^{l-1} EZ_{r,l}Z_{s,h} \\ &\quad + qf_{l-r} \sum_{x \neq 0} p_0^{r-s}(0, x) P_x(\tau_0 < \tau_x < \infty) - qf_{l-r} \sum_{x \neq 0} p_0^{r-s}(0, x) b(x) \end{aligned}$$

$$\begin{aligned}
&\leq \sum_{h=r+1}^{l-1} \sum_x p^{r-s}(0, x) p^{h-r}(x, 0) p^{l-h}(0, x) \\
&\quad + q f_{l-r} \sum_{x \neq 0} p_0^{r-s}(0, x) P_x(\tau_0 < \tau_x < \infty) P_x(\tau_0 < \infty).
\end{aligned}$$

The second term of V can be computed easily. Lemma 3.1 shows that

$$\begin{aligned}
&\sum_{j=2}^N \sum_{r=(j-1)m}^{jm-1} \sum_{s=0}^{(j-1)m-1} \sum_{l=jm+1}^{\infty} f_{l-r} \\
&\quad \times \sum_{x \neq 0} p_0^{r-s}(0, x) P_x(\tau_0 < \tau_x < \infty) P_x(\tau_0 < \infty) \\
&\leq C \sum_{j=2}^N \sum_{r=(j-1)m}^{jm-1} (jm - r)^{-\frac{1}{2}} \\
&\leq CNm^{\frac{1}{2}}.
\end{aligned}$$

We can calculate the remainder term in the following fashion.

(Case 1)

To estimate the first term of V, we split up the sum on h into two parts.

$$\begin{aligned}
&\sum_{j=2}^N \sum_{r=(j-1)m}^{jm-1} \sum_{s=0}^{(j-1)m-1} \sum_{l=jm+1}^{\infty} \sum_{h=r+1}^{l-1} \sum_x p^{r-s}(0, x) p^{h-r}(x, 0) p^{l-h}(0, x) \\
&\leq C \sum_{j=2}^N \sum_{r=(j-1)m}^{jm-1} \sum_{s=0}^{(j-1)m-1} \sum_{h=jm+1}^{\infty} (h - r)^{-\frac{3}{2}} \sum_x p^{r-s}(0, x) G(0, x) \\
&\quad + C \sum_{j=2}^N \sum_{r=(j-1)m}^{jm-1} \sum_{l=jm+1}^{\infty} \sum_{h=r+1}^{jm} (l - h)^{-\frac{3}{2}} \sum_x G(0, x) p^{h-r}(x, 0) \\
&\leq CN^{\frac{3}{2}} m.
\end{aligned}$$

(Case 2)

In this case, there is no need for dividing the summation on h .

$$\begin{aligned}
&\sum_{j=2}^N \sum_{r=(j-1)m}^{jm-1} \sum_{s=0}^{(j-1)m-1} \sum_{l=jm+1}^{\infty} \sum_{h=r+1}^{l-1} \sum_x p^{r-s}(0, x) p^{h-r}(x, 0) p^{l-h}(0, x) \\
&\leq \sum_{j=2}^N \sum_{r=(j-1)m}^{jm-1} \sum_{s=0}^{(j-1)m-1} \sum_x p^{r-s}(0, x) G(x, 0) G(0, x)
\end{aligned}$$

$$\begin{aligned}
&\leq \sum_{j=2}^N \sum_{r=(j-1)m}^{jm-1} \sum_x \{G(0, x) - G_{r-(j-1)m}(0, x)\} G(x, 0) G(0, x) \\
&= N \times o(m).
\end{aligned}$$

Hence

$$(4.4) \quad \text{Var} \left(\sum_{i=1}^N V_i^{(2)} \right) = \begin{cases} O(N^{\frac{3}{2}}m) & \text{Case 1} \\ N \times o(m) & \text{Case 2.} \end{cases}$$

Next, we will show the central limit theorem for the two cases separately. For the sake of convenience, put $\Delta_n = A_n - A_n^{(2)}$.

(Case 1) $\sum_x G(0, x)^2 G(x, 0) = \infty$

In this case, $\text{Var } Q_n \sim n\psi(n)$ and $\psi(n) \rightarrow \infty$ as $n \rightarrow \infty$. Put

$$m = m_n = \left\lfloor \frac{n}{\psi(n)} \right\rfloor, \quad N = N_n = \left\lfloor \frac{n}{m} \right\rfloor.$$

Since $\psi(n)$ is slowly varying, $m_n \rightarrow \infty$ and also $N_n \sim \psi(n) \rightarrow \infty$ as $n \rightarrow \infty$. By (4.2), (4.3), and (4.4), for any $\varepsilon > 0$, we obtain

$$\begin{aligned}
&P \left[\left| (\Psi_{Nm} - q^2 Nm) - \sum_{i=0}^N (\Delta_i - E\Delta_i) \right| > \varepsilon \sqrt{n\psi(n)} \right] \\
&\leq \varepsilon^{-2} \{n\psi(n)\}^{-1} \text{Var} \left[\sum_{i=0}^N (U_i^{(2)} - V_i^{(2)} - U_i) + (Z_{Nm} - q) \right] \\
&= O \left\{ \frac{N^{\frac{3}{2}}m}{n\psi(n)} \right\} \\
&= O \left\{ \frac{1}{\sqrt{\psi(n)}} \right\} \rightarrow 0
\end{aligned}$$

as n tends to infinity. We need to observe the residual part $(\Psi_n - \Psi_{Nm} - q^2n + q^2Nm)/\sqrt{n\psi(n)}$. Noting $Nm \geq n - m$, it can be derived that

$$P(|\Psi_n - \Psi_{Nm} - q^2n + q^2Nm| > \varepsilon \sqrt{n\psi(n)}) \leq \frac{\text{Var } \Psi_{n-Nm}}{\varepsilon^2 n\psi(n)} \leq \frac{Cm\psi(m)}{n\psi(n)}.$$

Since

$$\frac{m\psi(m)}{n\psi(n)} \sim \frac{n\psi(m)}{n\{\psi(n)\}^2} \leq \frac{1}{\psi(n)},$$

the right hand side converges to 0 as $n \rightarrow \infty$. Here we have used the property that $\psi(n)$ is non-decreasing. We have thus shown that

$$\text{Var} \left[\sum_{i=1}^N (U_i^{(2)} - V_i^{(2)} - U_i) \right], \text{Var} (\Psi_n - \Psi_{Nm}) = o\{n\psi(n)\}.$$

Recalling $E|Q_n - \Psi_n|^2 = O(n)$, we obtain

$$\text{Var} \left(\sum_{i=1}^N \Delta_i \right) \sim n\psi(n).$$

It still remains to check the Lindeberg-Feller condition. Δ_i are independent and each of their distributions coincides with that of $Q_m - 1$.

$$\begin{aligned} \sum_{i=1}^N E \left[\left(\frac{\Delta_i - E\Delta_i}{\sqrt{n\psi(n)}} \right)^2 ; |\Delta_i - E\Delta_i| > \varepsilon \sqrt{n\psi(n)} \right] \\ \leq \frac{N}{\varepsilon^2 n^2 \{\psi(n)\}^2} E|\Delta_i - E\Delta_i|^4 \\ \leq \frac{C\{\psi(m)\}^2}{\varepsilon^2 \{\psi(n)\}^3} \rightarrow 0 \end{aligned}$$

as $n \rightarrow \infty$ by Lemma 4.3. This finishes Case 1.

(Case 2) $\sum_x G(0, x)^2 G(x, 0) < \infty$

In this case, $\psi(n) \rightarrow \mu^2$ for some positive constant μ^2 and then $\text{Var } Q_n \sim \mu^2 n$. We now pick $m = m_n$ in such a way that $m \rightarrow \infty$ and $m/n \rightarrow 0$ as $n \rightarrow \infty$ and $N = [n/m]$. Hence we can obtain

$$\text{Var} \left[\sum_{i=0}^N (U_i^{(2)} - V_i^{(2)} - U_i) \right] = N \times o(m) = o(n),$$

which yields that

$$P \left[\left| (\Psi_{Nm} - q^2 Nm) - \sum_{i=1}^N (\Delta_i - E\Delta_i) \right| > \varepsilon \sqrt{n} \right] = o(1).$$

The residual part is negligible since $m/n \rightarrow 0$ as $n \rightarrow \infty$. Remembering $E|Q_n - \Psi_n|^2 = o(n)$, we see that these also show that

$$\text{Var} \left[\sum_{i=1}^N \Delta_i \right] \sim \mu^2 n.$$

By the same argument as in Case 1,

$$\begin{aligned} \sum_{i=1}^N E \left[\left(\frac{\Delta_i - E\Delta_i}{\sqrt{n}} \right)^2 ; |\Delta_i - E\Delta_i| > \varepsilon\sqrt{n} \right] &\leq \frac{N}{\varepsilon^2 n^2} E|\Delta_i - E\Delta_i|^4 \\ &\leq \frac{Cm}{\varepsilon^2 n} \longrightarrow 0 \end{aligned}$$

as $n \rightarrow \infty$. This shows the theorem for Case 2. $\blacksquare$

5. The variance of Q_n when $d = 2$

In calculating $\text{Var } Q_n$ for $d = 2$, we must split it up into many terms and derive the main part. The following lemmas are used in [9] or can be obtained similarly.

LEMMA 5.1 (Jain - Pruitt [9]). For $x \neq 0$ and $m \geq 1$, let

$$b_m(x) = P_x(\tau_0, \tau_x > m) - P_x(\tau_x > m)P_x(\tau_0 > m).$$

Then

$$b_m(x) = \sum_{j=1}^m p_0^j(x, 0) \left\{ - \sum_{i=m-j+1}^m f_i + \sum_{i=1}^{m-j} p_x^i(0, x) P_x(\tau_0, \tau_x > m - j - i) \right\}.$$

Observing its proof, we immediately have

LEMMA 5.2. Let $x \neq 0$ and $m \geq 1$. Then

$$\begin{aligned} P_0(\tau_x < \tau_0 \leq m) \\ = \sum_{j=1}^m \sum_{i=1}^{m-j} p_0^j(x, 0) p_x^i(0, x) \{ r_{m-j-i} + P_0(\tau_x < \tau_0 \leq m - j - i) \}. \end{aligned}$$

Classifying $\{\tau_0 \leq m\}$ by the last hitting time to 0 before time m , we can easily obtain

LEMMA 5.3. For $x \neq 0$,

$$P_x(\tau_0 \leq m < \tau_x) = \sum_{k=1}^m p_x^k(x, 0) P_0(\tau_x, \tau_0 > m - k).$$

LEMMA 5.4. For $x \neq 0$ and $\gamma \geq 0$, it holds that

$$\sum_{k=1}^m p_{0x}^k(0, x) r_{m-k}^\gamma = \sum_{k=1}^m p_x^k(0, x) r_{m-k}^{\gamma+1}$$

$$\begin{aligned}
& + \sum_{k=1}^m p_x^k(0, x) P_0(\tau_x < \tau_0 \leq m-k) r_{m-k}^\gamma \\
& - \sum_{k=1}^m \sum_{j=1}^{m-k} p_x^k(0, x) p_{0x}^j(0, 0) (r_{m-k-j}^\gamma - r_{m-k}^\gamma).
\end{aligned}$$

PROOF. By considering the first return to 0, we have

$$p_x^k(0, x) = p_{0x}^k(0, x) + \sum_{j=1}^{k-1} p_{0x}^j(0, 0) p_x^{k-j}(0, x),$$

and so

$$\sum_{k=1}^m p_x^k(0, x) r_{m-k}^\gamma = \sum_{k=1}^m p_{0x}^k(0, x) r_{m-k}^\gamma + \sum_{k=1}^m \sum_{j=1}^{k-1} p_{0x}^j(0, 0) p_x^{k-j}(0, x) r_{m-k}^\gamma.$$

Making the substitution $i = k - j$, the second part of the right hand side can be calculated as follows;

$$\begin{aligned}
\sum_{j=1}^{m-1} \sum_{i=1}^{m-j} p_{0x}^j(0, 0) p_x^i(0, x) r_{m-i-j}^\gamma &= \sum_{i=1}^{m-1} \sum_{j=1}^{m-i} p_x^i(0, x) p_{0x}^j(0, 0) r_{m-i}^\gamma \\
&+ \sum_{i=1}^{m-1} \sum_{j=1}^{m-i} p_x^i(0, x) p_{0x}^j(0, 0) (r_{m-i-j}^\gamma - r_{m-i}^\gamma),
\end{aligned}$$

which implies the assertion of the lemma. $\blacksquare$

We have another expression, that is,

$$\begin{aligned}
(5.1) \quad \sum_{k=1}^m p_{0x}^k(0, x) r_{m-k}^\gamma &= \sum_{k=1}^m p_x^k(0, x) r_{m-k}^{\gamma+1} \\
&- \sum_{k=1}^m \sum_{j=1}^{m-k} p_x^k(0, x) f_j(r_{m-k-j}^\gamma - r_{m-k}^\gamma) \\
&+ \sum_{k=1}^m \sum_{j=1}^{m-k} p_x^k(0, x) P_0(\tau_x < \tau_0 = j) r_{m-k-j}^\gamma.
\end{aligned}$$

The following two lemmas play an important role in estimating negligible parts.

LEMMA 5.5. For $x \neq 0$ and $\gamma \geq 0$,

$$\sum_{k=1}^m p_0^k(0, x) r_{m-k}^\gamma = \sum_{k=1}^m p^k(0, x) r_{m-k}^{\gamma+1} - \sum_{k=1}^m p^k(0, x) \sum_{j=1}^{m-k} f_j(r_{m-k-j}^\gamma - r_{m-k}^\gamma)$$

$$\leq \sum_{k=1}^m p^k(0, x) r_{m-k}^{\gamma+1}.$$

The proof of Lemma 5.5 is similar to that of Lemma 5.4 and given in [9]. For $n \geq 1$, $\alpha, \beta \geq 0$, and $h, k \geq 1$ satisfying $h + k \leq n$, let

$$T_{h,k}(n) = \sum_{\substack{1 \leq i_1 + i_2 + \dots + i_h \\ + j_1 + \dots + j_k \leq n \\ x \in \mathbb{Z}^2}} \prod_{\mu=1}^h p^{i_\mu}(0, x) \prod_{\rho=1}^k p^{j_\rho}(x, 0),$$

$$T_{h,k}^{\alpha,\beta}(n) = \sum_{\substack{1 \leq l + i_1 + i_2 + \dots + i_h \\ + j_1 + \dots + j_k \leq n \\ x \neq 0}} \prod_{\mu=1}^h p_x^{i_\mu}(0, x) \prod_{\rho=1}^k p_0^{j_\rho}(x, 0) r_l^\alpha r_{n - \sum_{\xi=1}^h i_\xi - \sum_{\eta=1}^k j_\eta - l}^\beta.$$

LEMMA 5.6. *It holds that*

$$T_{h,k}(n) = O(n), \quad T_{h,k}^{\alpha,\beta}(n) = O(n^2 r_n^{\alpha+\beta+h+k}).$$

PROOF. Firstly we will estimate $T_{h,k}(n)$. Since it is symmetric in $i_1, i_2, \dots, i_h$ and in $j_1, j_2, \dots, j_k$, the bound of $T_{h,k}(n)$ is

$$\begin{aligned} & C \sum_{\substack{1 \leq i_1 \leq i_2 \leq \dots \leq i_h \leq n \\ \dots \leq i_h \leq n}} \sum_{\substack{1 \leq j_1 \leq j_2 \leq \dots \leq j_k \leq n \\ \dots \leq j_k \leq n}} \sum_x p^{i_1}(0, x) p^{j_1}(x, 0) \prod_{\mu=2}^h i_\mu^{-1} \prod_{\rho=2}^k j_\rho^{-1} \\ & \leq C \sum_{i=1}^n \sum_{j=1}^n p^{i+j}(0, 0) \left(\log \frac{en}{i} \right)^{h-1} \left(\log \frac{en}{j} \right)^{k-1} \\ & \leq C \sum_{l=1}^{2n} \sum_{j=1}^{l-1} l^{-1} \left(\log \frac{en}{l-j} \right)^{h-1} \left(\log \frac{en}{j} \right)^{k-1} \\ & \leq C \sum_{l=1}^{2n} \left(\log \frac{e^2 n}{l} \right)^{h+k-2} = O(n). \end{aligned}$$

We will next compute the order of $T_{h,k}^{\alpha,\beta}(n)$. Since r_n is slowly varying and non-increasing, it holds that

$$\sum_{l=1}^m r_l^\alpha r_{m-l}^\beta \leq C m r_m^{\alpha+\beta}.$$

Then we have

$$T_{h,k}^{\alpha,\beta}(n) \leq C n r_n^{\alpha+\beta+h+k} T_{h,k}(n),$$

where Lemma 5.5 was used $(h+k)$ times at first, the summation on l was taken in the second step, and lastly Lemma 2.2 was applied. Hence we can conclude that $T_{h,k}^{\alpha,\beta}(n)$ is of order $n^2 r_n^{\alpha+\beta+h+k}$. ■

From now on, we will assume the following property.

ASSUMPTION [A]. The random walk is aperiodic with $EX_1 = 0$ and $E|X_1|^2 < \infty$.

Under this assumption, it holds that

$$r_n \sim \frac{c_1}{\log n}$$

by Jain and Pruitt [7], where $c_1 = 2\pi(\det \Xi)$, Ξ is the symmetric matrix satisfying $E(\theta, X_1)^2 = (\theta, \Xi^2 \theta)$ for any $\theta \in \mathbb{R}^2$, and $(\cdot, \cdot)$ is the standard inner product on $\mathbb{R}^2$.

To obtain the limit variance of Q_n , the asymptotic behavior of f_n is crucial, which Jain and Pruitt calculated by using the local limit theorem.

LEMMA 5.7 (Jain-Pruitt [9]). *Assume the property [A]. Then*

$$f_n \sim \frac{c_1}{n(\log n)^2}.$$

Employing this lemma, Jain and Pruitt derived that

$$(5.2) \quad \sum_{k=1}^m f_k(r_{m-k}^\gamma - r_m^\gamma) = O(r_m^{\gamma+2}).$$

When $\gamma = 1$, we can sharpen this estimate.

LEMMA 5.8. *Under Assumption [A],*

$$\sum_{k=1}^m f_k(r_{m-k} - r_m) = O(r_m^4).$$

PROOF. By Lemma 5.7, for $1 \leq k \leq m/2$,

$$r_{m-k} - r_m \leq \sum_{l=m-k+1}^m \frac{C}{l(\log l)^2} \leq \frac{Ck}{m(\log m)^2}$$

and so

$$\sum_{k=2}^{m/2} f_k(r_{m-k} - r_m) \leq \sum_{k=1}^{m/2} \frac{C}{m(\log k)^2(\log m)^2} \leq \frac{C}{(\log m)^4}.$$

On the other hand,

$$\sum_{k=m/2}^m f_k(r_{m-k} - r_m) = \sum_{k=m/2}^m \sum_{l=m-k+1}^m f_k f_l \leq \frac{C}{(\log m)^4}. \quad \blacksquare$$

Using the local limit theorem again, we have

LEMMA 5.9 (Jain-Pruitt [9]). *Under Assumption [A], it holds*

$$\lim_{n \rightarrow \infty} \sum_{\substack{i+j+k=n \\ x \neq 0}} p^i(0, x) p^j(x, 0) p^k(0, x) = 2c_1^{-2} K_1,$$

where

$$K_1 = - \int_0^1 \frac{\log x}{1-x+x^2} dx = 1.17195361935 \dots$$

The main result in this section is

THEOREM 5.10. *For an aperiodic two dimensional random walk with mean 0 and finite variance, it holds that*

$$\text{Var } Q_n \sim \frac{Ln^2}{(\log n)^6},$$

where

$$L = 16\pi^4 \left(8K_1 - \frac{2}{3}\pi^2 + 4 \right) (\det \Xi)^4.$$

PROOF. Recall the notation Y_j^i and Z_j^j in Section 3. Since we can express that $Q_n = \sum_{j=0}^n Y_j^0 Z_j^n$,

$$\text{Var } Q_n = \sum_{j=0}^n \text{Var } Y_j^0 Z_j^n + 2 \sum_{j=1}^n \sum_{i=0}^{j-1} \text{Cov}(Y_i^0 Z_i^n, Y_j^0 Z_j^n).$$

The first sum is $O(n)$, so we will estimate the second term. By the standard argument, it is easy to see that $EY_j^0 Z_j^n = r_j r_{n-j}$ and that for $i < j$,

$$EY_i^0 Z_i^n Y_j^0 Z_j^n = \sum_{x \neq 0} p_{0x}^{j-i}(0, x) P_x(\tau_0, \tau_x > i) P_x(\tau_0, \tau_x > n-j).$$

By the definition of $b_m(x)$,

$$\begin{aligned}
 (5.3) \quad & EY_i^0 Z_i^n Y_j^0 Z_j^n \\
 &= \sum_{x \neq 0} p_{0x}^{j-i}(0, x) P_x(\tau_0 > n-j) r_{n-j} r_i \\
 &\quad - \sum_{x \neq 0} p_{0x}^{j-i}(0, x) P_x(\tau_0 > n-j) r_{n-j} P_x(\tau_0 \leq i < \tau_x) \\
 &\quad + \sum_{x \neq 0} p_{0x}^{j-i}(0, x) b_{n-j}(x) P_x(\tau_0, \tau_x > i).
 \end{aligned}$$

Using (5.1), the summation on i and j of the first term of (5.3) is equal to

$$\begin{aligned}
 & \sum_{j=1}^n \sum_{i=0}^{j-1} \sum_{x \neq 0} p_x^{j-i}(0, x) P_x(\tau_0 > n-j) r_{n-j} r_i^2 \\
 & \quad - \sum_{j=1}^n \sum_{i=0}^{j-1} \sum_{x \neq 0} p_x^{j-i}(0, x) P_x(\tau_0 > n-j) r_{n-j} \sum_{l=1}^i f_l(r_{i-l} - r_i) \\
 & \quad + \sum_{j=1}^n \sum_{i=0}^{j-1} \sum_{x \neq 0} p_x^{j-i}(0, x) r_{n-j} \sum_{l=1}^i P_0(\tau_x < \tau_0 = l) r_{i-l} \\
 & \quad - \sum_{j=1}^n \sum_{i=0}^{j-1} \sum_{x \neq 0} p_x^{j-i}(0, x) r_{n-j} P_x(\tau_0 \leq n-j) \sum_{l=1}^i P_0(\tau_x < \tau_0 = l) r_{i-l} \\
 &= \text{I} - \text{II} + \text{III} - \text{IV}.
 \end{aligned}$$

The equality $p_x^k(0, x) = p_0^k(0, x)$ for every $k \geq 1$ shows that

$$\begin{aligned}
 \text{I} &= \sum_{j=1}^n \sum_{i=0}^{j-1} r_{n-i} r_{n-j} r_i^2, \\
 \text{II} &= \sum_{j=1}^n \sum_{i=0}^{j-1} r_{n-i} r_{n-j} \sum_{l=1}^i f_l(r_{i-l} - r_i).
 \end{aligned}$$

Since the summation on l is bounded by 1 and $r_{n-i} \leq 1$, the contribution for $n-j \leq n(\log n)^{-7}$ in II is $O\{n^2(\log n)^{-7}\}$, and when $n-j > n(\log n)^{-7}$, we can replace r_{n-j} with r_n by noting that $r_n \sim c_1(\log n)^{-1}$. Similarly r_{n-i} can be replaced with r_n . Then we have

$$\text{II} \sim r_n^2 \sum_{j=1}^n \sum_{i=0}^{j-1} \sum_{l=1}^i f_l(r_{i-l} - r_i)$$

$$\begin{aligned}
&= r_n^2 \sum_{i=1}^n \sum_{l=0}^{i-1} (n-i) f_{i-l}(r_l - r_i) \\
&\sim \frac{c_1^4}{(\log n)^2} \sum_{i=2}^{n-1} \sum_{l=2}^{i-1} \frac{(n-i) \log \frac{i}{l}}{(i-l) \{\log(i-l)\}^2 \log l \log i}.
\end{aligned}$$

Therefore we obtain

$$\text{II} \sim \frac{c_1^4 n^2}{(\log n)^6} \int_0^1 dx \int_0^x dy \frac{1-x}{x-y} \log \frac{x}{y} = \frac{\pi^2}{12} \frac{c_1^4 n^2}{(\log n)^6}.$$

Changing the order of summing over i and j and making the substitution $k = j - i$, we have

$$\begin{aligned}
\text{III} &= \sum_{i=0}^{n-1} \sum_{k=1}^{n-i} \sum_{x \neq 0} p_x^k(0, x) r_{n-k-i} \sum_{h=1}^i \sum_{l=1}^{h-1} p_{0x}^l(0, x) p_0^{h-l}(x, 0) r_{i-h} \\
&= \sum_{i=0}^{n-1} \sum_{k=1}^{n-i} \sum_{x \neq 0} p_x^k(0, x) r_{n-k-i} \sum_{h=1}^i \sum_{l=1}^{i-h} p_{0x}^l(0, x) p_0^h(x, 0) r_{i-h-l}.
\end{aligned}$$

Lemma 5.4 is applied to the summation on l . Thus

$$\begin{aligned}
\text{III} &= \sum_{k=0}^{n-1} \sum_{i=1}^{n-k} \sum_{x \neq 0} p_x^k(0, x) r_{n-k-i} \sum_{h=1}^i \sum_{l=1}^{i-h} p_x^l(0, x) p_0^h(x, 0) r_{i-h-l}^2 \\
&\quad + \sum_{k=0}^{n-1} \sum_{i=1}^{n-k} \sum_{x \neq 0} p_x^k(0, x) r_{n-k-i} \\
&\quad \quad \times \sum_{h=1}^i \sum_{l=1}^{i-h} p_x^l(0, x) p_0^h(x, 0) P_0(\tau_x < \tau_0 \leq i - h - l) r_{i-h-l} \\
&\quad - \sum_{k=0}^{n-1} \sum_{i=1}^{n-k} \sum_{x \neq 0} p_x^k(0, x) r_{n-k-i} \\
&\quad \quad \times \sum_{h=1}^i \sum_{l=1}^{i-h} \sum_{r=1}^{i-h-l} p_x^l(0, x) p_0^h(x, 0) p_{0x}^r(0, 0) (r_{i-h-l-r} - r_{i-h-l}).
\end{aligned}$$

By Lemma 5.6 and Lemma 5.8, the absolute value of the third part is bounded by

$$C \sum_{\substack{1 \leq k+i \leq n \\ 1 \leq l+h \leq i \\ x \neq 0}} p_x^k(0, x) p_x^l(0, x) p_0^h(x, 0) r_{n-k-i} r_{i-h-l}^4 \leq CT_{2,1}^{4,1}(n) = O(n^2 r_n^8).$$

By Lemma 5.2, the second part of $\mathbb{III}$ is not larger than

$$2 \sum_{\substack{1 \leq k+i \leq n \\ 1 \leq h+l+r+s \leq i \\ x \neq 0}} p_x^k(0, x) p_x^l(0, x) p_0^h(x, 0) p_0^r(x, 0) p_x^s(0, x) r_{n-k-i} r_{i-h-l},$$

which is bounded by $2T_{3,2}^{1,1}(n) = O(n^2 r_n^7)$. Hence

$$\mathbb{III} = \sum_{\substack{1 \leq k+i \leq n \\ 1 \leq l+h \leq i \\ x \neq 0}} p_x^k(0, x) p_x^l(0, x) p_0^h(x, 0) r_{n-k-i} r_{i-h-l}^2 + O(n^2 r_n^7).$$

Combining (5.2) with Lemma 5.5, we have

$$\sum_{k=1}^m p_0^k(0, x) r_{m-k}^\gamma = \sum_{k=1}^m p^k(0, x) \{r_{m-k}^{\gamma+1} + O(r_{m-k}^{\gamma+2})\}.$$

Therefore

$$\mathbb{III} \sim \sum_{\substack{0 \leq i+k+h+l \leq n \\ x \neq 0}} p^k(0, x) p^l(0, x) p^h(x, 0) r_{n-k-i-h-l}^2 r_i^4.$$

Lemma 5.9 implies that $\mathbb{III}$ is asymptotically equal to $c_1^4 K_1 n^2 (\log n)^{-6}$. We can show similarly that the term $\mathbb{IV}$ is of order $n^2 r_n^7$ by noting that $P_x(\tau_0 \leq m) = \sum_{k=1}^m p_0^k(x, 0)$.

Next we will estimate the summation of the second term of (5.3). By Lemma 5.3 and Lemma 5.4, it is equal to

$$\begin{aligned} & \sum_{i=0}^{n-1} \sum_{j=1}^{n-i} \sum_{x \neq 0} p_{0x}^j(0, x) r_{n-j-i} P_x(\tau_0 \leq i < \tau_x) \\ & - \sum_{j=1}^{n-1} \sum_{i=1}^j \sum_{x \neq 0} p_{0x}^i(0, x) P_x(\tau_0 \leq n-j) r_{n-j} P_x(\tau_0 \leq j-i < \tau_x) \\ & = \sum_{i=0}^{n-1} \sum_{j=1}^{n-i} \sum_{x \neq 0} p_x^j(0, x) r_{n-j-i}^2 P_x(\tau_0 \leq i < \tau_x) \\ & + \sum_{i=0}^{n-1} \sum_{j=1}^{n-i} \sum_{x \neq 0} p_x^j(0, x) r_{n-j-i} P_0(\tau_x < \tau_0 \leq n-j-i) P_x(\tau_0 \leq i < \tau_x) \\ & - \sum_{i=0}^{n-1} \sum_{j=1}^{n-i} \sum_{x \neq 0} p_x^j(0, x) P_x(\tau_0 \leq i < \tau_x) \sum_{k=1}^{n-j-i} p_{0x}^k(0, 0) (r_{n-j-i-k} - r_{n-j-i}) \end{aligned}$$

$$\begin{aligned}
& - \sum_{j=1}^{n-1} \sum_{i=1}^j \sum_{x \neq 0} p_{0x}^i(0, x) P_x(\tau_0 \leq n-j) r_{n-j} \sum_{k=1}^{j-i} p_x^k(x, 0) r_{j-i-k} \\
& + \sum_{j=1}^{n-1} \sum_{i=1}^j \sum_{x \neq 0} p_{0x}^i(0, x) P_x(\tau_0 \leq n-j) r_{n-j} \\
& \quad \times \sum_{k=1}^{j-i} p_x^k(x, 0) \sum_{h=1}^{j-i-k} p_0^h(0, x) P_x(\tau_0, \tau_x > j-i-k-h) \\
& = \text{I}' + \text{II}' - \text{III}' - \text{IV}' + \text{V}'.
\end{aligned}$$

Using Lemma 5.3 again,

$$\begin{aligned}
\text{I}' &= \sum_{i=0}^{n-1} \sum_{j=1}^{n-i} \sum_{x \neq 0} p_x^j(0, x) r_{n-j-i}^2 \sum_{k=1}^i p_x^k(x, 0) \{r_{i-k} - P_0(\tau_x \leq i-k < \tau_0)\} \\
&= \sum_{i=0}^{n-1} \sum_{j=1}^{n-i} \sum_{k=1}^i \sum_{x \neq 0} p_x^j(0, x) p_0^k(x, 0) r_{n-j-i}^2 r_{i-k} \\
&\quad - \sum_{i=0}^{n-1} \sum_{j=1}^{n-i} \sum_{k=1}^i \sum_{x \neq 0} p_x^j(0, x) p_0^k(x, 0) r_{n-j-i}^2 \sum_{h=1}^{i-k} p_0^h(0, x) r_{i-k-h} \\
&\quad + \sum_{i=0}^{n-1} \sum_{j=1}^{n-i} \sum_{k=1}^i \sum_{x \neq 0} p_x^j(0, x) p_0^k(x, 0) r_{n-j-i}^2 \sum_{h=1}^{i-k} p_0^h(0, x) P_x(\tau_0 \leq i-k-h < \tau_x).
\end{aligned}$$

The second part can be calculated in the same way as the first part of III and is asymptotically equal to $-c_1^4 K_1 n^2 (\log n)^{-6}$. Using Lemma 5.3 and dominating $P_0(\tau_0, \tau_x > m-k)$ by r_{m-k} , we have that a bound of the third term is

$$\sum_{\substack{1 \leq i+j \leq n \\ 1 \leq k+h+l \leq i \\ x \neq 0}} p_x^j(0, x) p_0^k(x, 0) p_0^h(0, x) r_{n-j-i}^2 p_x^l(x, 0) r_{i-k-h-l} \leq T_{2,2}^{1,2}(n),$$

which is $O(n^2 r_n^7)$. Hence we have

$$\text{I}' = \sum_{i=0}^{n-1} \sum_{j=1}^{n-i} \sum_{k=1}^i f_{j+k} r_{n-j-i}^2 r_{i-k} - K_1 \frac{c_1^4 n^2}{(\log n)^6} + o\left\{\frac{n^2}{(\log n)^6}\right\}.$$

Lemma 5.2 and Lemma 5.3 are applied to compute II' . Then

$$\text{II}' \leq \sum_{\substack{1 \leq i+j+l+k \leq n \\ x \neq 0}} p_x^j(0, x) r_{n-j-i} p_0^l(x, 0) p_x^k(0, x) r_{n-j-i-l-k} \sum_{h=1}^i p_0^h(x, 0) r_{i-h}$$

$$\begin{aligned}
& + \sum_{\substack{1 \leq i+j+l+k \leq n \\ x \neq 0}} p_x^j(0, x) r_{n-j-i} p_0^l(x, 0) p_x^k(0, x) \\
& \quad \times P_0(\tau_x < \tau_0 \leq n - j - i - l - k) \sum_{h=1}^i p_0^h(x, 0) r_{i-h}.
\end{aligned}$$

Since r_n is non-decreasing, especially $r_{n-j-i} \leq r_{n-j-i-l-k}$, the first part is $O(n^2 r_n^7)$. Using Lemma 5.2 again, we have a bound for the second part by $T_{3,3}^{1,1}(n) = O(n^2 r_n^8)$. The estimate of III' requires Lemma 5.8. By Lemma 5.3,

$$\begin{aligned}
\text{III}' & \leq C \sum_{i=0}^{n-1} \sum_{j=1}^{n-i} \sum_{x \neq 0} p_x^j(0, x) r_{n-j-i}^4 \sum_{h=1}^i p_x^h(x, 0) r_{i-h} \\
& = C \sum_{1 \leq i+j+h \leq n} f_{j+h} r_{n-j-i-h}^4 r_i \\
& \leq C n^2 r_n^7.
\end{aligned}$$

Summing first on i and applying Lemma 5.4, we have

$$\begin{aligned}
\text{IV}' & = \sum_{j=1}^n \sum_{k=0}^{j-1} \sum_{i=1}^{j-k} \sum_{x \neq 0} p_x^i(0, x) P_x(\tau_0 \leq n - j) r_{n-j} p_x^k(x, 0) r_{j-i-k}^2 \\
& \quad + \sum_{j=1}^n \sum_{k=0}^{j-1} \sum_{i=1}^{j-k} \sum_{x \neq 0} p_x^i(0, x) P_x(\tau_0 \leq n - j) r_{n-j} \\
& \quad \quad \times p_x^k(x, 0) P_0(\tau_x < \tau_0 \leq j - i - k) r_{j-i-k} \\
& \quad - \sum_{j=1}^n \sum_{k=0}^{j-1} \sum_{i=1}^{j-k} \sum_{x \neq 0} p_x^i(0, x) P_x(\tau_0 \leq n - j) r_{n-j} \\
& \quad \quad \times p_x^k(x, 0) \sum_{h=1}^{j-i-k} p_{0x}^h(0, 0) (r_{j-i-k-h} - r_{j-i-k}).
\end{aligned}$$

Using Lemma 5.2 and Lemma 5.6, we can prove the second part is of order $n^2 r_n^7$, and Lemma 5.8 shows that the third part is $O(n^2 r_n^7)$. The estimate of the first part of IV' is as follows.

$$\begin{aligned}
& \sum_{\substack{1 \leq j+h \leq n \\ 1 \leq i+k \leq j \\ x \neq 0}} p_x^i(0, x) p_0^h(x, 0) p_x^k(x, 0) r_{n-j} r_{j-i-k}^2 \\
& \sim r_n^2 \sum_{\substack{1 \leq j+h \leq n \\ 1 \leq i+k \leq j \\ x \neq 0}} p^i(0, x) p^h(x, 0) p^k(x, 0) r_{j-i-k}^4
\end{aligned}$$

$$\begin{aligned}
&\sim r_n^2 \sum_{\substack{1 \leq i+k+h+k \leq n \\ x \neq 0}} p^i(0, x) p^h(x, 0) p^k(x, 0) r_j^4 \\
&\sim K_1 \frac{c_1^4 n^2}{(\log n)^6}.
\end{aligned}$$

The bound of V' can be obtained in the same fashion by employing Lemma 5.5 and Lemma 5.6, and it is $O(n^2 r_n^7)$.

Now we will calculate the summation of the third term of (5.3). By Lemma 5.3 and Lemma 5.8, it is equal to

$$\begin{aligned}
&\sum_{j=1}^n \sum_{i=0}^{j-1} \sum_{x \neq 0} p_{0x}^{j-i}(0, x) b_{n-j}(x) r_i \\
&\quad - \sum_{j=1}^n \sum_{i=0}^{j-1} \sum_{x \neq 0} p_{0x}^{j-i}(0, x) b_{n-j}(x) P_x(\tau_0 \leq i < \tau_x) \\
&= \sum_{j=1}^n \sum_{i=1}^j \sum_{x \neq 0} p_x^i(0, x) b_{n-j}(x) r_{j-i}^2 \\
&\quad + \sum_{j=1}^n \sum_{i=1}^j \sum_{x \neq 0} p_x^i(0, x) b_{n-j}(x) P_0(\tau_x < \tau_0 \leq j-i) r_{j-i} \\
&\quad - \sum_{j=1}^n \sum_{i=1}^j \sum_{x \neq 0} p_x^i(0, x) b_{n-j}(x) \sum_{l=1}^{j-i} p_{0x}^l(0, 0) (r_{j-i-l} - r_{j-i}) \\
&\quad - \sum_{j=1}^n \sum_{i=1}^j \sum_{x \neq 0} p_{0x}^i(0, x) b_{n-j}(x) \sum_{l=1}^{j-i} p_x^l(x, 0) P_x(\tau_0, \tau_x > j-i-l) \\
&= \text{I}'' + \text{II}'' - \text{III}'' - \text{IV}'' .
\end{aligned}$$

Replacing r_i with r_n , we see that Theorem 4.2 in [9] implies that I'' is asymptotically equal to

$$\left(K_1 + \frac{1}{2} - \frac{\pi^2}{12} \right) \frac{c_1^4 n^2}{(\log n)^6}.$$

By Lemma 5.1, we have

$$\begin{aligned}
\text{II}'' &= \sum_{j=1}^n \sum_{i=1}^{n-j} \sum_{x \neq 0} p_x^i(0, x) b_j(x) r_{n-j-i} P_0(\tau_x < \tau_0 \leq n-j-i) \\
&= - \sum_{j=1}^n \sum_{i=1}^{n-j} \sum_{k=1}^j \sum_{x \neq 0} p_x^i(0, x) p_0^k(x, 0) r_{n-j-i} (r_{j-k} - r_j) P_0(\tau_x < \tau_0 \leq n-j-i)
\end{aligned}$$

$$\begin{aligned}
& + \sum_{j=1}^n \sum_{i=1}^{n-j} \sum_{k=1}^j \sum_{h=1}^{k-j} \sum_{x \neq 0} p_x^i(0, x) p_0^k(x, 0) p_0^h(x, 0) r_{n-j-i} \\
& \quad \times P_0(\tau_x < \tau_0 \leq n-j-i) P_x(\tau_0, \tau_x > j-k-h).
\end{aligned}$$

The absolute value of the first part is bounded by

$$C r_n^4 \sum_{\substack{1 \leq j+i+l+h \leq n \\ 1 \leq k \leq j \\ x \neq 0}} p^i(0, x) p^l(x, 0) p^h(0, x) p^k(x, 0) r_{j-k} (r_{j-k} - r_j),$$

where Lemma 5.2 and Lemma 5.5 was applied, and then it is of order $n^2 r_n^7$ since

$$\sum_{j=1}^{n-k} r_j (r_j - r_{j+k}) \leq \sum_{j=1}^n r_j (r_j - r_n) \leq \frac{Cn}{(\log n)^3}.$$

Moreover, we see similarly the second part of $\mathbb{I}''$ is $O(n^2 r_n^8)$, and Lemma 5.8 yields that $\mathbb{III}'' = O(n^2 r_n^7)$. To estimate $\mathbb{IV}''$, it is sufficient to have the following four bounds.

$$\begin{aligned}
& \sum_{j=1}^n \sum_{i=1}^j \sum_{x \neq 0} p_x^i(0, x) \sum_{k=1}^{n-j} p_0^k(x, 0) \sum_{h=n-j-k+1}^{n-j} f_h \sum_{l=1}^{j-i} p_x^l(x, 0) r_{j-i-l}^2 \\
& \leq \sum_{\substack{1 \leq j+k \leq n \\ 1 \leq i+l \leq j \\ x \neq 0}} p^i(0, x) p^l(x, 0) p^k(x, 0) r_{j-i-l}^4 r_{n-j-k} \sum_{h=n-j-k+1}^{n-j} f_h \\
& \leq C n r_n^7 T_{1,2}(n) = O(n^2 r_n^7), \\
& \sum_{\substack{1 \leq j+k \leq n \\ 1 \leq i+l \leq j \\ x \neq 0}} p_x^i(0, x) p_0^k(x, 0) p_x^l(x, 0) r_{j-i-l} P_0(\tau_x < \tau_0 \leq j-i-l) \sum_{h=n-j-k+1}^{n-j} f_h \\
& \leq C n r_n^8 T_{2,3}(n) = O(n^2 r_n^8), \\
& \sum_{\substack{1 \leq j+h+k \leq n \\ 1 \leq i+l \leq j \\ x \neq 0}} p_x^i(0, x) p_0^k(x, 0) p_x^h(0, x) p_x^l(x, 0) r_{j-i-l}^2 r_{n-j-k-h} \\
& \leq T_{2,2}^{2,1}(n) = O(n^2 r_n^7), \\
& \sum_{\substack{1 \leq j+h+k \leq n \\ 1 \leq i+l \leq j \\ x \neq 0}} p_x^i(0, x) p_0^k(x, 0) p_x^h(0, x) p_x^l(x, 0) P_0(\tau_x < \tau_0 \leq j-i-l) r_{j-i-l} r_{n-j-k-h} \\
& \leq T_{3,3}^{1,1}(n) = O(n^2 r_n^8).
\end{aligned}$$

Hence we have $\mathbb{V}'' = O(n^2 r_n^7)$. Therefore

$$\begin{aligned} \sum_{j=1}^n \sum_{i=0}^{j-1} \text{Cov}(Y_i Z_i^n, Y_j Z_j^n) &= \alpha_n - \beta_n + \left(4K_1 + \frac{1}{2} - \frac{1}{6}\pi^2\right) \frac{c_1^4 n^2}{(\log n)^6} \\ &\quad + o\left\{\frac{n^2}{(\log n)^6}\right\}, \end{aligned}$$

where

$$\begin{aligned} \alpha_n &= \sum_{j=1}^n \sum_{i=0}^{j-1} r_{n-i} r_{n-j} r_i (r_i - r_j), \\ \beta_n &= \sum_{i=0}^{n-1} \sum_{j=1}^{n-i} \sum_{k=1}^i f_{j+k} r_{n-j-i}^2 r_{i-k}. \end{aligned}$$

By the asymptotic behaviors of r_n and f_n , we have

$$\alpha_n = r_n^3 \sum_{j=1}^n \sum_{i=0}^{j-1} (r_i - r_j) + \left(\frac{15}{4} - \frac{1}{6}\pi^2\right) \frac{c_1^4 n^2}{(\log n)^6} + o\left\{\frac{n^2}{(\log n)^6}\right\}.$$

On the other hand,

$$\beta_n = r_n^3 \sum_{i=0}^{n-1} \sum_{j=1}^{n-i} (r_j - r_{j+i}) + \frac{9}{4} \frac{c_1^4 n^2}{(\log n)^6} + o\left\{\frac{n^2}{(\log n)^6}\right\}.$$

Hence we obtain that

$$\alpha_n - \beta_n \sim \left(\frac{3}{2} - \frac{1}{6}\pi^2\right) \frac{c_1^4 n^2}{(\log n)^6}.$$

Then we can conclude that

$$\text{Var } Q_n \sim \frac{L_1 c_1^4 n^2}{(\log n)^6},$$

where

$$(5.4) \quad L_1 = 2 \left(4K_1 - \frac{1}{3}\pi^2 + 2\right) = 4.79589 \dots$$

This completes the proof. $\blacksquare$

When Jain and Pruitt derived the variance of R_n , they first calculated it in the aperiodic case and next removed the assumption, namely they improved their result

for the periodic case. However, we have not been able to exclude the assumption of aperiodicity for our assertion in this theorem. For the simple random walk, the variance of Q_n can be computed directly in the same fashion as in Theorem 5.9.

THEOREM 5.11. *For the two dimensional simple random walk,*

$$\text{Var } Q_n \sim \frac{L_1 \pi^4 n^2}{(\log n)^6},$$

where L_1 is the same as (5.4).

6. The fluctuation for Q_n when $d = 2$

Throughout this section, we also assume that the random walk satisfies Assumption [A] introduced in Section 5. Our goal in this section is to establish the fluctuation result for Q_n when $d = 2$. For a strongly transient random walk, Jain and Orey [6] proved the central limit theorem for R_n . In 1986, Le Gall [12] investigated the number of distinct points at which a random walk intersects itself and proved that for an adapted random walk with $EX_1 = 0$ and $E|X_1|^2 < \infty$ (this case is recurrent),

$$\lim_{n \rightarrow \infty} \frac{(\log n)^2}{n} (R_n - ER_n) = -4\pi^2 (\det \Xi) \gamma(\mathcal{C})$$

in law, where Ξ^2 is the covariance matrix for X_1 ,

$$\gamma(\mathcal{C}) = \iint_{\mathcal{C}} \delta_0(W_t - W_s) ds dt - E \left[\iint_{\mathcal{C}} \delta_0(W_t - W_s) ds dt \right],$$

$\mathcal{C} = \{(s, t) \in \mathbb{R}^2; 0 \leq s < t \leq 1\}$, and $\{W_t\}_{t \geq 0}$ is a planar Brownian motion. To consider this type of theorem for Q_n , we must improve his observations. We will introduce some notations. For $0 \leq a < b$, let $S(a, b) = \{S_k; a \leq k \leq b\}$, $S^1(a, b)$ be the set of distinct sites visited once and only once by a random walk between time a and time b , and

$$\begin{aligned} I_n &= |S(0, n) \cap S(n, 2n)|, \\ L_n &= |S(0, n) \cap S^1(n, 2n)|, \\ M_n &= |S^1(0, n) \cap S(n, 2n)|, \end{aligned}$$

where $|A|$ means the number of elements belong to a set A . Recall the notations Y_j^i and Z_i^j ;

$$Y_j^i = \begin{cases} 1 & \text{if } S_j \neq S_\alpha \text{ for } i \leq \alpha < j, \\ 0 & \text{otherwise,} \end{cases}$$

$$Z_i^j = \begin{cases} 1 & \text{if } S_i \neq S_\alpha \text{ for } i < \alpha \leq j, \\ 0 & \text{otherwise.} \end{cases}$$

We can express I_n , L_n , and M_n by these indicator random variables;

$$\begin{aligned} I_n &= \sum_{j=n}^{2n} \sum_{i=0}^n Z_i^n Y_j^n \chi(S_i = S_j), \\ L_n &= \sum_{j=n}^{2n} \sum_{i=0}^n Z_i^n Y_j^n Z_j^{2n} \chi(S_i = S_j), \\ M_n &= \sum_{j=n}^{2n} \sum_{i=0}^n Y_i Z_i^n Y_j^n \chi(S_i = S_j), \end{aligned}$$

where $\chi(A)$ implies the indicator function of a set A . Before proving the fluctuation theorem for Q_n , we will give two lemmas, which will allow us to apply the result of Le Gall.

For $n \geq 1$, $\alpha \geq 0$, and $h, k \geq 1$ satisfying $h + k \leq n$, let

$$T_{h,k}^\alpha(n) = \sum_{\substack{1 \leq i_1 + i_2 + \dots + i_h \\ + j_1 + \dots + j_k \leq n \\ x \neq 0}} \prod_{\mu=1}^h p_x^{i_\mu}(0, x) \prod_{\rho=1}^k p_0^{j_\rho}(x, 0) r_{n - \sum_{\xi=1}^h i_\xi - \sum_{\eta=1}^k j_\eta}^\alpha.$$

LEMMA 6.1. *Under Assumption [A], it holds that*

$$T_{h,k}^\alpha(n) = O(nr_n^{\alpha+h+k}).$$

PROOF. Using Lemma 5.5, we have

$$T_{h,k}^\alpha(n) \leq C \sum_{\substack{1 \leq i_1 + i_2 + \dots + i_h \\ + j_1 + \dots + j_k \leq n \\ x \in \mathbb{Z}^2}} \prod_{\mu=1}^h p^{i_\mu}(0, x) \prod_{\rho=1}^k p^{j_\rho}(x, 0) r_{n - \sum_{\xi=1}^h i_\xi - \sum_{\eta=1}^k j_\eta}^{\alpha+h+k}.$$

Replacing $r_{n - \sum i_\xi - \sum j_\eta}$ with r_n , we obtain that

$$T_{h,k}^\alpha(n) \leq Cr_n^{\alpha+h+k} T_{h,k}(n),$$

which is of order $nr_n^{\alpha+h+k}$ by Lemma 5.6. ■

LEMMA 6.2. *Under Assumption [A], it holds that*

$$E|L_n - r_n I_n|^2, \quad E|M_n - r_n I_n|^2 = O(n^2 r_n^7).$$

PROOF. The estimate of M part can be obtained by considering the reversed random walk in the first assertion, and therefore we will prove the estimate only for the L_n case. For the sake of convenience, we put

$$\Sigma_{i,j}^n = Z_i^n Y_j^n Z_j^{2n} \chi(S_i = S_j), \quad \Theta_{i,j}^n = Z_i^n Y_j^n \chi(S_i = S_j).$$

Then

$$\begin{aligned} (6.1) \quad E|L_n - r_n I_n|^2 &= E \left| \sum_{j=n}^{2n} \sum_{i=0}^n (\Sigma_{i,j}^n - r_n \Theta_{i,j}^n) \right|^2 \\ &= \sum_{j=n}^{2n} \sum_{i=0}^n E(\Sigma_{i,j}^n - r_n \Theta_{i,j}^n)^2 \\ &\quad + 2 \sum_{\substack{n \leq h < j \leq 2n \\ 0 \leq k < i \leq n}} E(\Sigma_{i,j}^n - r_n \Theta_{i,j}^n)(\Sigma_{k,h}^n - r_n \Theta_{k,h}^n) \\ &\quad + 2 \sum_{\substack{n \leq h < j \leq 2n \\ 0 \leq i < k \leq n}} E(\Sigma_{i,j}^n - r_n \Theta_{i,j}^n)(\Sigma_{k,h}^n - r_n \Theta_{k,h}^n). \end{aligned}$$

For $i < j$, $E\Sigma_{i,j}^n = r_{n-j}f_{j-i}$ and $E\Theta_{i,j}^n = f_{j-i}$, which imply the first term is bounded by

$$\sum_{j=n}^{2n} \sum_{i=0}^n (E\Sigma_{i,j}^n + r_n^2 E\Theta_{i,j}^n) \leq Cn.$$

We thus concentrate our attention on the computation of the remaining two terms.

We begin with the second term. For $k < i < h < j$,

$$\begin{aligned} E\Sigma_{i,j}^n \Sigma_{k,h}^n &= \sum_{x \neq 0} p_0^{i-k}(0, x) p_{0x}^{h-i}(x, 0) p_{0x}^{j-h}(0, x) P_x(\tau_0, \tau_x > 2n - j), \\ E\Theta_{i,j}^n \Sigma_{k,h}^n &= \sum_{x \neq 0} p_0^{i-k}(0, x) p_{0x}^{h-i}(x, 0) p_{0x}^{j-h}(0, x) P_x(\tau_0 > 2n - j), \\ E\Sigma_{i,j}^n \Theta_{k,h}^n &= \sum_{x \neq 0} p_0^{i-k}(0, x) p_{0x}^{h-i}(x, 0) p_x^{j-h}(0, x) r_{2n-j}, \\ E\Theta_{i,j}^n \Theta_{k,h}^n &= \sum_{x \neq 0} p_0^{i-k}(0, x) p_{0x}^{h-i}(x, 0) p_x^{j-h}(0, x). \end{aligned}$$

Recalling the definition of $b_m(x)$, we have

$$\sum_{\substack{n \leq h < j \leq 2n \\ 0 \leq k < i \leq n}} (E\Sigma_{i,j}^n \Sigma_{k,h}^n - r_n E\Theta_{i,j}^n \Sigma_{k,h}^n)$$

$$\begin{aligned}
&= \sum_{\substack{n \leq h < j \leq 2n \\ 0 \leq k < i \leq n \\ x \neq 0}} p_0^{i-k}(0, x) p_{0x}^{h-i}(x, 0) p_{0x}^{j-h}(0, x) b_{2n-j}(x) \\
&\quad + \sum_{\substack{n \leq h < j \leq 2n \\ 0 \leq k < i \leq n \\ x \neq 0}} p_0^{i-k}(0, x) p_{0x}^{h-i}(x, 0) p_{0x}^{j-h}(0, x) P_x(\tau_0 > 2n - j)(r_{2n-j} - r_n) \\
&= \text{I} + \text{II}.
\end{aligned}$$

By Lemma 5.1,

$$\begin{aligned}
\text{I} &\leq \sum_{\substack{0 \leq k < i < h < j \leq 2n \\ x \neq 0}} p_0^{i-k}(0, x) p_{0x}^{h-i}(x, 0) p_0^{j-h}(0, x) \\
&\quad \times \sum_{1 \leq r+s \leq 2n-j} p_0^r(x, 0) p_x^s(0, x) r_{2n-j-r-s} \\
&\leq \sum_{\substack{0 \leq i+j+k \\ +h+r+s \leq 2n \\ x \neq 0}} p_0^i(0, x) p_{0x}^h(x, 0) p_0^j(0, x) p_0^r(x, 0) p_x^s(0, x) r_{2n-i-j-k-h-r-s} \\
&\leq \sum_{\substack{0 \leq i+j+k \\ +h+r+s \leq 2n \\ x \neq 0}} p_0^i(0, x) p_x^h(x, 0) p_0^j(0, x) p_0^r(x, 0) p_x^s(0, x) r_{2n-i-j-k-h-r-s}^2 \\
&\quad + \sum_{\substack{0 \leq i+j+k \\ +h+r+s \leq 2n \\ x \neq 0}} p_0^i(0, x) p_x^h(x, 0) p_0^j(0, x) p_0^r(x, 0) p_x^s(0, x) \\
&\quad \times P_0(\tau_x < \tau_0 < 2n - i - j - k - h - r - s).
\end{aligned}$$

The first term of I is bounded by $T_{3,2}^{0,2}(2n)$ and the second term is not larger than $2T_{4,4}^{0,0}(2n)$ by Lemma 5.2. Hence we have $\text{I} \leq Cn^2 r_n^7$, where Lemma 5.6 was applied.

$$\begin{aligned}
\text{II} &\leq \sum_{j=n}^{2n} \sum_{h=0}^{j-1} \sum_{i=0}^{h-1} \sum_{k=0}^{i-1} \sum_{x \neq 0} p_0^{i-k}(0, x) p_{0x}^{h-i}(x, 0) p_{0x}^{j-h}(0, x) (r_{2n-j} - r_{2n}) \\
&= \sum_{\substack{0 \leq i+j+k+h \leq 2n \\ x \neq 0}} p_0^i(0, x) p_{0x}^h(x, 0) p_{0x}^j(0, x) (r_{2n-i-j-k-h} - r_{2n}) \\
&\leq \sum_{\substack{0 \leq i+j+k+h \leq 2n \\ x \neq 0}} p_0^i(0, x) p_{0x}^h(x, 0) p_x^j(0, x) r_{2n-i-j-k-h} (r_{2n-i-j-k-h} - r_{2n}) \\
&\quad + \sum_{\substack{0 \leq i+j+k+h \leq 2n \\ x \neq 0}} p_0^i(0, x) p_{0x}^h(x, 0) p_x^j(0, x)
\end{aligned}$$

$$\begin{aligned} & \times P_0(\tau_x < \tau_0 \leq 2n - i - j - k - h)(r_{2n-i-j-k-h} - r_{2n}) \\ & = \Pi' + \Pi''. \end{aligned}$$

Replacing $r_{2n-i-j-k-h}$ with r_n , we have

$$\begin{aligned} \Pi' & \leq Cr_n \sum_{\substack{0 \leq i+j+k+h \leq 2n \\ x \neq 0}} p_0^i(0, x) p_{0x}^h(x, 0) p_x^j(0, x) (r_{2n-i-j-k-h} - r_{2n}) \\ & \leq Cr_n^2 \sum_{\substack{0 \leq i+j+k+h \leq 2n \\ x \neq 0}} p_0^i(0, x) p_x^h(x, 0) p_x^j(0, x) (r_{2n-i-j-k-h} - r_{2n}) \\ & \quad + Cr_n \sum_{\substack{0 \leq i+j+k+h \leq 2n \\ x \neq 0}} p_0^i(0, x) p_x^h(x, 0) p_x^j(0, x) \\ & \quad \times P_x(\tau_0 < \tau_x \leq 2n - i - j - k - h)(r_{2n-i-j-k-h} - r_{2n}). \end{aligned}$$

Hence the bound of the first part of Π' is

$$Cr_n^5 \sum_{\substack{0 \leq i+j+k+h \leq 2n \\ x \neq 0}} p^i(0, x) p^h(x, 0) p^j(0, x) (r_{2n-i-j-k-h} - r_{2n}),$$

which implies $O(n^2 r_n^7)$. The second part is not larger than $Cr_n T_{3,2}^{0,1} = O(n^2 r_n^7)$. Then Π' is of order $n^2 r_n^7$. Π'' can be estimated in the same way as Π' and we can conclude that $\Pi \leq Cn^2 r_n^7$. For $k < i \leq n \leq h < j \leq 2n$,

$$r_n E\Theta_{i,j}^n \Theta_{k,h}^n - E\Sigma_{i,j}^n \Theta_{k,h}^n = (r_n - r_{2n-j}) E\Theta_{i,j}^n \Theta_{k,h}^n < 0.$$

Hence the second term of (6.1) is bounded by $Cn^2 r_n^7$.

On the other hand, for $i < k < h < j$,

$$\begin{aligned} & E\Sigma_{i,j}^n \Sigma_{k,h}^n - r_n E\Theta_{i,j}^n \Sigma_{k,h}^n \\ & = \sum_{x \neq 0} p_0^{k-i}(0, x) p_{0x}^{h-k}(x, x) p_{0x}^{j-h}(x, 0) c_{2n-j}(x) \\ & \quad + (r_{2n-j} - r_n) \sum_{x \neq 0} p_0^{k-i}(0, x) p_{0x}^{h-k}(x, x) p_{0x}^{j-h}(x, 0) P_0(\tau_x > 2n - j), \end{aligned}$$

where $c_m(x) = P_0(\tau_0, \tau_x > m) - P_0(\tau_x > m)P_0(\tau_0 > m)$. $c_m(x)$ has an equality obtained by exchanging x with 0 in Lemma 5.1. Therefore

$$\sum_{j=n}^{2n} \sum_{h=n}^{j-1} \sum_{k=0}^n \sum_{i=0}^{k-1} (E\Sigma_{i,j}^n \Sigma_{k,h}^n - r_n E\Theta_{i,j}^n \Sigma_{k,h}^n)$$

$$\begin{aligned}
&\leq \sum_{\substack{n \leq h < j \leq 2n \\ 0 \leq i < k \leq n \\ x \neq 0}} p_0^{k-i}(0, x) p_x^{h-k}(x, x) p_{0x}^{j-h}(x, 0) \sum_{1 \leq l+r \leq 2n-j} p_x^l(0, x) p_0^r(x, 0) r_{2n-j-l-r} \\
&\quad + \sum_{\substack{n \leq h < j \leq 2n \\ 0 \leq i < k \leq n \\ x \neq 0}} (r_{2n-j} - r_n) p_0^{k-i}(0, x) p_x^{h-k}(x, x) p_{0x}^{j-h}(x, 0) \\
&= \text{III} + \text{IV}.
\end{aligned}$$

Taking the summation first on i and dominating it by 1, we have

$$\begin{aligned}
\text{III} &\leq \sum_{\substack{0 \leq k \leq n \\ n \leq h \leq 2n \\ 1 \leq j+l+r \leq 2n-h \\ x \neq 0}} f_{h-k} p_{0x}^j(x, 0) p_x^l(0, x) p_0^r(x, 0) r_{2n-j-h-l-r} \\
&\leq \sum_{\substack{0 \leq k \leq n \\ n \leq h \leq 2n}} f_{h-k} \{T_{1,2}^2(2n-h) + 2T_{2,3}^1(2n-h)\}.
\end{aligned}$$

By Lemma 2.2 and Lemma 6.1,

$$(6.2) \quad \text{III} \leq C n r_{2n}^5 \sum_{\substack{0 \leq k \leq n \\ n \leq h \leq 2n}} f_{h-k} = O(n^2 r_n^7).$$

The term IV is estimated as follows. Using Lemma 5.4,

$$\begin{aligned}
\text{IV} &= \sum_{h=n}^{2n} \sum_{k=0}^n \sum_{j=1}^{2n-h} \sum_{i=0}^{k-1} f_{h-k} \sum_{x \neq 0} p_0^i(0, x) p_{0x}^j(x, 0) (r_{2n-j-h} - r_n) \\
&\leq \sum_{h=n}^{2n} \sum_{k=0}^n \sum_{j=1}^{2n-h} \sum_{i=0}^{k-1} f_{h-k} \sum_{x \neq 0} p_0^i(0, x) p_0^j(x, 0) r_{2n-j-h} (r_{2n-j-h} - r_n) \\
&\quad + \sum_{h=n}^{2n} \sum_{k=0}^n \sum_{j=1}^{2n-h} \sum_{i=0}^{k-1} f_{h-k} \sum_{x \neq 0} p_0^i(0, x) p_0^j(x, 0) \\
&\quad \times P_x(\tau_0 < \tau_x \leq 2n-j-h) r_{2n-j-h} \\
&= \text{IV}' + \text{IV}''.
\end{aligned}$$

Since $\sum_{x \neq 0} p_0^i(0, x) p_0^j(x, 0) = f_{i+j}$,

$$\text{IV}' = \sum_{h=n}^{2n} \sum_{k=0}^n \sum_{j=1}^{2n-h} \sum_{i=0}^{k-1} f_{h-k} f_{i+j} r_{2n-j-h} (r_{2n-j-h} - r_n) \leq C n^2 r_n^7.$$

On the computation of IV'' , we take the summation first on i and dominate it by 1. By Lemma 5.2,

$$IV'' \leq \sum_{\substack{0 \leq k \leq n \\ n \leq h \leq 2n}} f_{h-k} \{T_{1,2}^2(2n-h) + T_{2,3}^1(2n-h)\}.$$

Applying the same argument as (6.2), we have $IV'' = O(n^2 r_n^7)$. Therefore IV is of order $n^2 r_n^7$. Since for $i < k \leq n \leq h < j \leq 2n$,

$$r_n E\Theta_{i,j}^n \Theta_{k,h}^n - E\Sigma_{i,j}^n \Theta_{k,h}^n = (r_n - r_{2n-j}) E\Theta_{i,j}^n \Theta_{k,h}^n < 0,$$

the third term of (6.1) is bounded by $Cn^2 r_n^7$. Therefore we have the first assertion of this lemma. $\blacksquare$

We are now ready to state and prove the fluctuation result for Q_n . We use the convenient notation $\langle \cdot \rangle$,

$$\langle X \rangle = X - EX,$$

for any random variable X .

THEOREM 6.3. *For a two dimensional aperiodic random walk with mean 0 and finite variance,*

$$\lim_{n \rightarrow \infty} \frac{(\log n)^3}{n} (Q_n - EQ_n) = -16\pi^3 (\det \Xi)^2 \gamma(C)$$

in the distribution sense, where Ξ and $\gamma(C)$ are the same as in the theorem of Le Gall.

PROOF. Let p be a fixed positive integer and

$$\begin{aligned} Q_n^{(i)} &= |S^1((i-1)2^{-p}n, i2^{-p}n)|, \\ I_n(k, j) &= |S((2j-2)2^{-k}n, (2j-1)2^{-k}n) \cap S((2j-1)2^{-k}n, 2j2^{-k}n)|, \\ L_n(k, j) &= |S((2j-2)2^{-k}n, (2j-1)2^{-k}n) \cap S^1((2j-1)2^{-k}n, 2j2^{-k}n)|, \\ M_n(k, j) &= |S^1((2j-2)2^{-k}n, (2j-1)2^{-k}n) \cap S((2j-1)2^{-k}n, 2j2^{-k}n)|, \end{aligned}$$

for $1 \leq i \leq 2^p$, $1 \leq k \leq p$, and $1 \leq j \leq 2^{k-1}$. We have

$$Q_n = \sum_{i=1}^{2^p} Q_n^{(i)} - \sum_{k=1}^p \sum_{j=1}^{2^{k-1}} \{L_n(k, j) + M_n(k, j)\}.$$

By Lemma 6.2,

$$E \left[\frac{(\log n)^6}{n^2} \left| \sum_{k=1}^p \sum_{j=1}^{2^{k-1}} \{L_n(k, j) + M_n(k, j) - 2r_{2^{-k}n} I_n(k, j)\} \right|^2 \right]$$

$$\begin{aligned}
&\leq \frac{(\log n)^6}{n^2} \left[\sum_{k=1}^p \sum_{j=1}^{2^{k-1}} \{E|L_n(k, j) + M_n(k, j) - 2r_{2^{-k}n}I_n(k, j)|^2\}^{\frac{1}{2}} \right]^2 \\
&\leq \frac{Cp^2(\log n)^6}{(\log 2^{-p}n)^7} \longrightarrow 0
\end{aligned}$$

as $n \rightarrow \infty$. Since

$$EI_n = \sum_{j=n}^{2n} \sum_{i=0}^n f_{j-i} \leq \frac{Cn}{(\log n)^2},$$

we obtain that

$$\begin{aligned}
&E \left[\frac{(\log n)^3}{n} \sum_{k=1}^p \sum_{j=1}^{2^{k-1}} (r_{2^{-k}n} - r_n) I_n(k, j) \right] \\
&\leq \frac{C(\log n)^3}{n} \sum_{k=1}^p \sum_{j=1}^{2^{k-1}} \frac{k2^{-k}n}{(\log 2^{-k}n)^3 \log n} \\
&\leq \frac{Cp^2(\log n)^2}{(\log 2^{-p}n)^3} \longrightarrow 0
\end{aligned}$$

as n tends to infinity. Therefore it is sufficient to consider

$$\sum_{i=1}^{2^p} \langle Q_n^{(i)} \rangle - 2r_n \sum_{k=1}^p \sum_{j=1}^{2^{k-1}} \langle I_n(k, j) \rangle.$$

Le Gall [12] showed that

$$\begin{aligned}
&\lim_{n \rightarrow \infty} \frac{(\log n)^2}{n} \sum_{k=1}^p \sum_{j=1}^{2^{k-1}} \langle I_n(k, j) \rangle \\
&= 4\pi^2(\det \Xi) \sum_{k=1}^p \sum_{j=1}^{2^{k-1}} \left\langle \iint_{A_j^k} \delta_0(W_t - W_s) ds dt \right\rangle
\end{aligned}$$

in law, where $A_j^k = [(2j-2)2^{-k}, (2j-1)2^{-k}) \times ((2j-1)2^{-k}, 2j2^{-k}] \in \mathbb{R}^2$, and by the definition of the renormalized self-intersection local time it holds that

$$\lim_{p \rightarrow \infty} \sum_{k=1}^p \sum_{j=1}^{2^{k-1}} \left\langle \iint_{A_j^k} \delta_0(W_t - W_s) ds dt \right\rangle = \gamma(C)$$

in $L^2(\Omega, \mathfrak{B}, P)$ (see [12]). On the other hand, noting that $\{Q_n^{(i)}\}$ is a sequence of independent identically distributed random variables and the distribution coincides with that of $Q_{2^{-p}n}$, we have by Theorem 5.10

$$\begin{aligned}
E \left[\frac{(\log n)^3}{n} \sum_{i=1}^{2^p} \langle Q_n^{(i)} \rangle \right]^2 &= \frac{(\log n)^6}{n^2} \sum_{i=1}^{2^p} \text{Var } Q_n^{(i)} \\
&\leq \frac{C 2^{-p} (\log n)^6}{(\log 2^{-p} n)^6} \\
&\leq C 2^{-p}
\end{aligned}$$

as n gets sufficiently large. Hence we can conclude

$$\lim_{n \rightarrow \infty} \frac{(\log n)^3}{n} \langle Q_n \rangle = -8c_1 \pi^2 (\det \Xi) \gamma(\mathcal{C})$$

in the distribution sense by choosing p sufficiently large in the beginning. ■

In the simple case, we can also show the fluctuation result similar to above theorem.

THEOREM 6.4. *For the two dimensional simple random walk,*

$$\lim_{n \rightarrow \infty} \frac{(\log n)^3}{n} (Q_n - EQ_n) = -4\pi^3 \gamma(\mathcal{C})$$

in the distribution sense.

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