Killing vector fields and characteristic forms

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1. Introduction

Let (M, g) be a compact orientable Riemannian manifold, $\dim M = n = 2m$, then the Riemannian metric determines the Riemannian connection ∇ uniquely. Let the corresponding connection form be ω , then the curvature form is $\Omega = d\omega - \omega \wedge \omega$. The structure group of the tangent bundle $T(M)$ of M is the special orthogonal group $SO(n)$, and the corresponding Lie algebra $\mathfrak{o}(n)$ is the algebra of real $(n \times n)$ -skew symmetric matrices. Let F be an $SO(n)$ -invariant polynomial of degree k defined on $\mathfrak{o}(n)$, then $F(\Omega)$ is a global closed $2k$ -form of M , so-called the characteristic form of M defined by F . The purpose of this paper is to calculate the integral

$$\int_M F(\Omega) \wedge \sigma,$$

where σ is an arbitrary closed $(n - 2k)$ -form of M .

When $\deg F = m$, this problem was solved by R. Bott [1] and P. Baum-J. Cheeger [2], but their method failed in the case $\deg F = k < m$. In this paper, we shall generalize their result to the case $k < m$, and give a formula for the above integral. All known results related to this problem, such as Pasternak [3], Lazarov-Pasternak [4], Gómez [5] and the others are contained in our result as particular cases.

We assume that all geometric objects appearing in this paper are C^∞ , and we discuss in the real framework.

2. The sets of singular points of the involutive vector fields

In this paper, we fix the ranges of the indices as follows

$$\begin{aligned} A, B, C, \dots &= 1, \dots, n; \\ a, b, c, \dots &= 1, \dots, 2k; \\ \alpha, \beta, \gamma, \dots &= 2k + 1, \dots, n; \\ \lambda, \mu, \nu, \dots &= 2k, 2k + 1, \dots, n. \end{aligned}$$

Given $(n - 2k + 1)$ C^∞ vector fields $\{u_{2k}, u_{2k+1}, \dots, u_n\}$ on the differentiable manifold M , $\dim M = n$, let S be the set of singular points of $\{u_\alpha\}$ ($\alpha = 2k + 1, \dots, n$):

$$S = \{x \in M : \text{the vectors } u_{2k+1}(x), \dots, u_n(x) \text{ are linearly dependent}\}.$$

According to the results of Pontrjagin [6, Th.1], S is the support of the dual cycle Dw_{2k+1} of the $(2k + 1)$ th Stiefel-Whitney cocycle w_{2k+1} of M ,

$$\dim S \geq \dim Dw_{2k+1} = n - 2k - 1.$$

Let $S(\varepsilon)$ be the ε -tubular neighbourhood of S in M . On $M \setminus S(\varepsilon)$, the vector fields $\{u_\alpha\}$ are linearly independent everywhere. Let N be the set of singular points of the vector fields $\{u_{2k}, u_{2k+1}, \dots, u_n\}$ on $M \setminus S(\varepsilon)$:

$$N = \{x \in M \setminus S(\varepsilon) : \text{the vectors } u_{2k}(x), u_{2k+1}(x), \dots, u_n(x) \text{ are linearly dependent}\},$$

$N = \bigcup_{i=1}^p N_i$, where N_i is the connected component of N . Denote $N_i(\varepsilon)$ to be the ε -tubular neighbourhood of N_i in $M \setminus S(\varepsilon)$.

The vector fields $\{u_\lambda\}$ ($\lambda = 2k, 2k + 1, \dots, n$) are called involutive, if there exists a set of C^∞ functions $C_{\lambda\mu}{}^\nu$ ($\lambda, \mu, \nu = 2k, 2k + 1, \dots, n$) defined on M such that

$$[u_\lambda, u_\mu] = \sum_\nu C_{\lambda\mu}{}^\nu u_\nu.$$

LEMMA 1. For $\forall p \in N_i$, $u_\lambda(p) \in T_p N_i$ ($\lambda = 2k, 2k + 1, \dots, n$); namely, the vectors $u_\lambda(p)$ are tangent to N_i .

PROOF. We use the method of Baum [7] to prove this lemma, but state it in a different version.

Suppose $u_{2k}(p) \notin T_p N_i$, there exists a coordinate chart U_p centered at p , and the local coordinates $(x^1, \dots, x^n)$ such that $u_{2k} = \frac{\partial}{\partial x^{2k}}$. Let $u_\alpha = \sum_A a_\alpha{}^A \frac{\partial}{\partial x^A}$, where $a_\alpha{}^A$ are C^∞ functions defined on U_p , we define another set of the vector fields u'_α as follows:

$$u'_\alpha = u_\alpha - a_\alpha^{2k} \frac{\partial}{\partial x^{2k}} = \sum_{A \neq 2k} a_\alpha{}^A \frac{\partial}{\partial x^A}.$$

Since $u'_\alpha = u_\alpha - a_\alpha^{2k} u_{2k}$, so that u'_α and $u_{2k} = \frac{\partial}{\partial x^{2k}}$ are involutive,

$$[u'_\alpha, u'_\beta] = \sum_\gamma C'_{\alpha\beta}{}^\gamma u'_\gamma + C'_\alpha{}^{2k} \frac{\partial}{\partial x^{2k}}.$$

But there is no term $\frac{\partial}{\partial x^{2k}}$ in u'_α , so that $[u'_\alpha, u'_\beta]$ doesn't involve the term $\frac{\partial}{\partial x^{2k}}$, and therefore $C_\alpha'^{2k} = 0$; it follows that the vector fields $\{u'_\alpha\}$ are involutive. According to the Frobenius theorem, there exist a small coordinate chart $V \subset N_p$ and the local coordinates $(y^1, \dots, y^n)$ such that $u'_\alpha = \sum_\beta a'_\alpha{}^\beta \frac{\partial}{\partial y^\beta}$, $u_{2k} = \frac{\partial}{\partial x^{2k}} = \frac{\partial}{\partial y^{2k}}$. Consider the integral curve of the vector field u_{2k} in V passing through p . It determines an one-parameter group of diffeomorphisms f_t . Let $x = f_t(p) \in V$, then since $u_{2k} \notin T_p N_i$, $x \in V \setminus N_i$ unless $x = p$. If $x \in V$, $x \neq p$, x is not a singular point, then the vectors $u_{2k}(x)$ and $u'_\alpha(x)$ are linearly independent. Define a diffeomorphism $g = f_t^{-1}$ of V , then the tangential mapping $g_*: T_x M \rightarrow T_p M$ is a linear isomorphism. Since $g(x) = p$, so that the vectors $u_{2k}(p)$ and $u'_\alpha(p)$ or $u_\alpha(p)$ are also linearly independent, this means that p is not a singular point of $\{u_\lambda\}$, contrary to the hypothesis, so that $u_{2k}(p) \in T_p N_i$. The above proof is satisfied by any vector field linearly dependent on $\{u_\alpha\}$, so that $u_\alpha(p) \in T_p N_i$. The lemma is proved.

COROLLARY. For $\forall p \in N_i$, there exists a coordinate chart U_p centered at p with local coordinates $(x^1, \dots, x^n)$ such that

$$u_\alpha = \sum_\beta C_\alpha^\beta \frac{\partial}{\partial x^\beta}, \quad \text{on } U_p \cap N_i,$$

where C_α^β are C^∞ functions defined on U_p and $\frac{\partial}{\partial x^\beta}$ are tangent to N_i .

PROOF. According to the above lemma, the vector fields $\{u_\lambda\}$ ($\lambda = 2k, 2k + 1, \dots, n$) are tangent to N_i . Restrict them to N_i , then since the vector fields $\{u_\lambda\}$ are involutive, and u_{2k} is linearly dependent on $\{u_\alpha\}$ on N_i , the vector fields $\{u_\alpha\}$ are involutive on N_i . There exist linear combinations $\{u'_\alpha\}$ of $\{u_\alpha\}$ such that $[u'_\alpha, u'_\beta] = 0$, and according to the Frobenius theorem, there exist the local coordinates $(x^1, \dots, x^n)$ in the coordinate chart U_p such that $u'_\alpha = \frac{\partial}{\partial x^\alpha}$ on $U_p \cap N_i$ and $\frac{\partial}{\partial x^\beta} = u'_\beta = \sum_\alpha b_\beta^\alpha u_\alpha$, and it follows that $u_\alpha = \sum_\beta a_\alpha^\beta u'_\beta = \sum_\beta a_\alpha^\beta \frac{\partial}{\partial x^\beta}$ where $(a_\alpha^\beta) = (b_\beta^\alpha)^{-1}$. Since u_α are tangent to N_i , $\frac{\partial}{\partial x^\beta}$ are also tangent to N_i .

3. The actions of the Killing vector fields

In the following, we use the notation $(v_1, \dots, v_p)$ to denote the linear space spanned by the vectors $v_1, \dots, v_p$.

Let (M, g) be a compact orientable Riemannian manifold, $\dim M = n = 2m$, then the Riemannian metric g determines uniquely the Riemannian connection ∇ of the tangent bundle $T(M)$, and the connection form of ∇ is ω , and the corresponding curvature form is given by $\Omega = d\omega - \omega \wedge \omega$.

On a coordinate chart U_p of M , let the coordinates be $(x^1, \dots, x^n)$, and $\{\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n}\}$ the local frame of the tangent bundle $T(M)$ on U_p , then the matrix representation of the Riemannian metric, the Riemannian connection and the

Riemannian curvature in U_p are given by

$$\begin{aligned}
g &= (g_{AB}), \quad g_{AB} = g_{BA} = g\left(\frac{\partial}{\partial x^A}, \frac{\partial}{\partial x^B}\right), \\
g^{-1} &= (g^{AB}), \\
\nabla\left(\frac{\partial}{\partial x^A}\right) &= \sum_B \omega_{AB} \frac{\partial}{\partial x^B}, \quad \omega = (\omega_{AB}), \\
\omega_{AB} &= \sum_C \Gamma_{AB}{}^C \frac{\partial}{\partial x^C}, \quad \Gamma_{AB}{}^C = \frac{1}{2} \sum_D g^{CD} \left(\frac{\partial g_{AD}}{\partial x^B} + \frac{\partial g_{DB}}{\partial x^A} - \frac{\partial g_{AB}}{\partial x^D} \right), \\
\Omega &= (\Omega_{AB}), \quad \Omega_{AB} = d\omega_{AB} - \sum_C \omega_{AC} \wedge \omega_{CB}.
\end{aligned}$$

The Riemannian connection keeps the Riemannian metric invariant; it means that $\nabla g = 0$ or $\nabla_Z g = 0$, for $\forall Z \in \Gamma(T(M))$, where $\nabla_Z = i(Z)\nabla$, in other words, for $\forall X, Y, Z \in \Gamma(T(M))$,

$$Z.g(X, Y) = g(\nabla_Z X, Y) + g(X, \nabla_Z Y).$$

Let $\{u_{2k}, u_{2k+1}, \dots, u_n\}$ be a set of involutive Killing vector fields; it means that they are involutive vector fields and also Killing vector fields, i.e. $L_{u_\lambda} g = 0$ ($\lambda = 2k, 2k+1, \dots, n$), where L_{u_λ} means the Lie's derivative with respect to the vector field u_λ . In other words, for $\forall X, Y, Z \in \Gamma(T(M))$,

$$u_\lambda.g(X, Y) = g(L_{u_\lambda} X, Y) + g(X, L_{u_\lambda} Y).$$

Let F be an $SO(n)$ -invariant polynomial of degree k ($k < m$), $F(\Omega)$ is a closed $2k$ -form of M . It is called the characteristic form of M defined by F . Let N_i be the connected component of the singular set N of the involutive Killing vector fields $u = \{u_\lambda\}$ ($\lambda = 2k, 2k+1, \dots, n$), we define the residue of u at the singular set N_i to be

$$\text{Res}_F(u, N_i) = \int_U F(\Omega) \wedge \sigma,$$

where U is an open set in $M \setminus S(\varepsilon)$ containing N_i , $U \supset N_i(\varepsilon) \supset N_i$ and N_i is the deformation retract of U .

Consider the subbundle E of $T(M)$; its fibre is defined to be

$$E_x = (u_{2k+1}(x), \dots, u_n(x))^\perp, \quad x \in U,$$

where the notation " $\perp$ " means the orthogonal complement in the tangent space $T_x(M)$ with respect to the Riemannian metric g .

The open set U can be covered by the coordinate charts in the Lemma 1, so that all problems in the following can be considered locally in U_p . Choose the local coordinates $(x^1, \dots, x^n)$ in U_p such that $\frac{\partial}{\partial x^a} \in \Gamma(E)$. Since $u_\alpha(x) \perp E_x$, so that

$$0 = g\left(\frac{\partial}{\partial x^a}, u_\alpha\right) = \sum_{\beta} a_{\alpha}^{\beta} g\left(\frac{\partial}{\partial x^a}, \frac{\partial}{\partial x^{\beta}}\right) = \sum_{\beta} a_{\alpha}^{\beta} g_{a\beta}.$$

Since $\{u_\alpha\}$ are linearly independent on U , $\det(a_{\alpha}^{\beta}) \neq 0$, then $g_{a\beta} = 0$. At the same time, $g^{a\beta} = 0$.

$$g = \left(\begin{array}{c|c} g_{ab} & 0 \\ \hline 0 & g_{\alpha\beta} \end{array} \right) = \left(\begin{array}{c|c} g^E & 0 \\ \hline 0 & g_{\alpha\beta} \end{array} \right),$$

where $g^E = (g_{ab})$ is the restriction of the Riemannian metric g to the subbundle E .

Let ∇^E be the restriction of the Riemannian connection ∇ to the subbundle E . On U_p the matrix representation of the connection form ω^E and the curvature form Ω^E are given by

$$\begin{aligned} \nabla^E \left(\frac{\partial}{\partial x^a} \right) &= \sum_b \omega_{ab} \frac{\partial}{\partial x^b}, \\ \omega^E &= (\omega_{ab}), \quad \omega_{ab} = \sum_A \Gamma_a^b{}_A dx^A, \\ \Omega^E &= d\omega^E - \omega^E \wedge \omega^E = (\Omega_{ab}^E), \\ \Omega_{ab}^E &= d\omega_{ab} - \sum_c \omega_{ac} \wedge \omega_{cb}. \end{aligned}$$

LEMMA 2. In the coordinate chart U_p , the metric tensor g_{ab} is a C^∞ functions of the coordinates $x^1, \dots, x^{2k}$, and $\omega^E = (\omega_{ab})$ is the Riemannian connection determined by the metric $g^E = (g_{ab})$; it means that

$$\omega_{ab} = \sum_c \Gamma_a^b{}_c dx^c, \quad \Gamma_a^b{}_c = \frac{1}{2} \sum_d g^{bd} \left(\frac{\partial g_{ad}}{\partial x^c} + \frac{\partial g_{cd}}{\partial x^a} - \frac{\partial g_{ac}}{\partial x^d} \right).$$

PROOF. Since $\{u_\alpha\}$ ($\alpha = 2k+1, \dots, n$) are Killing vector fields,

$$\begin{aligned} u_\alpha \cdot g_{ab} &= u_\alpha \cdot g \left(\frac{\partial}{\partial x^a}, \frac{\partial}{\partial x^b} \right) = g \left(\left[u_\alpha, \frac{\partial}{\partial x^a} \right], \frac{\partial}{\partial x^b} \right) + g \left(\frac{\partial}{\partial x^a}, \left[u_\alpha, \frac{\partial}{\partial x^b} \right] \right), \\ \sum_{\beta} a_{\alpha}^{\beta} \frac{\partial g_{ab}}{\partial x^{\beta}} &= g \left(\left[\sum_{\beta} a_{\alpha}^{\beta} \frac{\partial}{\partial x^{\beta}}, \frac{\partial}{\partial x^a} \right], \frac{\partial}{\partial x^b} \right) + g \left(\frac{\partial}{\partial x^a}, \left[\sum_{\beta} a_{\alpha}^{\beta} \frac{\partial}{\partial x^{\beta}}, \frac{\partial}{\partial x^b} \right] \right). \end{aligned}$$

Note that $\frac{\partial}{\partial x^\alpha} \perp \frac{\partial}{\partial x^a}$ or $\frac{\partial}{\partial x^b}$, so

$$\sum_{\beta} a_{\alpha}^{\beta} \frac{\partial g_{ab}}{\partial x^{\beta}} = \sum_{\beta} a_{\alpha}^{\beta} \left(g \left[\frac{\partial}{\partial x^{\beta}}, \frac{\partial}{\partial x^a} \right], \frac{\partial}{\partial x^b} \right) + g \left(\frac{\partial}{\partial x^a}, \left[\frac{\partial}{\partial x^{\beta}}, \frac{\partial}{\partial x^b} \right] \right).$$

Since $\det(a_{\alpha}^{\beta}) \neq 0$, we have

$$\begin{aligned} \frac{\partial g_{ab}}{\partial x^{\beta}} &= 0, \quad (a, b = 1, \dots, 2k; \beta = 2k+1, \dots, n). \\ \omega_{ab} &= \sum_c \Gamma_a^{\ b}{}_c dx^c + \sum_{\alpha} \Gamma_a^{\ b}{}_{\alpha} dx^{\alpha}, \end{aligned}$$

where

$$\begin{aligned} \Gamma_a^{\ b}{}_{\alpha} &= \frac{1}{2} \sum_{\alpha} g^{bd} \left(\frac{\partial g_{ad}}{\partial x^{\alpha}} + \frac{\partial g_{d\alpha}}{\partial x^a} + \frac{\partial g_{a\alpha}}{\partial x^d} \right) = 0, \\ \Gamma_a^{\ b}{}_c &= \frac{1}{2} \sum_d g^{bd} \left(\frac{\partial g_{dc}}{\partial x^a} + \frac{\partial g_{ad}}{\partial x^c} - \frac{\partial g_{ac}}{\partial x^d} \right) = 0. \end{aligned}$$

So that the metric $g^E = (g_{ab})$ determines the connection form $\omega^E = (\omega_{ab})$ uniquely, in other words, ∇^E is the Riemannian connection of the metric g^E .

COROLLARY. On U_p , the curvature form ω^E is determined by g^E uniquely.

LEMMA 3.

$$\text{Res}_F(u, N_i) = \int_U F(\Omega) \wedge \sigma = \int_U F(\Omega^E) \wedge \sigma.$$

PROOF. Since the structure group of the subbundle E is the reduction of the structure group of the tangent bundle $T(M)$, we can extend the connection ∇^E of E as the connection of $T(M)$ such that $\nabla^E \left(\frac{\partial}{\partial x^{\alpha}} \right) = 0$.

Use the Chern-Weil technique [8] to construct the homotopy between the connections ∇ and ∇^E of $T(M)$, let

$$\begin{aligned} \omega_t &= \omega^E + tu, \quad u = \omega - \omega^E, \\ \Omega_t &= d\omega_t - \omega_t \wedge \omega_t, \end{aligned}$$

then there exists a $(2k-1)$ -form τ on U_p such that

$$F(\Omega) - F(\Omega^E) = d\tau,$$

where

$$\tau = k \int_0^1 F(u, \underbrace{\Omega_t, \dots, \Omega_t}_{(k-1)}) dt.$$

$$\begin{aligned}\int_U F(\Omega) \wedge \sigma &= \int_U (F(\Omega^E) + d\tau) \wedge \sigma = \int_U F(\Omega^E) \wedge \sigma + \int_U d(\tau \wedge \sigma) \\ &= \int_U F(\Omega^E) \wedge \sigma,\end{aligned}$$

where $\int_U d(\tau \wedge \sigma) = \int_{\partial U} \tau \wedge \sigma = 0$.

On $U \setminus N_i(\varepsilon)$, consider the action of the Killing vector fields $\{u_\lambda\}$ on the subbundle E . Let v be the orthogonal projection of the Killing vector field u_{2k} on the subbundle E , i.e. $v = u_{2k} - \sum_\alpha c_\alpha u_\alpha$, where c_α are the solutions of the linear equations $\sum_\alpha c_\alpha g(u_\alpha, u_\beta) = g(u_{2k}, u_\beta)$. In the coordinate chart U_p , let $u_{2k} = \sum_A v_A \frac{\partial}{\partial x^A}$. Since $\frac{\partial}{\partial x^a} \perp u_\alpha$, $v = \sum_a v_a \frac{\partial}{\partial x^a}$, where v_a are C^∞ functions on U_p . The vector field v acts on the subbundle E as follows: Let $X \in \Gamma(E)$,

$$\begin{aligned}v &= \sum_a v_a \frac{\partial}{\partial x^a}, \quad X = \sum_b X_b \frac{\partial}{\partial x^b}, \\ L_v X &= [v, X] = \sum_{a,b} \left(v_a \frac{\partial X_b}{\partial x^a} - X_a \frac{\partial v_b}{\partial x^a} \right) \frac{\partial}{\partial x^b} \in \Gamma(E).\end{aligned}$$

LEMMA 4. On $U_p \setminus N_i(\varepsilon)$, the vector field v is the Killing vector field of the Riemannian metric g^E , i.e. $L_v g^E = 0$.

PROOF. Since u_λ is the Killing vector field of the Riemannian metric g , we have: For $\forall X, Y \in \Gamma(E)$,

$$\begin{aligned}u_{2k} \cdot g(X, Y) &= g([u_{2k}, X], Y) + g(X, [u_{2k}, Y]) \\ u_\alpha \cdot g(X, Y) &= g([u_\alpha, X], Y) + g(X, [u_\alpha, Y]).\end{aligned}$$

Since $X, Y \perp \frac{\partial}{\partial x^\beta}$, and $\det(a_\alpha^\beta) \neq 0$, it follows that

$$\frac{\partial}{\partial x^\beta} g(X, Y) = g\left(\left[\frac{\partial}{\partial x^\beta}, X\right], Y\right) + g\left(X, \left[\frac{\partial}{\partial x^\beta}, Y\right]\right);$$

but $[\frac{\partial}{\partial x^\beta}, X]$ and $[\frac{\partial}{\partial x^\beta}, Y] \in \Gamma(E)$, then we have

$$\begin{aligned}\frac{\partial}{\partial x^\beta} \cdot g^E(X, Y) &= g^E\left(\left[\frac{\partial}{\partial x^\beta}, X\right], Y\right) + g^E\left(X, \left[\frac{\partial}{\partial x^\beta}, Y\right]\right), \\ v \cdot g^E(X, Y) &= \left(u_{2k} - \sum_\alpha v_\alpha \frac{\partial}{\partial x^\alpha}\right) \cdot g(X, Y) \\ &= g([u_{2k}, X], Y) + g(X, [u_{2k}, Y]) \\ &\quad - \sum_\alpha v_\alpha g\left(\left[\frac{\partial}{\partial x^\alpha}, X\right], Y\right) + g\left(X, \left[\frac{\partial}{\partial x^\alpha}, Y\right]\right),\end{aligned}$$

Since $X, Y \perp \frac{\partial}{\partial x^\alpha}$, we have

$$\begin{aligned} v.g^E(X, Y) &= g([u_{2k}, X], Y) + g(X, [u_{2k}, Y]) - g\left(\left[\sum_{\alpha} v_{\alpha} \frac{\partial}{\partial x^{\alpha}}, X\right], Y\right) \\ &\quad - g\left(X, \left[\sum_{\alpha} v_{\alpha} \frac{\partial}{\partial x^{\alpha}}, Y\right]\right) = g([v, X], Y) + g(X, [v, Y]) \\ &= g^E([v, X], Y) + g^E(X, [v, Y]), \end{aligned}$$

it follows that $L_v g^E = 0$.

Note that ∇^E is the Riemannian connection determined uniquely by the metric g^E , according to Kobayashi-Nomizu ([9], p.237), the infinitesimal isometry must be an infinitesimal affine transformation, we can deduce from $L_v g^E = 0$ that $L_v \omega^E = 0$, and thus we have

LEMMA 5. *On $U \setminus N_i(\varepsilon)$, for $\forall Y \in \Gamma(E)$, the following formula holds*

$$L_v \cdot \nabla_Y^E - \nabla_Y^E \cdot L_v - \nabla_{[v, Y]}^E = 0.$$

PROOF. Since $L_v \omega^E = 0$, we have

$$\begin{aligned} 0 &= L_v \omega^E(Y) = i(v) d\omega^E(Y) + d(i(v)\omega^E)(Y) \\ &= d\omega^E(v, Y) + (d\omega^E(v))(Y) \\ &= v.\omega^E(Y) - Y.\omega^E(v) - \omega^E([v, Y]) + Y.\omega^E(v), \end{aligned}$$

and then

$$v.\omega^E(Y) - \omega^E([v, Y]) = 0.$$

On the other hand, for $\forall Z \in \Gamma(E)$,

$$\begin{aligned} &L_v(\nabla_Y^E Z) - \nabla_Y^E(L_v Z) - \nabla_{[v, Y]}^E Z \\ &= L_v(Y.Z + \omega^E(Y)Z) - Y.[v, Z] - \omega^E(Y)[v, Z] - [v, Y].Z - \omega^E([v, Y])(Z) \\ &= [v, Y.Z] + v.\omega^E(Y)Z + \omega^E(Y)[v, Y] - Y.v.Z + Y.Z.v \\ &\quad - \omega^E(Y)[v.Z] - v.Y.Z + Y.v.Z - \omega^E([v, Y])(Z) \\ &= (v.\omega^E(Y) - \omega^E([v, Y]))(Z) = 0. \end{aligned}$$

Now we introduce a new connection $\tilde{\nabla}$ in the subbundle E as follows:

$$\begin{aligned} \tilde{\nabla}_v &= L_v, \\ \tilde{\nabla}_Y &= \nabla_Y^E, \quad \text{if } g(v, Y) = 0. \end{aligned}$$

Let the connection form of $\tilde{\nabla}$ be $\tilde{\omega}$, and the corresponding curvature form be $\tilde{\Omega} = d\tilde{\omega} - \tilde{\omega} \wedge \tilde{\omega}$, then we have

THEOREM 1. (Vanishing theorem) On $U \setminus N_i(\varepsilon)$,

$$F(\tilde{\Omega}) = 0.$$

PROOF. For $\forall Y_r \in \Gamma(E)$ ($r = 1, \dots, 2k-1$), $Y_r \perp v$,

$$\tilde{\Omega}(v, Y) = \tilde{\nabla}_v \tilde{\nabla}_Y - \tilde{\nabla}_Y \tilde{\nabla}_v - \tilde{\nabla}_{[v, Y]} = L_v \cdot \nabla_Y^E - \nabla_Y^E \cdot L_v - \nabla_{[v, Y]}^E = 0,$$

$$F(\tilde{\Omega})(Y_1, \dots, Y_{2k-1}, v) = F(\tilde{\Omega}(Y_1, Y_2), \dots, \tilde{\Omega}(Y_{2k-1}, v)) = 0,$$

and the theorem is proved.

COROLLARY. There exists a $(2k-1)$ -form η on $U \setminus N_i(\varepsilon)$ such that

$$F(\Omega^E) = -d\eta.$$

PROOF. Construct the homotopy between the connections ∇^E and $\tilde{\nabla}$. Let

$$\omega_t = \omega^E + tu, \quad u = \tilde{\omega} - \omega^E,$$

$$\Omega_t = d\omega_t - \omega_t \wedge \omega_t,$$

according to the Chern-Weil technique

$$\eta = k \int_0^1 F(u, \underbrace{\Omega_t, \dots, \Omega_t}_{(k-1)}) dt.$$

In the following, we shall give an explicit formula for the transgressive form η . Define a 1-form π on $U \setminus N_i(\varepsilon)$ as follows: For $\forall X \in \Gamma(T(M))$,

$$\pi(X) = \frac{g(v, X)}{g(v, v)},$$

then $\pi(v) = 1$ and $\pi(Y) = 0$, if $g(v, Y) = 0$.

Consider an operator $A_v = L_v - \nabla_v^E$ on the subbundle E . $A_v(x)$ is a linear transformation on the fibre E_x . For $\forall X \in \Gamma(T(M))$,

$$u(X) \frac{\partial}{\partial x^a} = \tilde{\omega}(X) \frac{\partial}{\partial x^a} - \omega^E(X) \frac{\partial}{\partial x^a} = (\tilde{\nabla}_X - \nabla_X^E) \frac{\partial}{\partial x^a},$$

when $X = v$,

$$u(v) = (\tilde{\nabla}_v - \nabla_v^E) = (L_v - \nabla_v^E) = A_v = \pi(v) \otimes A_v.$$

If $Y \in \Gamma(T(M))$, $Y \perp v$, then

$$u(Y) = \tilde{\nabla}_Y - \nabla_Y^E = 0 = \pi(Y) \otimes A_v.$$

So we have

$$\begin{aligned} u &= \pi \otimes A_v. \\ \omega_t &= \omega^E + tu = \omega^E + t\pi \otimes A_v, \\ \Omega_t &= d\omega_t - \omega_t \wedge \omega_t = \Omega^E + td\pi \otimes A_v \pmod{\pi}, \\ \eta &= k \int_0^1 F(u, \Omega_t, \underbrace{\dots}_{(k-1)}, \Omega_t) dt = k \int_0^1 F(\pi \otimes A_v, \Omega^E + td\pi \otimes A_v, \underbrace{\dots}_{(k-1)}) dt \\ &= \sum_{p=0}^{k-1} C_k^p F(\Omega^E, \underbrace{\dots}_p, \Omega^E, A_v, \underbrace{\dots}_{(k-p-1)}, A_v) \wedge \pi \wedge (d\pi)^{k-p-1} \\ &= \text{The coefficient of } t^{k-1} \text{ in the Taylor expansion of the} \\ &\quad \text{expression } \left\{ F(t\Omega^E + A_v) \wedge \frac{\pi}{1 - td\pi} \right\}. \end{aligned}$$

4. The residue of the singular set of the involutive Killing vector fields

Let (M, g) be a compact orientable Riemannian manifold, $\dim M = n = 2m$, and $u = \{u_{2k}, u_{2k+1}, \dots, u_n\}$ be a set of involutive Killing vector fields defined on M . S is the singular set of $\{u_{2k+1}, \dots, u_n\}$, N is the singular set of $\{u_{2k}, u_{2k+1}, \dots, u_n\}$ in $M \setminus S(\varepsilon)$, $N = \bigcup_i N_i$, where N_i 's are connected components of M . Let $N_i(\varepsilon)$ be the ε -tubular neighbourhood of N_i in $M \setminus S(\varepsilon)$, and U be an open set in $M \setminus S(\varepsilon)$ containing N_i , $U \supset N_i(\varepsilon) \supset N_i$, N_i being the deformation retract of U . Note that U can be covered by the coordinate charts U_p in Lemma 1, so we can discuss our problem locally in U_p , then extend the results to whole U using the technique of partition of unity. In the following, we want to calculate the residue $\text{Res}_F(u, N_i)$.

Let E be a subbundle of the tangent bundle $T(M)$ on $U \setminus S(\varepsilon)$, with its fibre defined by

$$E_x = (u_{2k+1}(x), \dots, u_n(x))^\perp, \quad x \in U \setminus S(\varepsilon).$$

Let v be the orthogonal projection of the Killing vector field u_{2k} on the subbundle E , then N_i is the zero set of v . The operator $A_v = L_v - \nabla_v^E$ acts on the subbundle E , $A_v(x)$ can be considered as a linear transformation of the fibre E_x . For $\forall X, Y \in \Gamma(E)$

$$\begin{aligned} v.g^E(X, Y) &= g^E(L_v X, Y) + g^E(X, L_v Y), \\ v.g^E(X, Y) &= g^E(\nabla_v^E X, Y) + g^E(X, \nabla_v^E Y). \end{aligned}$$

Subtract these two expressions, we then get

$$g^E(A_v X, Y) + g^E(X, A_v Y) = 0,$$

so that the linear transformation $A_v(x)$ in the fibre E_x is skew symmetric with respect to the metric g^E .

On N_i , although $v = 0$, the linear transformation $A_v(x)$ is still meaningful. Since $A_v(x)$ is skew symmetric, the linear transformation $A_v(x)$ in the fibre E_x can be reduced to the following canonical form:

$$A_v(x)|_{x \in N_i} = \left(\begin{array}{ccc|ccc} 0 & a_1 & & & & \\ -a_1 & 0 & & & & \\ \hline & & \ddots & & & \\ & & & \boxed{\begin{array}{cc} 0 & a_{r_i} \\ -a_{r_i} & 0 \end{array}} & & \\ & & & & 0 & \ddots \\ & & & & & \ddots \\ & & & & & & 0 \end{array} \right) \left. \begin{array}{l} \\ \\ \\ \end{array} \right\} \begin{array}{l} 2r_i \\ \\ 2k - 2r_i \end{array}$$

Note that ∇^E is the Riemannian connection determined by g^E , it has no torsion, i.e.

$$\nabla_v^E X - \nabla_X^E v = L_v X,$$

so that $A_v X = -\nabla_X^E v$. Let $X \in E_x \cap T_x(N_i)$, extend it as a vector field on N_i , the integral curve of this vector field passing through x is contained in N_i , and the vector field $v = 0$ along this curve, hence $\nabla_X^E v = 0$. It means that: When $X \in E_x \cap T_x(N_i)$, $A_v(x)(X) = 0$.

Consider a subbundle $T^\perp(N_i)$ of the tangent bundle $T(M)$ on N_i , its fibre $T_x^\perp(N_i)$ is defined as $(T_x(N_i))^\perp$. According to the Lemma 1, $(u_{2k+1}(x), \dots, u_n(x)) \subset T_x(N_i)$, so that $T_x^\perp(N_i) \subset E_x = (u_{2k+1}(x), \dots, u_n(x))^\perp$, it means that $T^\perp(N_i)$ is the subbundle of E on N_i .

When $X \in T_x^\perp N_i$, locally $X(x) = \sum_a X^a(x) \frac{\partial}{\partial x^a}$, $X^a(x) \neq 0$.

$$\begin{aligned} A_v X(x) &= -\nabla_X^E v(x) = -(dv(X) + \omega^E(X).v)(x) \\ &= -dv(X)(x) = -\sum_a X^a(x) \frac{\partial v}{\partial x^a}(x). \end{aligned}$$

If the points of N_i are non-degenerate zeros of the vector field v , i.e. $\det\left(\frac{\partial v^b}{\partial x^a}(x)\right) \neq 0$, then $A_v X(x) \neq 0$. This fact tells us: $\dim T_x^\perp(N_i) = 2r_i$. We have also

LEMMA 6. *If N_i is the set of non-degenerate zeros of the vector field v , then $\dim N_i = n - 2r_i = 2(m - r_i)$.*

PROOF. $\dim N_i = \dim T_x(N_i) = n - \dim T_x^\perp(N_i) = n - 2r_i$.

Let the induced connection of the connection ∇^E of the bundle E to the subbundle $T^\perp(N_i)$ be $\nabla^\perp$, and the corresponding curvature form be $\Omega^\perp$. Define the operator $A_v^\perp$ as the restriction of the operator A_v to $T^\perp(N_i)$. Now we can prove our main result as follows:

THEOREM 2. *If N_i is the set of non-degenerate zeros of the vector field v , then*

$$\begin{aligned} \text{Res}_F(u, N_i) &= \int_U F(\Omega) \\ &= \int_{N_i} \left\{ \begin{array}{l} \text{The coefficient of } t^{k-r_i} \text{ in the Taylor expansion} \\ \text{of the expression } \frac{F(t\Omega^E + A_v)}{E(t\Omega^\perp + A_v^\perp)} \end{array} \right\} \wedge \sigma, \end{aligned}$$

where $E(\Omega^\perp)$ is the Euler characteristic form of the vector bundle $T^\perp(N_i)$, and σ is any closed $(n - 2k)$ -form defined on N_i .

PROOF. According to Lemma 3, we have

$$\begin{aligned} \int_U F(\Omega) \wedge \sigma &= \int_U F(\Omega^E) \wedge \sigma = \lim_{\varepsilon \rightarrow 0} \int_{U \setminus N_i(\varepsilon)} -d\eta \wedge \sigma \\ &= \lim_{\varepsilon \rightarrow 0} \int_{\partial N_i(\varepsilon)} \eta \wedge \sigma, \end{aligned}$$

Consider the ε -tubular neighbourhood $N_i(\varepsilon)$ as a sphere bundle on N_i , its fibre is an $(n - 2r_i - 1)$ -dimensional sphere $S_i(\varepsilon)$. According to the corollary of Theorem 1,

$$\begin{aligned} \int_U F(\Omega) \wedge \sigma &= \lim_{\varepsilon \rightarrow 0} \int_{N_i(\varepsilon)} \left\{ \begin{array}{l} \text{The coefficient of } t^{k-1} \text{ in the Taylor} \\ \text{expansion of the expression } F(t\Omega^E + A_v) \wedge \frac{\pi}{1 - td\pi} \end{array} \right\} \wedge \sigma \end{aligned}$$

For the integral in the parenthesis, we have

LEMMA 7.

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \int_{S_i(\varepsilon)} \frac{\pi}{1 - td\pi} &= \text{The coefficient of } t^{r_i-1} \text{ in the Taylor} \\ &\text{expansion of the expression } \frac{1}{E(t\Omega^\perp + A_v^\perp)}. \end{aligned}$$

PROOF. See Kobayashi [10, pp.72–75].

According to the above lemma, we have

$$\int_U F(\Omega) \wedge \sigma = \int_{N_i} \left\{ \begin{array}{l} \text{The coefficient } t^{(k-1)-(r_i-1)} \text{ in the} \\ \text{Taylor expansion of the expression } \frac{F(t\Omega^E + A_v)}{E(t\Omega^\perp + A_v^\perp)} \end{array} \right\} \wedge \sigma.$$

The theorem is proved.

5. The integral formula of the characteristic form

Let (M, g) be a compact orientable Riemannian manifold, $\dim M = n = 2m$, and $u = \{u_{2k}, u_{2k+1}, \dots, u_n\}$ be a set of involutive Killing vector fields defined on M . Let S be the singular set of the vector fields $\{u_{2k+1}, \dots, u_n\}$, we suppose that they are situated in general position, then $\dim S = n - 2k - 1$.

Note that the subbundle E is undefined on S , and $F(\Omega^E)$ is also undefined on S . But if the closed $(n - 2k)$ -form σ of M vanishes on S , then $F(\Omega^E) \wedge \sigma$ is still meaningful on S .

In this section, we want to calculate the integral $\int_M F(\Omega) \wedge \sigma$. We need the following:

LEMMA 8. *There exists another closed $(n - 2k)$ -form σ' on M and a sufficiently small ε such that $\sigma' \sim \sigma$ and $\sigma' = 0$ on the closure of the tubular neighbourhood $S(\frac{1}{4}\varepsilon)$; $\sigma' = \sigma$ on $M \setminus S(\frac{3}{4}\varepsilon)$.*

PROOF. Since σ is a closed $(n - 2k)$ -form of M and $\dim S = n - 2k - 1$, $\sigma|_S = 0$. For a sufficient small ε , the tubular neighbourhood $S(\varepsilon)$ of S is the deformation retract of S . Let $\pi : S(\varepsilon) \rightarrow S$ be the projection, then $\sigma|_{S(\varepsilon)} \sim \pi^*(\sigma|_S)$, so that $\sigma|_{S(\varepsilon)} \sim 0$, it means that: there exists a $(n - 2k - 1)$ -form θ on $S(\varepsilon)$ such that $\sigma|_{S(\varepsilon)} = d\theta$. Extend θ as a form of M . Constructs a C^∞ function ϕ on M such that $\phi|_{S(\frac{1}{4}\varepsilon)} = 1$, $\phi|_{M \setminus S(\frac{3}{4}\varepsilon)} = 0$, then we define $\sigma' = \sigma - d(\phi\theta)$. This closed form satisfies the requirement of the lemma.

COROLLARY.

$$\int_M F(\Omega) \wedge \sigma = \int_M F(\Omega) \wedge \sigma' = \int_M F(\Omega^E) \wedge \sigma'.$$

PROOF.

$$\int_M F(\Omega) \wedge \sigma = \int_M F(\Omega) \wedge (\sigma' + d(\phi\theta))$$

$$\begin{aligned}
&= \int_M F(\Omega) \wedge \sigma' + \int_M d(\phi\theta) \\
&= \int_M F(\Omega) \wedge \sigma' = \lim_{\varepsilon \rightarrow 0} \int_{M \setminus S_i(\varepsilon)} F(\Omega) \wedge \sigma' \\
&= \lim_{\varepsilon \rightarrow 0} \int_{M \setminus S_i(\varepsilon)} (F(\Omega^E) + d\tau) \wedge \sigma'.
\end{aligned}$$

When $\varepsilon \rightarrow 0$, $\sigma'|_{S_i(\varepsilon)} = 0$, so that

$$\begin{aligned}
\int F(\Omega) \wedge \sigma' &= \lim_{\varepsilon \rightarrow 0} \int_{M \setminus S_i(\varepsilon)} (F(\Omega^E) + d\tau) \wedge \sigma' \\
&= \int_M (F(\Omega^E) + d\tau) \wedge \sigma' = \int_M F(\Omega^E) \wedge \sigma' + \int_M d(\tau \wedge \sigma') \\
&= \int_M F(\Omega^E) \wedge \sigma'.
\end{aligned}$$

We obtain the integral formula of the general characteristic form as follows:

THEOREM 3.

$$\int_M F(\Omega) \wedge \sigma = \sum_{i=1}^p \text{Res}_F(u, N_i).$$

PROOF. According to the above corollary,

$$\begin{aligned}
\int_M F(\Omega) \wedge \sigma &= \int_M F(\Omega^E) \wedge \sigma' = \lim_{\varepsilon \rightarrow 0} \int_{M \setminus \bigcup_{i=1}^p N_i(\varepsilon)} -d\eta \wedge \sigma' \\
&= \sum_{i=1}^p \lim_{\varepsilon \rightarrow 0} \int_{\partial N_i(\varepsilon)} \eta \wedge \sigma',
\end{aligned}$$

Since $\partial N_i(\varepsilon) \cap S_i(\varepsilon) = \emptyset$, in the above integral, $\sigma' = \sigma$.

It follows from the proof of Theorem 2 that:

$$\begin{aligned}
\int_M F(\Omega) \wedge \sigma &= \sum_{i=1}^p \lim_{\varepsilon \rightarrow 0} \int_{\partial N_i(\varepsilon)} \eta \wedge \sigma = \sum_{i=1}^p \int_{U_i} F(\Omega^E) \wedge \sigma \\
&= \sum_{i=1}^p \lim_{\varepsilon \rightarrow 0} \int_{U_i} F(\Omega) \wedge \sigma = \sum_{i=1}^p \text{Res}_F(u, N_i).
\end{aligned}$$

The theorem is proved.

In the following, we shall show that: many known results are the particular cases of the above theorem.

(1) When $k = m$, then the Riemannian manifold (M, g) has only one Killing vector field u . In this case, the subbundle E coincides with the tangent bundle $T(M)$.

(a) If the singular points $N_i (i = 1, \dots, p)$ of the Killing vector field u are isolated, then $\dim N_i = 0$, $r_i = k = m$, so that the subbundle $T^\perp(N_i)$ also coincides with $T(M)$, $E(\Omega^\perp) = E(\Omega) = \sqrt{\det \Omega}$, so that

$$\int_M F(\Omega) = \sum_{i=1}^p \text{Res}_F(u, N_i) = \sum_{i=1}^p \frac{F(A_v(N_i))}{\sqrt{\det A_v(N_i)}}.$$

This is the result of Bott [1].

(b) If the singular points of u are non-isolated, let N_i ($i = 1, \dots, p$) be the connected components of the singular sets of u , and $\dim N_i = n - 2r_i$, then $T^\perp(N_i)$ is a subbundle of the tangent bundle $T(M)$.

$$\begin{aligned} \int_M F(\Omega) &= \sum_{i=1}^p \text{Res}_F(u, N_i) \\ &= \sum_{i=1}^p \int_{N_i} \left\{ \text{The coefficient of } t^{m-r_i} \text{ in the Taylor expansion} \right. \\ &\quad \left. \text{of the expression } \frac{F(t\Omega + A_v)}{E(t\Omega^\perp + A_v^\perp)} \right\}. \end{aligned}$$

This is the result of Baum-Cheeger [2].

(2) When $k < m$, but the involutive Killing vector fields $u = \{u_{2k}, u_{2k+1}, \dots, u_n\}$ are situated in general position, then $\dim N = \dim N_i = n - 2k = 2(m - k)$ and $r_i = k$, so that the subbundle $T^\perp(N_i)$ coincides with E ,

$$\begin{aligned} \int_M F(\Omega) \wedge \sigma &= \sum_{i=1}^p \text{Res}_F(u, N_i) \\ &= \sum_{i=1}^p \int_{N_i} \left\{ \text{The constant term in the Taylor expansion of} \right. \\ &\quad \left. \text{the expression } \frac{F(t\Omega^E + A_v)}{E(t\Omega^E + A_v)} \right\} \wedge \sigma \\ &= \sum_{i=1}^p \int_{N_i} \left(\frac{F(A_v)}{\sqrt{\det A_v}} \right) \sigma, \end{aligned}$$

This is the result of Lazarov-Pasternak [4].

(3) Let M be a compact orientable differentiable manifold, $\dim M = n = 2m$, with a $(n - 2k + 1)$ -dimensional compact Lie group G acting on it. We can introduce

a G -invariant metric g in M , then M becomes a Riemannian manifold. The basis of the Lie algebra of $G\{X_\lambda\}$ ($\lambda = 2k, 2k+1, \dots, n$) induces a set of involutive Killing vector fields $\{u_\lambda\}$ on the Riemannian manifold (M, g) . Then we obtain the result of Pasternak [3] from Theorem 3.

If the Lie group G acting on M is Abelian, then $[X_\lambda, X_\mu] = 0$, so we obtain the result of Gómez [5].

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