A decomposition theorem on differential polynomials of theta functions of high level

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Let $h$ and $g$ be two positive integers. We fix an element $\Omega$ of the Siegel upper half plane

$$H_g := \left\{ Z \in \mathbb{C}^{(g,g)} \mid Z = tZ, \quad \text{Im} Z > 0 \right\}$$

of degree $g$ once and for all. Let $\mathcal{M}$ be positive symmetric, even integral matrix of degree $h$. An entire function $f$ on $\mathbb{C}^{(h,g)}$ satisfying the transformation behaviour

$$f(W + \xi \Omega + \eta) = \exp \left\{ -\pi i \sigma(\mathcal{M}(\xi \Omega^t \xi + 2W^t \xi)) \right\} f(W)$$

for all $W \in \mathbb{C}^{(h,g)}$ and $(\xi, \eta) \in \mathbb{Z}^{(h,g)} \times \mathbb{Z}^{(h,g)}$ is called a theta function of level $\mathcal{M}$ with respect to $\Omega$. The set $T_{\mathcal{M}}(\Omega)$ of all theta functions of level $\mathcal{M}$ with respect to $\Omega$ forms a complex vector space of dimension $(\det \mathcal{M})^g$ with a canonical basis consisting of theta series

$$\vartheta^{(\mathcal{M})} \left[ \begin{array}{c} A \\ 0 \end{array} \right] (\Omega|W)$$

$$:= \sum_{N \in \mathbb{Z}^{(h,g)}} \exp \left\{ \pi i \sigma(\mathcal{M}((N + A)\Omega^t(N + A) + 2W^t(N + A))) \right\},$$

where $A$ runs over a complete system of representatives of the cosets $\mathcal{L}_{\mathcal{M}} := \mathcal{M}^{-1}\mathbb{Z}^{(h,g)}/\mathbb{Z}^{(h,g)}$.

We let

$$T(\Omega) := \sum_{\mathcal{M}} T_{\mathcal{M}}(\Omega)$$

be the graded algebra of theta functions, where $\mathcal{M} = (\mathcal{M}_{kl})$ $(1 \leq k, l \leq h)$ runs over the set $\mathbb{M}(h)$ of all positive symmetric, even integral $h \times h$ matrices with $\mathcal{M}_{kl} \neq 0$ for all $k, l$.

In this paper we prove the following decomposition theorem:

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The algebra of differential polynomials of theta functions has a canonical basis

$$\left\{ \left( \frac{\partial}{\partial W} \right)^J \vartheta(\mathcal{M}) \begin{bmatrix} A \\ 0 \end{bmatrix} (\Omega|W) \mid J \in \mathbb{Z}_{\geq 0}^{(h,g)}, \ A \in \mathcal{L}_\mathcal{M}, \ \mathcal{M} \in \mathcal{M}(h) \right\},$$

i.e., any differential polynomials of theta functions can be expressed uniquely as a linear combination of

$$\left( \frac{\partial}{\partial W} \right)^J \vartheta(\mathcal{M}) \begin{bmatrix} A \\ 0 \end{bmatrix} (\Omega|W) \ (J \in \mathbb{Z}_{\geq 0}^{(h,g)}, \ A \in \mathcal{L}_\mathcal{M}, \ \mathcal{M} \in \mathcal{M}(h))$$

with constant coefficients depending only on $\Omega$.

The key idea is quite similar to that of making transvectants in the classical invariant theory (cf. [M1], [M2]). However the Lie algebra is the Heisenberg Lie algebra instead of $sl_2$. The graded algebra $T(\Omega)$ of theta functions is embedded in the graded algebra $A(\Omega)$ of auxiliary theta functions in $(Z, W)$ with $Z, W \in \mathbb{C}^{(h,g)}$ with respect to $\Omega$ satisfying the following conditions:

1. A realization $\{\varepsilon_{ki}, D_{ma}, \Delta_{nb} \mid 1 \leq k, l, m, n \leq h, 1 \leq a, b \leq g\}$ (cf. see section 2 for detail) of the Heisenberg Lie algebra acts on $A(\Omega)$ as derivations.

2. $T(\Omega)$ is the subalgebra consisting of all the elements $\varphi \in A(\Omega)$ such that $\Delta_{nb}\varphi = 0$ for all $1 \leq n \leq h, 1 \leq b \leq g$.

3. The set

$$\left\{ \Delta^J \vartheta(\mathcal{M}) \begin{bmatrix} A \\ 0 \end{bmatrix} (\Omega|W) \mid J \in \mathbb{Z}_{\geq 0}^{(h,g)}, \ A \in \mathcal{L}_\mathcal{M}, \ \mathcal{M} \in \mathcal{M}(h) \right\}$$

forms a canonical basis of $A(\Omega)$.

4. The mapping

$$\Delta^J \vartheta(\mathcal{M}) \begin{bmatrix} A \\ 0 \end{bmatrix} (\Omega|W) \mapsto \left( \frac{\partial}{\partial W} \right)^J \vartheta(\mathcal{M}) \begin{bmatrix} A \\ 0 \end{bmatrix} (\Omega|W)$$

$(J \in \mathbb{Z}_{\geq 0}^{(h,g)}, \ A \in \mathcal{L}_\mathcal{M}$ and $\mathcal{M} \in \mathcal{M}(h))$ induces an algebra isomorphism of $A(\Omega)$ onto the algebra of differential polynomials of theta functions.

**Notations.** We denote by $\mathbb{Z}$, $\mathbb{R}$ and $\mathbb{C}$ the ring of integers, the field of real numbers, and the field of complex numbers, respectively. The symbol "$:="$ means that the expression on the right is the definition of that on the left. We denote by $\mathbb{Z}^+$ the set of all positive integers. $F^{(k,l)}$ denotes the set of all $k \times l$ matrices with entries in a commutative ring $F$. For any $M \in F^{(k,l)}$, $^tM$ denotes the transpose matrix of $M$. For $A \in F^{(k,k)}$, $\sigma(A)$ denotes the trace of $A$. For $A \in F^{(k,l)}$ and $B \in F^{(k,k)}$, we set $B[A] = ^tABA$. For a positive symmetric, even integral matrix $\mathcal{M}$ of degree $h$, $\mathcal{L}_\mathcal{M}$ denotes a complete system of representatives of the cosets.
\[ M^{-1} \mathbb{Z}^{(h,g)} / \mathbb{Z}^{(h,g)}. \]

\[ Z_{\geq 0}^{(h,g)} = \left\{ J = (J_{ka}) \in \mathbb{Z}^{(h,g)} \mid J_{ka} \geq 0 \text{ for all } k, a \right\}, \]

\[ |J| = \sum_{k,a} J_{ka}, \]

\[ J = (J_{ka}) \in \mathbb{Z}_{\geq 0}^{(h,g)} \text{ for all } k, a, \]

\[ Z = (Z_{ka}) \text{ and } W = (W_{ka}). \]

For \( J = (J_{ka}) \in \mathbb{Z}_{\geq 0}^{(h,g)} \), we set

\[ Z^J = Z_{11}^{J_{11}} \ldots Z_{hg}^{J_{hg}}, \quad W^J = W_{11}^{J_{11}} \ldots W_{hg}^{J_{hg}}. \]

For \( J = (J_{ka}) \in \mathbb{Z}_{\geq 0}^{(h,g)} \), we put

\[ \left( \frac{\partial}{\partial W} \right)^J = \left( \frac{\partial}{\partial W_{11}} \right)^{J_{11}} \ldots \left( \frac{\partial}{\partial W_{hg}} \right)^{J_{hg}}. \]

1. Auxiliary theta functions

We fix an element \( \Omega \) of \( H_g \) once and for all. Let \( M \) be a positive symmetric, even integral matrix of degree \( h \). An auxiliary theta function of level \( M \) with respect to \( \Omega \) means a function \( \varphi(Z, W) \) in complex variables \( (Z, W) \in \mathbb{C}^{(h,g)} \times \mathbb{C}^{(h,g)} \) such that

(a) \( \varphi(Z, W) \) is a polynomial in complex variables \( Z = (Z_{ka}) \) whose coefficients are entire functions in \( W = (W_{ka}) \), and

(b) for all \( (\xi, \eta) \in \mathbb{Z}^{(h,g)} \times \mathbb{Z}^{(h,g)} \) and \( (Z, W) \in \mathbb{C}^{(h,g)} \times \mathbb{C}^{(h,g)} \),

\[ \varphi(Z + \xi, W + \xi \Omega + \eta) = \exp \left\{ -\pi i \sigma(M(\xi \Omega^t \xi + 2W^t \xi)) \right\} \varphi(Z, W) \]

holds.

Let \( A_M(\Omega) \) be the vector space of auxiliary theta functions of level \( M \) with respect to \( \Omega \). We let

\[ A(\Omega) := \sum_M A_M(\Omega) \]

the graded algebra of auxiliary theta functions, where \( M = (M_{kl}) \) runs over the set of all positive symmetric, even integral \( h \times h \) matrices such that \( M_{kl} \neq 0 \) for all \( k, l \). We note that \( A(\Omega) \) contains the graded algebra \( T(\Omega) \) as the subalgebra of polynomials of degree zero in \( Z \).
We define the auxiliary theta series

$$\widetilde{\varphi}_J^{(M)} \begin{bmatrix} A \\ 0 \end{bmatrix} (\Omega|Z,W)$$

\begin{equation}
(1.1) \quad = (2\pi i)^{|J|} \sum_{N \in \mathbb{Z}^{(h,g)}} \prod_{k=1}^{h} \prod_{a=1}^{g} \left( \sum_{l=1}^{\mathcal{M}_{kl}(Z+N+A)_{la}} \right)^{J_{ka}} \times \exp \left\{ \pi i \sigma(\mathcal{M}((N+A)^t(N+A) + 2W^t(N+A))) \right\},
\end{equation}

where \( J = (J_{ka}) \in \mathbb{Z}_{\geq 0}^{(h,g)} \), \( \mathcal{M} = (\mathcal{M}_{kl}) \in \mathbb{M}(h) \), \( A \in \mathcal{L}_M \) and \((Z+N+A)_{la} = Z_{la} + N_{la} + A_{la}\).

**Lemma 1.1.** For each \( J \in \mathbb{Z}_{\geq 0}^{(h,g)} \) and \( \mathcal{M} \in \mathbb{M}(h) \), we have

$$\widetilde{\varphi}_J^{(M)} \begin{bmatrix} A \\ 0 \end{bmatrix} (\Omega|Z + \xi,W + \xi \Omega + \eta)$$

$$= \exp \left\{ -\pi i \sigma(\mathcal{M}(\xi \Omega^t \xi + 2W^t \xi)) \right\} \widetilde{\varphi}_J^{(M)} \begin{bmatrix} A \\ 0 \end{bmatrix} (\Omega|Z,W),$$

where \( A \in \mathcal{L}_M \) and \((\xi, \eta) \in \mathbb{Z}^{(h,g)} \times \mathbb{Z}^{(h,g)}\). In particular, \( \widetilde{\varphi}_J^{(M)} \begin{bmatrix} A \\ 0 \end{bmatrix} (\Omega|Z,W) \in \mathcal{A}_M(\Omega)\).

**Proof.** We observe that

$$(N+A)^t(N+A) + 2(W + \xi \Omega + \eta)^t(N+A)$$

$$= (N + \xi + A)^t(N + \xi + A) + 2W^t(N + \xi + A) - (N + A)^t \xi$$

$$+ \xi \Omega^t(N + A) + 2\eta^t(N + A) - (\xi \Omega^t \xi + 2W^t \xi).$$

In addition, it is easy to see that

$$\sigma(\mathcal{M}(N + A)^t \xi) = \sigma(\mathcal{M} \xi \Omega^t(N + A))$$

and

$$\sigma(\mathcal{M} \eta^t(N + A)) = \sigma(\mathcal{M} \eta^t N) + \sigma(\mathcal{M} A^t \eta) \in \mathbb{Z} \quad (because \ A \in \mathcal{L}_M).$$

The proof follows immediately from these facts.

For any \( k, a \in \mathbb{Z}^+ \) with \( 1 \leq k \leq h \), \( 1 \leq a \leq g \) and \( \mathcal{M} = (\mathcal{M}_{kl}) \in \mathbb{M}(h) \), we put

$$\partial(\mathcal{M}, Z, W)_{ka} := 2\pi i \sum_{l=1}^{h} \mathcal{M}_{kl} Z_{la} + \frac{\partial}{\partial W_{ka}}.$$
For each \( J = (j_{ka}) \in \mathbb{Z}_{\geq 0}^{(h,g)} \), we put

\[
(1.3) \quad \partial(\mathcal{M}, Z, W)^J := \partial(\mathcal{M}, Z, W)^{j_{11}} \cdots \partial(\mathcal{M}, Z, W)^{j_{ka}} \cdots \partial(\mathcal{M}, Z, W)^{j_{hg}}.
\]

Then we obtain the following.

**Lemma 1.2.** For each \( J \in \mathbb{Z}_{\geq 0}^{(h,g)} \), \( \mathcal{M} \in \mathbb{M}(h) \) and \( A \in \mathcal{L}_M \), we have

\[
(1.4) \quad \bar{\eta}^{(\mathcal{M})}_J \begin{bmatrix} A \\ 0 \end{bmatrix} (\Omega|Z, W) = \partial(\mathcal{M}, Z, W)^J \theta^{(\mathcal{M})} \begin{bmatrix} A \\ 0 \end{bmatrix} (\Omega|W).
\]

**Proof.** It is easy to compute that if \( \mathcal{M} = (\mathcal{M}_{kl}) \in \mathbb{M}(h) \),

\[
\partial(\mathcal{M}, Z, W)^{j_{ka}} \theta^{(\mathcal{M})} \begin{bmatrix} A \\ 0 \end{bmatrix} (\Omega|W)
= 2\pi i \sum_{N \in \mathbb{Z}^{(h,g)}} \left( \sum_{l=1}^{h} \mathcal{M}_{kl}(Z + N + A)_{la} \right)
\times \exp \left\{ 2\pi i \sigma(\mathcal{M}((N + A)\Omega^t(N + A) + 2W^t(N + A))) \right\}.
\]

The proof follows immediately from the fact that if \( J = (j_{ka}) \in \mathbb{Z}_{\geq 0}^{(h,g)} \) and \( \mathcal{M} = (\mathcal{M}_{kl}) \in \mathbb{M}(h) \),

\[
\partial(\mathcal{M}, Z, W)^{j_{ka}} \theta^{(\mathcal{M})} \begin{bmatrix} A \\ 0 \end{bmatrix} (\Omega|W)
= (2\pi i)^{j_{ka}} \sum_{N \in \mathbb{Z}^{(h,g)}} \left( \sum_{l=1}^{h} \mathcal{M}_{kl}(Z + N + A)_{la} \right)^{j_{ka}}
\times \exp \left\{ \pi i \sigma(\mathcal{M}((N + A)\Omega^t(N + A) + 2W^t(N + A))) \right\}.
\]

**Theorem 1.** For a fixed \( \mathcal{M} \in \mathbb{M}(h) \), the set

\[
\left\{ \eta^{(\mathcal{M})}_J \begin{bmatrix} A \\ 0 \end{bmatrix} (\Omega|Z, W) \mid J \in \mathbb{Z}_{\geq 0}^{(h,g)}, A \in \mathcal{L}_M \right\}
\]

is a basis of the vector space \( \mathcal{A}_M(\Omega) \) of auxiliary theta functions of level \( \mathcal{M} \) with respect to \( \Omega \).

**Proof.** According to Lemma 1.1, the functions \( \eta^{(\mathcal{M})}_J \begin{bmatrix} A \\ 0 \end{bmatrix} (\Omega|Z, W) \) \((J \in \mathbb{Z}_{\geq 0}^{(h,g)} \) and \( A \in \mathcal{L}_M \)) are contained in \( \mathcal{A}_M(\Omega) \) and it is obvious that they are
linearly independent. We give an ordering \( \prec \) on \( \mathbb{Z}_{\geq 0}^{(h,g)} \) as follows. For \( J, K \in \mathbb{Z}_{\geq 0}^{(h,g)} \), we write \( J \prec K \) if \( |J| \leq |K| \). We say that \( Z^K \) has higher degree in \( Z \) than \( Z^J \) if \( J \prec K \). Now we let \( \varphi(Z,W) = \sum_J Z^J f_J(W) \) be an element of \( A_M(\Omega) \) and let \( Z^K f_K(W) \) be one of the terms with highest degree \( K \) in \( Z \). Since \( \varphi \in A_M(\Omega) \), we obtain for each \( \xi, \eta \in \mathbb{Z}^{(h,g)} \)

\[
\sum_J (Z + \xi)^J f_J(W + \xi \Omega + \eta) = \exp \left\{ -\pi i \sigma(\mathcal{M}((\xi \Omega + \xi + 2W \xi)) \right\} \sum_J Z^J f_J(W).
\]

Comparing the coefficients of \( Z^K \), we get

\[
f_K(W + \xi \Omega + \eta) = \exp \left\{ -\pi i \sigma(\mathcal{M}((\xi \Omega + \xi + 2W \xi)) \right\} f_K(W)
\]

for each \( (\xi, \eta) \in \mathbb{Z}^{(h,g)} \times \mathbb{Z}^{(h,g)} \). Thus \( f_K \in T_M(\Omega) \) and so we obtain

\[
f_K(W) = \sum_{\alpha=1}^{(\det \mathcal{M})^g} c_\alpha \vartheta^{(\mathcal{M})} \left[ \begin{array}{c} A_{\alpha} \\ 0 \end{array} \right] (\Omega|W),
\]

where \( c_\alpha \in \mathbb{C} \) and \( A_{\alpha} \in L_M \). Therefore for suitable constants \( d_\alpha \in \mathbb{C} \) (\( \alpha = 1, \ldots, (\det \mathcal{M})^g \)), the function

\[
\varphi(Z,W) = \sum_{\alpha=1}^{(\det \mathcal{M})^g} d_\alpha \vartheta^{(\mathcal{M})} \left[ \begin{array}{c} A_{\alpha} \\ 0 \end{array} \right] (\Omega|Z,W)
\]

is an element of \( A_M(\Omega) \) without \( Z^K \)-term and all the new terms are of lower degree than \( K \) in \( Z \). Continuing this process successively, we can express \( \varphi(Z,W) \) as a linear combination of auxiliary theta functions \( \vartheta^{(\mathcal{M})}_J \left[ \begin{array}{c} A_{\alpha} \\ 0 \end{array} \right] (\Omega|Z,W) \) (\( J \in \mathbb{Z}_{\geq 0}^{(h,g)}, A_{\alpha} \in L_M \)).

2. **A realization of Heisenberg Lie algebra**

For each \( \mathcal{M} \in M(h) \), we let

\[
\sigma^{(\mathcal{M})} : A(\Omega) \rightarrow A_M(\Omega)
\]

be the projection operator of \( A(\Omega) \) onto \( A_M(\Omega) \). We define the differential operators

\[
\mathcal{E}_{kl} := \sum_{\mathcal{M} \in M(h)} \mathcal{M}_{kl} \sigma^{(\mathcal{M})}, \quad \mathcal{M} = (\mathcal{M}_{kl}),
\]

\( k, l = 1, \ldots, (\det \mathcal{M})^g \).
where $1 \leq k, l, m, n \leq h$ and $1 \leq a, b \leq g$.

For $J = (J_{ka}) \in \mathbb{Z}_{\geq 0}^{(h,g)}$, we put

$$D^J := D_{11}^{J_{11}} \cdots D_{hg}^{J_{hg}}, \quad \Delta^J := \Delta_{11}^{J_{11}} \cdots \Delta_{hg}^{J_{hg}}.$$

**Proposition 2.1.** Let $M = (M_{kl}) \in M(h)$ and $J = (J_{ka}) \in \mathbb{Z}_{\geq 0}^{(h,g)}$. Then we have

1. \[ D_{ma} \partial^M (A) (\Omega|Z, W) = \sum_{i=1}^{h} M_{mi} J_{ia} \partial^M (A) (\Omega|Z, W), \]
2. \[ \Delta_{nb} \partial^M (A) (\Omega|Z, W) = \partial^M (\Delta^J (A)) (\Omega|Z, W), \]
3. \[ \Delta^J (A) (\Omega|Z, W) = \Delta^J \partial^M (A) (\Omega|W), \]

where $A \in \mathcal{L}_M$, $1 \leq m, n \leq h$ and $1 \leq a, b \leq g$.

**Proof.** (1.1) follows from a direct application of $D_{ma}$ to (1.1). (2.2) and (2.3) follow immediately from Lemma 1.2, (1.4).

**Proposition 2.2.** $\varepsilon_{kl}, D_{ma}, \Delta_{nb} (1 \leq k, l, m, n \leq h, 1 \leq a, b \leq g)$ are derivations of $A(\Omega)$ such that

$$[\varepsilon_{kl}, D_{ma}] = [\varepsilon_{kl}, \Delta_{nb}] = [D_{ma}, \Delta_{nb}] = [\Delta_{ma}, \Delta_{nb}] = 0,$$

$$[D_{ma}, \Delta_{nb}] = \varepsilon_{ab} \varepsilon_{mn}.$$

**Proof.** According to Proposition 2.1, $\varepsilon_{kl}, D_{ma}, \Delta_{nb} (1 \leq k, l, m, n \leq h, 1 \leq a, b \leq g)$ map $A(\Omega)$ into itself. Since $A(\Omega) = \sum_{M \in M(h)} A_M(\Omega)$ is a graded algebra, $\varepsilon_{kl}, D_{ma}, \Delta_{nb} (1 \leq k, l, m, n \leq h, 1 \leq a, b \leq g)$ are derivations of $A(\Omega)$. An easy calculation yields the above commutation relations.

**Remark 2.3.** According to Proposition 2.2, $\{\varepsilon_{kl}, D_{ma}, \Delta_{nb} | 1 \leq k, l, m, n \leq h, 1 \leq a, b \leq g\}$ is a realization of the Heisenberg Lie algebra acting on $A(\Omega)$ as derivations (cf. [Y1], [Y2]).
PROPOSITION 2.4. The graded algebra $T(\Omega)$ of theta functions with respect to $\Omega$ is the subalgebra of $A(\Omega)$ consisting of $\varphi$ in $A(\Omega)$ such that $D_{ma}\varphi = 0$ ($1 \leq m \leq h$, $1 \leq a \leq g$).

PROOF. If $\varphi \in T(\Omega)$, $\varphi$ does not contain any variables $Z_{ma}$ and $D_{ma}\varphi = 0$ for any $m, a \in \mathbb{Z}^+$ with $1 \leq m \leq h$, $1 \leq a \leq g$.

Conversely, we assume that

$$
D_{ma}\left(\sum_{J,\alpha,M} c_{J,\alpha,M} \vartheta_j^{(M)} \begin{bmatrix} A_{\alpha,M} \\ 0 \end{bmatrix} (\Omega|Z,W)\right) = 0,
$$

where $A_{\alpha,M} \in \mathcal{L}_M$, $1 \leq m \leq h$ and $1 \leq a \leq g$. Then by (2.1), we get

$$
(*) \sum_{J,\alpha,M} \sum_{l=1}^{h} M_{ml} J_{la} c_{J,\alpha,M} \vartheta_{J-l\alpha}^{(M)} \begin{bmatrix} A_{\alpha,M} \\ 0 \end{bmatrix} (\Omega|Z,W) = 0.
$$

Since $\mathcal{M} = (M_{kl}) \in \mathbb{M}(h)$, $M_{kl} \neq 0$ for all $k, l$ with $1 \leq k, l \leq h$. Therefore if $J \neq (0, \ldots, 0)$, we get $c_{J,\alpha,M} = 0$ from the condition $(*)$. Hence this completes the proof. $\square$

THEOREM 2. $A(\Omega)$ has the direct sum decomposition

$$
(2.4) \quad A(\Omega) = \sum_{J \in \mathbb{Z}_{\geq 0}^{(h,g)}} \Delta^J T(\Omega) = \sum_{J \in \mathbb{Z}_{\geq 0}^{(h,g)}} \sum_{M \in \mathbb{M}(h)} \Delta^J T_M(\Omega)
$$

such that $\Delta^J$ induces a vector space isomorphism of $T_M(\Omega)$ onto $\Delta^J T_M(\Omega)$.

PROOF. The proof follows from (2.3) and the fact that

$$
\left\{ \vartheta_j^{(M)} \begin{bmatrix} A \\ 0 \end{bmatrix} (\Omega|Z,W) \bigg| J \in \mathbb{Z}_{\geq 0}^{(h,g)}, A \in \mathcal{L}_M, M \in \mathbb{M}(h) \right\},
$$

$$
\left\{ \vartheta_j^{(M)} \begin{bmatrix} A \\ 0 \end{bmatrix} (\Omega|Z,W) \bigg| A \in \mathcal{L}_M \right\}
$$

and

$$
\left\{ \vartheta_j^{(M)} \begin{bmatrix} A \\ 0 \end{bmatrix} (\Omega|W) \bigg| A \in \mathcal{L}_M \right\}
$$

are the bases of $A(\Omega)$, $\Delta^J T_M(\Omega)$ and $T_M(\Omega)$ respectively. $\square$

REMARK 2.5. We may express the inverse mapping of $\Delta^J : T_M(\Omega) \rightarrow \Delta^J T_M(\Omega)$ in terms of $D_{ma}, \Delta_{nb}$ ($1 \leq m, n \leq h$, $1 \leq a, b \leq g$). The expression is very complicated and so we omit it.
3. Decomposition theorem on differential polynomials of theta functions of high level

In this section, we prove the algebra isomorphism theorem.

**Theorem 3.** The replacement

\[ \Delta^J \varphi(W) \longrightarrow \left( \frac{\partial}{\partial W} \right)^J \varphi(W) \quad (J \in \mathbb{Z}_{\geq 0}^4, \varphi \in T(\Omega)) \]

induces a \( T(\Omega) \)-algebra isomorphism of \( A(\Omega) \) onto the algebra

\[ \mathbb{C}\left[ \cdots, \left( \frac{\partial}{\partial W} \right)^J \varphi(\mathcal{M}) \left[ A_{\alpha,\mathcal{M}} \right] (\Omega|W), \cdots \right], \quad A_{\alpha,\mathcal{M}} \in \mathcal{L}_{\mathcal{M}}, \mathcal{M} \in \mathcal{M}(h) \]

of differential polynomials of theta functions, namely

1. \( \quad G\left( \cdots, \Delta^J \varphi(\mathcal{M}) \left[ A_{\alpha,\mathcal{M}} \right] (\Omega|W), \cdots \right) = 0 \)

   if and only if \( G\left( \cdots, \left( \frac{\partial}{\partial W} \right)^J \varphi(\mathcal{M}) \left[ A_{\alpha,\mathcal{M}} \right] (\Omega|W), \cdots \right) = 0 \),

2. \( \quad G\left( \cdots, \Delta^J \varphi(\mathcal{M}) \left[ A_{\alpha,\mathcal{M}} \right] (\Omega|W), \cdots \right) = G\left( \cdots, \left( \frac{\partial}{\partial W} \right)^J \varphi(\mathcal{M}) \left[ A_{\alpha,\mathcal{M}} \right] (\Omega|W), \cdots \right) \)

   if and only if \( G\left( \cdots, \left( \frac{\partial}{\partial W} \right)^J \varphi(\mathcal{M}) \left[ A_{\alpha,\mathcal{M}} \right] (\Omega|W), \cdots \right) \in T(\Omega) \).

**Proof.** It is enough to assume that \( G\left( \cdots, \Delta^J \varphi(\mathcal{M}) \left[ A_{\alpha,\mathcal{M}} \right] (\Omega|W), \cdots \right) \) belongs to \( A_{\mathcal{M}}(\Omega) \) for some \( \mathcal{M} \in \mathcal{M}(h) \). Suppose

\[ G\left( \cdots, \Delta^J \varphi(\mathcal{M}) \left[ A_{\alpha,\mathcal{M}} \right] (\Omega|W), \cdots \right) = 0. \]

By putting \( Z = 0 \), we obtain

\[ G\left( \cdots, \left( \frac{\partial}{\partial W} \right)^J \varphi(\mathcal{M}) \left[ A_{\alpha,\mathcal{M}} \right] (\Omega|W), \cdots \right) = 0. \]

Conversely, we suppose that \( G\left( \cdots, \left( \frac{\partial}{\partial W} \right)^J \varphi(\mathcal{M}) \left[ A_{\alpha,\mathcal{M}} \right] (\Omega|W), \cdots \right) = 0. \)
According to Theorem 2, we may write

\[(3.1) \quad G \left( \cdots, \Delta^J \varphi(\mathcal{M}) \left[ \begin{array}{c} A_{\alpha,\mathcal{M}} \\ 0 \end{array} \right] (\Omega|W), \cdots \right) = \sum_K \Delta^K \phi_K(W),\]

where \( \phi_K \in T_{\mathcal{M}}(\Omega) \). Then we have

\[
\sum_K \left( \frac{\partial}{\partial W} \right)^K \phi_K(W) = G \left( \cdots, \Delta^J \varphi(\mathcal{M}) \left[ \begin{array}{c} A_{\alpha,\mathcal{M}} \\ 0 \end{array} \right] (\Omega|W), \cdots \right) \bigg|_{z=0} = G \left( \cdots, \left( \frac{\partial}{\partial W} \right)^J \varphi(\mathcal{M}) \left[ \begin{array}{c} A_{\alpha,\mathcal{M}} \\ 0 \end{array} \right] (\Omega|W), \cdots \right) = 0.
\]

Therefore it suffices to show \( \phi_K(W) = 0 \) under the condition

\[
\sum_K \left( \frac{\partial}{\partial W} \right)^K \phi_K(W) = 0 \quad \text{and} \quad \phi_K \in T_{\mathcal{M}}(\Omega).
\]

For each \( \xi \in \mathbb{Z}^{(h,\varrho)} \), we get

\[
\phi_K(W + \xi \Omega) = \exp \left\{ -\pi i \sigma(\mathcal{M}(\xi \Omega^t \xi + 2W^t \xi)) \right\} \phi_K(W)
\]

and

\[
\sum_K \left( \frac{\partial}{\partial W} \right)^K \phi_K(W + \xi \Omega) = \sum_K \left( \frac{\partial}{\partial W} \right)^K \left[ \exp \left\{ -\pi i \sigma(\mathcal{M}(\xi \Omega^t \xi + 2W^t \xi)) \right\} \phi_K(W) \right] = \exp \left\{ -\pi i \sigma(\mathcal{M}(\xi \Omega^t \xi + 2W^t \xi)) \right\} \times \sum_K \sum_P \binom{K}{P} \left( -2\pi i M \xi \right)^P \left( \frac{\partial}{\partial W} \right)^{K-P} \phi_K(W).
\]

Here if \( K = (K_{ma}) \) and \( P = (P_{ma}) \) in \( \mathbb{Z}_{\geq 0}^{(h,\varrho)} \), we put

\[
\binom{K}{P} := \binom{K_{11}}{P_{11}} \cdots \binom{K_{ma}}{P_{ma}} \cdots \binom{K_{hg}}{P_{hg}}
\]

and if \( \mathcal{M} = (M_{kl}) \) and \( \xi = (\xi_{ma}) \in \mathbb{Z}^{(h,\varrho)} \), we put

\[
(-2\pi i M \xi)^P := \left( -2\pi i \sum_{l=1}^h M_{1l} \xi_{1l} \right)^{P_{11}} \cdots \left( -2\pi i \sum_{l=1}^h M_{hl} \xi_{lg} \right)^{P_{hg}}.
\]
Thus we have

\[ (3.2) \quad \sum_K \sum_P \binom{K}{P} (-2\pi i M \xi)^P \left( \frac{\partial}{\partial W} \right)^{K-P} \phi_K(W) = 0 \]

for all \( \xi \in \mathbb{Z}^{(h,g)} \). Let \( K_0 \) be one of maximal \( K \) in the above sum (3.2). Then the coefficients of \( \xi^{K_0} \) in the polynomial relation (3.2) in \( \xi \) is given by \( C(M) \phi_{K_0}(W) \) with nonzero constant \( C(M) \neq 0 \). Thus we get \( \phi_{K_0}(W) = 0 \). Continuing this process successively, we have \( \phi_K = 0 \) for all \( K \) appearing in the sum (3.1). Hence from (3.1), we have

\[ G \left( \cdots, \Delta^J \varphi^{(M)} \left[ A_{\alpha,M} \right] (\Omega|W), \cdots \right) = 0. \]

We assume that

\[ G \left( \cdots, \Delta^J \varphi^{(M)} \left[ A_{\alpha,M} \right] (\Omega|W), \cdots \right) = G \left( \cdots, \left( \frac{\partial}{\partial W} \right)^J \varphi^{(M)} \left[ A_{\alpha,M} \right] (\Omega|W), \cdots \right). \]

Then we have, for any \( (\xi, \eta) \in \mathbb{Z}^{(h,g)} \times \mathbb{Z}^{(h,g)} \),

\[ G \left( \cdots, \left( \frac{\partial}{\partial W} \right)^J \varphi^{(M)} \left[ A_{\alpha,M} \right] (\Omega|W), \cdots \right) \bigg|_{W \mapsto W + \xi \Omega + \eta} = G \left( \cdots, \Delta^J \varphi^{(M)} \left[ A_{\alpha,M} \right] (\Omega|W), \cdots \right) \bigg|_{(Z,W) \mapsto (Z + \xi, W + \xi \Omega + \eta)} = \exp \left\{ -\pi i \sigma(M(\xi \Omega^t \xi + 2W \xi)) \right\} G \left( \cdots, \Delta^J \varphi^{(M)} \left[ A_{\alpha,M} \right] (\Omega|W), \cdots \right) \]

Therefore we obtain

\[ G \left( \cdots, \left( \frac{\partial}{\partial W} \right)^J \varphi^{(M)} \left[ A_{\alpha,M} \right] (\Omega|W), \cdots \right) \in T_M(\Omega). \]

Conversely, we assume that

\[ G \left( \cdots, \left( \frac{\partial}{\partial W} \right)^J \varphi^{(M)} \left[ A_{\alpha,M} \right] (\Omega|W), \cdots \right) \in T_M(\Omega). \]
Applying (1) to
\[ F \left( \ldots, \Delta^J \vartheta^{(\mathcal{M})} \left[ A_{\alpha,\mathcal{M}} \right] 0 \right) (\Omega|W), \ldots \) \\
= G \left( \ldots, \Delta^J \vartheta^{(\mathcal{M})} \left[ A_{\alpha,\mathcal{M}} \right] 0 \right) (\Omega|W), \ldots \)
we obtain
\[ F \left( \ldots, \Delta^J \vartheta^{(\mathcal{M})} \left[ A_{\alpha,\mathcal{M}} \right] 0 \right) (\Omega|W), \ldots \) = 0.
Hence we get
\[ G \left( \ldots, \Delta^J \vartheta^{(\mathcal{M})} \left[ A_{\alpha,\mathcal{M}} \right] 0 \right) (\Omega|W), \ldots \) \\
= G \left( \ldots, \left( \frac{\partial}{\partial W} \right)^J \vartheta^{(\mathcal{M})} \left[ A_{\alpha,\mathcal{M}} \right] 0 \right) (\Omega|W), \ldots \).

Combining Theorem 2 and Theorem 3, we obtain the decomposition theorem.

**THEOREM 4.** The algebra \( \mathbb{C} \left[ \ldots, \left( \frac{\partial}{\partial W} \right)^J \vartheta^{(\mathcal{M})} \left[ A_{\alpha,\mathcal{M}} \right] 0 \right) (\Omega|W), \ldots \) of differential polynomials of theta functions has a canonical linear basis
\[
(3.3) \quad \left\{ \left( \frac{\partial}{\partial W} \right)^J \vartheta^{(\mathcal{M})} \left[ A_{\alpha,\mathcal{M}} \right] 0 \right) (\Omega|W) \mid J \in \mathbb{Z}_{\geq 0}^{(h, g)}, A_{\alpha,\mathcal{M}} \in \mathcal{L}_{\mathcal{M}}, \mathcal{M} \in \mathcal{M}(h) \right\},
\]
namely, differential polynomials of theta functions are uniquely expressed as linear combinations of (3.3) with constant coefficients depending only on \( \Omega \).

**REMARK 3.1.** In [M3], Morikawa proved the decomposition theorem on differential polynomials of theta functions in the case that \( h = 1 \). He investigated the graded algebras of theta functions and of auxiliary theta functions:
\[
\Theta_0(\Omega) = \sum_{n=1}^{\infty} \Theta_0^{(n)}(\Omega), \quad \Theta(\Omega) = \sum_{n=1}^{\infty} \Theta^{(n)}(\Omega),
\]
where \( \Theta_0^{(n)}(\Omega) \) (resp. \( \Theta^{(n)}(\Omega) \)) denotes the vector space of theta functions (resp. auxiliary theta functions) of level \( n \) with respect to \( \Omega \). In this paper, when \( h = 1 \), we investigated the following graded algebras
\[
T(\Omega) = \sum_{n=1}^{\infty} \Theta_0^{(2n)}(\Omega), \quad A(\Omega) = \sum_{n=1}^{\infty} \Theta^{(2n)}(\Omega)
\]
of theta functions and of auxiliary theta functions of even level with respect to $\Omega$.

References