Asymptotic points of meromorphic functions on a maximal noncompact Riemann surface

By Qiu SHUXI*

(Received January 18, 1993)

§1. Introduction

Let $R \cup \alpha$ be a noncompact bordered Riemann surface with a compact border α (here and hereafter the case $\alpha = \emptyset$ is included, i.e., we consider an open Riemann surface R as a bordered surface $R \cup \emptyset$ with degenerate border $\emptyset$). For a function $w = f(p)$ in the class $M(R \cup \alpha)$ of non-constant single-valued meromorphic functions on $R \cup \alpha$, let $A_R(f, \beta)$ be the asymptotic set of f at the ideal boundary β of $R \cup \alpha$. Denote by $n_f(w)$ the valence function^[8] of f with the multiple points counted repeatedly, and by $n_f^*(w)$ the modified valence function of f with the multiple points counted only once.

In this paper we will give some extensions of the following proposition (see [1]):

PROPOSITION. *An open Riemann surface R is maximal if and only if for any $w = f(p) \in M(R)$, any disc G on the complex w -sphere and any component Δ of the inverse image $f^{-1}(G)$, where $f(\Delta) \neq G$, the following statements are true:*

- 1) $n_{f|_{\Delta}}^*(w) \equiv 1$ in $f(\Delta)$; and
- 2) If $1 < n_{f|_{\Delta}}^*(w) \equiv \text{constant} < +\infty$ in $f(\Delta)$, then $G \setminus f(\Delta)$ is not an isolated set,

where $f|_{\Delta}$ is the restriction of f to Δ .

In order to give the explanation of the proposition, a maximal open Riemann surface is constructed as follows:

Let I be the closed unit interval in the complex sphere $S = \{z \mid |z| \leq \infty\}$ and E be a Cantor set in I . Then $I \setminus E$ is a union of the disjoint open intervals I_{nj} ($n = 1, 2, \dots; j = 1, 2, \dots, 2^{n-1}$) in I , where $\text{mes}(I_{n+1, j'}) < \text{mes}(I_{nj})$ ($n = 1, 2, \dots$) ("mes" means the Lebesgue measure). Take four copies $S_i = S \setminus E$ with the slits along $I_{nj}^i = I_{nj}$ ($i = 1, 2, 3, 4$). Denote by I_{nj}^{i+} and I_{nj}^{i-} the upper-edge and the

*The author would like to express his gratefulness to professors M. Kishi, M. Ito, M. Sakai and T. Murai for their helpful discussions during his visit to Japan in 1992.

lower-edge of the slits along I_{nj}^i , respectively.

If n is odd, we identify I_{nj}^{1+} with I_{nj}^{2-} , I_{nj}^{1-} with I_{nj}^{2+} , I_{nj}^{3+} with I_{nj}^{4-} and I_{nj}^{3-} with I_{nj}^{4+} , respectively; if n is even, then identify I_{nj}^{1+} with I_{nj}^{2-} , I_{nj}^{2+} with I_{nj}^{3-} , I_{nj}^{3+} with I_{nj}^{4-} and I_{nj}^{4+} with I_{nj}^{1-} , respectively, where $j = 1, 2, \dots, 2^{n-1}$. Finally we get a Riemann surface W which consists of 4-sheet regular covering surface of $S \setminus E$.

If E is of zero measure, then $E \in N_D$ and W is non-prolongable according to the theorem in [2]. And we have

EXAMPLE 1. There exists a maximal open Riemann surface which is a finite-sheet covering surface over the complex sphere and has no branch points in it.

But we have also

EXAMPLE 2. There exists a prolongable open Riemann surface which is a finite-sheet covering surface over the complex sphere, and whose every ideal boundary, in Kerekjarto-Stoilow's sense, is positive point-like asymptotic point (see Definition 3).

PROOF. In constructing the above-mentioned surface W , if E is of positive measure and $E \in N_D$, then according to Theorem 4.1 in [5], W is prolongable because any peninsula of W over the upper half-plane of S is a disk of W with a crowded ideal boundary^[5]. Q.E.D.

It is pointed out that the Riemann surface mentioned above is different from the one we know well, for instance, Toki's example^[8], where the branch points are necessary for a covering surface with positive genus of the unit disc.

§2. Definitions for asymptotic points

Consider a simply connected analytic Jordan region G (briefly, nice region or nice neighborhood) on the complex sphere $K = \{w \mid |w| \leq \infty\}$. Let Δ be a non-compact component of $f^{-1}(G) = \{p \mid p \in R \cup \alpha, f(p) \in G\}$. Clearly, the condition

$$n_{f|\Delta}^*(w) \equiv \text{constant} < \infty \text{ in } f(\Delta)$$

implies that the Riemann surface over G , generated by $f|_{\Delta}$ on Δ , has no branch points and covers every point of $f(\Delta)$ the same number of times.

Let Δ be a noncompact subregion of $R \cup \alpha$, and δ its relative boundary in $R \cup \alpha$. Denote by $A_{\Delta}(f|_{\Delta}, \beta_{\Delta})$ the relative asymptotic set of f at the relative ideal boundary β_{Δ} of Δ , where $f|_{\Delta}$ is the restriction of f to Δ . For a point $w \in A_R(f, \beta)$, let

$$\Sigma(w) = \{\Delta \mid w \in A_{\Delta}(f|_{\Delta}, \beta_{\Delta}), \text{ distance } (w, f(\delta)) > 0\}.$$

Each Δ of $\Sigma(w)$ is called a w -peninsula.

DEFINITION 1. A sequence $\{\Delta_n\}$ in $\Sigma(w)$ is called a w -peninsula sequence if $\{\Delta_n\}$ satisfies the following:

- i) $\Delta_{n+1} \subset \Delta_n$, $n = 1, 2, \dots$; and
- ii) $\text{diameter } [f(\Delta_n)] \rightarrow 0$, $n \rightarrow \infty$.

Two w -peninsula sequences $\{\Delta_n\}$ and $\{\Delta'_m\}$ in $\Sigma(w)$ are said to be equivalent if for any n , there exists m such that $\Delta_n \supset \Delta'_m$ and vice versa. A class of the equivalent w -peninsula sequences is called a w -asymptotic point of f on $R \cup \alpha$.

For a w -asymptotic point $\sigma(w)$ of f , we denote $\sigma(w) = \{\Delta_n\}$ to mean that $\{\Delta_n\}$ is a determining sequence of $\sigma(w)$.

DEFINITION 2. Let $\sigma(w) = \{\Delta_n\}$ be a w -asymptotic point. If

$$\lim_{n \rightarrow \infty} \sup n_{f|\Delta_n}(w) = \infty,$$

then $\sigma(w)$ is called a singular w -asymptotic point. Otherwise, $\sigma(w)$ is called a finite w -asymptotic point of f .

We see easily that if $\sigma(w) = \{\Delta_n\}$ is a finite w -asymptotic point, then there is a number $N > 0$ such that for any $n > N$, the Riemann surface $\tilde{\Delta}_n$ generated by f from Δ_n is of the same finite number of sheets.

DEFINITION 3. Let $\sigma(w) = \{\Delta_n\}$ be a finite w -asymptotic point of f . If there exists an ideal boundary point ρ of $R \cup \alpha$ with a determining sequence $\{\Omega_n\}$ (briefly, $\rho = \{\Omega_n\}$) in Kerekjarto-Stoilow's sense such that for any n , there exists m such that $\Delta_n \supset \Omega_m$, then $\sigma(w)$ is called a point-like asymptotic point (briefly, pl-asymptotic point). Moreover, if each Ω_m of $\{\Omega_m\}$ is of positive genus, then $\sigma(w)$ is called a positive point-like asymptotic point (briefly, ppl-asymptotic point).

For an ideal boundary point $\rho = \{\Omega_m\}$ of $R \cup \alpha$ in Kerekjarto-Stoilow's sense, if the cluster set of f at ρ is composed of a single point $v \in K$, i.e., $C_R(f, \rho) = \{v\}$, then evidently there is a point-like asymptotic point corresponding to ρ .

Let Ω be a subregion of R . If there is a determining sequence $\{\Delta_n\}$ of an asymptotic point $\sigma(w)$ of f such that $\bigcup_{n=1}^{\infty} \Delta_n \subset \Omega$, then we say that Ω contains $\sigma(w)$.

DEFINITION 4. Let $\sigma(w) = \{\Delta_n\}$ be a w -asymptotic point of f . If for any n , Δ_n contains at least a branch point (resp. a pl-asymptotic point, a ppl-asymptotic point) of f , then $\sigma(w)$ is called a limit point of branch points (resp. pl-asymptotic points, ppl-asymptotic points) of f .

§3. Main results

THEOREM 1. Let $R \cup \alpha$ be a noncompact bordered Riemann surface with a

compact border α . Then $R \cup \alpha$ is maximal if and only if for any $w = f(p) \in M(R \cup \alpha)$, each asymptotic point of f is one of the following:

- i) a singular asymptotic point of f ;
- ii) a limit point of the branch points of f ;
- iii) a limit point of the ppl-asymptotic points of f .

From Theorem 1 we may easily obtain

THEOREM 2. *Let $R \cup \alpha$ be an maximal noncompact bordered Riemann surface spread over the complex w -sphere K . Then any ideal boundary ρ with a determining sequence $\{\Omega_n\}$, in Kerekjarto-Stoilow's sense, is one of the following:*

- i) a point with $\lim_{n \rightarrow \infty} \sup n_{\pi|\Omega_n}(w) = \infty$;
- ii) a limit point of the branch points of π ;
- iii) a limit point of the ppl-asymptotic points of π ,

where π is the projection mapping of R into K .

THEOREM 3. *Let $R \cup \alpha$ be a noncompact essentially maximal bordered Riemann surface spread over the complex w -sphere K . Then $R \cup \alpha$ is maximal if and only if for any noncompact sub-end $\Omega \cap \eta$ of $R \cup \alpha$, one of the following alternatives holds:*

- i) $\sup n_{\pi|\Omega}(w) = \infty$;
- ii) Ω contains at least a branch points of π ;
- iii) Ω contains at least a ppl-asymptotic point of π ,

where π is the projection mapping of R into K .

Incidentally, it should be pointed out that Tsuji's proposition on non-prolongable Riemann surfaces [6. p.505] is a particular situation of Theorem 3 since the Riemann surfaces in the proposition belong to O_G , of which each surface is essentially maximal.

§4. Asymptotic set and valence function

LEMMA 1. *Let γ be a nondegenerate continua nowhere dense in the complex sphere. For a point $z' \in \gamma$, if there exists a number $\varepsilon_0 > 0$ such that for any positive number $\varepsilon < \varepsilon_0$, $G(z', \varepsilon) \setminus \gamma$ is disconnected, where $G(z', \varepsilon)$ is the disc $\{z \mid |z - z'| < \varepsilon\}$, then z' is called a C -type point of γ . The set of all C -type points of γ is dense everywhere in γ .*

The proof is omitted.

LEMMA 2. *Let $w = f(p) \in M(R \cup \alpha)$ and assume $\sup n_f(w) < \infty$. If $A_R(f, \beta)$ contains a nondegenerate continua γ , then there is a subregion Δ of R which satisfies the following:*

- i) the asymptotic set $A_\Delta(f \mid \Delta, \beta_\Delta)$ of $f \mid \Delta$ at the relative ideal boundary β_Δ of Δ contains a nondegenerate continua; and
- ii) $n_{f|\Delta}(w) \equiv \text{constant} < \infty$ in $f(\Delta)$, except possibly for a totally disconnected set.

PROOF. Let $m = \sup n_f(w)$. Take a C -type point $w_0 \in \gamma$, then there exists a sufficiently small number $\varepsilon > 0$ such that $G_\varepsilon(w_0) \setminus \gamma$ is not connected, where $G_\varepsilon(w_0) = \{w \mid |w - w_0| < \varepsilon\}$. So there exists a noncompact component Δ_1 of $f^{-1}[G_\varepsilon(w_0)]$ such that the relative asymptotic set $A_{\Delta_1}(f \mid \Delta_1, \beta_{\Delta_1})$ contains a nondegenerate sub-continua of γ . If Δ_1 fulfils both conditions i) and ii), then the Lemma is proved. Otherwise, there exists at least a component of $A_{\Delta_1}(f \mid \Delta_1, \beta_{\Delta_1})$ which contains a nondegenerate continua γ_1 such that $\gamma_1 \cap f(\Delta_1) \neq \emptyset$. Choose a point $w_1 \in \gamma_1 \cap f(\Delta_1)$ and sufficiently small number $\varepsilon_1 > 0$ such that $G_{\varepsilon_1}(w_1) \setminus \gamma_1$ is not connected and there exists a compact component of $f^{-1}[G_{\varepsilon_1}(w_1)]$ in Δ_1 , where $G_{\varepsilon_1}(w_1) = \{w \mid |w - w_1| < \varepsilon_1\}$. Therefore, there is a noncompact component Δ_2 of $f^{-1}[G_{\varepsilon_1}(w_1)]$ such that $A_{\Delta_2}(f \mid \Delta_2, \beta_{\Delta_2})$ contains a nondegenerate sub-continua of γ_1 and $\sup n_{f|\Delta_2}(w) \leq m - 1$.

If Δ_2 satisfies both conditions i) and ii), then the Lemma is fulfilled. Otherwise, by repeating the same procedure for less than $m - 2$ times, we can finally get a subregion Δ of R such that both conditions i) and ii) are fulfilled. Q.E.D.

LEMMA 3. Let $w = f(p) \in M(R \cup \alpha)$ and assume $\sup n_f(w) < \infty$. If $A_R(f, \beta)$ is a totally disconnected set, then there exists a subregion Δ of R with a compact relative boundary in R such that $n_{f|\Delta}(w) \equiv \text{constant} < \infty$ in $f(\Delta)$, and $A_\Delta(f \mid \Delta, \beta_\Delta)$ is totally disconnected.

PROOF. Let $m = \sup n_f(w)$. Take $w \in A_R(f, \beta)$ and a sufficiently small nice neighborhood G of w with a compact boundary σ such that $\sigma \cap A_R(f, \beta) = \emptyset$ and each noncompact component Δ' of $f^{-1}(G)$ satisfies the condition $\Delta' \cap \alpha = \emptyset$. If no noncompact component Δ' of $f^{-1}(G)$ is what we want, take one of them and denote it by Δ_1 . Then, there is a point $w_1 \in A_R(f, \beta)$ such that $f^{-1}(w_1) \cap \Delta_1 = \{p_1, p_2, \dots, p_k\}$, $0 < k < m$. Choose a sufficiently small nice neighborhood G_1 of w_1 with a compact boundary σ_1 such that $\sigma_1 \cap A_R(f, \beta) = \emptyset$ and there are just k compact components of $f^{-1}(G_1)$ in Δ_1 . Evidently there is at least one noncompact component Δ_2 of $f^{-1}(G_1)$ such that $\sup n_{f|\Delta_2}(w) \leq m - k$. If Δ_2 does not satisfy the requirement, then there exists a point $w_2 \in A_R(f, \beta)$ such that $f^{-1}(w_2) \cap \Delta_2 = \{q_1, q_2, \dots, q_l\}$, $0 < l < m - k$. By repeating the procedure mentioned above at most $m - k - 1$ times, since $n_f(w) < \infty$, we can find the desired subregion of R .

Q.E.D.

LEMMA 4. Let $w = f(p) \in M(R \cup \alpha)$. If there is a noncompact subregion Ω of R such that $n_{f|\Omega}(w) \equiv \text{constant} < \infty$ in $f(\Omega)$ except for a non-empty totally

disconnected set E , then there is a subregion Δ with a compact relative boundary in R such that the relative asymptotic set of Δ is totally disconnected.

PROOF. From the given condition we have $E \subset f(\Omega) \cap A_\Omega(f | \Omega, \beta_\Omega)$, where β_Ω is the relative ideal boundary of Ω . Take arbitrary point $w \in E$. We can choose a sufficiently small nice neighborhood G of w with a compact boundary σ contained entirely in $f(\Omega) \setminus E$ such that there exists at least one compact component of $f^{-1}(G)$ in Ω and $G \cap A_\Omega(f | \Omega, \beta_\Omega)$ is non-empty and totally disconnected. As a result there is a noncompact component Δ of $f^{-1}(G)$ such that its asymptotic set is totally disconnected and its relative boundary in R is compact. Q.E.D.

From Lemmas 2, 3, and 4, we have

LEMMA 5. Let $w = f(p) \in M(R \cup \alpha)$ and assume $\sup n_f(w) < \infty$. Then there exists a noncompact subregion Δ of R such that $n_{f|\Delta}(w) \equiv \text{constant} < \infty$ in $f(\Delta)$.

Now we discuss a bordered Riemann surface which has a subregion Δ with property

$$n_{f|\Delta}^*(w) \equiv \text{constant} < \infty \text{ in } f(\Delta).$$

LEMMA 6. Let $w = f(p) \in M(R \cup \alpha)$ and G be a nice region on the complex sphere K . If there exists a noncompact component Δ of $f^{-1}(G)$ with $\Delta \cap \alpha = \emptyset$ such that

$$n_{f|\Delta}^*(w) \equiv \text{constant} < \infty \text{ in } f(\Delta),$$

then $(\Delta, f|_\Delta)$ is a regular covering surface of $f(\Delta)$. Furthermore, if $G \setminus f(\Delta)$ contains an isolated point, then $R \cup \alpha$ is prolongable.

PROOF. Let $m = n_{f|\Delta}^*(w)$. Suppose that $(\Delta, f|_\Delta)$ is not a regular covering surface of $f(\Delta)$, then $A_\Delta(f|_\Delta, \beta_\Delta) \cap f(\Delta) \neq \emptyset$. Choose a point $w_0 \in A_\Delta(f|_\Delta, \beta_\Delta) \cap f(\Delta)$ and a sufficiently small nice neighborhood G of w_0 such that there are m simply connected compact components of $f^{-1}(G)$ in Δ for $n_{f|\Delta}^*(w_0) = m$. But there must exist at least one noncompact component Δ_0 of $f^{-1}(G)$ in Δ for w_0 is an asymptotic value of f . Take arbitrary point $p_0 \in \Delta_0$ and let $w' = f(p_0)$, then we have $w' \in G$ and $n_{f|\Delta}^*(w') \geq m + 1$, which is absurd. Q.E.D.

LEMMA 7. Let $R \cup \alpha$ be maximal. Let $w = f(p) \in M(R \cup \alpha)$ and G be a nice region on the complex sphere K with $G \cap A_R(f, \beta) \neq \emptyset$. For a noncompact component Δ of $f^{-1}(G)$ with $\Delta \cap \alpha = \emptyset$, if

$$n_{f|\Delta}^*(w) \equiv \text{constant} < \infty \text{ in } f(\Delta),$$

then

- i) $f(\Delta)$ is not simply connected; and

ii) $(\Delta, f | \Delta)$ is neither the weakest nor the strongest covering surface of $f(\Delta)$.

PROOF. By Lemma 6, $(\Delta, f | \Delta)$ is a regular covering surface of $f(\Delta)$.

If $f(\Delta)$ is simply connected, then Δ is simply connected, too. So $R \cup \alpha$ is prolongable.

If $(\Delta, f | \Delta)$ is the weakest covering surface of $f(\Delta)$, then $f | \Delta$ is identity mapping and Δ is planar. So $R \cup \alpha$ is prolongable as Δ is not compact. Moreover, by i) and the condition for Δ we have that $(\Delta, f | \Delta)$ is not the strongest covering surface of $f(\Delta)$. Q.E.D.

LEMMA 8. Let $w = f(p) \in M(R \cup \alpha)$ and G be a nice region with $G \cap A_R(f, \beta) \neq \emptyset$ on the complex sphere K . If there exists a noncompact component Δ of $f^{-1}(G)$ with $\Delta \cap \alpha = \emptyset$ such that the following conditions are fulfilled:

- a) $n_{f|\Delta}^*(w) \equiv \text{constant} < \infty$ in $f(\Delta)$; and
- b) every component of $G \setminus f(\Delta)$ contains a nondegenerate continua, then $R \cup \alpha$ is prolongable.

PROOF. Suppose that $R \cup \alpha$ is maximal. We will derive a contradiction.

Notice that $G \setminus f(\Delta) \supset G \cap A_\Delta(f | \Delta, \beta_\Delta)$ in this case. Take one of the components of $G \setminus f(\Delta)$, then, obviously, its boundary contains a nondegenerate continua γ in G .

By Lemma 1, we can take a point $w_0 \in \gamma$ and a small enough positive number $\varepsilon_1 < 1$ such that $G'_1(w_0) \setminus \gamma$ is not connected, where $G'_1(w_0) = \{w \mid |w - w_0| < \varepsilon_1\}$. From condition b) we have that for any component D of $G'_1(w_0) \cap f(\Delta)$, each component Ω of $f^{-1}(D)$, which is contained entirely in Δ , satisfies conditions

$$n_{f|\Omega}^*(w) \equiv \text{constant} < \infty \text{ in } f(\Omega)$$

and $f(\Omega) = D$. So D is not simply connected by Lemma 7.

Now we choose one of the components of $G'_1(w_0) \cap f(\Delta)$ and denote it by D_1 . We draw an analytic Jordan curve σ_1 in D_1 such that σ_1 surrounds a part η_1 of the inner boundary of D_1 and σ_1 is contained entirely in D_1 . Denote by G_1 the simply connected region bounded by σ_1 in K which contains η_1 . By condition b), η_1 must contain a nondegenerate continua $\gamma_1 \subset G \cap A_\Delta(f | \Delta, \beta_\Delta)$.

Again by Lemma 1, we choose a point $w_1 \in \gamma_1$ and a small enough positive number $\varepsilon_2 < 1/2$ such that $G'_2(w_1) \setminus \gamma_1$ is not connected, where $G'_2(w_1) = \{w \mid |w - w_1| < \varepsilon_2\} \subset G_1$. By Lemma 7, any component of $G'_2(w_1) \cap f(\Delta)$ is not simply connected, whereby we take one of them and denote it by D_2 . We draw again an analytic Jordan curve σ_2 in D_2 such that σ_2 surrounds a part η_2 of the inner boundary of D_2 and σ_2 is contained entirely in D_2 . Denote by G_2 the simply connected region bounded by σ_2 in K which contains η_2 . By condition b), η_2 must contain a nondegenerate continua $\gamma_2 \subset G \cap A_\Delta(f | \Delta, \beta_\Delta)$. Evidently we have $G_2 \subset G'_2(w_1) \subset G_1$.

Consequently, by repeating the procedure mentioned above, we could get a sequence $\{G_n\}$ of simply connected subregions of K which satisfies the following properties:

- 1) $G_{n+1} \subset G_n$, and diameter $(G_n) < 1/n$;
- 2) $G_n \cap A_\Delta(f | \Delta, \beta_\Delta) \neq \emptyset$;
- 3) the boundary of G_n is a compact analytic Jordan curve σ_n , contained entirely in $G_{n-1} \cap f(\Delta)$, where $G_0 = G$, $n = 1, 2, \dots$.

In the union of the components of $f^{-1}(G_n)$ ($n = 1, 2, \dots$), by 1), 2) and 3), we can choose a sequence $\{\Delta_n \cup \delta_n\}$ of bordered subregions of R satisfying the following conditions:

- 1') Each Δ_n is a noncompact end of R whose relative boundary δ_n in R is compact and $f(\delta_n) = \sigma_n$; and
- 2') $\Delta_{n+1} \subset \Delta_n$, and diameter $[f(\Delta_n)] < 1/n$, $n = 1, 2, \dots$.

Evidently, $\bigcap_{n=1}^{\infty} \overline{f(\Delta_n)} = \bigcap_{n=1}^{\infty} \overline{G_n}$ contains only one point of K , which is an asymptotic value and consists of a component of $G \setminus f(\Delta)$. This is contradiction to b).
Q.E.D.

LEMMA 9. *Let $R \cup \alpha$ be maximal. Let $w = f(p) \in M(R \cup \alpha)$ and G be a nice region on the complex sphere K . If there exists a noncompact component Δ with $\Delta \cap \alpha = \emptyset$ of $f^{-1}(G)$ such that conditions*

- a) $G \cap A_\Delta(f | \Delta, \beta_\Delta) \neq \emptyset$; and
- b) $n_{f|\Delta}^*(w) \equiv \text{constant} < \infty$ in $f(\Delta)$

are fulfilled, then Δ contains at least a ppl-asymptotic point of f on $R \cup \alpha$.

PROOF. There exists at least a single-point component $\{w_0\}$ in $G \setminus f(\Delta)$ by Lemma 8. And w_0 is not isolated in $G \setminus f(\Delta)$ by Lemma 6. So for any sufficiently small disc G_ε centered at w_0 and any noncompact component Δ_ε of $f^{-1}(G_\varepsilon)$, $f(\Delta_\varepsilon)$ is not simply connected by Lemma 7.

By reasoning similarly to the proof of Lemma 8, we can get a sequence $\{\Delta_n \cup \delta_n\}$ of bordered sub-ends of R satisfying the conditions 1') and 2') in the proof of Lemma 8. Obviously, $\{\Delta_n\}$ is both a determining sequence of a point-like asymptotic point and that of an ideal boundary point of $R \cup \alpha$ in Kerekjarto-Stoilow's sense. Apparently, every Δ_n is not planar as $R \cup \alpha$ is maximal. So $\{\Delta_n\}$ is a ppl-asymptotic points of f .
Q.E.D.

§5. Proof of Theorem 1

The Sufficiency is obvious.

NECESSITY. Let $w \in A_R(f, \beta)$, and $\sigma(w) = \{\Delta_n\}$ be an asymptotic point of

f . Let

$$\lim_{n \rightarrow \infty} \sup n_{f|_{\Delta_n}}(w) < \infty,$$

where $f|_{\Delta_n}$ is the restriction of f to Δ_n . If $\sigma(w)$ is not a limit point of branch points of f , then there is a number $N > 0$ such that for any $n > N$, we have

$$\sup n_{f|_{\Delta_n}}(w) < \infty$$

and there are no branch points of f in Δ_n . By Lemma 5, there exists a noncompact subregion Δ'_n of Δ_n such that

$$n_{f|_{\Delta'_n}}^*(w) \equiv \text{constant} < \infty \text{ in } f(\Delta'_n).$$

By Lemma 9, Δ'_n contains at least a ppl-asymptotic point of f since $R \cup \alpha$ is maximal. Q.E.D.

References

- [1] Qiu Shuxi, Properties at the ideal boundary of maximal Riemann surfaces, Acta Sci. Nat. Univ., Xiamen, **27** (1988), 377–380.
- [2] J. Tamura, On a theorem of Tsuji, Japan J. Math., **29** (1959), 138–140.
- [3] M. Jurchescu, Bordered Riemann surfaces, Math. Ann., **143** (1961), 264–292.
- [4] H. Renggli, On maximal Riemann surfaces, Amer. J. Math., **88** (1966), 179–186.
- [5] M. Sakai, Continuations of Riemann surfaces, Canad. J. Math., **44** (2) (1992), 357–367.
- [6] M. Tsuji, Potential Theory in Modern Function Theory, Maruzen, Tokyo, 1959.
- [7] L.V. Ahlfors and L. Sario, Riemann Surfaces, Princeton University Press, 1960.
- [8] L. Sario and M. Nakai, Classification Theory of Riemann Surfaces, Springer, 1970.

DEPARTMENT OF MATHEMATICS
XIAMEN UNIVERSITY
XIAMEN 361005
P. R. OF CHINA