Substitution invariant Beatty sequences

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1. Introduction

We consider sequences (f_n) , where

$$f_n = [(n+1)\theta] - [n\theta], \quad n = 1, 2, \dots \quad (1)$$

with θ irrational and taken to satisfy $0 < \theta < 1$; plainly this may be assumed without loss of generality. Evidently (f_n) is a sequence of zeros and ones. Denote by w_0 and w_1 words on the alphabet $\{0, 1\}$; that is, finite strings in the letters 0 and 1. Then the sequence (f_n) is said to be invariant under the substitution W given by

$$W: 0 \longrightarrow w_0, \quad 1 \longrightarrow w_1,$$

if the infinite strings $f_\theta = f_1 f_2 f_3 \dots$ and $W(f_\theta) = w_{f_1} w_{f_2} w_{f_3} \dots$ coincide.

This situation was discussed by Brown [2] who displays non-trivial substitutions W for certain quadratic irrationals θ . Subsequently Crisp, Moran, Pollington and Shiue [3] gave a complete description of all θ admitting an invariant substitution. Their result agrees with the slightly different discussion of Ito and Yasutomi [5]. Crisp et al. obtain their result from the viewpoint of cutting sequences. Berstel and Séébold [1] give a shorter but similar proof by studying the composition of two particularly simple substitutions. In this note we provide a somewhat more mechanical but efficient and rather different argument.

2. Substitution invariance

Set $\theta = a_0 + \theta_0$, $a_0 = [\theta]$, and

$$1/\theta_{n-1} = a_n + \theta_n, \quad a_n = [1/\theta_{n-1}] \quad n = 1, 2, \dots,$$

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so that $\theta = [a_0, a_1, a_2, \dots]$ is the continued fraction expansion of θ . The n -th convergent $p_n/q_n = [a_0, a_1, \dots, a_n]$ of θ is given by the recurrence relations

$$\begin{aligned} p_n &= a_n p_{n-1} + p_{n-2} \quad (n = 0, 1, \dots), \quad p_{-2} = 0, \quad p_{-1} = 1 \\ q_n &= a_n q_{n-1} + q_{n-2} \quad (n = 0, 1, \dots), \quad q_{-2} = 1, \quad q_{-1} = 0. \end{aligned}$$

The Beatty sequence (1) has been studied and generalized by many authors. In particular, Fraenkel, Mushkin and Tassa show that

LEMMA 1 ([4]). Define the sequence (T_n) recursively by

$$T_0 = 0, \quad T_1 = 0^{a_1-1}1, \quad \text{and} \quad T_n = T_{n-1}^{a_n} T_{n-2} \quad \text{for } n \geq 2;$$

then T_n is a word of length q_n .

If θ is irrational,

$$f_1 f_2 f_3 \cdots = \lim_{n \rightarrow \infty} T_n.$$

If $\theta = p_N/q_N$ is rational, then without loss of generality one may take N even, and then

$$f_1 f_2 f_3 \cdots = T_N^\infty \quad (= T_N T_N \cdots).$$

In the light of this description of the sequence (f_n) the following is our main result:

THEOREM. Let θ be irrational, with characteristic word $f_\theta = \lim T_n$ and let W be a substitution on the alphabet $\{0, 1\}$. Then (i) $W(f_\theta) = f_\theta$ if and only if there is a nonnegative integer m such that $W : T_i \rightarrow T_{i+m}$ for each $i = 1, 2, \dots$ and (ii) there is such a substitution only if $a_{i+m} = a_i$ for $i = 3, 4, \dots$ and either $a_{2+m} = a_2$ or $a_1 = 1$.

PROOF. Plainly, if $T_i \rightarrow T_{m+i}$ for all $i = 1, 2, \dots$, then by Lemma 1

$$f_\theta = \lim_{n \rightarrow \infty} T_n \rightarrow \lim_{n \rightarrow \infty} T_{n+m} = f_\theta.$$

Conversely, suppose that $W(f_\theta) = f_\theta$. We fix a sufficiently large l and observe that there are $\lambda \leq \mu \leq \nu \leq \tau$ so that we may write

$$T_l \rightarrow T_\lambda^{c_\lambda} V_\lambda, \quad T_{l+1} \rightarrow T_\mu^{c_\mu} V_\mu, \quad T_{l+2} \rightarrow T_\nu^{c_\nu} V_\nu, \quad T_{l+3} \rightarrow T_\tau^{c_\tau} V_\tau, \dots$$

Here the $c_\lambda, c_\mu, \dots$ are positive integers and the remnants $V_\lambda, V_\mu, \dots$ are finite, possibly empty, words too short to allow replacement of any of the parameters $\lambda, \dots$ or $c_\lambda, \dots$ by a greater one. Plainly, invariance entails that

$$T_{l+2} = T_{l+1}^{a_{l+2}} T_l \rightarrow (T_\mu^{c_\mu} V_\mu)^{a_{l+2}} T_\lambda^{c_\lambda} V_\lambda = T_\nu^{c_\nu} V_\nu.$$

We interrupt the argument to mention the simple but very effective

LEMMA 2. *If the words u and v commute then there is a word w and non-negative integers m and n such that $u = w^m$ and $v = w^n$.*

PROOF. Suppose not. Let u be the shortest nonempty word for which there is a counterexample v , and let v be the shortest nonempty counterexample for that u . Then u is no longer than v . Looking at the last $|v|$ symbols in $uv = vu$ we see that $v = v'u$ for some, plainly nonempty, word v' shorter than v . We have $uv'u = v'u^2$, so $uv' = v'u$; thus, u commutes with v' . By hypothesis, w exists such that $u = w^m$ and $v' = w^n$. Then $v = w^{m+n}$, contradicting the hypothesis.

We are indebted to Gerry Myerson for this efficient formulation of the argument. Returning to the main argument, we note that if θ is irrational then T_i is not periodic — that is, it is not a power of some shorter word — so, by Lemma 2, T_i cannot commute with any nonempty shorter word, or, indeed, with any longer word that is not a power of T_i .

Suppose first that $\mu < \nu$. We note, incidentally, that this cannot lose generality by increasing l if necessary, for, we cannot have $T_\mu(l) = T_\nu(l)$ for every l . Then $T_\mu^{a_{\mu+1}}T_{\mu-1}$ is a prefix of T_ν and thus

$$(T_\mu^{c_\mu}V_\mu)^{a_{l+2}} \dots = T_\mu^{a_{\mu+1}}T_{\mu-1} \dots$$

It follows that V_μ must be empty, for the apparent alternative $V_\mu = T_{\mu-1}$ and $a_{\mu+1} = c_\mu$ is excluded because then $T_\mu^{c_\mu}V_\mu = T_{\mu+1}$, contradicting the condition defining the remnant V_μ .

On the other hand, suppose that $T_\mu = T_\nu$. Then

$$(T_\mu^{c_\mu}V_\mu)^{a_{l+2}}T_\lambda^{c_\lambda}V_\lambda = T_\mu^{c_\nu}V_\nu,$$

and if V_μ is not empty then $c_\mu = c_\nu$ and, because V_ν is shorter than T_μ , necessarily $a_{l+2} = 1$ so $V_\mu T_\lambda^{c_\lambda}V_\lambda = V_\nu$. Clearly we must have $\lambda < \mu$, since $c_\lambda = 0$ contradicts the definition of λ . Plainly then, we cannot have $T_\nu = T_\tau$, for, as just explained, that leads to $\mu < \nu$. On the other hand, $\nu < \tau$ entails V_ν is empty, which is absurd. Thus, in fact, always $\mu < \nu$.

The preceding argument similarly entails that both V_λ and V_ν are empty, so it follows that

$$T_\mu^{c_\mu a_{l+2}}T_\lambda^{c_\lambda} = T_\nu^{c_\nu}.$$

Since $\mu < \nu$, then necessarily $\lambda = \mu - 1 = \nu - 2$ and $c_\mu a_{l+2} = a_{\mu+1}$, $c_\lambda = c_\nu = 1$. Arguing similarly with l replaced by $l + 1$ we see that also $c_\mu = 1$ whence $a_{l+2} = a_{\mu+1} = a_{\lambda+2}$. That is, we have

$$W : T_l \longrightarrow T_\lambda, \quad T_{l+1} \longrightarrow T_{\lambda+1}, \quad T_{l+2} \longrightarrow T_{\lambda+2}, \dots$$

and $a_{l+2} = a_{\lambda+2}$, $a_{l+3} = a_{\lambda+3}, \dots$

Suppose however that $T_{l-1} \rightarrow U_{\lambda-1}$. Then

$$T_{l+1} = T_l^{a_{l+1}} T_{l-1} \rightarrow T_\lambda^{a_{l+1}} U_{\lambda-1} = T_\lambda^{a_{\lambda+1}} T_{\lambda-1}.$$

Clearly $a_{l+1} > a_{\lambda+1}$ is impossible, whereas if $a_{l+1} < a_{\lambda+1}$, then we must have $|U_{\lambda-1}| > q_\lambda = |T_\lambda|$. Thus, necessarily $a_{l+1} = a_{\lambda+1}$. Hence also $T_{l-1} \rightarrow T_{\lambda-1}$. Repeating this argument, we see that $T_{l-2} \rightarrow T_{\lambda-2}, \dots, T_1 \rightarrow T_{\lambda-l+1}$ and that $a_l = a_\lambda, \dots, a_3 = a_{\lambda-l+3}$.

COROLLARY. *An irrational θ admits a substitution W , so that $W(f_\theta) = f_\theta$, if and only if (a) $\theta = [0, 1, a_2, \overline{a_3, \dots, a_n}] > 1/2$ and $a_n \geq a_2$, when the string $f_\theta = f_1 f_2 f_3 \dots$ is invariant under the substitution*

$$W : 0 \rightarrow T_{n-1}^{a_n - a_2} T_{n-2}, \quad 1 \rightarrow T_{n-1},$$

or (b) $\theta = [0, a_1, \overline{a_2, \dots, a_n}] < 1/2$ and $a_n + 1 \geq a_1 \geq 2$, when the string $f_\theta = f_1 f_2 f_3 \dots$ is invariant under the substitution

$$W : 0 \rightarrow T_{n-1}, \quad 1 \rightarrow T_{n-1}^{a_n + 1 - a_1} T_{n-2}.$$

PROOF. It is easy to check case (a) and to confirm that

$$\begin{aligned} W : T_1 = 1 & \rightarrow T_{n-1} \\ W : T_2 = 1^{a_2} 0 & \rightarrow T_{n-1}^{a_2} T_{n-1}^{a_n - a_2} T_{n-2} = T_n \\ W : T_3 = T_2^{a_3} T_1 & \rightarrow T_n^{a_3} T_{n-1} = T_n^{a_n + 1} T_{n-1} = T_{n+1} \\ & \vdots \\ W : T_i & \rightarrow T_{n+i-2}. \end{aligned}$$

The other case is similarly straightforward.

3. Some substitutions

It is not too difficult to verify that the Corollary matches the result of Crisp et al. Let R , G_k , H_k denote the substitutions

$$\begin{aligned} R : 0 & \rightarrow 1, \quad 1 \rightarrow 0, \\ G_k : 0 & \rightarrow 1^k 0, \quad 1 \rightarrow 1, \\ H_k : 0 & \rightarrow 1, \quad 1 \rightarrow 1^k 0. \end{aligned}$$

To verify the correspondence between our present results and those of [3] we need just the following

PROPOSITION. If $\theta > 1/2$, then

$$\begin{aligned} H_{a_2} \circ H_{a_3} \circ \cdots \circ H_{a_{n-1}} \circ G_{a_n-a_2}(1) &= T_{n-1}, \\ H_{a_2} \circ H_{a_3} \circ \cdots \circ H_{a_{n-1}} \circ G_{a_n-a_2}(0) &= T_{n-1}^{a_n-a_2} T_{n-2}, \end{aligned}$$

where $a_n \geq a_2$; whereas if $\theta < 1/2$,

$$\begin{aligned} R \circ H_{a_1-1} \circ H_{a_2} \circ \cdots \circ H_{a_{n-1}} \circ G_{a_n+1-a_1} \circ R(0) &= T_{n-1}, \\ R \circ H_{a_1-1} \circ H_{a_2} \circ \cdots \circ H_{a_{n-1}} \circ G_{a_n+1-a_1} \circ R(1) &= T_{n-1}^{a_n+1-a_1} T_{n-2}, \end{aligned}$$

where $a_n + 1 \geq a_1 \geq 2$.

PROOF. We shall check only the first relation. The others are confirmed in much the same way.

Since $G_{a_n-a_2}(1) = 1$, it is sufficient to show that

$$H_{a_2} \circ H_{a_3} \circ \cdots \circ H_{a_{n-1}}(1) = T_{n-1}.$$

When $n = 3$, $1^{a_2}0 = T_2$. When $n = 4$, $(1^{a_2}0)^{a_3}1 = T_2^{a_3}T_1 = T_3$.

If $H_{a_2} \circ H_{a_3} \circ \cdots \circ H_{a_{n-3}}(1) = T_{n-3}$ and $H_{a_2} \circ H_{a_3} \circ \cdots \circ H_{a_{n-2}}(1) = T_{n-2}$, then

$$\begin{aligned} H_{a_2} \circ H_{a_3} \circ \cdots \circ H_{a_{n-1}}(1) &= H_{a_2} \circ H_{a_3} \circ \cdots \circ H_{a_{n-2}}(1^{a_{n-1}-1}0) \\ &= T_{n-2}^{a_{n-1}-1} H_{a_2} \circ H_{a_3} \circ \cdots \circ H_{a_{n-3}}(1) \\ &= T_{n-2}^{a_{n-1}-1} T_{n-3} = T_{n-1}. \end{aligned}$$

On the other hand, Brown [2] uses the substitution

$$h_k : 0 \longrightarrow 0^{k-1}1, \quad 1 \longrightarrow 0^{k-1}10.$$

Then we have

$$\begin{aligned} h_{a_1} \circ \cdots \circ h_{a_n}(0) &= T_n, \\ h_{a_1} \circ \cdots \circ h_{a_n}(1) &= T_n T_{n-1}. \end{aligned}$$

Thus, it is clear that $f_1 f_2 f_3 \cdots$ is invariant under the substitution $k_{a_1} \circ \cdots \circ h_{a_n}$ when $\theta = [0, \overline{a_1, \dots, a_n}]$ (Theorem 1 [2]). He also considers the block-to-block substitution

$$s \longrightarrow C_1, \quad t \longrightarrow C_2,$$

where s and t are finite strings of 0's and 1's and C_1 and C_2 are finite strings of s 's and t 's. If we allow this kind of substitution, we can leave $f_1 f_2 f_3 \cdots$ with

$$\theta = [0, a_1, \dots, a_l, \overline{a_{l+1}, \dots, a_{l+m}}]$$

invariant under the substitution

$$T_l \longrightarrow T_{l+m}, \quad T_l T_{l-1} \longrightarrow T_{l+m} T_{l+m-1}$$

(See [2], Theorem 2). Indeed, $T_n \longrightarrow T_{n+m}$ holds for any integer $n \geq 1$.

Ito and Yasutomi deal with similar cases to Brown's, but their definition of the Beatty sequence includes the term f_0 , and that makes a considerable difference in the present context. Since $\theta_{2n+1} = 0T_{2n}0^{-1}$ and $\theta_{2n+2} = 0T_{2n}^{-1}T_{2n+1}T_{2n}0^{-1}$ for any non-negative integer n (see [5], p.299 concerning this notation), it immediately follows that $f_0 f_1 f_2 f_3 \cdots$ is invariant under the substitution

$$\gamma_{a_1, a_2} \circ \cdots \circ \gamma_{a_{2n-1}, a_{2n}} : 0 \longrightarrow 0T_{2n}0^{-1}, \quad 1 \longrightarrow 0T_{2n-1}T_{2n}0^{-1}$$

when $\theta = [0, \overline{a_1, a_2, \dots, a_{2n-1}, a_{2n}}]$. In fact, for any positive integer i we have $T_i \longrightarrow 0T_{2n}^{-1}T_{2n+i}T_{2n}0^{-1}$ but $0T_i \longrightarrow 0T_{2n+i}T_{2n}0^{-1}$.

References

- [1] J. Berstel and P. Séébold, A characterization of Sturmian morphisms, *Mathematical Foundations of Computer Science, 1993* (Gdańsk, 1993), 281–290, *Lecture Notes in Comp. Sci.*, **711**, Springer, Berlin, 1993.
- [2] T.C. Brown, A characterization of the quadratic irrationals, *Canad. Math. Bull.*, **34** (1991), 36–41.
- [3] D. Crisp, W. Moran, A. Pollington and P. Shiue, Substitution invariant cutting sequences, *J. Théorie des Nombres de Bordeaux*, **5** (1993), 123–137.
- [4] A.S. Fraenkel, M. Mushkin and U. Tassa, Determination of $[n\theta]$ by its sequence of differences, *Canad. Math. Bull.*, **21** (1978), 441–446.
- [5] Sh. Ito and S. Yasutomi, On continued fractions, substitutions and characteristic sequences $[nx + y] - [(n-1)x + y]$, *Japan. J. Math.*, **16** (1990), 287–306.

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