Non-smooth hyperbolic mixed problems of the second order

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1. Introduction

It is well known that any mathematical model of arbitrary dynamical theory should satisfy the requirement of the existence of solutions to the corresponding initial and initial-boundary value problems. Unfortunately the known existence theorems are not strong enough to describe global solutions in elasticity or thermoelasticity, in the general, multidimensional and nonlinear case. In some special cases, for example for special materials and very smooth and small initial data one can prove the global in time existence (cf. for example [10, 25, 32]) but generally, the global solvability in the class of smooth functions is not expected. Furthermore the experience with formation of singularities (cf. [9, 13, 26, 27]) suggests that a sufficiently general, non-linear theory of evolution problems should admit the non-classical solutions, i.e. the solutions with discontinuities in some derivatives occurring in the corresponding equations. Unfortunately almost all papers concerning the existence of solutions for nonlinear, multidimensional hyperbolic and hyperbolic-parabolic problems of the second order are restricted to the classical solutions, cf. for example [1, 3–4, 6–8, 10–12, 14–15, 17–18, 22–23, 25, 28–32]. An exceptional character is found in the papers [16, 21] but it seems that the results contained there do not cover the systems of elasticity or thermoelasticity.

The main purpose of this article is to present some methods of proving the local in time existence results for initial-boundary value problems for second order quasilinear hyperbolic systems modeled on the system of elasticity. The solutions we consider are relatively regular but they may possess singularities in derivatives of the second order and the coefficients of the investigated quasi-linear systems are not necessarily Lipschitz-continuous. As a consequence the methods developed in the papers quoted above are not applicable and we are forced to apply a new technique. In the case of the initial problem (cf. [5]) it is possible to give relatively short proof of the existence of non-classical solutions for a class of systems generalizing the systems of elasticity and thermoelasticity, using the methods from the book [34].

The case of the mixed, initial-boundary value problem is more complicated

and therefore in the present paper we restrict ourselves to the hyperbolic system of the type of elasticity. The proof of the existence of solutions to the non-linear mixed problem (cf. Section 4) is based on some quite delicate results concerning the corresponding linear problem. This is the reason for which the largest part of this article is devoted to the analysis of the linear, hyperbolic, mixed problems with non-smooth coefficients (cf. Sections 2–3). Sections 5, 6 contain some technical material frequently used in the main text. A more precise description of the contents can be found in the table of contents.

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2. Regularization procedure for hyperbolic mixed problems

2.1. Introductory remarks

In some methods of proving the existence of solutions to evolution problems

it is necessary to approximate the initial values, the coefficients or the right hand side by smooth functions. When the mixed, initial-boundary value problems are considered, an essential trouble occurs. The usual approximation procedure leads to smooth data, which do not satisfy the boundary or compatibility conditions. To overcome this difficulty, some additional ideas are needed, cf. for example [8, 15, 22–23].

In the present section we consider the linear, second order, mixed problems with homogeneous Dirichlet boundary conditions and nonhomogeneous Neumann conditions. Our main purpose is to present some smoothing procedure, which preserves the boundary and compatibility conditions and which is applicable in the case when the coefficients are less smooth than those in [1, 4, 8, 15, 17–18, 22–23, 28–32]. In the next section we shall show that the presented procedure is in an accordance with some energy method.

2.2. Basic assumptions

In this section we shall consider the two basic mixed problems for linear, hyperbolic systems of the second order, the Neumann and Dirichlet problems:

$$\begin{aligned}
 (2.2.1) \quad & u^{(2)} = a\partial^2 u + f, & \text{in } \Omega \times I, \\
 & b\partial u = g, & \text{on } \partial\Omega \times I, \quad (\text{Neumann b.c.}), \\
 & u = 0, & \text{on } \partial\Omega \times I, \quad (\text{Dirichlet b.c.}), \\
 & u(0) = u^0, \quad u^{(1)}(0) = u^1, & \text{in } \Omega.
 \end{aligned}$$

Here $\Omega \subset R^n$, $n \geq 2$ is a bounded domain with C^∞ and compact boundary $\partial\Omega$, $I = (0, T)$ denotes a fixed, finite interval, $u^{(k)}$, $k = 0, 1, \dots$, denotes the k -th derivative with respect to $t \in I$. Furthermore

$$\begin{aligned}
 (2.2.2) \quad & a\partial^2 u = \sum_{p,q=1}^{n+1} a_{p,q} \partial_p \partial_q u, \\
 & b\partial u = \sum_{p,q=1}^n \nu_p a_{p,q} \partial_q u,
 \end{aligned}$$

where $\partial_p = \partial/\partial x_p$ denotes the partial derivative with respect to x_p , for $p = 1, \dots, n$, $\partial_{n+1}\varphi = \varphi$ and $\nu = (\nu_1, \dots, \nu_n)$ denotes the unit outward normal to the boundary, which is considered as a $C_0^\infty(R^n)$ vector field. The unknown function $u = u(x, t)$ and the data $u^i = u^i(x)$, $i = 0, 1$, $f = f(x, t)$, $g = g(x', t)$, $t \in I$, $x \in \Omega$, $x' \in \partial\Omega$ take values in R^m . The symbols $a_{p,q} = a_{p,q}(x, t)$, $p, q = 1, \dots, n+1$, $x \in \Omega$, $t \in I$ denote given, real, $m \times m$ matrices which are symmetric for $p, q \neq n+1$.

We shall assume that a real number s and an integer L are given which satisfy

the relations

$$(2.2.3) \quad \frac{n}{2} < s \leq L-1,$$

$$r_j + \frac{3}{2} \text{ are noninteger for } j = 0, \dots, L-1,$$

$$\text{where } r_j = \frac{s}{L} \left(L - j - \frac{1}{2} \right), \quad j = 0, \dots, L-1.$$

To formulate our regularity hypotheses let us introduce some additional notations. If $k = 0, 1, \dots$ and $r \in R$ then one can define the anisotropic Sobolev spaces

$$(2.2.4) \quad H^{r,k}(\Omega \times I) = \{u : u \in L^2(I, H^r(\Omega)), u^{(k)} \in L^2(I, H^0(\Omega))\},$$

with the norm

$$(2.2.5) \quad \|u\|_{H^{r,k}(\Omega \times I)}^2 = \int_I (\|u(t)\|_{H^r(\Omega)}^2 + \|u^{(k)}(t)\|_{H^0(\Omega)}^2) dt.$$

Here $H^r(\Omega)$ and $\|\cdot\|_{H^r(\Omega)}$ denote the usual Sobolev space and its norm respectively, as defined for example in [2], Chapt. 2 or [20], Chapt. 1, Sect. 9.1. The symbol $L^\alpha(I, X)$ denotes the space of all strongly measurable functions of t with values in X , which are integrable with the power α for $1 \leq \alpha < \infty$ and are essentially bounded for $\alpha = \infty$.

We shall assume that the data occurring in (2.2.1) satisfy the following conditions

$$(2.2.6) \quad f \in H^{s,L}(\Omega \times I),$$

$$(2.2.7) \quad \begin{aligned} &\text{the boundary function } g \text{ is a trace on } \partial\Omega \times I \\ &\text{of some function } \bar{g} \in H^{s+1,L}(\Omega \times I), \end{aligned}$$

$$(2.2.8) \quad \begin{aligned} &\partial a \in H^{s,L-1}(\Omega \times I), \quad a^{(L)} \in L^2(I, H^0(\Omega)), \\ &a = \{a_{p,q} : p, q = 1, \dots, n+1\}, \quad \partial = (\partial_1, \dots, \partial_n), \\ &a^{(j)}(0) \in H^{r_j+1}(\Omega), \quad j = 0, \dots, L-1, \end{aligned}$$

$$(2.2.9) \quad \begin{aligned} &u^j \in H^{r_j+2}(\Omega), \quad j = 0, \dots, L-1, \\ &u^L \in H^1(\Omega) \cap H^{r_{L-2}}(\Omega), \quad u^{L+1} \in H^{r_{L-1}}(\Omega). \end{aligned}$$

Here $u^2, \dots, u^{L+1}$ are defined with the use of the recursive formula

$$(2.2.10) \quad u^{j+2} = \sum_{i=0}^j \binom{j}{i} a^{(i)}(0) \partial^2 u^{j-i} + f^{(j)}(0), \quad j = 0, \dots, L-1.$$

Lemma 5.3.2 from Section 5.3, below shows that (2.2.10) is meaningful if (2.2.6)–(2.2.9) are satisfied. Similarly, using Lemma 5.3.3 one can show that if (2.2.6)–(2.2.9) are satisfied then the compatibility conditions are meaningful:

$$(2.2.11) \quad \sum_{i=0}^j \binom{j}{i} b^{(i)}(0) \partial u^{j-i} = g^{(j)}(0), \quad \text{on } \partial\Omega, \quad j = 0, \dots, L-1,$$

for the Neumann b.c.,

$$u^j = 0, \quad \text{on } \partial\Omega, \quad j = 0, \dots, L, \quad \text{for the Dirichlet b.c.}$$

It is clear that the regularity assumptions (2.2.6)–(2.2.9) and the compatibility conditions (2.2.10), (2.2.11) are not sufficient for solvability of (2.2.1). We shall assume also that the operator $a\partial^2$ is elliptic i.e. there exists $\delta > 0$ such that

$$(2.2.12) \quad \delta I_m \sum_{p=1}^n (\xi^p)^2 \leq \sum_{p,q=1}^n a_{p,q}(x, t) \xi^p \xi^q \leq \delta^{-1} I_m \sum_{p=1}^n (\xi^p)^2,$$

for arbitrary $x \in \Omega$, $t \in I$, $(\xi_1, \dots, \xi_n) \in R^n$. Here I_m denotes the unit $m \times m$ matrix.

2.3. Formulation of the regularization theorem

Our main purpose is to construct a family of functions

$$(2.3.1) \quad \begin{aligned} & f_\alpha, \bar{g}_\alpha, a_\alpha, u_\alpha^j, \quad j = 0, \dots, L+1, \\ & a_\alpha = \{a_{p,q,\alpha} : p, q = 1, \dots, n+1\}, \quad \alpha \in (0, 1), \end{aligned}$$

satisfying for $\alpha \in (0, 1)$ the regularity conditions

$$(2.3.2) \quad \begin{aligned} & f_\alpha, \bar{g}_\alpha, a_\alpha \in C^\infty(\bar{\Omega} \times \bar{I}), \\ & u_\alpha^j \in C^\infty(\bar{\Omega}), \quad j = 0, \dots, L+1, \end{aligned}$$

and the compatibility conditions

$$(2.3.3) \quad \sum_{i=0}^j \binom{j}{i} b_\alpha^{(i)}(0) \partial u_\alpha^{j-i} = g_\alpha^{(j)}(0), \quad \text{on } \partial\Omega, \quad j = 0, \dots, L-1,$$

for the Neumann b.c.,

$$u_\alpha^j = 0, \quad \text{on } \partial\Omega, \quad j = 0, \dots, L, \quad \text{for the Dirichlet b.c.,}$$

where g_α denotes the trace of $\bar{g}_\alpha$ on $\partial\Omega \times I$ and

$$b_\alpha \partial u = \sum_{p,q=1}^n \nu_p a_{p,q,\alpha} \partial_q u,$$

$$u_\alpha^{j+2} = \sum_{i=0}^j \binom{j}{i} a_\alpha^{(i)}(0) \partial^2 u_\alpha^{j-i} + f_\alpha^{(j)}(0), \quad j = 0, \dots, L-1.$$

The approximated coefficients a_α must also satisfy the ellipticity condition

$$(2.3.4) \quad \delta_0 I_m \sum_{p=1}^n \xi_p^2 \leq \sum_{p,q=1}^n a_{p,q,\alpha}(x,t) \xi_p \xi_q \leq \delta_0^{-1} I_m \sum_{p=1}^n \xi_p^2,$$

for some constant $\delta_0 > 0$ and arbitrary $\alpha \in (0, 1)$, $x \in \Omega$, $t \in I$, $(\xi_1, \dots, \xi_n) \in R^n$.

We are interested in approximating the original data in the following spaces as $\alpha \rightarrow 0$,

$$(2.3.5) \quad \begin{aligned} f_\alpha &\rightarrow f && \text{in } H^{s,L}(\Omega \times I), && \bar{g}_\alpha &\rightarrow \bar{g}, && \text{in } H^{s+1,L}(\Omega \times I), \\ \partial a_\alpha &\rightarrow \partial a && \text{in } H^{s,L-1}(\Omega \times I), && a_\alpha &\rightarrow a && \text{in } H^{s+1,L}(\Omega \times I), \end{aligned}$$

$$(2.3.6) \quad a_\alpha^{(j)}(0) \rightarrow a^{(j)}(0) \quad \text{in } H^{r_j+1}(\Omega), \quad j = 0, \dots, L-1,$$

$$(2.3.7) \quad \begin{aligned} u_\alpha^j &\rightarrow u^j && \text{in } H^{r_j+2}(\Omega), && j = 0, \dots, L-1, \\ u_\alpha^L &\rightarrow u^L && \text{in } H^1(\Omega) \cap H^{r_{L-2}}(\Omega), \\ u_\alpha^{L+1} &\rightarrow u^{L+1} && \text{in } H^{r_{L-1}}(\Omega). \end{aligned}$$

THEOREM 2.3.1. *If the conditions (2.2.3), (2.2.6)–(2.2.12) are fulfilled, then there exists a family (2.3.1) with the properties (2.3.2)–(2.3.7).*

2.4. Proof of the regularization theorem

First consider the mollification d_α with respect to the x, t variables of the function a . Let a_α^j be a family of smooth functions such that $a_\alpha^j \rightarrow a^{(j)}(0)$ as $\alpha \rightarrow 0$ in $H^{r_j+1}(\Omega)$, $j = 0, \dots, L-1$. By the trace theory, cf. Lemma 5.2.2, Section 5.2 below, there exists a smooth function e_α , such that $e_\alpha^{(j)}(0) = a_\alpha^j - d_\alpha^{(j)}(0)$ and

$$(2.4.1) \quad \begin{aligned} \|e_\alpha\|_{H^{s+1,L}(\Omega \times I)} &\leq C \sum_{j=0}^{L-1} \|a_\alpha^j - d_\alpha^{(j)}(0)\|_{H^{s_j+1}(\Omega)}, \\ \|\partial e_\alpha\|_{H^{s,L-1}(\Omega \times I)} &\leq C \sum_{j=0}^{L-2} \|a_\alpha^j - d_\alpha^{(j)}(0)\|_{H^{s_j+1}(\Omega)}, \end{aligned}$$

$$s_j^1 = \frac{s+1}{L} \left(L - j - \frac{1}{2} \right), \quad j = 0, \dots, L-1,$$

$$s_j^2 = \frac{s}{L-1} \left(L - 1 - j - \frac{1}{2} \right), \quad j = 0, \dots, L-2.$$

If we put $a_\alpha = d_\alpha + e_\alpha$, then $a_\alpha^{(j)}(0) = a_\alpha^j \rightarrow a^{(j)}(0)$ in $H^{r_j+1}(\Omega)$, $j = 0, \dots, L-1$ as $\alpha \rightarrow 0$. The continuity of the trace operator implies that $d_\alpha^{(j)}(0) - a^{(j)}(0) \rightarrow 0$ in $H^{s_j^1}(\Omega)$ and $H^{s_j^2+1}(\Omega)$ respectively as $\alpha \rightarrow 0$ for j as before. As a consequence we obtain the convergence $\|e_\alpha\|_{H^{s+1,L}(\Omega)} \rightarrow 0$, $\|\partial e_\alpha\|_{H^{s,L-1}(\Omega)} \rightarrow 0$ as $\alpha \rightarrow 0$ and therefore the function a_α converges as in (2.3.5), (2.3.6). Applying mollifiers to the remaining data one can construct families

$$(2.4.2) \quad \bar{f}_\alpha, \bar{g}_\alpha, a_\alpha, \bar{u}_\alpha^j, \quad j = 0, \dots, L+1, \quad \alpha \in (0, 1),$$

possessing the regularity as in (2.3.2) and converging as in (2.3.5)–(2.3.7). Since the approximated data (2.4.2) generally speaking do not satisfy the boundary and compatibility conditions, we have to improve them in an appropriate way. For this purpose let us define u_α^0 as a solution of the boundary value problem

$$(2.4.3) \quad \begin{aligned} \bar{u}_\alpha^2 + \gamma \bar{u}_\alpha^0 &= a_\alpha(0) \partial^2 u_\alpha^0 + \gamma u_\alpha^0 + \bar{f}_\alpha(0) \quad \text{in } \Omega, \\ b_\alpha(0) \partial u_\alpha^0 &= g_\alpha(0), \quad \text{on } \partial\Omega \text{ for the Neumann b.c.,} \\ u_\alpha^0 &= 0, \quad \text{on } \partial\Omega \text{ for Dirichlet b.c.} \end{aligned}$$

Note that from the inequality $r_0 + 1 > n/2$ it follows $H^{r_0+1} \subset C^0$, and (2.3.6) implies the convergence $a_\alpha(0) \rightarrow a(0)$ in $C^0(\Omega)$ as $\alpha \rightarrow 0$. Therefore, for sufficiently small $\alpha \in (0, 1)$, $a_\alpha(0) \partial^2$ satisfies the ellipticity condition (2.3.4) and one can choose a constant $\gamma > 0$ such that the problem (2.4.3) is uniquely solvable in $C^\infty(\bar{\Omega})$.

Next, assuming that $u_\alpha^0, \dots, u_\alpha^{j-1}$ are given, let us define u_α^j , $j \leq L-1$ as the solution to the elliptic boundary value problem

$$(2.4.4) \quad \begin{aligned} \bar{u}_\alpha^{j+2} + \gamma \bar{u}_\alpha^j &= a_\alpha(0) \partial^2 u_\alpha^j + \gamma u_\alpha^j + \bar{f}_\alpha^{(j)}(0) \\ &\quad + \sum_{i=1}^j \binom{j}{i} a_\alpha^{(i)}(0) \partial^2 u_\alpha^{j-i}, \quad \text{in } \Omega, \\ b_\alpha(0) \partial u_\alpha^j + \sum_{i=1}^j \binom{j}{i} b_\alpha^{(i)}(0) \partial u_\alpha^{j-i} &= g_\alpha^{(j)}(0), \\ &\quad \text{on } \partial\Omega \text{ for the Neumann b.c.,} \\ u_\alpha^j &= 0, \quad \text{on } \partial\Omega \text{ for the Dirichlet b.c.,} \end{aligned}$$

where γ is a constant chosen in such a way, that both problems (2.4.3), (2.4.4) are uniquely solvable in $C^\infty(\bar{\Omega})$. Finally let us put

$$(2.4.5) \quad u_\alpha^j = \bar{u}_\alpha^j, \quad j = L, L+1.$$

Now let $\hat{f}_\alpha^j$ be the functions of the form

$$(2.4.6) \quad \hat{f}_\alpha^j = u_\alpha^{j+2} - \bar{u}_\alpha^{j+2} + \gamma(u_\alpha^j - \bar{u}_\alpha^j), \quad j = 0, \dots, L-1.$$

Since $\widehat{f}_\alpha^j \in C^\infty(\overline{\Omega})$, according to Lemma 5.2.2 from Section 5.2, there exists a function $\widehat{f}_\alpha$ with the properties

$$(2.4.7) \quad \widehat{f}_\alpha \in C^\infty(\overline{\Omega} \times \overline{I}), \quad \widehat{f}_\alpha^{(j)}(0) = \widehat{f}_\alpha^j, \quad j = 0, \dots, L-1, \quad \alpha \in (0, 1),$$

$$(2.4.8) \quad \|\widehat{f}_\alpha\|_{H^{s,L}(\Omega \times I)} \leq c \sum_{j=0}^{L-1} \|\widehat{f}_\alpha^j\|_{H^{r_j}(\Omega)},$$

for some positive constant c . If we define

$$(2.4.9) \quad f_\alpha = \overline{f}_\alpha - \widehat{f}_\alpha,$$

then the corresponding family (2.3.1) will satisfy the compatibility condition (2.3.3).

It is sufficient to show that f_α and u_α^j satisfy (2.3.5), (2.3.7). Subtracting the equations (2.2.10), (2.2.11) and (2.4.4) side by side we obtain for $j = 0, \dots, L-1$,

$$(2.4.10) \quad \begin{aligned} (u^{j+2} - \overline{u}_\alpha^{j+2}) + \gamma(u^j - \overline{u}_\alpha^j) &= a_\alpha(0)\partial^2(u^j - u_\alpha^j) + \gamma(u^j - u_\alpha^j) \\ &\quad + (a(0)\partial^2 - a_\alpha(0)\partial^2)u^j + f^{(j)}(0) - \overline{f}_\alpha^{(j)}(0) \\ &\quad + \sum_{i=1}^j \binom{j}{i} [a_\alpha^{(i)}(0)\partial^2(u^{j-i} - u_\alpha^{j-i}) \\ &\quad + (a^{(i)}(0)\partial^2 - a_\alpha^{(i)}(0)\partial^2)u^{j-i}], \quad \text{in } \Omega, \end{aligned}$$

$$(2.4.11) \quad \begin{aligned} &b_\alpha(0)\partial(u^j - u_\alpha^j) + (b(0)\partial - b_\alpha(0)\partial)u^j \\ &\quad + \sum_{i=1}^j \binom{j}{i} [b_\alpha^{(i)}(0)\partial(u^{j-i} - u_\alpha^{j-i}) + (b^{(i)}(0)\partial - b_\alpha^{(i)}(0)\partial)u^{j-i}] \\ &= g^{(j)}(0) - g_\alpha^{(j)}(0) \quad \text{on } \partial\Omega, \quad \text{for the Neumann b.c.,} \\ &u_\alpha^j = 0 \quad \text{on } \partial\Omega, \quad \text{for the Dirichlet b.c.} \end{aligned}$$

Since for $p, \theta = 2$ the Besov spaces $B_{p,\theta}^s$ reduce to the Sobolev spaces H^s (cf. [24], Sect. 9.3), the elliptic estimate (6.1.1) from Section 6.1 gives for $j = 0, \dots, L-1$,

$$(2.4.12) \quad \|u^j - u_\alpha^j\|_{H^{r_j+2}(\Omega)} \leq c(\delta_0, \|a_\alpha(0)\|_{H^{r_0+1}(\Omega)})(F_{j,\alpha} + G_{j,\alpha}),$$

where

$$\begin{aligned}
 F_{j,\alpha} &= \sum_{i=1}^6 \|F_{j,\alpha}^i\|_{H^{r_j}(\Omega)}, \\
 F_{j,\alpha}^1 &= u^{j+2} - \bar{u}_\alpha^{j+2}, \quad F_{j,\alpha}^2 = \gamma(u^j - \bar{u}_\alpha^j), \\
 F_{j,\alpha}^3 &= (a(0)\partial^2 - a_\alpha(0)\partial^2)u^j, \\
 F_{j,\alpha}^4 &= \sum_{i=1}^j \binom{j}{i} a_\alpha^{(i)}(0)\partial^2(u^{j-i} - u_\alpha^{j-i}), \\
 F_{j,\alpha}^5 &= \sum_{i=1}^j \binom{j}{i} (a^{(i)}(0)\partial^2 - a_\alpha^{(i)}(0)\partial^2)u^{j-i}, \\
 F_{j,\alpha}^6 &= f^{(j)}(0) - \bar{f}_\alpha^{(j)}(0), \\
 (2.4.13) \quad G_{j,\alpha} &= \sum_{i=1}^4 \|G_{j,\alpha}^i\|_{H^{r_j+1-1/2}(\partial\Omega)}, \quad \text{for the Neumann b.c.,} \\
 G_{j,\alpha} &= 0, \quad \text{for the Dirichlet b.c.,} \\
 G_{j,\alpha}^1 &= (b(0)\partial - b_\alpha(0)\partial)u^j, \\
 G_{j,\alpha}^2 &= \sum_{i=1}^j \binom{j}{i} b_\alpha^{(i)}(0)\partial(u^{j-i} - u_\alpha^{j-i}), \\
 G_{j,\alpha}^3 &= \sum_{i=1}^j \binom{j}{i} (b^{(i)}(0)\partial - b_\alpha^{(i)}(0)\partial)u^{j-i}, \\
 G_{j,\alpha}^4 &= g^{(j)}(0) - g_\alpha^{(j)}(0).
 \end{aligned}$$

Using Lemmas 5.3.2 and 5.3.3 we can check that $F_{j,\alpha}$, $G_{j,\alpha}$ converge to 0 as $\alpha \rightarrow 0$, $j = 0, \dots, L-1$ and consequently u_α^j satisfies (2.3.7). To prove that f_α satisfies (2.3.5) we have to check that $\widehat{f}_\alpha \rightarrow 0$ in $H^{s,L}(\Omega \times I)$ as $\alpha \rightarrow 0$. According to (2.4.8) it is sufficient to prove that $\widehat{f}_\alpha^j \rightarrow 0$ in $H^{r_j}(\Omega)$, $j = 0, \dots, L-1$ and the last relation is a consequence of (2.4.6), (2.3.7) and Lemma 5.3.4.

3. Energy estimates for non-smooth Cauchy-Dirichlet problem

3.1. Introductory remarks

The energy estimates for hyperbolic mixed problems play a fundamental role in the existence theorems in both the linear and the nonlinear case. In the latter case, the regularity of coefficients depends on the regularity of solutions. Since the solutions are usually assumed to be functions of $t \in I \subset \mathbb{R}$ with values in some

Sobolev space $H^s(\Omega)$, $\Omega \subset R^n$, $s \in R$, it is natural to describe the regularity of coefficients in the same language.

The first multidimensional hyperbolic estimates, applicable to the quasilinear mixed problems of the second order, were obtained in [18]. Contemporary results are much more optimal. For example in [8] and [15] the norm

$$(3.1.1) \quad \max_{t \in I} \sum_{k=0}^{L+1} \|\partial_t^k u(t)\|_{H^{L+1-k}(\Omega)}, \quad L \geq [n/2] + 2, \quad L - \text{integer},$$

of the solution to the linear, second order Cauchy-Dirichlet problem was estimated by the data and a constant depending on the norm

$$(3.1.2) \quad \operatorname{ess\,sup}_{t \in I} \sum_{k=0}^L \|\partial_t^k a(t)\|_{H^{L-k}(\Omega)}$$

of the coefficients $a = \{a_{p,q} : p, q = 1, \dots, n+1\}$, cf. (3.2.2). From the embedding theorems it follows that the coefficients with finite norm (3.1.2) are Lipschitz-continuous on $\Omega \times I$ and the solutions with finite norm (3.1.1) are the classical ones, i.e. they possess the second derivatives continuous on $\overline{\Omega} \times \overline{I}$. To include the nonclassical solutions into considerations it is necessary to investigate the linear problems with coefficients less regular than those in [8, 15].

In Section 2 we have proved that some class of such nonregular problems can be approximated by the smooth ones. Now we are interested in an appropriate energy estimate. In (3.3.3) below we formulate precisely what kind of the estimate we have in mind. It is important that in our approach it is sufficient to assume that the following norm

$$(3.1.3) \quad |a|^2 = \int_I (\|a(t)\|_{H^{\bar{s}+1}(\Omega)}^2 + \|\partial_t^{L-1} a(t)\|_{H^1(\Omega)}^2 + \|\partial_t^L a(t)\|_{H^0(\Omega)}^2) dt$$

of coefficients $a = \{a_{p,q}\}$ is finite, where $\bar{s} \in R$, $n/2 < \bar{s} \leq L-1$.

One can easily see that the functions with finite norm (3.1.3) are not necessarily Lipschitz-continuous on $\Omega \times I$. If the assumptions of both the present section and Section 2 are satisfied, then using (3.3.3) it is possible to prove that the sequence of smooth solutions corresponding to all regularized problems from Section 2 is uniformly bounded in some sufficiently strong norm. Consequently, one can obtain a solution to the nonregularized problem as a weak limit of solutions to the regularized one. The corresponding argument is given in Section 4. From the form of the inequality (3.3.3) it follows that the obtained solution is not necessarily the classical one. The relaxation of the regularity is possible due the fact that in the proof of (3.3.3), uniformly with respect to $t \in I$, elliptic estimates are replaced by the corresponding estimates with the L^2 dependence on t , (cf. Section 6 below). To

overcome some technical difficulties connected with this approach, we also replace the usual norms of the solution by the weighted ones, cf. (3.3.1), (3.3.3).

3.2. Basic assumptions

Let us consider the linear, initial-boundary value problem

$$(3.2.1) \quad \begin{aligned} u^{(2)} &= a\partial^2 u + f, & \text{in } \Omega \times I, \\ u &= 0, & \text{on } \partial\Omega \times I, \\ u(0) &= u^0, \quad u^{(1)}(0) = u^1, & \text{on } \Omega. \end{aligned}$$

Here we use the same notations as in Section 2.2, and the operator $a\partial^2$ is defined as before:

$$(3.2.2) \quad a\partial^2 u = \sum_{p,q=1}^{n+1} a_{p,q} \partial_p \partial_q u.$$

We shall assume that real numbers s , $\bar{s}$ and an integer L are given which satisfy the following relations:

$$(3.2.3) \quad \begin{aligned} \frac{n}{2} < \bar{s} < s \leq L-1, \quad s - \bar{s} \leq \frac{1}{2} \\ s_j + \frac{3}{2} \text{ are nonintegers for } j = 0, \dots, L-1, \\ \text{where } s_j = \frac{s}{L-1}(L-1-j). \end{aligned}$$

Let the conditions (2.2.6), (2.2.8), (2.2.10), (2.2.12) from Section 2.2 be fulfilled and additionally

$$(3.2.4) \quad \begin{aligned} u^j &\in H_0^1(\Omega) \cap H^{r_j+2}(\Omega), \quad j = 0, \dots, L-1, \\ u^L &\in H_0^1(\Omega) \cap H^{r_{L-2}}(\Omega), \quad u^{L+1} \in H^{r_{L-1}}(\Omega). \end{aligned}$$

Here $u^2, \dots, u^{L+1}$ are defined with the use of the recursive formula (2.2.10) and the space $H_0^1(\Omega)$ is defined as follows

$$H_0^1(\Omega) = \{u \in H^1(\Omega) : u = 0 \text{ on } \partial\Omega \text{ in the sense of the trace}\}.$$

REMARK. The assumption (3.2.4) contains both the regularity and the compatibility conditions.

3.3. The basic estimate

Let us assume that $\lambda \geq 0$ is a constant, $t \in I$ and introduce the following notations.

$$\begin{aligned}
(u(t), v(t))_{0,\lambda} &= \int_{\Omega} u(t)v(t)e^{-2\lambda t} dx, \\
(u, v)_{0,\lambda,t} &= \int_0^t (u(\tau), v(\tau))_{0,\lambda} d\tau, \\
|v(t)|_{0,\lambda}^2 &= (v(t), v(t))_{0,\lambda}, \\
|v|_r &\text{-the norm of the Sobolev space } H^r(\Omega) = H^r, \\
|v(t)|_{r,\lambda}^2 &= |v(t)|_r^2 e^{-2\lambda t}, \\
(3.3.1) \quad |v|_{0,1,\lambda}^2 &= |v^{(1)}|_{0,\lambda}^2 + |v|_{1,\lambda}^2, \\
|v|_{0,1,\lambda,t}^2 &= \int_0^t |v(\tau)|_{0,1,\lambda}^2 d\tau, \\
\|v\|_{0,1,\lambda,t}^2 &= \operatorname{ess\,sup}_{\tau \in (0,t)} |v(\tau)|_{0,1,\lambda}^2, \\
\|v\|_{r,\lambda,t}^2 &= \operatorname{ess\,sup}_{\tau \in (0,t)} |v(\tau)|_{r,\lambda}^2, \\
|v|_{r,\lambda,t}^2 &= \int_0^t |v(\tau)|_{r,\lambda}^2 d\tau, \\
\|u^j\|_{0,1}^2 &= |u^{j+1}|_0^2 + |u^j|_1^2, \quad j = 0, \dots, L.
\end{aligned}$$

Let us recall that the symbols L , s_k and δ were introduced in (3.2.3) and (2.2.12) respectively. Let us introduce the norm of coefficients which is a little weaker than that introduced in the formula (3.1.3), (cf. Section 5.1)

$$\begin{aligned}
\|a\|^2 &= \max_{t \in I} |a(t)|_{\bar{s}_0^1+1}^2 + \operatorname{ess\,sup}_{t \in I} \left(\sum_{j=1}^{L-2} |a^{(j)}(t)|_{\bar{s}_j^1+1}^2 + |a^{(L-1)}(t)|_{\bar{s}_{L-1}^2}^2 \right) \\
(3.3.2) \quad &+ \int_I \left\{ \sum_{j=0}^{L-2} |a^{(j)}(t)|_{\bar{s}_j+1}^2 + |a^{(L-1)}(t)|_{\bar{s}_{L-2}}^2 + |a^{(L)}(t)|_0^2 \right\} dt, \\
\bar{s}_j &= \frac{\bar{s}}{L-1} (L-1-j), \quad \bar{s}_j^1 = \frac{\bar{s}}{L-1} \left(L-1-j-\frac{1}{2} \right), \\
\bar{s}_j^2 &= \frac{\bar{s}+1}{L} \left(L-j-\frac{1}{2} \right).
\end{aligned}$$

The main purpose of the present section is to prove the following estimate.

THEOREM 3.3.1. *If the assumptions (2.2.6), (2.2.8), (2.2.12), (3.2.3), (3.2.4) are satisfied then there exist positive constants $C = C(\delta)$ and $\lambda_0 = \lambda_0(\delta, \|a\|)$, such*

that the inequality

$$\begin{aligned}
 & \sum_{0 \leq j \leq L} \lambda^{2j} (\|u^{(L-j)}\|_{0,1,\lambda,t}^2 + \lambda |u^{(L-j)}|_{0,1,\lambda,t}^2) \\
 & + \lambda \sum_{0 \leq l \leq k \leq L-1} \lambda^{2l} |u^{(k-l)}|_{s_k+2,\lambda,t}^2 \\
 (3.3.3) \quad & \leq C(\delta) \left\{ \sum_{0 \leq j \leq L} \lambda^{2j} \left[\|u^{(L-j)}\|_{0,1}^2 + \left(\int_0^t |f^{(L-j)}(\tau)|_0 e^{-\lambda\tau} d\tau \right)^2 \right] \right. \\
 & \left. + \sum_{0 \leq l \leq k \leq L-1} \lambda^{2l+1} |f^{(k-l)}|_{s_k,\lambda,t}^2 \right\}
 \end{aligned}$$

holds true for arbitrary $\lambda \geq \lambda_0$ and arbitrary u satisfying (3.2.1) and the relations

$$\begin{aligned}
 & u^{(j+1)} \in L^\infty(I, H^0(\Omega)), \\
 & u^{(j)} \in L^\infty(I, H_0^1(\Omega)), \\
 (3.3.4) \quad & u^{(j)} \in L^2(I, H^{s_j+2}(\Omega) \cap H_0^1(\Omega)), \\
 & j = 0, \dots, L-1.
 \end{aligned}$$

3.4. First step of the proof

Assume temporarily that u is arbitrarily smooth with respect to t and differentiate the equation

$$(3.4.1) \quad u^{(2)} - a\partial^2 u + \gamma u = f + \gamma u,$$

$L-j$ times, $0 \leq j \leq L$ with respect to t . Here γ denotes some constant to be determined later, cf. (3.4.6). Multiplying both sides by $u^{(L-j+1)} e^{-2\lambda t}$ and integrating over Ω , we obtain

$$\begin{aligned}
 (3.4.2) \quad & (u^{(L-j+2)}, u^{(L-j+1)})_{0,\lambda} - (a\partial^2 u^{(L-j)}, u^{(L-j+1)})_{0,\lambda} \\
 & + \gamma(u^{(L-j)}, u^{(L-j+1)})_{0,\lambda} = (f^{(L-j)} + F_1^{L-j}, u^{(L-j+1)})_{0,\lambda} \\
 & + \gamma(u^{(L-j)}, u^{(L-j+1)})_{0,\lambda}, \quad 0 \leq j \leq L,
 \end{aligned}$$

where

$$(3.4.3) \quad F_1^{L-j} = \sum_{i=1}^{L-j} \binom{L-j}{i} a^{(i)} \partial^2 u^{(L-j-i)}, \quad 0 \leq j \leq L-1, \quad F_1^0 = 0.$$

Integrating (3.4.2) over $(0, t)$ and using the formula

$$\frac{d}{dt} \{(v, v)_{0,\lambda}\} = -2\lambda(v, v)_{0,\lambda} + 2(v, v^{(1)})_{0,\lambda},$$

we get for $0 \leq j \leq L$

$$\begin{aligned}
 & |u^{(L-j+1)}(t)|_{0,\lambda}^2 - |u^{(L-j+1)}(0)|_0^2 + 2\lambda |u^{(L-j+1)}|_{0,\lambda,t}^2 \\
 & + a_\gamma(\partial u^{(L-j)}(t), \partial u^{(L-j)}(t)) - a_\gamma(\partial u^{(L-j)}(0), \partial u^{(L-j)}(0)) \\
 & + 2\lambda \int_0^t a_\gamma(\partial u^{(L-j)}(\tau), \partial u^{(L-j)}(\tau)) d\tau \\
 & = 2(f^{(L-j)} + F_1^{L-j}, u^{(L-j+1)})_{0,\lambda,t} + F_{2,t}^{L-j}(u^{(L-j)}, u^{(L-j)}).
 \end{aligned}
 \tag{3.4.4}$$

Here

$$\begin{aligned}
 a_\gamma(\partial v, \partial v) &= \sum_{p,q=1}^n (a_{p,q} \partial_q v, \partial_p v)_{0,\lambda} + \gamma(v, v)_{0,\lambda}, \\
 F_{2,t}^j(u^{(j)}, u^{(j)}) &= \sum_{p,q=1}^n (a_{p,q}^{(1)} \partial_q u^{(j)}, \partial_p u^{(j)})_{0,\lambda,t} + \gamma(u^{(j)}, u^{(j+1)})_{0,\lambda,t} \\
 &+ \sum_{p,q=1}^n ((\partial_p a_{p,q}) \partial_q u^{(j)}, u^{(j+1)})_{0,\lambda,t} \\
 &+ \sum_{\substack{p=n+1 \\ \text{or } q=n+1}} (a_{p,q} \partial_p \partial_q u^{(j)}, u^{(j+1)})_{0,\lambda,t}
 \end{aligned}
 \tag{3.4.5}$$

and $\gamma = \gamma(\delta, \|a\|)$ is an appropriately chosen constant so that

$$a_\gamma(\partial v, \partial v) \geq \frac{1}{2} \delta |\partial v|_0^2 \quad \text{for arbitrary } v \in H_0^1(\Omega).
 \tag{3.4.6}$$

From (3.4.4) and (3.4.6) we deduce

$$\begin{aligned}
 & |u^{(L-j)}(t)|_{0,1,\lambda}^2 + 2\lambda |u^{(L-j)}|_{0,1,\lambda,t}^2 \\
 & \leq C_j(\delta) \{ \|u^{(L-j)}\|_{0,1}^2 + 2(f^{(L-j)} + F_1^{L-j}, u^{(L-j+1)})_{0,\lambda,t} \\
 & \quad + F_{2,t}^{L-j}(u^{(L-j)}, u^{(L-j)}) \}, \quad 0 \leq j \leq L.
 \end{aligned}
 \tag{3.4.7}$$

Now, since (3.4.7) makes sense for u satisfying (3.3.4), using mollifiers with respect to t and Lemmas 5.5.1, 5.5.2 from Section 5.5 below, we can drop the additional regularity assumption concerning u . To be more precise, applying the smoothing operation from Section 5.5 to both sides of (3.4.1) and repeating the previous argument we can obtain a relation analogous to (3.4.7), with u replaced by u_σ and f replaced by $f_\sigma + I_\sigma$. Here we use the same notations as in Section 5.5 with $v = \partial^2 u$. The estimates from the remaining part of the present section, combined with Lemmas 5.5.1, 5.5.2 show that if $\sigma \rightarrow 0$ then we obtain (3.4.7) for u satisfying (3.3.4).

Multiplying both sides of (3.4.7) by λ^{2j} and summing over $0 \leq j \leq L$ we get

$$(3.4.8) \quad E_1^t + 2\lambda E_2^t \leq C_1 \sum_{k=3}^6 E_k^t,$$

with some positive constant C_1 and

$$(3.4.9) \quad E_k^t = \sum_{0 \leq j \leq L} \lambda^{2j} e_{k,L-j}^t, \quad k = 1, \dots, 6,$$

where

$$(3.4.10) \quad \begin{aligned} e_{1,j}^t &= \|u^{(j)}\|_{0,1,\lambda,t}^2, & e_{2,j}^t &= |u^{(j)}|_{0,1,\lambda,t}^2, \\ e_{3,j}^t &= \|u^j\|_{0,1}^2, & e_{4,j}^t &= 2(f^{(j)}, u^{(j+1)})_{0,\lambda}, \\ e_{5,j}^t &= 2(F_1^j, u^{(j+1)})_{0,\lambda}, & e_{6,j}^t &= F_{2,t}^j(u^{(j)}, u^{(j)}). \end{aligned}$$

The term E_4^t can be estimated without difficulties:

$$(3.4.11) \quad C_1 E_4^t \leq c_0 E_7^t + 4^{-1} E_1^t,$$

where E_7^t is defined as in (3.4.9) with $k = 7$ and

$$(3.4.12) \quad e_{7,L-j}^t = \left(\int_0^t |f^{(L-j)}|_0 e^{-\lambda\tau} d\tau \right)^2.$$

Next let us note that from Lemma 5.1.1, Sect. 5.1 below it follows that if $\partial a \in L^2(I, H^{\bar{s}}(\Omega))$, $\partial a^{(L-1)} \in L^2(I, H^0(\Omega))$, then for some constant $C > 0$,

$$(3.4.13) \quad \begin{aligned} a^{(i)} &\in L^2(I, H^{\bar{s}_i+1}(\Omega)), \quad i = 0, \dots, L-1, \\ \text{with } \bar{s}_i &= \frac{\bar{s}}{L-1}(L-1-i), \\ \sum_{i=0}^{L-1} \|a^{(i)}\|_{L^2(I, H^{\bar{s}_i+1}(\Omega))} &\leq C \left(\|a\|_{L^2(I, H^{\bar{s}+1}(\Omega))} + \|a^{(L-1)}\|_{L^2(I, H^1(\Omega))} \right). \end{aligned}$$

Similarly, if $\partial^2 u \in L^2(I, H^s(\Omega))$, $\partial^2 u^{(L-1)} \in L^2(I, H^0(\Omega))$, where $\partial^2 = \{\partial^{|\alpha|} : |\alpha| = 2\}$, then

$$(3.4.14) \quad \begin{aligned} u^{(i)} &\in L^2(I, H^{s_i+2}(\Omega)), \quad i = 0, \dots, L-1, \quad s_i = \frac{s}{L-1}(L-1-i), \\ \sum_{i=0}^{L-1} \|u^{(i)}\|_{L^2(I, H^{s_i+2}(\Omega))} &\leq C \left(\|u\|_{L^2(I, H^{s+2}(\Omega))} + \|u^{(L-1)}\|_{L^2(I, H^2(\Omega))} \right), \end{aligned}$$

for some constant $C > 0$. The relations

$$\begin{aligned}
 (3.4.15) \quad & \bar{s}_i + 1 + s_{L-i} > \bar{s} + 1 - \frac{s}{L-1} \geq \bar{s} > \frac{n}{2}, \\
 & \bar{s}_i \geq 0, \quad s_{L-i} \geq 0, \\
 & 1 \leq i \leq L-j, \quad 0 \leq j \leq L-1, \quad i \leq L-2, \\
 & \bar{s}_{L-2} + s_1 > \bar{s} > \frac{n}{2},
 \end{aligned}$$

combined with Lemmas 5.3.1, 5.3.5 below, imply

$$\begin{aligned}
 (3.4.16) \quad & |a^{(i)} \partial^2 u^{(L-j-i)}|_0 \leq c |a^{(i)}|_{\bar{s}_i+1} |u^{(L-j-i)}|_{s_{L-i}+2}, \\
 & \quad i, j \text{ as in (3.4.15)}, \\
 & |a^{(L-1)} \partial^2 u^{(1-j)}|_0 \leq c |a^{(L-1)}|_{\bar{s}_{L-2}} |u^{(1-j)}|_{s_1+2}, \quad j = 0, 1, \\
 & |a^{(L)} \partial^2 u|_0 \leq c |a^{(L)}|_0 |u|_{s+2}.
 \end{aligned}$$

As a consequence we obtain the following estimate for E_5^t

$$\begin{aligned}
 (3.4.17) \quad & C_1 E_5^t \leq c_0 \|a\|^2 E_8^t + 4^{-1} E_1^t, \quad (\|a\| \text{ is defined in (3.3.2)}), \\
 & E_8^t = \sum_{\substack{0 \leq j \leq L-1 \\ 1 \leq i \leq L-j}} \lambda^{2j} |u^{(L-j-i)}|_{s_{L-i}+2, \lambda, t}^2 + |u|_{s+2, \lambda, t}^2.
 \end{aligned}$$

To estimate the three-linear terms contained in E_6^t , one can use the inequalities

$$\begin{aligned}
 (3.4.18) \quad & \int_0^t (bv, w)_{0, \lambda} d\tau \leq |b|_{L^2(I, H^r(\Omega))} |v|_{0, \lambda, t} \|w\|_{0, \lambda, t} \\
 & \leq c |b|_{L^2(I, H^r(\Omega))}^2 |v|_{0, \lambda, t}^2 + 4^{-1} \|w\|_{0, \lambda, t}^2, \quad \left(r > \frac{n}{2}\right),
 \end{aligned}$$

which imply

$$(3.4.19) \quad C_1 E_6^t \leq c_0 (\|a\| + \gamma)^2 E_2^t + 4^{-1} E_1^t.$$

Combining (3.4.8), (3.4.11), (3.4.17) and (3.4.19) we see that if λ is sufficiently large so as to satisfy

$$(3.4.20) \quad c_0 (\|a\| + \gamma)^2 \leq \lambda,$$

then

$$(3.4.21) \quad 4^{-1} E_1^t + \lambda E_2^t \leq C^1(\delta) (E_3^t + E_7^t + \|a\|^2 E_8^t),$$

for some positive constant $C^1(\delta)$ independent of u and a .

3.5. Application of elliptic estimates with L^2 dependence on t

Now we shall use the ellipticity of the operator $a\partial^2$. For this purpose let us differentiate the equation

$$(3.5.1) \quad a\partial^2 u = u^{(2)} - f,$$

$k-l$ times with respect to t , where $0 \leq l \leq k \leq L-1$. We obtain

$$(3.5.2) \quad \begin{aligned} a\partial^2 u^{(k-l)} &= u^{(k-l+2)} - f^{(k-l)} - F_1^{k-l}, \\ F_1^{k-l} &= \sum_{i=1}^{k-l} \binom{k-l}{i} a^{(i)} \partial^2 u^{(k-l-i)}, \quad \text{if } 0 \leq l < k \leq L-1, \\ F_1^{k-l} &= 0, \quad \text{if } l = k. \end{aligned}$$

In the special case $p = \theta = 2$ the Besov spaces $B_{p,\theta}^s$ reduce to the Sobolev spaces H^s , cf. [24], Sect. 9.3, so the estimate (6.2.1) from Section 6.2 implies

$$(3.5.3) \quad \begin{aligned} |u^{(k-l)}|_{s_k+2,\lambda,t}^2 &\leq C_l^k \left\{ |u^{(k-l+2)}|_{s_k,\lambda,t}^2 + |f^{(k-l)}|_{s_k,\lambda,t}^2 \right. \\ &\quad \left. + |F_1^{k-l}|_{s_k,\lambda,t}^2 + \overline{C}_l^k(\|a\|) |u^{(k-l)}|_{0,\lambda,t}^2 \right\}, \\ &\quad \text{for } 0 \leq l \leq k \leq L-1 \end{aligned}$$

and for some constants $C_l^k = C_l^k(\delta) > 0$, $\overline{C}_l^k(\|a\|) > 0$. Note that from the relation (3.5.12), below it follows $a \in C^0(I, H^{\overline{s}_0^1+1}(\Omega))$. The relations $\overline{s}_0^1 + 1 = \overline{s} + 1 - \frac{1}{2} \frac{\overline{s}}{L-1} \geq \overline{s} + \frac{1}{2} \geq s_0 = s > \frac{n}{2}$, imply that we can apply Lemma 6.2.2 with $s^0 = \overline{s}_0^1$ and $s = s_k$, $k = 0, \dots, L-1$.

Multiplying both sides of (3.5.3) by $\varepsilon_l^k \lambda^{2l+1}$, where ε_l^k are some constants satisfying the inequalities $0 < \varepsilon_l^k \leq 1$, and summing side by side we get

$$(3.5.4) \quad G_1^t \leq \sum_{m=2}^7 G_m^t,$$

where

$$\begin{aligned} G_m^t &= \lambda \sum_{0 \leq l \leq k \leq L-1} \lambda^{2l} g_{m,k,l}^t, \quad m = 1, 5, 6, 7, \\ G_2^t &= \lambda \sum_{0 \leq l \leq L-1} \lambda^{2l} g_{2,L-1,l}^t, \quad (k = L-1), \\ G_3^t &= \lambda \sum_{0 \leq l \leq L-2} \lambda^{2l} g_{3,L-2,l}^t, \quad (k = L-2), \end{aligned}$$

$$\begin{aligned}
(3.5.5) \quad G_4^t &= \lambda \sum_{0 \leq l \leq k \leq L-3} \lambda^{2l} g_{4,k,l}^t, \quad (k \leq L-3), \\
g_{1,k,l}^t &= \varepsilon_l^k |u^{(k-l)}|_{s_k+2,\lambda,t}^2, \\
g_{2,L-1,l}^t &= \varepsilon_1^{L-1} C_l^{L-1} |u^{(L+1-l)}|_{s_{L-1},\lambda,t}^2, \\
g_{3,L-2,l}^t &= \varepsilon_l^{L-2} C_l^{L-2} |u^{(L-l)}|_{s_{L-2},\lambda,t}^2, \\
g_{4,k,l}^t &= \varepsilon_l^k C_l^k |u^{(k-l+2)}|_{s_k,\lambda,t}^2, \\
g_{5,k,l}^t &= \varepsilon_l^k C_l^k |f^{(k-l)}|_{s_k,\lambda,t}^2, \\
g_{6,k,l}^t &= \varepsilon_l^k C_l^k \bar{C}_l^k (\|a\|) |u^{(k-l)}|_{0,\lambda,t}^2, \\
g_{7,k,l}^t &= \varepsilon_l^k C_l^k |F_1^{k-l}|_{s_k,\lambda,t}^2.
\end{aligned}$$

Observe, that since $s_{L-1} = 0$, $s_{L-2} \leq 1$, the terms G_2^t , G_3^t , can be estimated as follows

$$(3.5.6) \quad G_2^t + G_3^t \leq 4^{-1} \lambda E_2^t,$$

provided

$$(3.5.7) \quad \varepsilon_l^k C_l^k \leq 4^{-1}, \quad k = L-2, L-1, \quad 0 \leq l \leq k.$$

If we assume that the constants ε_l^k , C_l^k satisfy the condition

$$(3.5.8) \quad \varepsilon_l^k C_l^k \leq 4^{-1} \varepsilon_l^{k+2} C_l^{k+2}, \quad 0 \leq l \leq k \leq L-3,$$

then

$$(3.5.9) \quad G_4^t \leq 4^{-1} G_1^t.$$

To estimate G_6^t , let us assume that λ is sufficiently large to satisfy

$$(3.5.10) \quad \varepsilon_l^k C_l^k \bar{C}_l^k (\|a\|) \leq 4^{-1} \lambda^{2(L-(k-l))}, \quad 0 \leq l \leq k \leq L-1.$$

We obtain

$$(3.5.11) \quad G_6^t \leq 4^{-1} \lambda E_2^t.$$

Lemma 5.1.2 implies, that if we have $\partial a \in L^2(I, H^{\bar{s}}(\Omega))$, $\partial a^{(L-1)} \in L^2(I, H^0(\Omega))$, then

$$\begin{aligned}
(3.5.12) \quad & a^{(j)} \in C^0(I, H^{\bar{s}_j^1+1}(\Omega)), \\
& \bar{s}_j^1 = \frac{\bar{s}}{L-1} \left(L-1-j-\frac{1}{2} \right), \quad j = 0, \dots, L-2, \\
& \sum_{j=0}^{L-2} \|a^{(j)}\|_{C^0(I, H^{\bar{s}_j^1+1}(\Omega))} \leq c \|a\|,
\end{aligned}$$

for some positive constant c .

Using Lemma 5.3.1 we also can check that the inequalities

$$\begin{aligned}
 \bar{s}_i^1 + 1 + s_{k-i}^1 - s_k &> \bar{s} - \frac{s}{L-1} + 1 \geq \bar{s} > \frac{n}{2}, \\
 \bar{s}_i^1 + 1 - s_k &> \bar{s} - s + 1 - \frac{1}{2} \frac{\bar{s}}{L-1} \geq 0, \\
 (3.5.13) \quad s_{k-i}^1 - s_k &= \frac{s}{L-1} \left(i - \frac{1}{2} \right) \geq 0, \\
 s_j^1 &= \frac{s}{L-1} \left(L-1-j-\frac{1}{2} \right), \quad 0 \leq l < k \leq L-1, \\
 1 \leq i \leq k-l, \quad i &\leq L-2,
 \end{aligned}$$

imply for $i \leq L-2$

$$\begin{aligned}
 |a^{(i)} \partial^2 u^{(k-l-i)}|_{s_k, \lambda, t}^2 &\leq c |a^{(i)}|_{\bar{s}_i^1+1}^2 \sum_{i=1}^{k-l} |\partial^2 u^{(k-l-i)}|_{s_{k-i}^1, \lambda, t}^2, \\
 (3.5.14) \quad 0 \leq l < k &\leq L-1.
 \end{aligned}$$

Applying the interpolation inequality

$$(3.5.15) \quad |v|_{\sigma_1(1-\theta)+\sigma_2\theta} \leq c |v|_{\sigma_1}^{1-\theta} |v|_{\sigma_2}^{\theta},$$

with $\theta = \frac{1}{2}$, $\sigma_1 = s_j$, $\sigma_2 = s_{j+1}$ and using the fact that $\frac{1}{2}s_j + \frac{1}{2}s_{j+1} = s_j^1$, we can obtain the estimate

$$(3.5.16) \quad |v|_{s_j^1}^2 \leq c |v|_{s_j} |v|_{s_{j+1}} \leq c(\lambda^{-1} |v|_{s_j}^2 + \lambda |v|_{s_{j+1}}^2).$$

If $i = L-1$ then the relations $a^{(L)} \in L^2(I, H^0(\Omega))$, $a \in L^2(I, H^{\bar{s}+1}(\Omega))$ give

$$\begin{aligned}
 a^{(j)} &\in C^0(I, H^{\bar{s}_j^2}(\Omega)), \\
 (3.5.12)' \quad \bar{s}_j^2 &= \frac{\bar{s}+1}{L} \left(L-j-\frac{1}{2} \right), \quad j = 0, \dots, L-1, \\
 \sum_{j=0}^{L-1} \|a^{(j)}\|_{C^0(I, H^{\bar{s}_j^2}(\Omega))} &\leq c \|a\|,
 \end{aligned}$$

and we have

$$(3.5.14)' \quad |a^{(L-1)} \partial^2 u|_0^2 \leq c |a^{(L-1)}|_{\bar{s}_{L-1}^2}^2 |\partial^2 u|_{\bar{s}-\bar{s}_{L-1}^2}^2,$$

so we can apply (3.5.15) to ∂u with $\sigma_1 = \bar{s} + 1$, $\sigma_2 = \bar{s} + 1 - \frac{\bar{s}+1}{L}$:

$$\begin{aligned}
 |\partial u|_{\bar{s}+1-\bar{s}_L^2-1}^2 &= |\partial u|_{(\sigma_1+\sigma_2)/2}^2 \\
 (3.5.16)' \quad &\leq c(\lambda^{-1}|\partial u|_{\bar{s}+1}^2 + \lambda|\partial u|_{\bar{s}+1-(\bar{s}+1)/L}^2) \\
 &\leq c(\lambda^{-1}|u|_{\bar{s}_0+2}^2 + \lambda|u|_{\bar{s}_1+2}^2) \leq c(\lambda^{-1}|u|_{s_0+2}^2 + \lambda|u|_{s_1+2}^2).
 \end{aligned}$$

As a consequence we get the following estimate for G_7^t

$$\begin{aligned}
 G_7^t &\leq c\|a\|^2(G_8^t + G_9^t), \\
 (3.5.17) \quad G_m^t &= \lambda \sum_{\substack{0 \leq l < k \leq L-1 \\ 1 \leq i \leq k-l}} \varepsilon_l^k C_l^k g_{m,k,l,i}^t, \quad m = 8, 9, \\
 g_{8,k,l,i}^t &= \lambda^{2l-1} |u^{(k-l-i)}|_{s_{k-i}+2,\lambda,t}^2, \\
 g_{9,k,l,i}^t &= \lambda^{2l+1} |u^{(k-l-i)}|_{s_{k-i+1}+2,\lambda,t}^2.
 \end{aligned}$$

It is not difficult to see that if λ is sufficiently large so as to satisfy

$$(3.5.18) \quad c\|a\|^2 \leq 8^{-1} \lambda \min\{\varepsilon_l^k C_l^k : 0 \leq l < k \leq L-1\},$$

then

$$(3.5.19) \quad G_7^t \leq 4^{-1} G_1^t.$$

Combining (3.5.4), (3.5.6), (3.5.9), (3.5.11) and (3.5.19) we see that

$$(3.5.20) \quad 2^{-1} G_1^t \leq G_5^t + 2^{-1} \lambda E_2^t.$$

Formulae (3.4.21) and (3.5.20) give

$$\begin{aligned}
 (3.5.21) \quad &4^{-1} E_1^t + 2^{-1} \lambda E_2^t + 2^{-1} G_1^t \\
 &\leq C_1(\delta) \{E_3^t + E_7^t + G_5^t\} + C_2(\delta) \|a\|^2 E_8^t.
 \end{aligned}$$

If λ is sufficiently large so as to satisfy

$$(3.5.22) \quad C_2(\delta) \|a\|^2 \leq 4^{-1} \lambda \min\{\varepsilon_l^k : 0 \leq l \leq k \leq L-1\},$$

then the last term in (3.5.21) can be estimated by $4^{-1} G_1^t$ and we obtain

$$(3.5.23) \quad 4^{-1} E_1^t + 2^{-1} \lambda E_2^t + 4^{-1} G_1^t \leq C(\delta) \{E_3^t + E_7^t + G_5^t\}.$$

The proof of Theorem 3.3.1 will be complete if we prove the estimate (6.2.1). The corresponding argument is postponed to Section 6.2.

4. Existence theorems for non-smooth Cauchy-Dirichlet problem

4.1. Introductory remarks

The local, smooth solutions to mixed problems for nonlinear, second order hyperbolic equations and systems were investigated for example in [1, 3–4, 8, 15, 17–18, 22–23, 30–31, 34]. In the mentioned papers all second derivatives of solutions must be continuous on $\Omega \times I$, where Ω denotes some domain in R^n and I is a time interval. Since the initially smooth solutions may develop singularities in finite time (cf. [27], Sect. 10, and the references given there) the question of the existence of less smooth solutions arises naturally.

In the present section we consider the local solutions to the mixed, Cauchy-Dirichlet problem for the quasilinear, hyperbolic system of the second order, without assuming that they are of class $C^2(\Omega \times I)$. Instead we assume that some second derivatives of solutions belong to the space $L^2(I, H^s(\Omega))$, $s > \frac{n}{2}$. The basic idea of the proof is similar to that presented in [8] but in our approach, the auxiliary theorems concerning the linear problem used in [8] are replaced by corresponding results from previous sections.

4.2. Existence theorem for the linear problem

Let us consider the linear, Cauchy-Dirichlet problem (3.2.1) with all assumptions formulated in Section 3.2. Let us assume that $\lambda \geq 0$ is a constant, $t \in I$ and recall that in (3.3.1) we have defined the basic norms used in this article. Let us also recall that the symbols L , s_k , $\bar{s}$ and δ were introduced in (3.2.3) and (2.2.12) respectively and define

$$\begin{aligned}
 \|a\|_T^2 = & \max_{t \in (0, T)} |a^{(j)}(t)|_{\bar{s}_j^1+1}^2 + \operatorname{ess\,sup}_{t \in (0, T)} \left(\sum_{j=1}^{L-2} |a^{(j)}(t)|_{\bar{s}_j^1+1}^2 + |a^{(L-1)}(t)|_{\bar{s}_{L-1}^2}^2 \right) \\
 & + \int_0^T \left\{ \sum_{j=0}^{L-2} |a^{(j)}(t)|_{\bar{s}_j+1}^2 + |a^{(L-1)}(t)|_{\bar{s}_{L-2}}^2 + |a^{(L)}(t)|_0^2 \right\} dt, \\
 (4.2.1) \quad \bar{s}_j = & \frac{\bar{s}}{L-1} (L-1-j), \quad \bar{s}_j^1 = \frac{\bar{s}}{L-1} \left(L-1-j-\frac{1}{2} \right), \\
 \bar{s}_j^2 = & \frac{\bar{s}+1}{L} \left(L-j-\frac{1}{2} \right).
 \end{aligned}$$

The energy inequality we have proved in Section 3 has the following consequence.

THEOREM 4.2.1. *If the assumptions (2.2.6), (2.2.8), (2.2.12), (3.2.3), (3.2.4) are satisfied then there exist positive constants $C = C(\delta)$ and $\Lambda = \Lambda(\delta, \|a\|_T)$, such*

that the inequality

$$\begin{aligned}
 & \sum_{0 \leq j \leq L} (\|u^{(L-j)}\|_{0,1,0,t}^2 + |u^{(L-j)}|_{0,1,0,t}^2) + \sum_{0 \leq l \leq k \leq L-1} |u^{(k-l)}|_{s_k+2,0,t}^2 \\
 (4.2.2) \quad & \leq C(\delta) \Lambda^{2L}(\delta, \|a\|_T) e^{2\Lambda(\delta, \|a\|_T)T} \left\{ \sum_{0 \leq j \leq L} \left(\|u^{(L-j)}\|_{0,1}^2 \right. \right. \\
 & \quad \left. \left. + \left(\int_0^t |f^{(L-j)}(\tau)|_0 d\tau \right)^2 \right) + \sum_{0 \leq l \leq k \leq L-1} |f^{(k-l)}|_{s_k,0,t}^2 \right\}
 \end{aligned}$$

holds true for arbitrary u satisfying (3.2.1) and the relations

$$\begin{aligned}
 (4.2.3) \quad & u^{(j+1)} \in L^\infty(I, H^0(\Omega)), \\
 & u^{(j)} \in L^\infty(I, H_0^1(\Omega)), \\
 & u^{(j)} \in L^2(I, H^{s_j+2}(\Omega) \cap H_0^1(\Omega)), \quad j = 0, \dots, L-1.
 \end{aligned}$$

Now we shall prove the following corollary from (4.2.2).

THEOREM 4.2.2. *If the assumptions (2.2.6), (2, 2, 8), (2.2.12), (3.2.3), (3.2.4) are satisfied then there exist unique solution of the problem (3.2.1) satisfying the conditions*

$$\begin{aligned}
 (4.2.4) \quad & u^{(j)} \in C^0(I, H^0(\Omega)), \quad j = 0, \dots, L+1, \\
 & u^{(j)} \in C^0(I, H_0^1(\Omega)), \quad j = 0, \dots, L, \\
 & u^{(j)} \in L^2(I, H^{s_j+2}(\Omega) \cap H_0^1(\Omega)), \quad j = 0, \dots, L-1.
 \end{aligned}$$

PROOF. Using the regularization procedure described in Section 2 we obtain the family of smooth problems

$$\begin{aligned}
 (4.2.5) \quad & u_\alpha^{(2)} = a_\alpha \partial^2 u_\alpha + f_\alpha, \quad \text{in } \Omega \times I, \\
 & u_\alpha = 0, \quad \text{on } \partial\Omega \times I, \\
 & u_\alpha(0) = u_\alpha^0, \quad u_\alpha^{(1)}(0) = u_\alpha^1, \quad \text{on } \Omega,
 \end{aligned}$$

with $\alpha \in (0, 1)$. By the results of [8, 15], for $\alpha \in (0, 1)$ the problem (4.2.5) possesses a solution with the regularity

$$\begin{aligned}
 (4.2.6) \quad & u_\alpha^{(j)} \in C^0(I, H^{L+1-j}(\Omega)), \quad j = 0, \dots, L+1, \\
 & u_\alpha^{(j)}(t) \in H_0^1(\Omega), \quad j = 0, \dots, L, \quad t \in I.
 \end{aligned}$$

The inequality (4.2.2) shows that the family $\{u_\alpha\}$, $\alpha \in (0, 1)$ is bounded in the norm from the left hand side of (4.2.2). Consequently, we may extract a sequence (u_μ) such that

$$(4.2.7) \quad \begin{aligned} u_\mu^{(j)} &\rightarrow u^{(j)} \quad \text{weakly in } L^2(I, H^{s_j+2}(\Omega) \cap H_0^1(\Omega)), \\ &\quad j = 0, \dots, L-1, \\ u_\mu^{(j)} &\rightarrow u^{(j)} \quad \star \text{ weakly in } L^\infty(I, H_0^1(\Omega)), \\ &\quad j = 0, \dots, L, \\ u_\mu^{(j)} &\rightarrow u^{(j)} \quad \star \text{ weakly in } L^\infty(I, H^0(\Omega)), \\ &\quad j = 0, \dots, L+1. \end{aligned}$$

By the compactness lemma, cf. [19], Sect. 5.2, Chapt. 1, for some subsequence (u_ν) and some $n/2 < r < s$ we have

$$(4.2.8) \quad u_\nu \rightarrow u \quad \text{strongly in } L^2(I, H^{r+2}(\Omega)).$$

Since $a_\nu \partial^2 u_\nu - a \partial^2 u = (a_\nu - a) \partial^2 u_\nu + a(\partial^2 u_\nu - \partial^2 u)$, the limit function u satisfies the first two conditions (3.2.1). Choose a sequence $\sigma_k \rightarrow 0$ and consider the mollified problems corresponding to σ_k and σ_m . Using the notations from Sect. 5.5 and replacing v by $\partial^2 u$, u_{σ_k} by u_k , f_{σ_k} by f_k , I_{σ_k} by I_k and similarly for m we obtain

$$(4.2.9) \quad \begin{aligned} u_k^{(2)} &= a \partial^2 u_k + f_k + I_k, \quad \text{in } \Omega \times I, \\ u_k &= 0, \quad \text{on } \partial\Omega \times I, \end{aligned}$$

$$(4.2.10) \quad \begin{aligned} u_m^{(2)} &= a \partial^2 u_m + f_m + I_m, \quad \text{in } \Omega \times I, \\ u_m &= 0, \quad \text{on } \partial\Omega \times I, \end{aligned}$$

The difference $u_k - u_m$ satisfies

$$(4.2.11) \quad \begin{aligned} (u_k - u_m)^{(2)} &= a \partial^2 (u_k - u_m) + f_k - f_m + I_k - I_m, \quad \text{in } \Omega \times I, \\ u_k - u_m &= 0, \quad \text{on } \partial\Omega \times I. \end{aligned}$$

Repeating the argument from Section 3.4 we obtain the inequality

$$(4.2.12) \quad \begin{aligned} &|(u_k - u_m)^{(L-j)}(t)|_{0,1,\lambda}^2 + 2\lambda |(u_k - u_m)^{(L-j)}|_{0,1,\lambda,t}^2 \\ &\leq C_j(\delta) \left\{ |u_k^{(L-j)}(0) - u_m^{(L-j)}(0)|_{0,1,0}^2 + 2((f_k - f_m)^{(L-j)} \right. \\ &\quad + F_{1,k,m}^{L-j} + (I_k - I_m)^{(L-j)}, (u_k - u_m)^{(L-j+1)})_{0,\lambda,t} \\ &\quad \left. + F_{2,t,k,m}^{L-j}((u_k - u_m)^{(L-j)}, (u_k - u_m)^{(L-j)}) \right\}, \quad 0 \leq j \leq L, \end{aligned}$$

where

$$(4.2.13) \quad \begin{aligned} F_{1,k,m}^{L-j} &= \sum_{i=1}^{L-j} \binom{L-j}{i} a^{(i)} \partial^2 (u_k - u_m)^{(L-j-i)}, \\ 0 \leq j &\leq L-1, \quad F_{1,k,m}^0 = 0, \end{aligned}$$

$$(4.2.14) \quad \begin{aligned} F_{2,t,k,m}^j(u^{(j)}, u^{(j)}) &= \sum_{p,q=1}^n (a_{p,q}^{(1)} \partial_q u^{(j)}, \partial_p u^{(j)})_{0,\lambda,t} + \gamma(u^{(j)}, u^{(j+1)})_{0,\lambda,t} \\ &+ \sum_{p,q=1}^n ((\partial_p a_{p,q}) \partial_q u^{(j)}, u^{(j+1)})_{0,\lambda,t} \\ &+ \sum_{\substack{p=n+1 \\ \text{or } q=n+1}} (a_{p,q} \partial_p \partial_q u^{(j)}, u^{(j+1)})_{0,\lambda,t}. \end{aligned}$$

The inequalities (3.4.16), (3.4.18) and the Lemmas 5.5.1, 5.5.2 show that the right hand side in (4.2.12) tends to zero if $k, m \rightarrow \infty$. Since the constant λ remains bounded as $k, m \rightarrow \infty$ we see that $u_k^{(j)} - u_m^{(j)} \rightarrow 0$ as $k, m \rightarrow \infty$ in the complete norm $\|\cdot\|_{0,1,0,T}$ and therefore there exists a limit function u of the sequence u_k with the properties (4.2.4). The uniqueness is also a consequence of (4.2.2).

4.3. Existence theorem for the quasi-linear problem

Now let us consider the quasilinear, initial-boundary value problem

$$(4.3.1) \quad \begin{aligned} u^{(2)} &= a(\partial u) \partial^2 u + f, & \text{in } \Omega \times I, \\ u &= 0, & \text{on } \partial\Omega \times I, \\ u(0) &= u^0, \quad u^{(1)}(0) = u^1, & \text{on } \Omega, \end{aligned}$$

where we use the same notations as in Section 2.2 and where

$$(4.3.2) \quad a(\partial u) \partial^2 u = \sum_{p,q=1}^{n+1} a_{p,q}(\partial u) \partial_p \partial_q u.$$

We shall assume that (2.2.6), (3.2.3), (3.2.4) are valid and the matrix functions $a(\cdot) = \{a_{p,q}(\cdot) : p, q = 1, \dots, n+1\}$, defined in some neighbourhood U_0 of the range of ∂u^0 are arbitrarily smooth for all possible p, q and symmetric if $p, q \neq n+1$. Let us define the function $a^{(i)}(0)$ as the value of the derivative $\partial_t^i \{a(\partial u)\}$ at $t = 0$. Using Lemma 5.4.4 one can show that the recursive definition (2.2.10) makes sense for such $a^{(i)}(0)$ and therefore the functions $u^2, \dots, u^{L+1}$ are well defined provided (3.2.4) is satisfied, (cf. the argument leading to the relation (4.3.9), below). We shall assume that $u^j, j = 0, \dots, L+1$ satisfy (3.2.4) and additionally

$$(4.3.3) \quad u^j \in H^{s_j^2}(\Omega), \quad s_j^2 = \frac{s+2}{L+2} \left(L+2-j-\frac{1}{2} \right), \quad j = 0, \dots, L+1.$$

In the quasilinear case the ellipticity assumption may be formulated as follows:

$$(4.3.4) \quad \delta I_m \sum_{p=1}^n (\xi^p)^2 \leq \sum_{p,q=1}^n a_{p,q}(U) \xi^p \xi^q \leq \delta^{-1} I_m \sum_{p=1}^n (\xi^p)^2,$$

for some constant $\delta > 0$ for arbitrary $\xi = (\xi^1, \dots, \xi^n) \in R^n$ and arbitrary $U \in U_0$.

THEOREM 4.3.1. *If the assumptions (2.2.6), (3.2.3), (3.2.4) with the modifications described above and (4.3.3), (4.3.4) are satisfied then there exists unique, local solution of the problem (4.3.1) with the regularity*

$$(4.3.5) \quad \begin{aligned} u^{(L+1)} &\in L^\infty(I', H^0(\Omega)), \quad u^{(L)} \in L^\infty(I', H_0^1(\Omega)), \\ u^{(j)} &\in L^2(I', H^{s_j+2}(\Omega) \cap H_0^1(\Omega)), \\ j &= 0, \dots, L-1, \quad I' = (0, T'). \end{aligned}$$

PROOF. We adopt the method of the proof from [8]. For $M, T > 0$, define $Z(M, T)$ as the set of functions w satisfying the following conditions

$$(4.3.6) \quad \begin{aligned} w^{(L+1)} &\in L^\infty(I, H^0(\Omega)), \quad w^{(L)} \in L^\infty(I, H_0^1(\Omega)), \\ w^{(j)} &\in L^2(I, H^{s_j+2}(\Omega) \cap H_0^1(\Omega)), \\ j &= 0, \dots, L-1, \quad I = (0, T), \\ w^{(k)}(0) &= u^k, \quad k = 0, \dots, L+1, \\ \text{ess sup}_{t \in (0, T)} (|w^{(L+1)}(t)|_0^2 + |w^{(L)}(t)|_1^2) &+ \int_0^T (|w^{(L-1)}|_2^2 + |w|_{s+2}^2) d\tau \leq M^2. \end{aligned}$$

The trace theory, cf. Sect. 5.2, implies that $Z(M, T)$ is nonempty for M sufficiently large. If $w \in Z(M, T)$, then one can consider the linear problem

$$(4.3.7) \quad \begin{aligned} u^{(2)} &= a(\partial w) \partial^2 u + f, & \text{in } \Omega \times I, \\ u &= 0, & \text{on } \partial\Omega \times I, \\ u(0) &= u^0, \quad u^{(1)}(0) = u^1, & \text{on } \Omega. \end{aligned}$$

Using Lemmas 5.4.6, 5.4.8 and 5.4.9 one can check that the functions

$$(4.3.8) \quad \begin{aligned} a^{(L)}(\partial w) &\in L^\infty(I, H^0(\Omega)), \quad a^{(L-1)}(\partial w) \in L^2(I, H^1(\Omega)), \\ a(\partial w) &\in L^2(I, H^{s+1}(\Omega)) \end{aligned}$$

are bounded uniformly with respect to w in corresponding spaces. It is clear that $\partial w^{(j)}(0) = \partial u^j \in H^{r_j+1}(\Omega)$, $j = 0, \dots, L-1$ and $\partial_t^j \{a(\partial w)\}(0)$ is equal to the sum of expressions of the form $b_{l_1 \dots l_k}(\partial w(0)) \partial w^{(l_1)}(0) \dots \partial w^{(l_k)}(0)$ for some smooth

functions $b_{l_1 \dots l_k}$ and for $l_1 + \dots + l_k = j$, $1 \leq k \leq j$. Using Lemmas 5.4.4, 5.4.5 we can check that

$$(4.3.9) \quad \partial_t^j \{a(\partial w)\}(0) \in H^{r_j+1}(\Omega), \quad j = 0, \dots, L-1.$$

By the assumptions of the Theorem 4.3.1 the data in (4.3.7) satisfy the remaining assumptions of the Theorem 4.2.2, so the problem (4.3.7) is uniquely solvable on the interval $I = (0, T)$. Let us denote by $\mathcal{T}$ the map which carries $w \in Z(M, T)$ into the solution u of the problem (4.3.7). We shall show that if M and T are appropriately chosen then $\mathcal{T}$ maps $Z(M, T)$ into itself. For this purpose we shall use Theorem 4.2.1. Choose $M_0 > 0$ such that

$$(4.3.10) \quad \begin{aligned} M_0^2 &\geq 4N_1 C(\delta) \Lambda^{2L}(\delta, \|a(\partial w)(0)\|_0), \\ \|a(\partial w)(0)\|_0^2 &= \sum_{j=0}^{L-2} |a^{(j)}(\partial w)(0)|_{s_j^1+1}^2, \\ N_T &= \sum_{0 \leq j \leq L} \left\{ \|u^{L-j}\|_{0,1}^2 + \left(\int_0^T |f^{(L-j)}(\tau)|_0 d\tau \right)^2 \right\} \\ &\quad + \sum_{0 \leq l \leq k \leq L-1} |f^{(k-l)}|_{s_k, 0, T}^2. \end{aligned}$$

The boundedness of the coefficients in the norms of the spaces listed in (4.3.8) combined with the relations

$$(4.3.11) \quad \begin{aligned} \max_{t \in [0, T]} |v(t) - v(0)|_0 &\leq T \max_{t \in [0, T]} |v^{(1)}(t)|_0, \\ |v(t) - v(0)|_\beta &\leq c |v(t) - v(0)|_0^{1-\beta/m} |v(t) - v(0)|_m^{\beta/m}, \\ \beta &\in (0, m), \\ \int_0^T |v(t)|_r^2 dt &\leq T \operatorname{ess\,sup}_{t \in (0, T)} |v(t)|_r^2, \end{aligned}$$

imply that $\|a(\partial w)\|_{T'} \rightarrow \|a(\partial w)(0)\|_0$ as $T' \rightarrow 0$. One can assume that Λ depends continuously on $\|a(\partial w)\|_T$ and $\|a(\partial w)\|_T$ depends continuously on $w \in Z(M, T)$, so if $T' \in (0, 1]$ is sufficiently small then

$$(4.3.12) \quad \begin{aligned} e^{2\Lambda(\delta, \|a(\partial w)\|_{T'})T'} &\leq 2, \\ C(\delta) N_{T'} \Lambda^{2L}(\delta, \|a(\partial w)\|_{T'}) &\leq 2C(\delta) N_1 \Lambda^{2L}(\delta, \|a(\partial w)(0)\|_0). \end{aligned}$$

The relations (4.2.2), (4.3.10)–(4.3.12) show that $\mathcal{T}$ maps $Z(M_0, T')$ into itself for all sufficiently small T' .

It is sufficient to check that $\mathcal{T}$ is contractive for sufficiently small T' with respect to some complete norm. To this end for $w, \bar{w} \in Z(M_0, T')$ let us put $u = \mathcal{T}w$, $\bar{u} = \mathcal{T}\bar{w}$, $U = u - \bar{u}$, $W = w - \bar{w}$. The function U is a solution of the problem

$$(4.3.13) \quad \begin{aligned} U^{(2)} &= a(\partial w)\partial^2 U + [a(\partial \bar{w}) - a(\partial w)]\partial^2 \bar{u}, & \text{in } \Omega \times (0, T'), \\ U &= 0, & \text{on } \partial\Omega \times (0, T'), \\ U(0) &= 0, \quad U^{(1)}(0) = 0, & \text{on } \Omega. \end{aligned}$$

Repeating some arguments from Section 3.4 and using the same notations we obtain for $t \in (0, T')$ and $\Lambda = \Lambda(\delta, \|a(\partial w)\|_{T'})$

$$(4.3.14) \quad \begin{aligned} &|U^{(1)}(t)|_{0,1,\Lambda}^2 + 2\Lambda|U^{(1)}|_{0,\Lambda,t}^2 + a_\gamma(\partial U(t), \partial U(t)) \\ &+ 2\Lambda \int_0^t a_\gamma(\partial U(\tau), \partial U(\tau))d\tau \\ &= 2([a(\partial \bar{w}) - a(\partial w)]\partial^2 \bar{u}, U^{(1)})_{0,\Lambda,t} + F_{2,t}^0(U, U). \end{aligned}$$

As in Sect. 3.4 one can prove (cf. formula (3.4.21)) that for Λ equal to $\Lambda(\delta, \|a(\partial w)\|_{T'})$

$$(4.3.15) \quad \begin{aligned} \frac{1}{4}e^{-2\Lambda T'} \|U\|_{0,1,0,T'}^2 &\leq \frac{1}{4}\|U\|_{0,1,\lambda,T'}^2 + \lambda|U|_{0,1,\lambda,T'}^2 \\ &\leq C(\delta, M_0) \left(\int_0^{T'} |a(\partial \bar{w}) - a(\partial w)|_0 |\partial^2 \bar{u}|_s d\tau \right)^2 \\ &\leq C(\delta, M_0) \left(\int_0^{T'} |a(\partial \bar{w}) - a(\partial w)|_0^2 d\tau \right) \left(\int_0^{T'} |\partial^2 \bar{u}|_s^2 d\tau \right). \end{aligned}$$

Applying the mean-value theorem and using the fact that $u, \bar{u}, w, \bar{w} \in Z(M_0, T')$ we get easily

$$(4.3.16) \quad \begin{aligned} \|U\|_{0,1,0,T'}^2 &\leq C(\delta, M_0)e^{2\Lambda(\delta, \|a(\partial w)\|_{T'})T'} \\ &\times \left(\int_0^{T'} \int_0^1 |da((1-\theta)\partial \bar{w} + \theta\partial w)|_s^2 d\theta d\tau \right) \left(\int_0^{T'} |\partial^2 \bar{u}|_s^2 d\tau \right) \|W\|_{0,1,0,T'}^2 \\ &\leq T' C(\delta, M_0)e^{2\Lambda(\delta, \|a(\partial w)\|_{T'})T'} \\ &\times \max_{\tau \in [0, T']} \int_0^1 |da((1-\theta)\partial \bar{w} + \theta\partial w)|_s^2 d\theta \left(\int_0^{T'} |\partial^2 \bar{u}|_s^2 d\tau \right) \|W\|_{0,1,0,T'}^2. \end{aligned}$$

Since the functions in (4.3.8) are uniformly bounded in corresponding norms, from (4.3.16) it follows that the map $\mathcal{T}$ is contractive for sufficiently small T' with respect

to the norm $\|\cdot\|_{0,1,0,T'}$. The unique fixed point of this map gives the solution to the problem (4.3.1).

5. Function spaces and product estimates

5.1. Some properties of Sobolev spaces

In this section we collect some facts concerning the function spaces which are used in this paper. Let us begin from the remark that Theorem 2.3 of [20], Chapt. 1 has the following consequence.

LEMMA 5.1.1. *If $w \in L^2(I, H^\sigma(\Omega))$, $w^{(M)} \in L^2(I, H^0(\Omega))$, $\sigma \geq 0$, $M = 1, 2, \dots$, then*

$$(5.1.1) \quad \begin{aligned} w^{(j)} &\in L^2(I, H^{\sigma_j}(\Omega)), \quad \sigma_j = \frac{\sigma}{M}(M-j), \quad j = 0, \dots, M, \\ \sum_{j=0}^M \|w^{(j)}\|_{L^2(I, H^{\sigma_j}(\Omega))} &\leq c(\|w\|_{L^2(I, H^\sigma(\Omega))} + \|w^{(M)}\|_{L^2(I, H^0(\Omega))}), \end{aligned}$$

with some positive constant c .

Similarly from Theorem 3.1 of [20], Chapt. 1 we can infer

LEMMA 5.1.2. *If $w \in L^2(I, H^\sigma(\Omega))$, $w^{(M)} \in L^2(I, H^0(\Omega))$, $\sigma \geq 0$, $M = 1, 2, \dots$, then*

$$(5.1.2) \quad \begin{aligned} w^{(j)} &\in C^0(I, H^{\sigma_j^1}(\Omega)), \quad \sigma_j^1 = \frac{\sigma}{M} \left(M - j - \frac{1}{2} \right), \\ j &= 0, \dots, M-1, \\ \sum_{j=0}^{M-1} \|w^{(j)}\|_{C^0(I, H^{\sigma_j^1}(\Omega))} &\leq c(\|w\|_{L^2(I, H^\sigma(\Omega))} + \|w^{(M)}\|_{L^2(I, H^0(\Omega))}), \end{aligned}$$

with some positive constant c .

5.2. Some properties of traces

The theory of traces from [20], Chapt. 1, Section 3.2 has the following consequences.

LEMMA 5.2.1. *If $f \in L^2(I, H^\sigma(\Omega))$, $f^{(M)} \in L^2(I, H^0(\Omega))$, $\sigma \geq 0$, $M = 1, 2, \dots$, then there exist traces $f^{(0)}(0), \dots, f^{(M-1)}(0)$ of f at $t = 0$ satisfying the conditions*

$$(5.2.1) \quad \begin{aligned} f^{(j)}(0) &\in H^{\sigma_j^1}(\Omega), \quad j = 0, \dots, M-1, \quad (\text{cf. (5.1.2)}), \\ \sum_{j=0}^{M-1} \|f^{(j)}(0)\|_{H^{\sigma_j^1}(\Omega)} &\leq C(\|f\|_{L^2(I, H^\sigma(\Omega))} + \|f^{(M)}\|_{L^2(I, H^0(\Omega))}), \end{aligned}$$

$$C = \text{const.} > 0.$$

LEMMA 5.2.2. If σ_j^1 are as before then for every set $\{f^0, \dots, f^{M-1}\}$ of functions with the properties

$$(5.2.2) \quad f^j \in H^{\sigma_j^1}(\Omega), \quad j = 0, \dots, M-1,$$

there exists a function $f \in L^2(I, H^\sigma(\Omega))$ such that $f^{(M)} \in L^2(I, H^0(\Omega))$, satisfying the relations

$$(5.2.3) \quad \begin{aligned} f^{(j)}(0) &= f^j, \quad j = 0, \dots, M-1, \\ \|f\|_{L^2(I, H^\sigma(\Omega))} + \|f^{(M)}\|_{L^2(I, H^0(\Omega))} &\leq C \sum_{j=0}^{M-1} \|f^j\|_{H^{\sigma_j^1}(\Omega)}, \\ C &= \text{const.} > 0. \end{aligned}$$

If $f^j \in C^\infty(\bar{\Omega})$, $j = 0, \dots, M-1$ then one can find $f \in C^\infty(\bar{\Omega} \times \bar{I})$ satisfying the relation (5.2.3).

5.3. Some product estimates

Assume that Ω is a domain in R^n with C^∞ and compact boundary and let $G = \Omega$ or $G = \partial\Omega$. For $s > 0$, $p, \theta > 1$, s non-integer, denote by $B_{p,\theta}^s(G)$ the Besov space of functions for which the norm defined in (5.3.5) below is finite, (the more general description of Besov spaces can be found in [35], Sect. 2.3.1, 2.5.1 and 4.2.1).

LEMMA 5.3.1. Suppose p, θ, s_1, s_2 , are arbitrary real numbers, s is real, non-integer and the following conditions are satisfied

$$(5.3.1) \quad \begin{aligned} 1 < p < \infty, \quad 1 < \theta < \infty, \\ s_i \geq s > 0, \quad i = 1, 2, \quad s_1 + s_2 - s - \frac{n}{p} > 0, \end{aligned}$$

where n denotes the dimension of G , then the relations

$$(5.3.2) \quad u_i \in B_{p,\theta}^{s_i}(G), \quad i = 1, 2,$$

imply $u_1 u_2 \in B_{p,\theta}^s(G)$ and

$$(5.3.3) \quad \|u_1 u_2\|_{B_{p,\theta}^s(G)} \leq C \|u_1\|_{B_{p,\theta}^{s_1}(G)} \|u_2\|_{B_{p,\theta}^{s_2}(G)},$$

for some positive constant C . Furthermore, there exist a positive constant C and small positive numbers $\varepsilon_1, \varepsilon_2$, such that $s_i - \varepsilon_i > 0$, $i = 1, 2$, and for arbitrary u_1, u_2 satisfying (5.3.2)

$$\|u_1 u_2\|_{B_{p,\theta}^s(G)} \leq C \|u_1\|_{B_{p,\theta}^{s_1-\varepsilon_1}(G)} \|u_2\|_{B_{p,\theta}^{s_2-\varepsilon_2}(G)}, \quad \text{if } s_i > s, \quad i = 1, 2,$$

$$\begin{aligned}
(5.3.4) \quad & \|u_1 u_2\|_{B_{p,\theta}^s(G)} \leq C(\|u_1\|_{B_{p,\theta}^{s_1-\varepsilon_1}(G)} \|u_2\|_{B_{p,\theta}^{s_2-\varepsilon_2}(G)} + \|u_1\|_{C^0(G)} \|u_2\|_{B_{p,\theta}^{s_2}(G)}), \\
& \text{if } s_1 > s, \quad s_2 = s, \\
& \|u_1 u_2\|_{B_{p,\theta}^s(G)} \leq C(\|u_1\|_{B_{p,\theta}^{s_1-\varepsilon_1}(G)} \|u_2\|_{B_{p,\theta}^{s_2-\varepsilon_2}(G)} + \|u_1\|_{B_{p,\theta}^{s_1}(G)} \|u_2\|_{C^0(G)}), \\
& \text{if } s_1 = s, \quad s_2 > s, \\
& \|u_1 u_2\|_{B_{p,\theta}^s(G)} \leq C(\|u_1\|_{B_{p,\theta}^{s_1-\varepsilon_1}(G)} \|u_2\|_{B_{p,\theta}^{s_2-\varepsilon_2}(G)} \\
& \quad + \|u_1\|_{C^0(G)} \|u_2\|_{B_{p,\theta}^{s_2}(G)} + \|u_1\|_{B_{p,\theta}^{s_1}(G)} \|u_2\|_{C^0(G)}), \\
& \text{if } s_1 = s, \quad s_2 = s.
\end{aligned}$$

PROOF. The case $G = \partial\Omega$ can be reduced to the case of an open domain in R^{n-1} by using the local charts, cf. [36], Sect. 3.2.2. Therefore we restrict ourselves to the case of an open domain $\Omega \subset R^n$. In this case, $B_{p,\theta}^s(\Omega)$ can be characterized as the completion of the set $C^\infty(\Omega)$ in the norm

$$\begin{aligned}
(5.3.5) \quad & \|u\|_{B_{p,\theta}^s(\Omega)} = \|u\|_{L_p(\Omega)} + |u|_{B_{p,\theta}^s(\Omega)}, \\
& |u|_{B_{p,\theta}^s(\Omega)} = \sum_{|\alpha|=s} \left(\int_{R^n} \|\Delta_h D^\alpha u\|_{L_p(\Omega_h)}^\theta |h|^{-n-\{s\}\theta} dh \right)^{1/\theta},
\end{aligned}$$

where

$$\begin{aligned}
(5.3.6) \quad & [s] = \text{the integer part of } s, \quad \{s\} = s - [s], \\
& \Omega_h = \{x \in R^n : x + h \in \Omega\}, \\
& (\Delta_h \phi)(x) = \phi(x + h) - \phi(x), \\
& D^\alpha = \partial_1^{\alpha_1} \dots \partial_n^{\alpha_n}, \quad \alpha = (\alpha_1, \dots, \alpha_n), \quad |\alpha| = \alpha_1 + \dots + \alpha_n.
\end{aligned}$$

Since the product estimates in the L_p norm are simple (cf. (5.3.28)) it is sufficient to estimate the terms $|v_k w_{[s]-k}|_{B_{p,\theta}^{\{s\}}(\Omega)}$, $k = 0, \dots, [s]$, where $v_k \in B_{p,\theta}^{s_1-k}(\Omega)$ is an arbitrary term of the form $D^\alpha u_1$, $|\alpha| = k$ and $w_{[s]-k} \in B_{p,\theta}^{s_2-[s]+k}(\Omega)$ is an arbitrary term of the form $D^\beta u_2$, $|\beta| = [s] - k$. According to the formula

$$(5.3.7) \quad \Delta_h(vw) = (\Delta_h v)(\Delta_h w) + v\Delta_h w + (\Delta_h v)w,$$

we have to estimate the integrals

$$\begin{aligned}
(5.3.8) \quad & I_1 = \left(\int_{R^n} \|(\Delta_h v_k)(\Delta_h w_{[s]-k})\|_{L_p(\Omega_h)}^\theta |h|^{-n-\{s\}\theta} dh \right)^{1/\theta}, \\
& I_2 = \left(\int_{R^n} \|v_k \Delta_h w_{[s]-k}\|_{L_p(\Omega_h)}^\theta |h|^{-n-\{s\}\theta} dh \right)^{1/\theta},
\end{aligned}$$

$$I_3 = \left(\int_{R^n} \|(\Delta_h v_k) w_{[s]-k}\|_{L_p(\Omega_h)}^\theta |h|^{-n-\{s\}\theta} dh \right)^{1/\theta}.$$

To consider I_1 let us first note, that if $k \leq [s]$, $r_1, r_2 > 1$, $\frac{1}{r_1} + \frac{1}{r_2} = 1$, and s_1, s_2, s satisfy (5.3.1), then one can choose the numbers

$$(5.3.9) \quad q_1, q_2 > 1, \quad \frac{1}{q_1} + \frac{1}{q_2} = 1, \quad \varepsilon_1, \varepsilon_2 > 0,$$

satisfying the conditions

$$(5.3.10) \quad \begin{aligned} s_1 - k - \frac{\{s\}}{r_1} - \varepsilon_1 &> \frac{n}{p} \frac{1}{q_2}, \\ s_2 - [s] + k - \frac{\{s\}}{r_2} - \varepsilon_2 &> \frac{n}{p} \frac{1}{q_1}. \end{aligned}$$

This is the consequence of the fact that the rectangle $0 < x_1 < a_1$, $0 < x_2 < a_2$ and the straight line $x_1 + x_2 = \frac{n}{p}$ have a non-empty intersection for $a_1 + a_2 > \frac{n}{p}$. If $a_1 = s_1 - k - \frac{\{s\}}{r_1} - \varepsilon_1$, $a_2 = s_2 - [s] + k - \frac{\{s\}}{r_2} - \varepsilon_2$, $x_1 = \frac{n}{p} \frac{1}{q_2}$, $x_2 = \frac{n}{p} \frac{1}{q_1}$, then one can check that for sufficiently small $\varepsilon_1, \varepsilon_2 > 0$, the inequality $a_1 + \varepsilon_1 \geq s_1 - [s] - \frac{\{s\}}{r_1} > s_1 - s \geq 0$, implies $a_1 > 0$ and the inequality $a_2 + \varepsilon_2 \geq s_2 - [s] - \frac{\{s\}}{r_2} > s_2 - s \geq 0$, implies $a_2 > 0$. It is also clear that for sufficiently small $\varepsilon_1, \varepsilon_2 > 0$, from the inequality $(a_1 + \varepsilon_1) + (a_2 + \varepsilon_2) = s_1 + s_2 - s > \frac{n}{p}$ it follows that $a_1 + a_2 > \frac{n}{p}$.

The relations (5.3.10) can be written in the form

$$(5.3.11) \quad \begin{aligned} s_1 - k - \varepsilon_1 - \frac{n}{p} &> \frac{\{s\}}{r_1} - \frac{n}{pq_1}, \\ s_2 - [s] + k - \varepsilon_2 - \frac{n}{p} &> \frac{\{s\}}{r_2} - \frac{n}{pq_2}, \end{aligned}$$

and (5.3.11) implies the embeddings

$$(5.3.12) \quad B_{p,\theta}^{s_1-k-\varepsilon_1}(\Omega) \subset B_{pq_1, r_1\theta}^{\frac{\{s\}}{r_1}}(\Omega), \quad B_{p,\theta}^{s_2-[s]+k-\varepsilon_2}(\Omega) \subset B_{pq_2, r_2\theta}^{\frac{\{s\}}{r_2}}(\Omega),$$

cf. [36], Theorem 3.3.1. Using Hölder's inequality and (5.3.12) we deduce that

$$(5.3.13) \quad \begin{aligned} I_1 &\leq \left(\int \| \Delta_h v_k \|_{L_{pq_1}(\Omega_h)}^\theta |h|^{-\frac{1}{r_1}(n+\{s\}\theta)} \right. \\ &\quad \times \left. \| \Delta_h w_{[s]-k} \|_{L_{pq_2}(\Omega_h)}^\theta |h|^{-\frac{1}{r_2}(n+\{s\}\theta)} dh \right)^{\frac{1}{\theta}} \\ &\leq \left(\int \| \Delta_h v_k \|_{L_{pq_1}(\Omega_h)}^{r_1\theta} |h|^{-(n+\{s\}\theta)} dh \right)^{\frac{1}{r_1\theta}} \end{aligned}$$

$$\begin{aligned}
& \times \left(\int \|\Delta_h w_{[s]-k}\|_{L_{pq_2}^{r_2\theta}(\Omega_h)} |h|^{-(n+\{s\}\theta)} dh \right)^{\frac{1}{r_2\theta}} \\
& \leq \|v_k\|_{B_{pq_1, r_1\theta}^{\{s\}}(\Omega)} \|w_{[s]-k}\|_{B_{pq_2, r_2\theta}^{\{s\}}(\Omega)} \\
& \leq C \|v_k\|_{B_{p,\theta}^{s_1-k-\varepsilon_1}(\Omega)} \|w_{[s]-k}\|_{B_{p,\theta}^{s_2-[s]+k-\varepsilon_2}(\Omega)} \\
& \leq C \|u_1\|_{B_{p,\theta}^{s_1-\varepsilon_1}(\Omega)} \|u_2\|_{B_{p,\theta}^{s_2-\varepsilon_2}(\Omega)}.
\end{aligned}$$

To estimate I_2 in the case $k > 0$ or $s_2 - s > 0$ we first choose the numbers (5.3.9) in such way, that

$$(5.3.14) \quad s_1 - k - \varepsilon_1 \geq \frac{n}{p} \frac{1}{q_2}, \quad s_2 - s + k - \varepsilon_2 \geq \frac{n}{p} \frac{1}{q_1}.$$

The last relations are equivalent to

$$(5.3.15) \quad s_1 - k - \varepsilon_1 - \frac{n}{p} \geq -\frac{n}{pq_1}, \quad s_2 - [s] + k - \varepsilon_2 - \frac{n}{p} \geq \{s\} - \frac{n}{pq_2},$$

and therefore the embeddings

$$(5.3.16) \quad B_{p,\theta}^{s_1-k-\varepsilon_1}(\Omega) \subset L_{pq_1}(\Omega), \quad B_{p,\theta}^{s_2-[s]+k-\varepsilon_2}(\Omega) \subset B_{pq_2,\theta}^{\{s\}}(\Omega),$$

are true. Thus in the case $k > 0$ or $s_2 - s > 0$ we have

$$\begin{aligned}
I_2 & \leq \left(\int \|v_k\|_{L_{pq_1}^\theta(\Omega_h)} \|\Delta_h w_{[s]-k}\|_{L_{pq_2}^\theta(\Omega_h)} |h|^{-(n+\{s\}\theta)} dh \right)^{\frac{1}{\theta}} \\
& \leq \|v_k\|_{L_{pq_1}(\Omega)} \|w_{[s]-k}\|_{B_{pq_2,\theta}^{\{s\}}(\Omega)} \\
(5.3.17) \quad & \leq C \|v_k\|_{B_{p,\theta}^{s_1-k-\varepsilon_1}(\Omega)} \|w_{[s]-k}\|_{B_{p,\theta}^{s_2-[s]+k-\varepsilon_2}(\Omega)} \\
& \leq C \|u_1\|_{B_{p,\theta}^{s_1-\varepsilon_1}(\Omega)} \|u_2\|_{B_{p,\theta}^{s_2-\varepsilon_2}(\Omega)}.
\end{aligned}$$

In the case $k = 0$ and $s_2 - s = 0$, the last part of (5.3.1) implies $s_1 > \frac{n}{p}$ and consequently $B_{p,\theta}^{s_1}(\Omega) \subset C^0(\Omega)$. Therefore

$$(5.3.18) \quad I_2 \leq \|v_0\|_{C^0(\Omega)} \|w_{[s]}\|_{B_{p,\theta}^{\{s\}}(\Omega)} \leq \|u_1\|_{C^0(\Omega)} \|u_2\|_{B_{p,\theta}^{s_2}(\Omega)}.$$

The argument leading to the estimate for I_3 is analogous. If $k < [s]$ or $s_1 > s$, one can choose the numbers (5.3.9) satisfying

$$(5.3.19) \quad s_1 - k - \{s\} - \varepsilon_1 > \frac{n}{p} \frac{1}{q_2}, \quad s_2 - [s] + k - \varepsilon_2 > \frac{n}{p} \frac{1}{q_1},$$

which imply

$$(5.3.20) \quad s_1 - k - \varepsilon_1 - \frac{n}{p} > \{s\} - \frac{n}{pq_1}, \quad s_2 - [s] + k - \varepsilon_2 - \frac{n}{p} > -\frac{n}{pq_2},$$

and consequently

$$(5.3.21) \quad B_{p,\theta}^{s_1-k-\varepsilon_1}(\Omega) \subset B_{pq_1,\theta}^{\{s\}}(\Omega), \quad B_{p,\theta}^{s_2-[s]+k-\varepsilon_2}(\Omega) \subset L_{pq_2}(\Omega).$$

Therefore

$$(5.3.22) \quad \begin{aligned} I_3 &\leq \left(\int \|\Delta_h v_k\|_{L_{pq_1}(\Omega_h)}^\theta \|w_{[s]-k}\|_{L_{pq_2}(\Omega_h)}^\theta |h|^{-(n+\{s\}\theta)} dh \right)^{\frac{1}{\theta}} \\ &\leq \|v_k\|_{B_{pq_1,\theta}^{\{s\}}(\Omega)} \|w_{[s]-k}\|_{L_{pq_2}(\Omega)} \\ &\leq C \|v_k\|_{B_{p,\theta}^{s_1-k-\varepsilon_1}(\Omega)} \|w_{[s]-k}\|_{B_{p,\theta}^{s_2-[s]+k-\varepsilon_2}(\Omega)} \\ &\leq C \|u_1\|_{B_{p,\theta}^{s_1-\varepsilon_1}(\Omega)} \|u_2\|_{B_{p,\theta}^{s_2-\varepsilon_2}(\Omega)}. \end{aligned}$$

If $k = [s]$ and $s_1 = s$ then (5.3.1) implies $s_2 > \frac{n}{p}$, so we have $B_{p,\theta}^{s_2}(\Omega) \subset C^0(\Omega)$ and

$$(5.3.23) \quad I_3 \leq \|w_0\|_{C^0(\Omega)} \|v_{[s]}\|_{B_{p,\theta}^{\{s\}}(\Omega)} \leq \|u_2\|_{C^0(\Omega)} \|u_1\|_{B_{p,\theta}^{s_1}(\Omega)}.$$

The proof is complete.

The first assertion of the last lemma can be formulated more simply:

$$(5.3.24) \quad \begin{aligned} B_{p,\theta}^{s_1}(G) B_{p,\theta}^{s_2}(G) &\subset B_{p,\theta}^s(G), \quad \text{for } s_i \geq s > 0, \\ s_1 + s_2 - s - \frac{n}{p} &> 0, \quad s\text{-non-integer.} \end{aligned}$$

Using the same convention we can formulate some simple consequences of Lemma 5.3.1 in the special case when $p, \theta = 2$ and consequently $B_{p,\theta}^s = H^s$.

LEMMA 5.3.2. For L, r_i as in (2.2.3) and $0 \leq i \leq j \leq L-1$, the following embedding is true

$$(5.3.25) \quad H^{r_i+1}(\Omega) H^{r_{j-i}}(\Omega) \subset H^{r_j}(\Omega).$$

PROOF. According to Lemma 5.3.1 it is sufficient to note that $r_i + 1 \geq r_j$, $r_{j-i} \geq r_j$ and $r_i + 1 + r_{j-i} - r_j = s - \frac{1}{2} \frac{s}{L} + 1 > \frac{n}{2}$.

LEMMA 5.3.3. If the assumptions of Lemma 5.3.2 are fulfilled then

$$(5.3.26) \quad H^{r_i+1-\frac{1}{2}}(\partial\Omega) H^{r_{j-i}+1-\frac{1}{2}}(\partial\Omega) \subset H^{r_j+1-\frac{1}{2}}(\partial\Omega).$$

PROOF. The embedding (5.3.26) is a consequence of the relations $r_i \geq r_j$, $r_{j-i} \geq r_j$ and $r_i + 1 - 1/2 + r_{j-i} + 1 - 1/2 - r_j - 1 + 1/2 - (n-1)/2 = r_i + 1 + r_{j-i} - r_j - n/2 > 0$, (cf. the proof of the preceding lemma).

LEMMA 5.3.4. *If r_j , L , are as before and $j = 0, \dots, L-3$, then*

$$(5.3.27) \quad H^{r_{j+2}+2}(\Omega) \subset H^{r_j}(\Omega).$$

PROOF. It is sufficient to check that $r_{j+2} + 2 \geq r_j$, $j = 0, \dots, L-3$, and one can easily show, that the last relation is equivalent to $1 \geq s/L$ and is true by assumption (2.2.3).

Let us note that in the last three lemmas we do not assume that s is a non-integer. To cover the case of an integer s we give also some version of Lemma 5.3.1 concerning the usual Sobolev spaces.

LEMMA 5.3.5. *If $s > 0$ is an integer and s_1, s_2 , are arbitrary real numbers satisfying the conditions (5.3.1), then all the statements of Lemma 5.3.1 remain valid, provided that the spaces $B_{p,\theta}^s(G)$ and $B_{p,\theta}^{s_i}(G)$ are replaced by the Sobolev spaces $H_p^s(G)$ and $H_p^{s_i}(G)$ respectively. If $s = 0$ then the estimates (5.3.4) must be replaced by*

$$(5.3.28) \quad \begin{aligned} \|u_1 u_2\|_{L_p(G)} &\leq C \|u_1\|_{H_p^{s_1-\varepsilon_1}(G)} \|u_2\|_{H_p^{s_2-\varepsilon_2}(G)}, & s_1 > 0, \quad s_2 > 0, \\ \|u_1 u_2\|_{L_p(G)} &\leq \|u_1\|_{L_p(G)} \|u_2\|_{C^0(G)}, & s_1 = 0, \quad s_2 > 0, \\ \|u_1 u_2\|_{L_p(G)} &\leq \|u_1\|_{C^0(G)} \|u_2\|_{L_p(G)}, & s_1 > 0, \quad s_2 = 0, \end{aligned}$$

for some positive constants $C, \varepsilon_1, \varepsilon_2$ such that $s_1 - \varepsilon_1 > 0$ and $s_2 - \varepsilon_2 > 0$ if $s_1, s_2 > 0$.

Recall that if $\mathcal{S}'(R^n)$ denotes the space of tempered distributions on R^n and $\mathcal{F}$ denotes the Fourier transform on $\mathcal{S}'(R^n)$ then

$$H_p^s(R^n) = \{f \in \mathcal{S}'(R^n) : \|f\|_{H_p^s(R^n)} = \|\mathcal{F}^{-1}(1 + |x|^2)^{s/2} \mathcal{F}f\|_{L^p(R^n)} < \infty\}$$

and $H_p^s(\Omega)$ can be defined as the space of restrictions to Ω of all functions from $H_p^s(R^n)$. If s is a non-negative integer, then the space $H_p^s(\Omega)$ can be endowed with the norm

$$\|f\|_{H_p^s(\Omega)}^p = \|f\|_{L^p(\Omega)}^p + \sum_{|\alpha|=s} \|D^\alpha f\|_{L^p(\Omega)}^p.$$

PROOF OF LEMMA 5.3.5. We have to estimate the terms

$$(5.3.29) \quad I = \|v_k w_{s-k}\|_{L_p(G)}, \quad k = 0, \dots, s,$$

where

$$(5.3.30) \quad \begin{aligned} v_k &= D^\alpha u_1 \in H_p^{s_1-k}(G), \quad |\alpha| = k \\ w_{s-k} &= D^\beta u_2 \in H_p^{s_2-(s-k)}(G), \quad |\beta| = s-k. \end{aligned}$$

In the case $s_i > s \geq 0$, $i = 1, 2$, one can choose the numbers (5.3.9), such that

$$(5.3.31) \quad s_1 - k - \varepsilon_1 > \frac{n}{p} \frac{1}{q_2}, \quad s_2 - s + k - \varepsilon_2 > \frac{n}{p} \frac{1}{q_1},$$

or equivalently

$$(5.3.32) \quad s_1 - k - \varepsilon_1 - \frac{n}{p} > -\frac{n}{pq_1}, \quad s_2 - s + k - \varepsilon_2 - \frac{n}{p} > -\frac{n}{pq_2}.$$

The embedding theorem, cf. [35], Sect. 4.6.1, gives

$$(5.3.33) \quad \begin{aligned} I &\leq \|v_k\|_{L_{pq_1}(G)} \|w_{s-k}\|_{L_{pq_2}(G)} \\ &\leq C \|v_k\|_{H_p^{s_1-k-\varepsilon_1}(G)} \|w_{s-k}\|_{H_p^{s_2-s+k-\varepsilon_2}(G)} \\ &\leq C \|u_1\|_{H_p^{s_1-\varepsilon_1}(G)} \|u_2\|_{H_p^{s_2-\varepsilon_2}(G)}. \end{aligned}$$

If one of the numbers s_i is equal to s , say $s_2 = s$ and $s_1 > s$ then $s_1 > n/p$ and consequently $u_1 \in C^0(G)$. In this case

$$(5.3.34) \quad \|u_1 u_2\|_{H_p^s(G)} \leq \|u_1\|_{C^0(G)} \|u_2\|_{H_p^{s_2}(G)} + \tilde{I},$$

where

$$(5.3.35) \quad \tilde{I} = \sum_{\substack{|\beta| < s \\ |\alpha| + |\beta| \leq s}} \|D^\alpha u_1 D^\beta u_2\|_{L_p(G)}, \quad \text{if } s > 0, \quad \tilde{I} = 0, \quad \text{if } s = 0.$$

If $s = 0$, then we obtain the third inequality from (5.3.28). If $s > 0$, then $s_1 - |\alpha| > 0$, $s_2 - |\beta| > 0$ and $(s_1 - |\alpha|) + (s_2 - |\beta|) \geq s_1 > n/p$, so the argument from the previous case, with s replaced by 0, s_1 replaced by $s_1 - |\alpha|$ and s_2 replaced by $s_2 - |\beta|$ can be applied to $\tilde{I}$. In the case $s_1 = s_2 = s > 0$ we have

$$(5.3.36) \quad \begin{aligned} \|u_1 u_2\|_{H_p^s(G)} &\leq \|u_1\|_{C^0(G)} \|u_2\|_{H_p^{s_2}(G)} + \|u_2\|_{C^0(G)} \|u_1\|_{H_p^{s_1}(G)} \\ &\quad + \sum_{\substack{|\alpha| < s, |\beta| < s \\ |\alpha| + |\beta| \leq s}} \|D^\alpha u_1 D^\beta u_2\|_{L_p(G)}, \end{aligned}$$

and the last expression can be estimated as before.

5.4. Some further product estimates

In this section we shall show some further auxiliary facts used in the main text. We start from the lemma which can be obtained by using Lemma 5.3.1 and a standard inductive argument.

LEMMA 5.4.1. *If $\sigma_i \geq s \geq 0$, $i = 1, \dots, l$ and for arbitrary $j_1 < \dots < j_k$, $1 < k \leq l$ the inequalities $\sigma_{j_1} + \dots + \sigma_{j_k} - (k-1)\frac{n}{2} > s$ are satisfied, then the embedding $H^{\sigma_1}(\Omega) \dots H^{\sigma_l}(\Omega) \subset H^s(\Omega)$ is continuous.*

PROOF. It is sufficient to show that if $l > 2$ and the assertion of the lemma is true for $l-1$ then it is true for l . In the case $\min\{\sigma_1, \dots, \sigma_{l-1}\} = s$ we have for some $i < l$, $\sigma_i = s$ so $\sigma_i + \sigma_l - \frac{n}{2} > s$ implies $\sigma_l - \frac{n}{2} > 0$. By the inductive hypothesis $H^{\sigma_1} \dots H^{\sigma_{l-1}} \subset H^s$ and from the inequality $s + \sigma_l - \frac{n}{2} > s$ it follows $H^s H^{\sigma_l} \subset H^s$. In the case $\min\{\sigma_1, \dots, \sigma_{l-1}\} > s$ let us put $\bar{s} = \min_{1 \leq k \leq l-1} \min_{i_1 < \dots < i_k} \{\sigma_{i_1} + \dots + \sigma_{i_k} - (k-1)\frac{n}{2}\} > \bar{s} - \varepsilon > s$, where ε is a sufficiently small positive number. By the inductive hypothesis $H^{\sigma_1} \dots H^{\sigma_{l-1}} \subset H^{\bar{s}-\varepsilon}$. On the other hand for some $i_1 < \dots < i_k$, $1 \leq k \leq l-1$, we have $\bar{s} - \varepsilon + \sigma_l - \frac{n}{2} = \sigma_{i_1} + \dots + \sigma_{i_k} - (k-1)\frac{n}{2} + \sigma_l - \frac{n}{2} - \varepsilon > s$ so $H^{\bar{s}-\varepsilon} H^{\sigma_l} \subset H^s$.

The next three lemmas are simple consequences of Lemma 5.4.1.

LEMMA 5.4.2. *If the assumption (3.2.3) is satisfied and $s_i^3 = \frac{s+1}{L}(L-i-\frac{1}{2})$, $i = 0, \dots, L-1$, then for arbitrary $j_1, \dots, j_k \geq 1$, such that $j_1 + \dots + j_k \leq L$, the embedding $H^{s_{j_1}^3}(\Omega) \dots H^{s_{j_k}^3}(\Omega) \subset H^0(\Omega)$ is continuous.*

PROOF. Let us observe that for $1 < i \leq k$, $s_{j_1}^3 + \dots + s_{j_i}^3 = \frac{s+1}{L}(iL - (j_1 + \dots + j_i) - \frac{i}{2})$ and therefore $s_{j_1}^3 + \dots + s_{j_i}^3 - (i-1)\frac{n}{2} = (s+1) - \frac{s+1}{L}(j_1 + \dots + j_i) + ((i-1) - \frac{s+1}{L}\frac{i}{2}) + (i-1)(s - \frac{n}{2}) > 0$.

LEMMA 5.4.3. *If the assumption (3.2.3) is satisfied and $s_i^3 = \frac{s+1}{L}(L-i-\frac{1}{2})$, $s_i^4 = \frac{s+1}{L}(L-i)$, $i = 1, \dots, L-1$ then for arbitrary $j_1, \dots, j_k \geq 1$, such that $j_1 + \dots + j_k \leq L-1$, the embedding $H^{s_{j_1}^3}(\Omega) \dots H^{s_{j_{k-1}}^3}(\Omega) H^{s_{j_k}^4}(\Omega) \subset H^0(\Omega)$ is continuous.*

PROOF. If $\sigma_{j_1} = s_{j_1}^3, \dots, \sigma_{j_i} = s_{j_i}^3$, then one can apply the previous argument. Assume that one of the numbers σ_{j_l} is equal to $s_{j_l}^4 - 1$. We have for $1 < i \leq k$, $\sigma_{j_1} + \dots + \sigma_{j_i} - (i-1)\frac{n}{2} = \frac{s+1}{L}(iL - (j_1 + \dots + j_i) - \frac{i}{2}) + \frac{1}{2}\frac{s+1}{L} - 1 - (i-1)\frac{n}{2} \geq (s+1) - \frac{s+1}{L}L + \frac{3}{2}\frac{s+1}{L} - 1 + ((i-1) - \frac{s+1}{L}\frac{i}{2}) + (i-1)(s - \frac{n}{2}) > 0$. Note that by (3.2.3) $L \geq 3$ and therefore $s+1 > L-1 \geq \frac{2}{3}L$.

LEMMA 5.4.4. *If the assumption (3.2.3) is satisfied and $r_i = \frac{s}{L}(L-i-\frac{1}{2})$, $i = 1, \dots, L-1$, then for arbitrary $j_1, \dots, j_k \geq 1$, such that $j_1 + \dots + j_k \leq j \leq L-1$, the embedding $H^{r_{j_1}+1}(\Omega) \dots H^{r_{j_k}+1}(\Omega) \subset H^{r_j+1}(\Omega)$ is continuous.*

PROOF. Similarly as before, for $1 < i \leq k$, $r_{j_1} + 1 + \dots + r_{j_i} + 1 = \frac{s}{L}(iL - (j_1 + \dots + j_i) - \frac{i}{2}) + i$ and therefore $r_{j_1} + 1 + \dots + r_{j_i} + 1 - (i-1)\frac{n}{2} \geq \frac{s}{L}(L - j - \frac{1}{2}) + (i-1)(1 - \frac{1}{2}\frac{s}{L}) + (i-1)(s - \frac{n}{2}) + 1 > r_j + 1$.

Using the extension operator and Theorem 4 from [33] (observe that for bounded domains the assumption $b(0) = 0$ is not necessary) we can prove.

LEMMA 5.4.5. *If $b \in C^\infty$, G is an open, bounded domain in R^μ with the cone property and $v \in H^\sigma(G)$, $\sigma > \frac{\mu}{2}$, then $b(v) \in H^\sigma(G)$ and $|b(v)|_{H^\sigma(G)} \leq c(1 + |v|_{H^\sigma(G)}^\sigma)$ for some constant $c > 0$.*

Let us note that by (3.2.3) we have $L \geq s+1 > \frac{n+1}{2}$ and therefore $H^{s+1,L}(\Omega \times I) \subset H^{s+1}(\Omega \times I)$ (cf. (2.2.4)). As a consequence we get.

LEMMA 5.4.6. *If $b \in C^\infty$, $\partial u \in H^{s+1,L}(\Omega \times I)$, then we have $b(\partial u) \in H^{s+1}(\Omega \times I)$ and $|b(\partial u)|_{H^{s+1}(\Omega \times I)} \leq c(1 + |\partial u|_{H^{s+1}(\Omega \times I)}^{s+1})$ for some positive constant c .*

Recall that from Lemma 5.1.2 it follows that for $\partial u \in H^{s+1,L}(\Omega \times I)$ we have $\partial u^{(j)} \in C^0(I, H^{s_j^3}(\Omega))$, $j = 0, \dots, L-1$. Combining this fact with Lemma 5.4.5 we obtain.

LEMMA 5.4.7. *If s_0^3 is defined as in Lemma 5.4.2 and $b \in C^\infty$, $\partial u \in H^{s+1,L}(\Omega \times I)$, then we have $b(\partial u) \in C^0(I, H^{s_0^3}(\Omega))$ and for some positive constant c ,*

$$\max_{t \in I} |b(\partial u(t))|_{H^{s_0^3}(\Omega)} \leq c \left(1 + \max_{t \in I} |\partial u(t)|_{H^{s_0^3}(\Omega)}^{s_0^3} \right).$$

Consider the derivatives with respect to t of the composite function $a(\partial u)$, where $a \in C^\infty$. Using mollifiers one can check that the chain rule remains valid provided that the corresponding products make sense. For example $\partial_t^L \{a(\partial u)\}$ is formally equal to the sum of the term of the form $b(\partial u)\partial u^{(L)}$ and the terms of the form $b(\partial u)\partial u^{(j_1)} \dots \partial u^{(j_i)}$, where $j_1 + \dots + j_i = L$, $j_1, \dots, j_i \leq L-1$. If $\partial u^{(L)} \in L^\infty(I, H^0(\Omega))$, $\partial u \in H^{s+1,L}(\Omega \times I)$, then using Lemma 5.4.7 one can prove that the expression of the first kind belongs to $L^\infty(I, H^0(\Omega))$. To prove that the expressions of the second kind belong to $L^\infty(I, H^0(\Omega))$ it is sufficient to apply Lemmas 5.4.2, 5.4.7 and Lemma 5.1.2. Thus, we have proved the following:

LEMMA 5.4.8. *If $\partial u^{(L)} \in L^\infty(I, H^0(\Omega))$, $\partial u \in H^{s+1,L}(\Omega \times I)$ and $a(\cdot) \in C^\infty$ then $\partial_t^L \{a(\partial u)\} \in L^\infty(I, H^0(\Omega))$ and the $L^\infty(I, H^0(\Omega))$ norm of $\partial_t^L \{a(\partial u)\}$ can be estimated by some polynomial depending only on the norms $\max_{t \in I} |\partial u^{(0)}|_{s_0^3}, \dots, \max_{t \in I} |\partial u^{(L-1)}|_{s_{L-1}^3}$, and the $L^\infty(I, H^0(\Omega))$ norm of $\partial u^{(L)}$.*

Now consider $\partial \partial_t^{L-1} \{a(\partial u)\}$. Using formal differentiation we can write this derivative as a sum of the expressions of the form $b(\partial u)\partial^2 u^{(L-1)}$ and the expressions

of the form $b(\partial u)\partial u^{(j_1)}\dots\partial u^{(j_{i-1})}\partial^2 u^{(j_i)}$, where $j_1+\dots+j_i=L-1$, $j_1,\dots,j_i\leq L-2$. Since $\partial^2 u^{(L-1)}\in L^2(I, H^0(\Omega))$ and by Lemma 5.4.7 $b(\partial u)\in C^0(I, H^{s_0^3}(\Omega))$, the first kind of these expressions belongs to $L^2(I, H^0(\Omega))$. To prove the same for the second kind of expressions it is sufficient to apply Lemma 5.4.3.

LEMMA 5.4.9. *If $\partial^2 u^{(L-1)}\in L^2(I, H^0(\Omega))$, $\partial u\in H^{s+1,L}(\Omega\times I)$ and $a(\cdot)\in C^\infty$ then $\partial_t^{L-1}\{a(\partial u)\}\in L^2(I, H^1(\Omega))$ and the $L^2(I, H^1(\Omega))$ norm of $\partial_t^{L-1}\{a(\partial u)\}$ can be estimated by some polynomial depending only on the norms*

$$\max_{t\in I} |\partial u^{(0)}|_{s_0^3}, \dots, \max_{t\in I} |\partial u^{(L-1)}|_{s_{L-1}^3}, \|\partial^2 u^{(1)}\|_{L^2(I, H^{s_1^4-1}(\Omega))}, \dots, \|\partial^2 u^{(L-2)}\|_{L^2(I, H^{s_{L-2}^4-1}(\Omega))}.$$

5.5. Convergence of commutators

Assume that s, L are the same as in (3.2.3) and $\rho\in C_0^\infty([-2, -1])$ is such that $\int \rho(\tau)d\tau=1$, $\rho\geq 0$. Put $\rho_\sigma(\tau)=\sigma^{-1}\rho(\sigma^{-1}\tau)$, $\sigma>0$, $\tau\in[0, t]$, $v_\sigma(\tau)=\int \rho_\sigma(\tau-r)v(r)dr$, $I_\sigma(\tau)=(av)_\sigma(\tau)-a(\tau)v_\sigma(\tau)$.

LEMMA 5.5.1. *If a satisfies (2.2.8) and $v\in H^{s,L-1}(\Omega\times I)$, (cf. (2.2.4)) then*

$$(5.5.1) \quad \int_0^t |I_\sigma^{(L)}(\tau)|_0 d\tau \rightarrow 0 \quad \text{as } \sigma \rightarrow 0, \quad \text{for } t\in(0, T).$$

PROOF. Assume that $0<\sigma<\sigma_0$ and σ_0 is sufficiently small to satisfy $0<t<T-2\sigma_0$. Note that for $\tau\in[0, t]$, the support of the function $r\rightarrow\rho_\sigma(\tau-r)$ is contained in $[\tau+\sigma, \tau+2\sigma]\subset(0, T)$. Using the well known properties of the convolution we get

$$(5.5.2) \quad I_\sigma^{(L)}(\tau) = \sum_{i=0}^L \binom{L}{i} I_\sigma^i(\tau),$$

$$I_\sigma^i(\tau) = (a^{(i)}v^{(L-i)})_\sigma(\tau) - a^{(i)}(\tau)v_\sigma^{(L-i)}(\tau).$$

Assume first that $i>0$. Repeating the argument leading to (3.4.16), (with $j=0$) we get for $s_i, i=1, \dots, L-1$ as in (3.2.3) and $s_L=-1$,

$$(5.5.3) \quad |I_\sigma^i(\tau)|_0 \leq c \int \rho_\sigma(\tau-r) |a^{(i)}(r) - a^{(i)}(\tau)|_{s_i+1} |v^{(L-i)}(r)|_{s_{L-i}} dr,$$

and by Schwarz's inequality

$$\int_0^t |I_\sigma^i(\tau)|_0 d\tau \leq c \left(\int_0^t \int \rho_\sigma(\tau-r) |a^{(i)}(r) - a^{(i)}(\tau)|_{s_i+1}^2 dr d\tau \right)^{1/2}$$

$$(5.5.4) \quad \times \left(\int_0^t \int \rho(\tau - r) |v^{(L-i)}(r)|_{s_{L-i}}^2 dr d\tau \right)^{1/2}.$$

The second integral in the right hand side of (5.5.4) is bounded since the function $w = |v^{(L-i)}|_{s_{L-i}}^2$ belongs to $L^1(I)$ and consequently w_σ is bounded in $L^1(I)$. Using the inequality $\rho_\sigma \leq c\sigma^{-1}$, one can estimate the first integral by

$$(5.5.5) \quad \begin{aligned} & c\sigma^{-1} \int_0^t \int_{\tau+\sigma}^{\tau+2\sigma} |a^{(i)}(r) - a^{(i)}(\tau)|_{s_i+1}^2 dr d\tau \\ &= c\sigma^{-1} \int_0^t \int_\sigma^{2\sigma} |a^{(i)}(\tau + \omega) - a^{(i)}(\tau)|_{s_i+1}^2 d\omega d\tau \\ &= c\sigma^{-1} \int_\sigma^{2\sigma} \int_0^t |a^{(i)}(\tau + \omega) - a^{(i)}(\tau)|_{s_i+1}^2 d\tau d\omega \\ &\leq c \sup_{\sigma \leq \omega \leq 2\sigma} \int_0^t |a^{(i)}(\tau + \omega) - a^{(i)}(\tau)|_{s_i+1}^2 d\tau. \end{aligned}$$

The last term in (5.5.5) converges to zero as $\sigma \rightarrow 0$ since $a^{(i)} \in L^2(I, H^{s_i+1}(\Omega))$, $i = 1, \dots, L$.

Now consider

$$(5.5.6) \quad \begin{aligned} I_\sigma^0 &= (av^{(L)})_\sigma - a(v^{(L)})_\sigma \\ &= \partial_\tau [(av^{(L-1)})_\sigma - a(v^{(L-1)})_\sigma] = J_1 + J_2 + J_3, \end{aligned}$$

where

$$(5.5.7) \quad \begin{aligned} J_1(\tau) &= \int \partial_r(\rho_\sigma(\tau - r))(a(\tau) - a(r))(v^{(L-1)}(r) - v^{(L-1)}(\tau))dr, \\ J_2(\tau) &= - \int \rho_\sigma(\tau - r)a^{(1)}(r)(v^{(L-1)}(r) - v^{(L-1)}(\tau))dr, \\ J_3(\tau) &= \int \rho_\sigma(\tau - r)(a^{(1)}(r) - a^{(1)}(\tau))v^{(L-1)}(r)dr. \end{aligned}$$

Since $s_1 + 1 = s - \frac{s}{L-1} + 1 > \frac{n}{2}$, we have

$$(5.5.8) \quad \begin{aligned} |J_1(\tau)|_0 &\leq c \int |\partial_r(\rho_\sigma(\tau - r))(a(\tau) - a(r))|_{s_1+1} \\ &\quad \times |(v^{(L-1)}(r) - v^{(L-1)}(\tau))|_0 dr. \end{aligned}$$

Using the estimate for the difference quotient from Lemma 5.5.3, below and the inequality

$$(5.5.9) \quad |\omega \partial_\omega(\rho_\sigma(\omega))| \leq c\sigma^{-1},$$

we obtain

$$\begin{aligned}
 & \int_0^t \int_{\tau+\sigma}^{\tau+2\sigma} |\partial_r(\rho_\sigma(\tau-r))| |a(\tau) - a(r)|_{s_1+1} |v^{(L-1)}(r) - v^{(L-1)}(\tau)|_0 dr d\tau \\
 &= \int_0^t \int_{\tau+\sigma}^{\tau+2\sigma} |\tau-r| |\partial_r(\rho_\sigma(\tau-r))| \\
 &\quad \times |(\tau-r)^{-1}(a(\tau) - a(r))|_{s_1+1} |v^{(L-1)}(r) - v^{(L-1)}(\tau)|_0 dr d\tau \\
 (5.5.10) \quad & \leq c\sigma^{-1} \int_\sigma^{2\sigma} \left(\int_0^t |\omega^{-1}(a(\tau+\omega) - a(\tau))|_{s_1+1}^2 d\tau \right)^{1/2} \\
 &\quad \times \left(\int_0^t |v^{(L-1)}(\tau+\omega) - v^{(L-1)}(\tau)|_0^2 d\tau \right)^{1/2} d\omega \\
 & \leq c \left(\int_0^T |a^{(1)}(r)|_{s_1+1}^2 dr \right) \\
 &\quad \times \sup_{\sigma \leq \omega \leq 2\sigma} \left(\int_0^t |v^{(L-1)}(\tau+\omega) - v^{(L-1)}(\tau)|_0^2 d\tau \right)^{1/2}.
 \end{aligned}$$

Since $v^{(L-1)} \in L^2(I, H^0(\Omega))$, the last term in (5.5.10), and consequently J_1 converges to zero as $\sigma \rightarrow 0$. To prove the same for J_3 , it is sufficient to repeat the argument leading to (5.5.3)–(5.5.5), with $i = 1$, and to consider J_2 it is sufficient to replace in this argument $|a^{(i)}(r) - a^{(i)}(\tau)|_{s_1+1}$ by $|v^{(L-1)}(r) - v^{(L-1)}(\tau)|_0$ and $|v^{(L-i)}(r)|_{s_{L-i}}$ by $|a^{(1)}(r)|_{s_1+1}$.

From the proof of Lemma 5.5.1 the next result follows easily.

LEMMA 5.5.2. *The assertion of Lemma 5.5.1 remains valid if the number L in (5.5.1) is replaced by arbitrary $j = 0, \dots, L-1$.*

For completeness we also give the proof of the following result.

LEMMA 5.5.3. *If $t \in (0, T)$, $0 < h_0 < T - t$ and*

$$(5.5.11) \quad a, a^{(1)} \in L^p((0, T), X), \quad p \geq 1,$$

where X is a Banach space with the norm $\|\cdot\|_X$, then

$$(5.5.12) \quad \sup_{0 < h < h_0} \int_0^t \|h^{-1}(a(\tau+h) - a(\tau))\|_X^p d\tau \leq \int_0^T \|a^{(1)}(r)\|_X^p dr.$$

PROOF. Since the functions satisfying (5.5.11) are absolutely continuous with respect to t , we have for $0 < h < h_0$ and $\tau \in [0, t]$

$$(5.5.13) \quad h^{-1}(a(\tau+h) - a(\tau)) = h^{-1} \int_\tau^{\tau+h} a^{(1)}(r) dr.$$

Hölder's inequality gives

$$\begin{aligned}
 \|h^{-1}(a(\tau+h) - a(\tau))\|_X^p &= h^{-1} \left\| \int_{\tau}^{\tau+h} a^{(1)}(r) dr \right\|_X^p \\
 (5.5.14) \qquad \qquad \qquad &\leq h^{-1} \int_{\tau}^{\tau+h} \|a^{(1)}(r)\|_X^p dr.
 \end{aligned}$$

Integrating over $(0, t)$ we get

$$\begin{aligned}
 \int_0^t \|h^{-1}(a(\tau+h) - a(\tau))\|_X^p d\tau &\leq h^{-1} \int_0^t \int_{\tau}^{\tau+h} \|a^{(1)}(r)\|_X^p dr d\tau \\
 (5.5.15) \qquad \qquad \qquad &= h^{-1} \int_0^h \int_0^t \|a^{(1)}(\tau+\omega)\|_X^p d\tau d\omega.
 \end{aligned}$$

Since

$$\begin{aligned}
 \int_0^t \|a^{(1)}(\tau+\omega)\|_X^2 d\tau &= \int_{\omega}^{t+\omega} \|a^{(1)}(r)\|_X^2 dr \\
 (5.5.16) \qquad \qquad \qquad &\leq \int_0^{t+h} \|a^{(1)}(r)\|_X^2 dr \leq \int_0^T \|a^{(1)}(r)\|_X^2 dr,
 \end{aligned}$$

the proof is complete.

6. Some elliptic estimates

6.1. Elliptic estimates independent of t

Fix the numbers $1 < p, \theta < \infty$ and denote by $B_{p,\theta}^s(G)$, $s > 0$, $G \subset R^k$ the Besov space, defined for example in [36], p. 192, (cf. also Sect. 5.3). In this section we shall show that there exist constants $C(\delta)$ and $C(\|a\|_{B_{p,\theta}^{s^0+1}(\Omega)})$ such that for arbitrary $u \in B_{p,\theta}^{s+2}(\Omega)$ the following inequality holds true

$$\begin{aligned}
 \|u\|_{B_{p,\theta}^{s+2}(\Omega)} &\leq C(\delta) \left\{ \|a\partial^2 u\|_{B_{p,\theta}^s(\Omega)} + \|b\partial u\|_{B_{p,\theta}^{s+1-\frac{1}{p}}(\partial\Omega)} \right. \\
 (6.1.1) \qquad \qquad \qquad &\quad \left. + C(\|a\|_{B_{p,\theta}^{s^0+1}(\Omega)}) \|u\|_{L_p(\Omega)} \right\}.
 \end{aligned}$$

Here by assumption

$$\begin{aligned}
 (6.1.2) \qquad \qquad \qquad &s \text{ and } s+1-\frac{1}{p} \text{ are non-integer and} \\
 &0 \leq s \leq s^0, \quad \text{for some } s^0 \text{ satisfying } s^0+1 > \frac{n}{p}.
 \end{aligned}$$

If s is an integer (6.1.1) remains valid with $B_{p,\theta}^s(\Omega)$ replaced by $H_p^s(\Omega)$ and $B_{p,\theta}^s(\partial\Omega)$ replaced by $B_{p,p}^s(\partial\Omega)$. The operators $a\partial^2$ and $b\partial$ are defined in (2.2.2) and $a\partial^2$

satisfies (2.2.12). The estimate (6.1.1) corresponds to the Neumann problem. In the case of the homogeneous Dirichlet problem it is sufficient to drop the boundary term in (6.1.1). It is of a fundamental importance that we consider coefficients $a = \{a_{p,q}\}$ with the regularity

$$(6.1.3) \quad a \in B_{p,\theta}^{s^0+1}(\Omega).$$

We shall sketch a proof of (6.1.1) in the special case $\Omega = R_+^n$, assuming u is sufficiently smooth with small support. The usual procedure of localization allows for extending our result to the general case of smooth domain with compact boundary. The general scheme of the proof is the same as in the Appendix of [15].

Assume that u has small support in which the oscillations of the coefficients $a = \{a_{p,q}\}$ are small. Recall, that in the constant coefficients case the following inequality is true for sufficiently smooth u

$$(6.1.4) \quad \|u\|_{B_{p,\theta}^{s+2}(R_+^n)} \leq C(\delta) \left\{ \|a\partial^2 u\|_{B_{p,\theta}^s(R_+^n)} + \|b\partial u\|_{B_{p,\theta}^{s+1-\frac{1}{p}}(R_+^{n-1})} + \|u\|_{L_p(R_+^n)} \right\},$$

cf. [36], Sect. 4.2.3. Therefore it is sufficient to prove the following perturbation estimate.

LEMMA 6.1.1. *For arbitrary $\varepsilon > 0$, there exist $\rho > 0$ such that if $u \in B_{p,\theta}^{s+2}(R_+^n)$ is a function with support in the ball $|x| < \rho$, the real numbers s , p , θ satisfy (6.1.2) and $1 < p$, $\theta < \infty$, then the following inequality holds true*

$$(6.1.5) \quad \begin{aligned} & \|a\partial^2 u - a_0\partial^2 u\|_{B_{p,\theta}^s(R_+^n)} + \|b\partial u - b_0\partial u\|_{B_{p,\theta}^{s+1-\frac{1}{p}}(R_+^{n-1})} \\ & \leq \varepsilon \|u\|_{B_{p,\theta}^{s+2}(R_+^n)} + C(\varepsilon, \|a\|_{B_{p,\theta}^{s^0+1}(R_+^n)}) \|u\|_{L_p(R_+^n)}, \end{aligned}$$

where $a_0\partial^2$ and $b_0\partial$ are the constant coefficients operators obtained from $a\partial^2$ and $b\partial$ by freezing the coefficients at $x = 0$.

PROOF. Let us assume that C denotes the constant from (5.3.4) and

$$(6.1.6) \quad C \sup_{|x|<\rho} |a(x) - a(0)| \leq \frac{1}{4}\varepsilon.$$

The relation (6.1.6) is satisfied for sufficiently small ρ since the inequality $s^0 + 1 > n/p$ implies $B_{p,\theta}^{s^0+1}(R_+^n) \subset C^0(R_+^n)$, cf. [35], p. 328. From Lemma 5.3.1 it follows

$$\begin{aligned} J_1 &= \|a\partial^2 u - a_0\partial^2 u\|_{B_{p,\theta}^s(R_+^n)} \\ &\leq C\|a - a_0\|_{C^0(R_+^n)} \|u\|_{B_{p,\theta}^{s+2}(R_+^n)} \end{aligned}$$

$$\begin{aligned}
(6.1.7) \quad & + C \|a - a_0\|_{B_{p,\theta}^{s_0+1}(R_+^n)} \|u\|_{B_{p,\theta}^{s+2-\varepsilon_2}(R_+^n)} \\
& \leq \frac{1}{4} \varepsilon \|u\|_{B_{p,\theta}^{s+2}(R_+^n)} \\
& + C \|a - a_0\|_{B_{p,\theta}^{s_0+1}(R_+^n)} \left(\varepsilon_2 \|u\|_{B_{p,\theta}^{s+2}(R_+^n)} + C_{\varepsilon_2} \|u\|_{L_p(R_+^n)} \right).
\end{aligned}$$

Putting $\varepsilon_2 = \frac{1}{4} \varepsilon C^{-1} (1 + \|a - a_0\|_{B_{p,\theta}^{s_0+1}(R_+^n)})^{-1}$, we obtain

$$(6.1.8) \quad J_1 \leq \frac{1}{2} \varepsilon \|u\|_{B_{p,\theta}^{s+2}(R_+^n)} + C(\varepsilon, \|a - a_0\|_{B_{p,\theta}^{s_0+1}(R_+^n)}) \|u\|_{L_p(R_+^n)}.$$

The boundary term can be estimated analogously because

$$\begin{aligned}
(6.1.9) \quad J_2 &= \|b \partial u - b_0 \partial u\|_{B_{p,\theta}^{s+1-\frac{1}{p}}(R^{n-1})} \\
&\leq C \|b - b_0\|_{C^0(R^{n-1})} \|\partial u\|_{B_{p,\theta}^{s+1-\frac{1}{p}}(R^{n-1})} \\
&+ C \|b - b_0\|_{B_{p,\theta}^{s_0+1-\frac{1}{p}}(R^{n-1})} \|\partial u\|_{B_{p,\theta}^{s+1-\frac{1}{p}-\varepsilon_2}(R^{n-1})} \\
&\leq \frac{1}{4} \varepsilon \|u\|_{B_{p,\theta}^{s+2}(R_+^n)} + C \|b - b_0\|_{B_{p,\theta}^{s_0+1}(R_+^n)} \|u\|_{B_{p,\theta}^{s+2-\varepsilon_2}(R_+^n)}.
\end{aligned}$$

6.2. Elliptic estimates with L^2 dependence on t

The present section is devoted to the estimate of the form

$$\begin{aligned}
(6.2.1) \quad \int_0^t e^{-2\lambda\tau} \|u(\tau)\|_{B_{p,\theta}^{s+2}(\Omega)}^2 d\tau &\leq C(\delta) \left\{ \int_0^t e^{-2\lambda\tau} \|a \partial^2 u(\tau)\|_{B_{p,\theta}^s(\Omega)}^2 d\tau \right. \\
&+ \int_0^t e^{-2\lambda\tau} \|B^d u(\tau)\|_{B_{p,\theta}^{s+2-d-1/p}(\partial\Omega)}^2 d\tau \\
&\left. + C(\|a\|_{C^0(I, B_{p,\theta}^{s_0+1}(\Omega))}) \int_0^t e^{-2\lambda\tau} \|u(\tau)\|_{L_p(\Omega)}^2 d\tau \right\},
\end{aligned}$$

where u is an arbitrary function satisfying the inclusion $u \in L^2(I, B_{p,\theta}^{s+2}(\Omega))$, and the symbols $C(\delta)$, $C(\|a\|_{C^0(I, B_{p,\theta}^{s_0+1}(\Omega))})$ denote some positive constants depending on the indicated parameters. The operator $a \partial^2$ is the same as in (2.2.2) and B^d , $d = 0, 1$ denotes the Dirichlet or Neumann boundary operator

$$\begin{aligned}
(6.2.2) \quad B^0 u &= u, \\
B^1 u &= \sum_{p,q=1}^n \nu_p a_{p,q} \partial_q u,
\end{aligned}$$

where $\nu = (\nu_1, \dots, \nu_n)$ is the unit, outward, normal vector to the boundary $\partial\Omega$. Recall that $\partial\Omega$ is assumed to be C^∞ and compact. Additionally, we assume that

$$(6.2.3) \quad \begin{aligned} &s, s+2-d-1/p \text{ are non-integers,} \\ &0 \leq s \leq s^0, \quad \frac{n}{p} < s^0 + 1, \end{aligned}$$

and the coefficients $a = \{a_{p,q}\}$ have the regularity

$$(6.2.4) \quad a \in C^0(I, B_{p,\theta}^{s^0+1}(\Omega)).$$

We repeat some statements from the previous section to enable the reader to control the regularity requirements concerning the coefficients. Thus, as in the previous section, we shall sketch the proof of (6.2.1) in special case $\Omega = R_+^n$, assuming u is smooth, with small support in which the oscillations of the coefficients $a_{p,q}$ are small. If $a_0 = \{a_{p,q}(x_0, t_0)\}$, $B_0^d = B^d(x_0, t_0)$, i.e. the variables of the coefficients are fixed at some (x_0, t_0) , then from [36], Sect. 4.2.3 we can deduce the following inequality for $u \in B_{p,\theta}^{s+2}(R_+^n)$

$$(6.2.5) \quad \begin{aligned} \|u\|_{B_{p,\theta}^{s+2}(R_+^n)}^2 &\leq C(\delta) \left(\|a_0 \partial^2 u\|_{B_{p,\theta}^s(R_+^n)}^2 \right. \\ &\quad \left. + \|B_0^d u\|_{B_{p,\theta}^{s+2-d-1/p}(R^{n-1})}^2 + \|u\|_{L_p(R_+^n)}^2 \right). \end{aligned}$$

Multiplying both sides of (6.2.5) by $e^{-2\lambda\tau}$ and integrating over $(0, t)$ we see that

$$(6.2.6) \quad \begin{aligned} \int_0^t e^{-2\lambda\tau} \|u(\tau)\|_{B_{p,\theta}^{s+2}(R_+^n)}^2 d\tau &\leq C(\delta) \left\{ \int_0^t e^{-2\lambda\tau} \|a_0 \partial^2 u(\tau)\|_{B_{p,\theta}^s(R_+^n)}^2 d\tau \right. \\ &\quad \left. + \int_0^t e^{-2\lambda\tau} \|B_0^d u(\tau)\|_{B_{p,\theta}^{s+2-d-1/p}(R^{n-1})}^2 d\tau \right. \\ &\quad \left. + \int_0^t e^{-2\lambda\tau} \|u(\tau)\|_{L_p(R_+^n)}^2 d\tau \right\}. \end{aligned}$$

Inequality (6.2.6) shows that to obtain (6.2.1) for $\Omega = R_+^n$ it is sufficient to prove the perturbation estimate from the following lemma.

LEMMA 6.2.1. *For all, sufficiently small $\varepsilon > 0$, there exist $\rho > 0$, such that if $u \in L^2(I, B_{p,\theta}^{s+2}(R_+^n))$ has support in*

$$(6.2.7) \quad K = \{(x, t) \in R_+^n \times I : |x| < \rho, |t - t_0| < \rho\},$$

then for arbitrary s, p, θ , $1 < p, \theta < \infty$ satisfying (6.2.3) and for some positive

constant $C(\varepsilon, \|a\|_{C^0(I, B_{p,\theta}^{s_0+1}(R_+^n))})$,

$$\begin{aligned}
 (6.2.8) \quad & \int_0^t e^{-2\lambda\tau} \|a\partial^2 u(\tau) - a_0\partial^2 u(\tau)\|_{B_{p,\theta}^s(R_+^n)}^2 d\tau \\
 & + \int_0^t e^{-2\lambda\tau} \|B^d u(\tau) - B_0^d u(\tau)\|_{B_{p,\theta}^{s+2-d-1/p}(R_+^{n-1})}^2 d\tau \\
 & \leq \varepsilon \int_0^t e^{-2\lambda\tau} \|u(\tau)\|_{B_{p,\theta}^{s+2}(R_+^n)}^2 d\tau \\
 & + C(\varepsilon, \|a\|_{C^0(I, B_{p,\theta}^{s_0+1}(R_+^n))}) \int_0^t e^{-2\lambda\tau} \|u(\tau)\|_{L_p(R_+^n)}^2 d\tau,
 \end{aligned}$$

where $a_0\partial^2$ and B_0^d denote the operators with constant coefficients obtained from $a\partial^2$ and B^d respectively, by freezing the coefficients at $(x, t) = (0, t_0)$, $t_0 \in I$.

PROOF. Assume that ε is a small positive number, C is the constant from Lemma 5.3.1 and

$$(6.2.9) \quad C \sup_{(x,t) \in K} |a - a_0| \leq 4^{-1}\varepsilon,$$

for some K as in (6.2.7). Similarly as for (6.1.6), (6.2.9) is satisfied for sufficiently small ρ . Using Lemma 5.3.1 we get for some $\varepsilon_2 > 0$

$$\begin{aligned}
 (6.2.10) \quad I_1 &= \int_0^t e^{-2\lambda\tau} \|a\partial^2 u(\tau) - a_0\partial^2 u(\tau)\|_{B_{p,\theta}^s(R_+^n)}^2 d\tau \\
 &\leq C \int_0^t e^{-2\lambda\tau} \|a - a_0\|_{C^0(K)} \|u(\tau)\|_{B_{p,\theta}^{s+2}(R_+^n)}^2 d\tau \\
 &\quad + C \int_0^t e^{-2\lambda\tau} \|a - a_0\|_{B_{p,\theta}^{s_0+1}(R_+^n)} \|u(\tau)\|_{B_{p,\theta}^{s+2-\varepsilon_2}(R_+^n)}^2 d\tau \\
 &\leq 4^{-1}\varepsilon \int_0^t e^{-2\lambda\tau} \|u(\tau)\|_{B_{p,\theta}^{s+2}(R_+^n)}^2 d\tau \\
 &\quad + C \|a - a_0\|_{C^0(B_{p,\theta}^{s_0+1}(R_+^n))} \left(\varepsilon_2 \int_0^t e^{-2\lambda\tau} \|u(\tau)\|_{B_{p,\theta}^{s+2}(R_+^n)}^2 d\tau \right. \\
 &\quad \left. + C_{\varepsilon_2} \int_0^t e^{-2\lambda\tau} \|u(\tau)\|_{L_p(R_+^n)}^2 d\tau \right).
 \end{aligned}$$

For sufficiently small ε we can put

$$(6.2.11) \quad \varepsilon_2 = 4^{-1}\varepsilon C^{-1} (1 + \|a - a_0\|_{C^0(I, B_{p,\theta}^{s_0+1}(R_+^n))})^{-1}$$

and consequently we have

$$(6.2.12) \quad \begin{aligned} I_1 &\leq 2^{-1}\varepsilon \int_0^t e^{-2\lambda\tau} \|u(\tau)\|_{B_{p,\theta}^{s+2}(R_+^n)}^2 d\tau \\ &\quad + C(\varepsilon, \|a - a_0\|_{C^0(I, B_{p,\theta}^{s_0+1}(R_+^n))}) \int_0^t e^{-2\lambda\tau} \|u(\tau)\|_{L_p(R_+^n)}^2 d\tau. \end{aligned}$$

In the case of the Dirichlet boundary condition, the boundary term in (6.2.8) vanishes so it is sufficient to consider the boundary term corresponding to the Neumann problem. Applying Lemma 5.3.1 with $G = \partial R_+^n = R^{n-1}$ we obtain

$$(6.2.13) \quad \begin{aligned} I_2 &= \int_0^t e^{-2\lambda\tau} \|B^1 u(\tau) - B_0^1 u(\tau)\|_{B_{p,\theta}^{s+1-1/p}(R^{n-1})}^2 d\tau \\ &\leq C \int_0^t e^{-2\lambda\tau} \|a - a_0\|_{C^0(K_0)}^2 \|u(\tau)\|_{B_{p,\theta}^{s+2-1/p}(R^{n-1})}^2 d\tau \\ &\quad + C \int_0^t e^{-2\lambda\tau} \|a - a_0\|_{B_{p,\theta}^{s_0+1-1/p}(R^{n-1})}^2 \|u(\tau)\|_{B_{p,\theta}^{s+2-1/p}(R^{n-1})}^2 d\tau, \end{aligned}$$

where $K_0 = K \cap \{x \in R^n : x_n = 0\}$. Since arbitrary $w \in B_{p,\theta}^r(R_+^n)$, $r > 1/p$, $p, \theta > 1$, satisfies the inequality

$$\|w\|_{B_{p,\theta}^{r-1/p}(R^{n-1})} \leq C \|w\|_{B_{p,\theta}^r(R_+^n)}, \quad c = \text{const} > 0,$$

we can repeat the preceding argument to obtain for I_2 the same estimate as for I_1 .

It is worth noting that to estimate I_1 , the relation

$$(6.2.14) \quad s \in [0, s^0 + 1]$$

is sufficient while to estimate I_2 we need (6.2.3).

The standard procedure of localization and deformation extends the obtained estimate to the case of general Ω .

LEMMA 6.2.2. *If $\Omega \subset R^n$ is a bounded domain with a C^∞ and compact boundary $\partial\Omega$, $I \subset R^1$ is a finite interval and the assumptions (2.2.12), (6.2.3), (6.2.4) are fulfilled, then the estimate (6.2.1) is valid for arbitrary $u \in L^2(I, B_{p,\theta}^{s+2}(\Omega))$. If u satisfies additionally the homogeneous Dirichlet condition, then the estimate (6.2.1) with dropped boundary term remains valid for a satisfying (6.2.14).*

Using Lemma 5.3.5 one also can prove the following modified version of Lemma 6.2.2.

LEMMA 6.2.3. *If s is an integer then all the statements of Lemma 6.2.2 remain valid provided that the Besov spaces $B_{p,\theta}^s(\Omega)$, $B_{p,\theta}^{s+2-d-1/p}(\partial\Omega)$ are replaced by the Sobolev space $H_p^s(\Omega)$ and the Besov space $B_{p,p}^{s+2-d-1/p}(\partial\Omega)$ respectively.*

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