Hypersurfaces with parallel difference tensor

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§1. Introduction

One of the most attractive results in classical affine differential geometry is the Theorem of Pick and Berwald, stating that a non degenerate hypersurface has vanishing cubic form if and only if it is a non degenerate hyperquadric. In this way, non degenerate quadratic hypersurfaces are characterized in a differential geometric way. In the last decade, several generalizations of this result have been given, starting with the work of Nomizu and Pinkall on surfaces whose cubic form \( \nabla C \) is parallel with respect to the induced connection \( \nabla \) and the work of Magid and Nomizu on surfaces whose cubic form is parallel with respect to Levi Civita connection \( \hat{\nabla} \) of the affine metric. Their results are the following:

**Theorem A** [MN]. Let \( M^2 \) be a non degenerate affine surface in \( \mathbb{R}^3 \) with \( \hat{\nabla} C = 0 \). Then either \( M \) is an open part of a non degenerate quadric (i.e. \( C = 0 \)) or \( M \) is affine equivalent to an open part of one of the following surfaces:

1. \( xyz = 1 \),
2. \( x(y^2 + z^2) = 1 \),
3. \( z = xy + \frac{1}{3}y^3 \) (the Cayley surface).

**Theorem B** [NP]. Let \( M^2 \) be a non degenerate affine surface in \( \mathbb{R}^3 \) with \( \nabla C = 0 \). Then either \( M \) is an open part of a non degenerate quadric or \( M \) is affine equivalent to an open part of the Cayley surface.

A partial extension of Theorem B for arbitrary \( n \) can be given as follows.

**Theorem C** [DV1]. Let \( M^n \) be a non degenerate affine hypersurface in \( \mathbb{R}^{n+1} \)
with $\nabla C = 0$. Then either $M$ is an open part of a non degenerate quadric or $M$ is an improper affine sphere given by the graph of a polynomial function of degree 3 with constant Hessian determinant. In particular, if $M^n$ is locally strongly convex, then $M^n$ is an open part of a quadric.

The classification of polynomial functions of degree 3 in 3 variables with constant Hessian determinant is done by the second author, giving the following result.

**Theorem D** [V]. Let $M^3$ be a non degenerate affine hypersurface in $\mathbb{R}^4$ with $\nabla C = 0$. Then $M$ is affine equivalent to an open part of one of the following hypersurfaces:

1. a non degenerate quadric,
2. $x_4 = x_1 x_2 + \frac{1}{2} x_3^2 + \frac{1}{3} x_1^3$,
3. $x_4 = x_1 x_2 + \frac{1}{2} x_3^2 + x_1^2 x_3$.

A partial generalization of Theorem A is given by the following result.

**Theorem E** [DVY]. Let $M$ be a non degenerate affine hypersurface in $\mathbb{R}^{n+1}$ with $\nabla C = 0$. Then $M$ is a locally homogeneous affine sphere.

In fact, the conclusion that $M$ is an affine sphere holds under weaker conditions on $C$, see for instance [Sch], [Si] and [NS]. The actual determination of which locally homogeneous affine spheres satisfy $\nabla C = 0$ is done only for $n = 3, 4$ and $M^n$ locally strongly convex. Examples in [DV3] show that a complete classification will be rather difficult.

**Theorem F** [DV2, DVY]. Let $M^n$ be a locally strongly convex affine hypersurface in $\mathbb{R}^{n+1}$ with $\nabla C = 0$ and $n = 3, 4$. Then either $M$ is a part of a locally strongly convex quadric or $M$ is affine equivalent to an open part of one of the following hypersurfaces:

1. $xyzw = 1$, $(n = 3)$,
2. $(y^2 - z^2 - w^2)^3 x^2 = 1$, $(n = 3)$,
3. $xyzwt = 1$, $(n = 4)$,
4. $(y^2 - z^2 - w^2 - t^2)^2 x = 1$, $(n = 4)$,
5. $(z^2 - w^2 - t^2)^3 (xy)^2 = 1$, $(n = 4)$.

We want to quote one final result, which deals with proper affine spheres, where the affine normal is chosen to point away from the center of the sphere.

**Theorem G** [W]. Let $M^n$ be a non degenerate affine hypersurface in $\mathbb{R}^{n+1}$ with $\nabla C = 0$, such that $M$ is a proper affine sphere with flat affine metric, which is Lorentzian (with the convention mentioned above). Then $M$ is affine equivalent to an open part of the hypersphere $(x_1^2 + x_2^2)x_3 \cdots x_{n+1} = 1$. 
On the other hand, the difference tensor \( K \) on a non degenerate hypersurface is defined by \( K(X, Y) = \nabla_X Y - \nabla_Y X \). The cubic form \( C = \nabla h \) is related to the difference tensor by the affine metric \( h \) as follows: \( h(K_X Y, Z) = -\frac{1}{2}C(X, Y, Z) \). Hence \( \nabla C = 0 \) is equivalent to \( \nabla K = 0 \). However, \( \nabla C = 0 \) does not have to be equivalent to \( \nabla K = 0 \). In this paper we study non degenerate hypersurfaces satisfying \( \nabla K = 0 \). Our first result is the following.

**Theorem 1.1.** Let \( M^n \) be a non degenerate affine hypersurface in \( \mathbb{R}^{n+1} \) with \( \nabla K = 0 \). If \( K \neq 0 \) at some point, then \( M^n \) is an improper affine sphere whose affine metric is flat. Further there exists a number \( k \in \{2, \ldots, n\} \) such that \( K^{k-1} \neq 0 \) and \( K^k = 0 \). \( M^n \) is given as the graph of a polynomial of degree \( k + 1 \) with constant Hessian.

As two corollaries, we have the following.

**Corollary 1.1.** Let \( M^n \) be a locally strongly convex affine hypersurface in \( \mathbb{R}^{n+1} \) with \( \nabla K = 0 \). Then \( K = 0 \), so \( M^n \) is a part of a non degenerate quadric.

**Corollary 1.2.** Let \( M^2 \) be a non degenerate affine surface in \( \mathbb{R}^3 \). Then the following assertions are equivalent:

1. \( \nabla K = 0 \),
2. \( \nabla C = 0 \),
3. \( \nabla K = 0 \) and \( J = 0 \),
4. \( M^2 \) is a part of a quadric or the Cayley surface.

Corollary 1.2 is due to Jelonek [J]. In Sections 4 and 6, we will consider the 2 extremal cases, namely we consider hypersurfaces for which the number \( k \) is equal to 2 or \( n \). In Section 5, we obtain complete classifications of the Lorentzian hypersurfaces satisfying \( \nabla K = 0 \). Finally, in Section 7, we consider the case \( k = n - 1 \) and reduce it to the case \( k = n \). In Section 2 we give a brief introduction to affine hypersurface theory, merely to fix notations. For more information the reader is referred to [NS].

### §2. Preliminaries

Let \( M^n \) be a connected differentiable \( n \)-dimensional hypersurface of the equiaffine space \( \mathbb{R}^{n+1} \) equipped with its usual flat connection \( D \) and a parallel volume element \( \omega \), given by the determinant. We allow \( M \) to be immersed by an immersion \( f \), but we will not denote the immersion if there is no confusion possible. Let \( \xi \) be an arbitrary local transversal vector field to \( f(M) \). For any vector fields \( X, Y, X_1, \ldots, X_n \), we write

\[
D_X Y = \nabla_X Y + h(X, Y)\xi, \\
\theta(X_1, \ldots, X_n) = \omega(X_1, \ldots, X_n, \xi),
\]
thus defining an affine connection $\nabla$, a symmetric $(0,2)$-type tensor $h$, called the second fundamental form, and a volume element $\theta$. $M$ is said to be non degenerate if $h$ is non degenerate (and this condition is independent of the choice of transversal vector field $\xi$). If $M$ is non degenerate it is known that there is a unique choice (up to sign) of transversal vector field such that the induced connection $\nabla$, the induced second fundamental form $h$ and the induced volume element $\theta$ satisfy the following conditions:

(i) $\nabla \theta = 0$

(ii) $\theta = \omega_h$,

where $\omega_h$ is the metric volume element induced by $h$. $\nabla$ is called the induced affine connection, $\xi$ the affine normal and $h$ the affine metric.

The conditions (i) and (ii) imply the apolarity condition which is $\nabla \omega_h = 0$. A non degenerate immersion equipped with this special transversal vector field is called a Blaschke immersion. Through this paper, $M$ is always assumed to be a Blaschke immersion. Condition (i) implies that $D_X \xi$ is tangent to $M$ for any tangent vector $X$ to $M$. Hence we can define a $(1,1)$-tensor field $S$, called the affine shape operator, by $D_X \xi = -S X$.

Let $\overset{\circ}{\nabla}$ denote the Levi Civita connection of the affine metric $h$. The difference tensor $K$ is defined by $K(X,Y) = K_X Y = \nabla_X Y - \overset{\circ}{\nabla}_X Y$ for tangent vector fields $X$ and $Y$. Notice that $K$ is symmetric in $X$ and $Y$. If we define the cubic form $C$ by $C(X,Y,Z) = (\nabla h)(X,Y,Z)$, then the Codazzi equation says that $C$ is totally symmetric. Moreover, we have the following relation

\begin{equation}
(2.1) \quad h(K_X Y, Z) = -\frac{1}{2} C(X,Y,Z),
\end{equation}

such that $K_X$ is a symmetric operator w.r.t. $h$. The Pick invariant $J$ is defined by $J = \frac{1}{n(n-1)} h(K,K)$. The curvature tensors $R$ and $\overset{\circ}{R}$ of $\nabla$ and $\overset{\circ}{\nabla}$ are related to $S$ and $K$ by

\begin{equation}
(2.2) \quad R(X,Y) Z = h(Y,Z) SX - h(X,Z) SY,
\end{equation}

\begin{equation}
(2.3) \quad \overset{\circ}{R}(X,Y) Z = \frac{1}{2} (h(Y,Z) SX - h(X,Z) SY + h(SY,Z) X - h(SX,Z) Y) - [K_X, K_Y] Z.
\end{equation}

Finally, the apolarity condition implies that trace $K_X = 0$ for all $X$.

If $S = \lambda I$, then $M$ is called an affine sphere. If $M$ is an affine sphere and $n \geq 2$ then $\lambda$ is constant. $M$ is called a proper affine sphere if $\lambda \neq 0$ and an improper affine sphere if $\lambda = 0$. If $M$ is a proper affine sphere, then the affine normal $\xi$ satisfies $\xi = -\lambda (f - p)$, where $f$ is the position vector and $p$ is a fixed point in $\mathbb{R}^{n+1}$, called the center of $M$. If $M$ is an improper affine sphere, so $S = 0$, then $\nabla$ is flat and the
affine normal $\xi$ is constant. After changing coordinates so that $\xi = (0, \ldots, 0, 1)$, it is clear that $M$ is given by $x_{n+1} = F(x_1, \ldots, x_n)$. It turns out that then $(x_1, \ldots, x_n)$ are $\nabla$-flat coordinates on $M$, that

$$h \left( \frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j} \right) = \frac{\partial^2 F}{\partial x_i \partial x_j}$$

and that the Hessian of $F$ satisfies $\det \left[ \frac{\partial^2 F}{\partial x_i \partial x_j} \right] = \pm 1$. For later use, we remark that (2.4) easily implies that

$$\left( \nabla^{k-2} h \right) \left( \frac{\partial}{\partial x_{i_1}}, \ldots, \frac{\partial}{\partial x_{i_k}} \right) = \frac{\partial^k F}{\partial x_{i_1} \cdots \partial x_{i_k}}.$$

Conversely, if $M$ is given as a graph $x_{n+1} = F(x_1, \ldots, x_n)$, then $M$ is an improper affine sphere with affine normal $\xi = (0, \ldots, 0, 1)$ if and only if the Hessian of $F$ equals $\pm 1$.

Further, if $M$ is an improper affine sphere with affine normal $\xi$ and $(x_1, \ldots, x_n)$ are $\nabla$-parallel coordinates on $M$, then the position vector $f$ satisfies

$$\frac{\partial^2 f}{\partial x_i \partial x_j} = h \left( \frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j} \right) \xi.$$

After an affine transformation, we may assume that $f(0) = 0$ and that $\{f_*(\frac{\partial}{\partial x_1})(0), \ldots, f_*(\frac{\partial}{\partial x_n})(0), \xi\}$ is the standard basis of $\mathbb{R}^{n+1}$. With these initial conditions, the equation (2.6) can be solved, and one easily obtains that $M$ is given in standard coordinates by $x_{n+1} = F(x_1, \ldots, x_n)$, where $F$ is the function determined by (2.4) and the initial conditions $\frac{\partial F}{\partial x_1}(0) = 0$ and $F(0) = 0$. We will always assume that these initial conditions are satisfied. We remark that, if $\det \left[ h(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}) \right] = \pm 1$, this affine transformation is equiaffine, at least after, if necessary, changing the orientation of $\mathbb{R}^{n+1}$. In the remaining part of the paper, when we say that two hypersurfaces are equivalent, we mean that they are equivalent by an affine transformation whose linear part has determinant equal to $\pm 1$.

§3. Hypersurfaces with parallel difference tensor

Let $M^n$ be a non degenerate hypersurface of $\mathbb{R}^{n+1}$. The following formula follows immediately from the definitions of $K$ and $C$ and (2.1).

$$h((\nabla X K)(Y, Z), W) = -\frac{1}{2} (\nabla X C)(Y, Z, W) + 2h(K_X K_Y Z, W).$$

**Lemma 3.1.** $\nabla K$ is totally symmetric (i.e. $K$ is a Codazzi tensor with respect to $\nabla$) if and only if $S = \lambda I$ and $[K_X, K_Y] = 0$ for each $X$ and $Y$. 
PROOF. Interchanging $X$ and $Y$ in (3.1) and subtracting gives us that $\nabla K$ is totally symmetric if and only if

$$-\frac{1}{2}(R(X,Y) \cdot h)(Z,W) + 2h([K_X,K_Y]Z,W) = 0.$$  

Since the first term is symmetric in $Z$ and $W$ and the second one is skew-symmetric in $Z$ and $W$, we obtain that $\nabla K$ is totally symmetric if and only if $(R(X,Y) \cdot h)(Z,W) = 0$ and $h([K_X,K_Y]Z,W) = 0$. Now $(R(X,Y) \cdot h)(Z,W) = 0$ if and only if $\nabla C$ is totally symmetric, and hence, by [BNS], if and only if $M$ is an affine sphere. $\square$

REMARK 3.1. From (2.3) it is easy to see that Lemma 3.1 can also be formulated as "$\nabla K$ is totally symmetric if and only if $S=\lambda I$ and $(M, h)$ has constant sectional curvature $\lambda$". In this form it was communicated to us by Jelonek (personal communication).

REMARK 3.2. We further remark that $[K_X,K_Y] = 0$ implies that $J = 0$. For surfaces both statements are even equivalent. Hence for $n = 2$, Lemma 3.1 is the same as [J, Corollary 2].

LEMMA 3.2. If $\nabla K = 0$ and $K \neq 0$, then $S = 0$ and $h$ is flat.

PROOF. From Lemma 3.1 we obtain that $S = \lambda I$ and that $K_X$ and $K_Y$ commute. If $\lambda = 0$, then from (2.3) we obtain that $h$ is flat. We now assume that $\lambda \neq 0$ at some point $p$. From $\nabla K = 0$, we obtain that $R \cdot K = 0$, which means that

$$(3.2) \quad R(X,Y)KZW - K_{R(X,Y)Z}W - K_ZR(X,Y)W = 0.$$  

We take a tangent basis $\{e_1,\ldots,e_n\}$ at $p$ satisfying $h(e_i,e_j) = \delta_{ij}e_i$ ($e_i = \pm 1$). Taking $X = e_i$ and $Y = Z = W = e_j$ ($i \neq j$) in (3.2) and applying (2.2), we obtain that

$$h(K_{e_j}e_j,e_j)e_i - h(K_{e_j}e_j,e_i)e_j - 2e_jK_{e_i}e_j = 0.$$  

Taking the component in the direction of $e_j$, we immediately see that $h(K_{e_i}e_j,e_j) = 0$ for all $j \neq i$. From the apolarity condition then follows that also $h(K_{e_j}e_j,e_j) = 0$. Hence, $K(e_j,e_j) = 0$, where $\{e_1,\ldots,e_n\}$ is an arbitrary orthonormal basis. Obviously this implies that $K = 0$ at $p$, and hence $K = 0$ everywhere, which is a contradiction. $\square$

LEMMA 3.3. If $[K_Y,K_Z] = 0$ for all $Y$ and $Z$, then $K_X$ is nilpotent for each $X$.

PROOF. Take any tangent vector $X$ at a point $p$. We now extend $K_X$ complex linearly to the complexification $\mathbb{C} \otimes T_pM$. Let $\{Y_1^1,\ldots,Y_1^1,\ldots,Y_k^1,\ldots,Y_k^k\}$,
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\[ \sum_{i=1}^{k} r_i = n \] be a basis for \( C \otimes T_p M \) such that \( K_X \) takes the Jordan normal form, in particular

\[ K_X y^i_j = \lambda_i y^i_j + \epsilon_j^i y^i_{j+1}, \quad 1 \leq j \leq r_i, \]

where \( \lambda_i \in \mathbb{C} \) and \( \epsilon_j^i \in \{0,1\} \). The apolarity condition for \( K_X \) then implies that \( \sum_{i=1}^{k} r_i \lambda_i = 0 \).

For simplicity we put \( y^i_{r_i+k} = 0 \) for \( k > 0 \). Using the assumption of the lemma, we find that

\[ K_{K_X} x y^i_j = K_{y^i_j} K_X x = K_X K_{y^i_j} x = K_X K_X y^i_j \]

\[ = K_X (\lambda_i y^i_j + \epsilon_j^i y^i_{j+1}) = \lambda_i^2 y^i_j + 2\lambda_i \epsilon_j^i y^i_{j+1} + \epsilon_j^i + 1 y^i_{j+2}. \]

Hence the apolarity condition for \( K_{K_X} \) implies that \( \sum_{i=1}^{k} r_i \lambda_i^2 = 0 \).

Similarly we obtain that for every natural number \( m \), we have \( \sum_{i=1}^{k} r_i \lambda_i^m = 0 \), which clearly implies that \( \lambda_i = 0 \). Hence \( (K_X)^n = 0 \) for all \( X \).

**Remark 3.3.** Assume that \( K_Y \) and \( K_Z \) commute for all \( Y \) and \( Z \). Then the tensor \( T_m \) defined by

\[ T_m(X_1, \ldots, X_{m+2}) = h(K_{X_1} K_{X_2} \cdots K_{X_m} X_{m+1}, X_{m+2}) \]

is totally symmetric. Hence \( T_m \) vanishes identically if and only if \( (K_v)^m v = 0 \) for all vectors \( v \). Denote by \( k \) the smallest number such that the symmetric tensor \( T_k \) is identically zero at the point \( p \). Then for any tangent vector \( v \) at \( p \), we have \( (K_v)^k v = 0 \) and there exists a tangent vector at \( p \) such that \( h((K_u)^{k-1} u, u) \neq 0 \).

**Lemma 3.4.** If \( \nabla K = 0 \) and \( K \neq 0 \), then for any natural number \( m \), we have

\[ (\nabla^m h)(X_1, \ldots, X_m, Z, W) = (-2)^m h(K_{X_1} K_{X_2} \cdots K_{X_m} Z, W), \]

for all tangent vector fields \( X_1, \ldots, X_n, Z, W \).

**Proof.** From (2.1), we get that the lemma is true for \( m = 1 \). Let us now assume that the lemma is satisfied for all values \( 1 \leq m \leq r - 1 \). Let \( x_1, \ldots, x_{r+2} \) be tangent vectors at a point \( p \) and extend these vectors to local vector fields \( X_1, \ldots, X_{r+2} \) such that \( \nabla X_i X_j(p) = 0 \). Then \( \hat{\nabla} X_i X_j(p) = -K_{x_i} x_j \) and we get that

\[ (\nabla^r h)(x_1, \ldots, x_{r+2}) = x_1(\nabla^{r-1} h)(X_2, \ldots, X_{r+2}) \]

\[ = (-2)^{r-1} x_1 h(K_{X_2} \cdots K_{X_r} X_{r+1}, X_{r+2}) \]

\[ = (-2)^{r-1} [h(-K_{x_1} (K_{x_2} \cdots K_{x_r} x_{r+1}), x_{r+2}) \]

\[ + h(K_{x_2} \cdots K_{x_r} x_{r+1}, K_{x_1} x_{r+2})] \]

\[ = (-2)^{r} h(K_{x_1} K_{x_2} \cdots K_{x_r} x_{r+1}, x_{r+2}). \]
Proof of Theorem 1.1. Let us assume that $M$ is not a quadric. Then we know that $M$ is an improper affine sphere with flat affine metric. Let $k$ be as in Remark 3.3. Then, by Lemma 3.4, we get that $\nabla^k h = 0$. Hence, from (2.5), $M$ is the graph of a polynomial of degree $k + 1$ with constant Hessian.

Proof of Corollary 1.1. If $\nabla K = 0$, then from Remark 3.2, it follows that the Pick invariant $J$ must vanish. If $M$ is locally strongly convex, this means that $M$ is a quadric.

Before proving Corollary 1.2, we first prove the following proposition.

Proposition 3.1. Let $M^n$ be a hypersurface in $\mathbb{R}^{n+1}$ with $\nabla K = 0$. Then the following conditions are equivalent:

1. $K^2 = 0$,
2. $\nabla K = 0$,
3. $\nabla^2 h = 0$, i.e. $\nabla C = 0$.

Proof. Formula (3.1) proves the equivalence of (1) and (3). Next, since $\nabla X K = -K_X \cdot K$ and, by Lemma 3.1, $K_U$ and $K_V$ always commute, we have that $(\nabla X K)(Y, Z) = K_X K_Y Z$. Hence we also obtain the equivalence of (1) and (2).

Proof of Corollary 1.2. If $n = 2$, then the number $k$ in Theorem 1.1 is either 1 or 2. So either $K = 0$ or from Proposition 3.1, also $\nabla K = 0$. Comparison of Theorem A and Theorem B then completes the proof.

§4. The case that $K^2 = 0$

Throughout this section, we will assume that $M$ is a hypersurface with $\nabla K = 0$, $K^2 = 0$ and $K \neq 0$. We define

$$I_1(p) = \text{span}\{K_{xy} \mid x, y \in T_p M\},$$

and denote by $r$ the dimension of this vector space. In the following lemmas, we will investigate properties of this space. These properties will be essential to obtain a classification of the hypersurfaces with $\nabla K = 0$ and $K^2 = 0$. The first lemma is trivial.

Lemma 4.1. If $K^2 = 0$, then vector space $I_1(p)$ is a null space, i.e. for all $v, w \in I_1(p)$, we have $h(v, w) = 0$.

We need the following lemma from linear algebra. Since we could not find it in the literature, we provide a proof.

Lemma 4.2. Let $h$ be a non degenerate metric on $\mathbb{R}^n$ and let $\{v_1, \ldots, v_r\}$ be
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a basis of a null space V. Then there exists a null space spanned by \( \{w_1, \ldots, w_r\} \) such that \( h(v_i, w_j) = \delta_{ij} \).

**Proof.** Consider the linear subspaces \( I \) and \( J \) of \( \mathbb{R}^n \) containing \( V \), defined by

\[
I = \{v \mid h(v, v_1) = 0\}, \\
J = \{v \mid h(v, v_2) = \cdots = h(v, v_r) = 0\}.
\]

Since \( v_1, v_2, \ldots, v_r \) are linearly independent, \( J \not\subset I \). Hence there exists a vector \( w \in J \setminus I \). Rescaling \( w \) if necessary, we may assume that \( h(w, v_2) = \cdots = h(w, v_r) = 0 \) and \( h(w, v_1) = 1 \). Then \( w_1 = w - \frac{1}{2} h(w, w)v_1 \) satisfies

\[
h(w_1, v_2) = h(w_1, v_3) = \cdots = h(w_1, v_r) = 0, \\
h(w_1, w_1) = 0, \quad h(v_1, w_1) = 1.
\]

Now we notice that the space spanned by \( \{v_1, w_1\} \) is non degenerate. So \( h \) defines a non degenerate metric on \( \{v_1, w_1\}^\perp \). By the definition of \( w_1 \), it is clear that \( \{v_2, \ldots, v_r\} \) is a basis of a null space in \( \{v_1, w_1\}^\perp \). Induction on \( n \) completes the proof.

It is clear that \( 2r \leq n \) and we put \( s = n - 2r \). Let \( \{v_1, \ldots, v_r\} \) be a basis of \( I_1(p) \) and let \( \{w_1, \ldots, w_r\} \) be as in Lemma 4.2. Since \( h \), restricted to the \((2r)\)-dimensional space spanned by \( \{v_1, \ldots, v_r, w_1, \ldots, w_r\} \), is non degenerate, we can choose a pseudo-orthonormal basis \( u_1, \ldots, u_s \) of \( \{v_1, \ldots, v_r, w_1, \ldots, w_s\}^\perp \), i.e. we have \( h(u_a, u_b) = \delta_{ab} \), where \( \epsilon_a = \pm 1 \).

Since \( K^2 = 0 \) and \( v_i \in I_1(p) \), we have that \( K_{vi} = 0 \) for all \( i \). Moreover, since \( h(K(u_a, w_i), w_k) = h(K(w_i, w_k), u_a) = 0 \), \( h(K(u_a, u_b), w_k) = h(K(u_a, w_k), u_b) = 0 \) and \( h(K(u_a, u_b), u_c) = 0 \), we obtain that also \( K_{u_a} = 0 \) for all \( a \). Hence the only non zero components of \( K \) are given by

\[
K_{w_i, w_j} = \sum_{k=1}^{r} A_{ij}^k u_k,
\]

where \( A_{ij}^k = h(K_{w_i, w_j}, w_k) \) are numbers which are symmetric in \( i, j \) and \( k \).

**Theorem 4.3.** Let \( M^n \) be a hypersurface in \( \mathbb{R}^{n+1} \) satisfying \( \nabla K = 0 \) and \( K^2 = 0 \). Then either \( M \) is an open part of a quadric or there exists a number \( r \) with \( 2r \leq n \) and \( M \) is equivalent to

\[
z = \sum_{i=1}^{r} v_i w_i + \frac{1}{2} \sum_{a=1}^{s} \epsilon_a u_a^2 - \frac{1}{3} \sum_{i,j,k=1}^{r} A_{ij}^k w_i w_j w_k,
\]
where $A_{ij}^k$, $i, j, k \in \{1, \ldots, r\}$ are totally symmetric constants, $\epsilon_a = \pm 1$ and $(v_1, \ldots, v_r, w_1, \ldots, w_r, u_1, \ldots, u_s, 2)$ are equiaffine coordinates on $\mathbb{R}^{n+1}$. The converse is also true.

Remark 4.1. In Theorem 4.3, one can replace the conditions "$\nabla K = 0$ and $K^2 = 0$" by "$\nabla K = 0$ and $\tilde{\nabla} K = 0$" or even by "$\nabla K = 0$ and $\nabla C = 0$", as follows from Proposition 3.1.

Proof. Let us assume that $M$ is not an open part of a quadric. Then, by Lemma 3.2, we know that $\nabla$ is flat. Let $p \in M$ and consider the basis $\{v_1, \ldots, v_r, w_1, \ldots, w_r, u_1, \ldots, u_s\}$ as constructed above. We can extend this basis to $\nabla$-parallel vector fields, which determine $\nabla$-flat coordinates, denoted also by $(v_1, \ldots, v_r, w_1, \ldots, w_r, u_1, \ldots, u_s)$. We may assume that $p$ has coordinates 0. Since $\nabla K = 0$, we then have that $K_{u_i} = 0$ and $K_{v_i} = 0$ and that

$$K_{w_i} \frac{\partial}{\partial w_j} = \sum_{k=1}^r A_{ij}^k \frac{\partial}{\partial v_k},$$

where $A_{ij}^k$ are the constants determined by (4.1). From (2.1), we then immediately have the following formulas.

$$h\left(\frac{\partial}{\partial v_i}, \frac{\partial}{\partial u_p}\right) = h\left(\frac{\partial}{\partial w_i}, \frac{\partial}{\partial u_p}\right) = h\left(\frac{\partial}{\partial v_i}, \frac{\partial}{\partial v_j}\right) = h\left(\frac{\partial}{\partial w_i}, \frac{\partial}{\partial v_j}\right) = 0,$$

$$h\left(\frac{\partial}{\partial v_i}, \frac{\partial}{\partial w_j}\right) = \delta_{ij}, \quad h\left(\frac{\partial}{\partial u_p}, \frac{\partial}{\partial u_q}\right) = \epsilon_p \delta_{pq}.$$

Putting $\alpha_{ij} = h\left(\frac{\partial}{\partial w_i}, \frac{\partial}{\partial w_j}\right)$, again from (2.1) we obtain that $\frac{\partial}{\partial w_k} \alpha_{ij} = \frac{\partial}{\partial u_p} \alpha_{ij} = 0$ and

$$\frac{\partial}{\partial w_k} \alpha_{ij} = (-2)h\left(K_{\frac{\partial}{\partial w_k}} \frac{\partial}{\partial w_i}, \frac{\partial}{\partial w_j}\right) = (-2)A_{ij}^k.$$

Integrating this, we obtain that $\alpha_{ij} = (-2)\sum_k A_{ij}^k w_k$. Having determined all components of the affine metric, we can determine the function $F$ using (2.4) and obtain that $M$ is as given in the theorem.

Remark 4.2. It is clear from the proof of the theorem that we still have the possibility to change the variables $w_i$ by a (regular) linear transformation (provided we also change the variables $v_j$ by the appropriate linear transformation). Indeed, the part of the function $F$ which contains $w_i$-coordinates can be any homogeneous polynomial of degree 3. This enables us to reduce the number of hypersurfaces obtained in the previous theorem. For example if $n = 2$ then $r = 1$ and $s = 0$ and we only get the Cayley surface described by

$$z = v_1 w_1 - \frac{1}{3} w_1^3.$$
If \( n = 3 \), then \( r = 1 \) and \( s = 1 \) and we have the hypersurface

\[
z = v_1w_1 + \frac{1}{2} \epsilon_1 u_1^2 - \frac{1}{3} w_1^3,
\]

which is (2) of Theorem D. Finally, if \( n = 4 \) then either \( r = 1 \) and \( s = 2 \) or \( r = 2 \) and \( s = 0 \). So using the classification of homogeneous polynomials of degree 3 in 2 variables, we obtain the following hypersurfaces:

\[
\begin{align*}
z &= v_1w_1 + \frac{1}{2} (\epsilon_1 u_1^2 + \epsilon_2 u_2^2) - \frac{1}{3} w_1^3, \\
z &= v_1w_1 + v_2w_2 - \frac{1}{3} (u_1^2 + u_2^2), \\
z &= v_1w_1 + v_2w_2 - \frac{1}{3} w_1^2, \\
z &= v_1w_1 + v_2w_2 - \frac{1}{3} w_1^2(w_1 + w_2).
\end{align*}
\]

\section{The Lorentzian case}

Throughout this section, we will assume that \( M^n \) is an affine hypersurface in \( \mathbb{R}^{n+1} \) with \( \nabla K = 0 \) and whose induced affine metric is Lorentzian, if necessary after changing the sign of \( \xi \). The main aim of this section is to prove the following theorem:

**Theorem 5.1.** Let \( M^n \) be an affine hypersurface in \( \mathbb{R}^{n+1} \) with \( \nabla K = 0 \) and whose induced affine metric is Lorentzian. Then \( M \) is equivalent to one of the following hypersurfaces.

1. a non degenerate quadric with Lorentzian affine metric,
2. the hypersurface \( x_{n+1} = \frac{1}{2} \sum_{i=3}^{n} x_i^2 + x_1x_2 - \frac{1}{3} x_2^3 \),
3. the hypersurface \( x_{n+1} = \frac{1}{2} \sum_{i=4}^{n} x_i^2 + x_1x_3 + \frac{1}{2} x_2^2 - x_1^2x_2 + \frac{1}{6} x_1^4 \).

The proof of this theorem will be divided in several lemmas.

**Lemma 5.2.** Under the assumptions of Theorem 5.1, we have \( K^3 = 0 \).

**Proof.** Let \( p \in M \). As in Remark 3.3, let \( k \) be the smallest number such that \( K^k = 0 \). Let \( u \) be a vector such that \( h((K_u)^{k-1}u, u) \neq 0 \). Assume that \( k > 3 \) and define

\[
x = (K_u)^{k-1}u, \quad y = (K_u)^{k-2}u.
\]

Since \( K_u x = 0 \) and \( K_u y = x \neq 0 \), we see that \( x \) and \( y \) are linearly independent vectors. Then, since we assumed \( k > 3 \), we have

\[
h(x, x) = h((K_u)^{k-1}u, (K_u)^{k-1}u) = h(u, (K_u)^{2k-2}u) = 0,
\]

\[
h(x, y) = h((K_u)^{k-1}u, (K_u)^{k-2}u) = h(u, (K_u)^{2k-3}u) = 0,
\]

\[
h(y, y) = h((K_u)^{k-2}u, (K_u)^{k-2}u) = h(u, (K_u)^{2k-4}u) = 0,
\]

\[
h(x, y) = h((K_u)^{k-1}u, (K_u)^{k-2}u) = h(u, (K_u)^{2k-3}u) = 0.
\]
\[
\begin{align*}
    h(x, y) &= h((K_u)^{k-1}u, (K_u)^{k-2}u) = h(u, (K_u)^{2k-3}u) = 0, \\
    h(y, y) &= h((K_u)^{k-2}u, (K_u)^{k-2}u) = h(u, (K_u)^{2k-4}u) = 0.
\end{align*}
\]

Hence the vector space spanned by \(x\) and \(y\) is a 2-dimensional null space in \(T_pM\). This is impossible since the affine metric is supposed to be Lorentzian.

**Lemma 5.3.** Let \(M\) be as in Theorem 5.1. If \(K^2 \neq 0\), then there exists a basis \(\{y_1, \ldots, y_n\}\) such that the only non zero components of the affine metric are

\[
h(y_1, y_3) = h(y_2, y_2) = h(y_i, y_i) = 1, \quad i > 3,
\]

and the only non zero components of the difference tensor are

\[
K(y_1, y_1) = y_2, \quad K(y_1, y_2) = y_3.
\]

**Proof.** There exists a vector \(u\) such that \(h(K_u K_u u, u) = \alpha \neq 0\). We define

\[
y_1 = u, \quad y_2 = K_u u, \quad y_3 = K_u K_u u.
\]

Then we put \(x = y_1 + \beta_1 y_2 + (\beta_2 - \frac{1}{2} \beta_1^2) y_3\), for some numbers \(\beta_1\) and \(\beta_2\). Let \(x_1 = x, \ x_2 = K_x x, \) and \(x_3 = K_x K_x x\). We get that \(x_2 = y_2 + 2 \beta_1 y_3\) and \(x_3 = y_3\). Then,

\[
h(x_1, x_1) = h(y_1, y_1) + 2 \beta_1 h(y_1, y_2) + 2 \beta_2 \alpha,
\]

and

\[
h(x_1, x_2) = h(y_1, y_2) + 3 \beta_1 \alpha,
\]

showing that we can choose \(\beta_1\) and \(\beta_2\) such that \(x_1\) is a null vector and \(h(x_1, x_2) = 0\).

So, we may assume this holds for \(u\), and then we have

\[
\begin{align*}
    h(y_1, y_1) &= 0, \quad h(y_1, y_2) = 0, \quad h(y_1, y_3) = \alpha, \\
    h(y_2, y_2) &= \alpha, \quad h(y_2, y_3) = 0, \quad h(y_3, y_3) = 0.
\end{align*}
\]

Since \(M\) is Lorentzian, the above formulas imply that \(\alpha\) is positive. So, by rescaling \(y_1\), we may assume that \(\alpha = 1\).

Clearly, the vector space \(V\) spanned by \(y_1, y_2\) and \(y_3\) is non degenerate and the restriction of the metric to this space is Lorentzian. Hence, the metric restricted to the vector space \(W = \{y_1, y_2, y_3\}\) is positive definite. By its construction, \(V\) is an invariant subspace of \(K_{y_1}\). Since \(K_{y_1}\) is symmetric, this implies that \(W\) is also an invariant subspace of \(K_{y_1}\). Since by Lemma 3.3, \(K_{y_1}\) is nilpotent, this implies that \(K_{y_1} w = 0\), for all \(w \in W\). Hence, we have

\[
\begin{align*}
    K(y_1, w) &= 0, \\
    K(y_2, w) &= K(K_{y_1} y_1, w) = K(y_1, K_{y_1} w) = 0, \\
    K(y_3, w) &= K(K_{y_1} K_{y_1} y_1, w) = K(K_{y_1} y_1, K_{y_1} w) = 0,
\end{align*}
\]
showing that \( K_w w = 0 \) for \( w \in W \) and \( v \in V \). But this also implies that \( V \) is an invariant subspace of \( K_w \). Hence \( W \) is also an invariant subspace of \( K_w \). Again since \( K_w \) is nilpotent and the metric restricted to \( W \) is positive definite, this implies that \( K_w w' = 0 \) for \( w' \in W \). Taking any orthonormal basis \( \{y_4, \ldots, y_n\} \) of \( W \), finishes the proof of the lemma.

PROOF OF THEOREM 5.1. If \( K = 0 \), then \( M \) is a quadric. If \( K^2 = 0 \), then \( M \) is one of the examples of Theorem 4.3. Since the affine metric is Lorentzian, we find that \( M \) has to be given as in (2). Therefore, we may assume that \( K^2 \neq 0 \) and \( K^3 = 0 \). Let \( p \in M \) and take a basis as in Lemma 5.3. We can extend these vector to \( \nabla \)-parallel vector fields \( Y_1, \ldots, Y_n \), with corresponding coordinates \( (x_1, \ldots, x_n) \), \( p \) corresponding to 0. Then, since \( \nabla K = 0 \), we have that

\[
K(Y_1, Y_1) = Y_2, \quad K(Y_1, Y_2) = Y_3,
\]

and that all other components of \( K \) are zero. Let \( \alpha_{ij} = h(Y_i, Y_j) \). Then \( \frac{\partial}{\partial x_i} \alpha_{jk} = (-2) h(K_{ij} Y_j, Y_k) \) is symmetric in \( i, j \) and \( k \). Therefore \( \alpha_{jk} \) is constant if either \( j > 2 \) or \( k > 2 \) and all \( \alpha_{ij} \) depend only on \( x_1 \) and \( x_2 \). Hence \( \alpha_{13} = 1, \alpha_{23} = 0 \) and \( \alpha_{ij} = \delta_{ij} \), for \( i, j > 3 \). Since \( \alpha_{22} = \alpha_{13} \), we obtain \( \alpha_{22} = 1 \). Further, we have

\[
\frac{\partial}{\partial x_1} \alpha_{12} = -2\alpha_{13} = -2, \\
\frac{\partial}{\partial x_2} \alpha_{12} = 0,
\]

so \( \alpha_{12} = -2x_1 \). Finally

\[
\frac{\partial}{\partial x_1} \alpha_{11} = -2\alpha_{12} = 4x_1, \\
\frac{\partial}{\partial x_2} \alpha_{11} = -2\alpha_{13} = -2.
\]

Hence \( \alpha_{11} = -2x_2 + 2x_1^2 \). So we have completely determined \( h \), and, as in Section 2, we know that \( M \) is the graph of a function \( F \), determined by (2.4). So we immediately obtain that \( F = \frac{1}{2} \sum_{i=4}^n x_i^2 + G(x_1, x_2, x_3) \). Since \( F_{33} = 0 \), we obtain that \( G = x_1 x_3 + B(x_1, x_2) \). Then from \( B_{22} = 1, B_{12} = -2x_1 \) and \( B_{11} = -2x_2 + 2x_1^2 \), it follows that \( B = \frac{1}{2} x_2^2 - x_1^2 x_2 + \frac{1}{6} x_1^4 \). So \( M \) is given by (3).

\[\square\]

§6. The case that \( K^{n-1} \neq 0 \)

Throughout this section, we will assume that \( M \) is an affine hypersurface with \( \nabla K = 0 \) and \( K^{n-1} \neq 0 \). Then \( K^n = 0 \). Let \( p \in M \) and let \( u \in T_p M \) such that \( h((K_u)^{n-1} u, u) \neq 0 \). We define

\[
y_1 = u, \\
y_i = (K_u)^{i-1} u, \quad 2 \leq i \leq n.
\]
By our assumption, we have $y_n \neq 0$. Since $K_u y_i = y_{i+1}$ and $K_u y_n = 0$, we get that $y_1, \ldots, y_n$ are linearly independent. Since $K_u$ is a symmetric operator with respect to $h$, we also get that

$$h(y_i, y_j) = \begin{cases} h(y_{i+j-1}, y_1), & i + j \leq n + 1 \\ 0, & i + j > n + 1. \end{cases}$$

We put $\alpha_\ell = h(y_\ell, y_1)$, for $1 \leq \ell \leq n$. We have chosen $u$ such that $\alpha_n \neq 0$, so by rescaling $u$, and if necessary changing the sign of $\xi$, we may assume that $\alpha_n = 1$.

**Lemma 6.1.** Let $M$ be a hypersurface of $\mathbb{R}^{n+1}$ with $\nabla K = 0$ and $K^{n-1} \neq 0$. Let $p \in M$, then there exists a basis $\{x_1, \ldots, x_n\}$ such that

$$K_{x_1} x_i = x_{i+1}, \quad i < n,$$
$$K_{x_1} x_n = 0,$$

and moreover

$$h(x_i, x_j) = \begin{cases} 0, & i + j \neq n + 1 \\ 1, & i + j = n + 1. \end{cases}$$

**Proof.** Let $y_1, \ldots, y_n$ be the basis of $T_p M$ as constructed before. Let $\beta$ be an arbitrary real number. We put $x_1 = y_1 + \beta y_2$ and $x_i = K_{x_1} x_{i-1}$ for all $i = 2, \ldots, n$. Then

$$x_2 = y_2 + 2\beta y_3 + \ldots,$$
$$x_3 = y_3 + 3\beta y_4 + \ldots,$$
$$\vdots$$
$$x_{n-1} = y_{n-1} + (n-1)\beta y_n,$$
$$x_n = y_n.$$

Hence $h(x_1, x_{n-1}) = h(y_1, y_{n-1}) + n\beta$. So by choosing $\beta$ appropriately, we may assume that $\alpha_{n-1} = 0$.

Next we assume that we have a basis $\{y_1, \ldots, y_n\}$ for which $\alpha_n = 1$ and $\alpha_{n-1} = \cdots = \alpha_{n-i} = 0$, for some $i > 0$. Then again we put $x_1 = y_1 + \beta y_{2+i}$ and $x_i = K_{x_1} x_{i-1}$ for all $i = 2, \ldots, n$. Then

$$x_2 = y_2 + 2\beta y_{3+i} + \ldots,$$
$$x_3 = y_3 + 3\beta y_{4+i} + \ldots,$$
$$\vdots.$$
Hypersurfaces with parallel difference tensor

\[x_{n-i-1} = y_{n-i-1} + (n - i - 1)\beta y_n,\]
\[x_{n-i} = y_{n-i},\]
\[\vdots\]
\[x_n = y_n.\]

Then we get that \(h(x_1, x_n) = 1,\)

\[h(x_1, x_{n-j}) = h(y_1, y_{n-j}) = \alpha_{n-j} = 0, \quad 1 \leq j \leq i,\]

and

\[h(x_1, x_{n-i-1}) = h(y_1, y_{n-i-1}) + (n - i)\beta \alpha_i,\]

showing that we can choose \(f_i\) in such a way that we may also assume that \(\alpha_{n-i-1} = 0.\) Continuing this process we complete the proof.

Now, we fix a point \(p \in M\) and use the basis constructed in the previous lemma. We proceed as in the previous cases, so we extend to \(\nabla\)-parallel vector fields \(X_1, \ldots, X_n,\) with corresponding coordinates \((x_1, \ldots, x_n),\) the point \(p\) corresponding to 0. Then we have \(K_{X_i} X_i = X_{i+1}\) if \(i < n\) and \(K_{X_i} X_n = 0.\) This implies that \(K_{X_i} X_j = K_{X_{i+1}}^{i+j-1} X_1,\) so \(K_{X_i} X_j = X_{i+j}\) if \(i + j \leq n\) and \(K_{X_i} X_j = 0\) if \(i + j > n.\) Further we immediately obtain that \(h(X_i, X_j) = h(X_1, X_{i+j} - 1)\) if \(i + j \leq n + 1\) and \(h(X_i, X_j) = 0\) if \(i + j > n + 1.\) Putting \(\alpha_{ij} = h(X_i, X_j)\) and \(\alpha_i = h(X_1, X_i),\) we obtain that

\[\alpha_{ij} = \begin{cases} 
\alpha_{i+j-1}, & i + j \leq n + 1 \\
0, & i + j > n + 1.
\end{cases}\]

From (2.1) we find \(\frac{\partial}{\partial x_i}\alpha_{jk} = (-2)h(K_{X_i} X_j, X_k),\) which implies that

\[(6.1) \quad \frac{\partial}{\partial x_i}\alpha_{jk} = \begin{cases} 
(-2)\alpha_{i+j+k-1} & i + j + k \leq n + 1 \\
0 & i + j + k > n + 1.
\end{cases}\]

We know that \(M\) is the graph of the function \(F,\) determined by (2.4), which, by Theorem 1.1, is a polynomial of degree \(n + 1.\) Moreover, we have the following formula

\[\frac{\partial^k F}{\partial x_{i_1} \cdots \partial x_{i_k}} = \frac{\partial^{k-2} \alpha_{i_k-1} i_k}{\partial x_{i_1} \cdots \partial x_{i_k-2}},\]

so from (6.1) and the initial conditions \(\alpha_n(0) = 1\) and \(\alpha_1(0) = \cdots = \alpha_{n-1}(0) = 0,\) we obtain that

\[(6.2) \quad \frac{\partial^k F}{\partial x_{i_1} \cdots \partial x_{i_k}}(0) = \begin{cases} 
(-2)^{k-2}, & \text{if } i_1 + \cdots + i_k = n + 1, \\
0, & \text{if } i_1 + \cdots + i_k \neq n + 1.
\end{cases}\]
From (6.2) we obtain that (using for instance Taylor expansion)

\begin{equation}
F = \sum_{k=2}^{n+1} (-2)^{k-2} \frac{1}{k!} \left( \sum_{i_1, \ldots, i_k = 1}^{n} x_{i_1} x_{i_2} \cdots x_{i_k} \right).
\end{equation}

Summarizing, we have proved the following theorem.

**Theorem 6.2.** Let \( M^n \) be an affine hypersurface in \( \mathbb{R}^{n+1} \) with \( \nabla K = 0 \) and \( K^{n-1} \neq 0. \) Then \( M \) is equivalent to the improper hypersphere given by \( x_{n+1} = F(x_1, \ldots, x_n), \) where \( F \) is given by (6.3).

As illustration, we list the functions \( F \) for dimensions \( n = 2, 3, 4. \) If \( n = 2, \) then \( F \) is given by

\[ F(x_1, x_2) = x_1 x_2 - \frac{1}{3} x_1^3, \]

so \( M \) is the Cayley surface. If \( n = 3 \) then

\[ F(x_1, x_2, x_3) = x_1 x_3 + \frac{1}{2} x_2^2 - x_1^2 x_2 + \frac{1}{6} x_1^4, \]

which is (3) of Theorem 5.1 (for \( n = 3 \); this was to be expected, since every indefinite 3-dimensional metric is Lorentzian up to a sign. For \( n = 4 \) we obtain a new example, the function \( F \) being given by

\[ F(x_1, x_2, x_3, x_4) = x_1 x_4 + x_2 x_3 - x_1^2 x_2 - x_1 x_2^2 + \frac{2}{3} x_1^3 x_2 - \frac{1}{15} x_1^5. \]

Note that in \( F, \) the component of largest degree is always \( \frac{(-2)^{n-1}}{(n+1)!} x_1^{n+1}. \)

§7. The case that \( K^{n-2} \neq 0 \) and \( K^{n-1} = 0. \)

So we will assume that \( M \) is an affine hypersurface with \( \nabla K = 0, \) \( K^{n-2} \neq 0 \) and \( K^{n-1} = 0. \) In the beginning we proceed along the lines of Section 6, so we omit most of the details. Let \( p \in M \) and let \( u \in T_p M \) such that \( h((K_u)^{n-2} u, u) \neq 0. \) We define as in the previous section

\[ y_1 = u, \]
\[ y_i = (K_u)^{i-1} u, \quad 2 \leq i \leq n - 1, \]

and obtain that they are linearly independent vectors, spanning an \((n-1)\)-dimensional linear subspace \( B(p) \). Since the proof of Lemma 6.1 also applies here, we may assume that \( h(y_i, y_j) = 0 \) if \( i + j \neq n \) and \( h(y_i, y_{n-i}) = 1. \) Therefore \( B(p) \) is non degenerate, so it has an orthogonal complement, spanned by a vector \( y_n, \) which can
be taken to satisfy $h(y_n, y_n) = \varepsilon = \pm 1$. Since $B(p)$ is invariant under $K_u$, $y_n$ must be an eigenvector, so by the nilpotency of $K_u$, we have that $K_u y_n = 0$.

We have proved that $K_{y_n} y_1 = 0$, which implies for all $i < n$ that

$$K_{y_n} y_i = K_{y_n} K_{y_1}^{i-1} y_1 = K_{y_1}^{i-1} K_{y_n} y_1 = 0.$$

So $B(p)$ is a part of the kernel of $K_{y_n}$. Hence $y_n$ is an eigenvector of $K_{y_n}$, which again by the nilpotency of $K_{y_n}$, implies that $K_{y_n} y_n = 0$. So, simply $K_{y_n} = 0$.

As before, we fix a point $p \in M$ and use the basis constructed above and extend this basis to $\nabla$-parallel vector fields $X_1, \ldots, X_n$, with corresponding coordinates $(x_1, \ldots, x_n)$, the point $p$ corresponding to $0$. For all $i, j$ we then have that

$$X_i h(X_j, X_n) = (-2) h(K X_i, X_n, X_j) = 0,$$

so $h(X_n, X_n) = \varepsilon$ and $h(X_n, X_k) = 0$ for all $k < n$. Further $X_n h(X_i, X_j) = 0$ for all $i, j < n$. From this it is easily seen that the polynomial function $F$, of which $M$ is the graph, splits into

$$F(x_1, \ldots, x_{n-1}, x_n) = \frac{1}{2} \varepsilon x_n^2 + G(x_1, \ldots, x_{n-1}),$$

and that the hypersurface $M_1^n$ of $\mathbb{R}^n$ given by $x_n = G(x_1, \ldots, x_{n-1})$, satisfies $\nabla_1 K_1 = 0$ and $K_1^{n-2} \neq 0$ – we denote all objects of $M_1$ with subscript $1$. So $G$ is the function given by (6.3), after replacing $n+1$ by $n$. We have proved the following theorem.

**Theorem 7.1.** Let $M^n$ be an affine hypersurface in $\mathbb{R}^{n+1}$ with $\nabla K = 0$, $K^{n-2} \neq 0$ and $K^{n-1} = 0$. Then $M$ is equivalent to the improper hypersphere given by $x_{n+1} = F(x_1, \ldots, x_n)$, where $F$ is given by

$$F(x_1, \ldots, x_{n-1}, x_n) = \frac{1}{2} \varepsilon x_n^2 + \sum_{k=2}^{n} \frac{(-2)^{k-2}}{k!} \left( \sum_{i_1, \ldots, i_k = 1}^{n-1} x_{i_1} x_{i_2} \cdots x_{i_k} \right).$$

**Remark.** From the results in this paper, it is easy to see that we have in particular obtained a complete classification of hypersurfaces $M^n$ of $\mathbb{R}^{n+1}$ with $\nabla K = 0$ for dimensions $n = 2, 3, 4$.

**References**


