2-graded decompositions of exceptional Lie algebras $\mathfrak{g}$ and group realizations of $\mathfrak{g}_{\text{ev}}, \mathfrak{g}_0$ Part I, $G = G_2, F_4, E_6$

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(Received December 20, 1996)

(from Tokyo Journal of Mathematics)

The 2-graded decompositions of simple Lie algebras $\mathfrak{g}$,

$$\mathfrak{g} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2, \quad [\mathfrak{g}_k, \mathfrak{g}_l] \subset \mathfrak{g}_{k+l}$$

are classified and the types of subalgebras $\mathfrak{g}_{\text{ev}} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_2, \mathfrak{g}_0$ are determined. The following table is the results of $\mathfrak{g}_{\text{ev}}, \mathfrak{g}_0$ for exceptional Lie algebras $\mathfrak{g}$ of type G_2, F_4 and E_6 (Kaneyuki [2]). (The order is arranged for our uses).

$\mathfrak{g}$	$\mathfrak{g}_{\text{ev}}$	$\mathfrak{g}_0$	$\dim \mathfrak{g}_1$	$\dim \mathfrak{g}_2$
$\mathfrak{g}_2^C$	$\mathfrak{sl}(2, C) \oplus \mathfrak{sl}(2, C)$	$C \oplus \mathfrak{sl}(2, C)$	4	1
$\mathfrak{g}_2(2)$	$\mathfrak{sl}(2, \mathbf{R}) \oplus \mathfrak{sl}(2, \mathbf{R})$	$\mathbf{R} \oplus \mathfrak{sl}(2, \mathbf{R})$	4	1
$\mathfrak{f}_4^C$	$\mathfrak{sl}(2, C) \oplus \mathfrak{sp}(3, C)$	$C \oplus \mathfrak{sp}(3, C)$	14	1
$\mathfrak{f}_4(4)$	$\mathfrak{sl}(2, \mathbf{R}) \oplus \mathfrak{sp}(3, \mathbf{R})$	$\mathbf{R} \oplus \mathfrak{sp}(3, \mathbf{R})$	14	1
$\mathfrak{f}_4^C$	$\mathfrak{so}(9, C)$	$C \oplus \mathfrak{so}(7, C)$	8	7
$\mathfrak{f}_4(4)$	$\mathfrak{so}(4, 5)$	$\mathbf{R} \oplus \mathfrak{so}(3, 4)$	8	7
$\mathfrak{f}_4(-20)$	$\mathfrak{so}(8, 1)$	$\mathbf{R} \oplus \mathfrak{so}(7)$	8	7
$\mathfrak{e}_6^C$	$\mathfrak{sl}(2, C) \oplus \mathfrak{sl}(6, C)$	$C \oplus \mathfrak{sl}(6, C)$	20	1
$\mathfrak{e}_6(6)$	$\mathfrak{sl}(2, \mathbf{R}) \oplus \mathfrak{sl}(6, \mathbf{R})$	$\mathbf{R} \oplus \mathfrak{sl}(6, \mathbf{R})$	20	1
$\mathfrak{e}_6(2)$	$\mathfrak{sl}(2, \mathbf{R}) \oplus \mathfrak{su}(3, 3)$	$\mathbf{R} \oplus \mathfrak{su}(3, 3)$	20	1
$\mathfrak{e}_6(-14)$	$\mathfrak{sl}(2, \mathbf{R}) \oplus \mathfrak{su}(1, 5)$	$\mathbf{R} \oplus \mathfrak{su}(1, 5)$	20	1
$\mathfrak{e}_6^C$	$\mathfrak{sl}(2, C) \oplus \mathfrak{sl}(6, C)$	$\mathfrak{sl}(2, C) \oplus C \oplus \mathfrak{sl}(5, C)$	20	5
$\mathfrak{e}_6(6)$	$\mathfrak{sl}(2, \mathbf{R}) \oplus \mathfrak{sl}(6, \mathbf{R})$	$\mathfrak{sl}(2, \mathbf{R}) \oplus \mathbf{R} \oplus \mathfrak{sl}(5, \mathbf{R})$	20	5
$\mathfrak{e}_6^C$	$C \oplus \mathfrak{so}(10, C)$	$C \oplus C \oplus \mathfrak{so}(8, C)$	16	8
$\mathfrak{e}_6(6)$	$\mathbf{R} \oplus \mathfrak{so}(5, 5)$	$\mathbf{R} \oplus \mathbf{R} \oplus \mathfrak{so}(4, 4)$	16	8
$\mathfrak{e}_6(2)$	$i\mathbf{R} \oplus \mathfrak{so}(6, 4)$	$i\mathbf{R} \oplus \mathbf{R} \oplus \mathfrak{so}(5, 3)$	16	8
$\mathfrak{e}_6(-14)$	$i\mathbf{R} \oplus \mathfrak{so}(2, 8)$	$i\mathbf{R} \oplus \mathbf{R} \oplus \mathfrak{so}(1, 7)$	16	8
$\mathfrak{e}_6(-26)$	$\mathbf{R} \oplus \mathfrak{so}(9, 1)$	$\mathbf{R} \oplus \mathbf{R} \oplus \mathfrak{so}(8)$	16	8

Now, for the exceptional universal Lie groups G of type G_2 , F_4 and E_6 , we realize the subgroups G_{ev} , G_0 of G corresponding to the subalgebras $\mathfrak{g}_{\text{ev}}$, $\mathfrak{g}_0$ of $\mathfrak{g} = \text{Lie } G$. Our results are as follows.

G	G_{ev}	G_0
G_2^C	$(Sp(1, C) \times Sp(1, C))/\mathbf{Z}_2$	$(C^* \times Sp(1, C))/\mathbf{Z}_2$
$G_{2(2)}$	$(Sp(1, \mathbf{R}) \times Sp(1, \mathbf{R}))/\mathbf{Z}_2 \times 2$	$(\mathbf{R}^+ \times Sp(1, \mathbf{R})) \times 2$
F_4^C	$(Sp(1, C) \times Sp(3, C))/\mathbf{Z}_2$	$(C^* \times Sp(3, C))/\mathbf{Z}_2$
$F_{4(4)}$	$(Sp(1, \mathbf{R}) \times Sp(3, \mathbf{R}))/\mathbf{Z}_2 \times 2$	$(\mathbf{R}^+ \times Sp(3, \mathbf{R})) \times 2$
F_4^C	$Spin(9, C)$	$(C^* \times Spin(7, C))/\mathbf{Z}_2$
$F_{4(4)}$	$spin(4, 5)$	$(\mathbf{R}^+ \times spin(3, 4)) \times 2$
$F_{4(-20)}$	$Spin(8, 1)$	$(\mathbf{R}^+ \times Spin(7)) \times 2$
E_6^C	$(Sp(1, C) \times SL(6, C))/\mathbf{Z}_2$	$(C^* \times SL(6, C))/\mathbf{Z}_2$
$E_{6(6)}$	$(Sp(1, \mathbf{R}) \times SL(6, \mathbf{R}))/\mathbf{Z}_2 \times 2$	$(\mathbf{R}^+ \times SL(6, \mathbf{R})) \times 2$
$E_{6(2)}$	$(Sp(1, \mathbf{R}) \times SU(3, 3))/\mathbf{Z}_2 \times 2$	$(\mathbf{R}^+ \times SU(3, 3)) \times 2$
$E_{6(-14)}$	$(Sp(1, \mathbf{R}) \times SU(1, 5))/\mathbf{Z}_2 \times 2$	$(\mathbf{R}^+ \times SU(1, 5)) \times 2$
E_6^C	$(Sp(1, C) \times SL(6, C))/\mathbf{Z}_2$	$(Sp(1, C) \times C^* \times SL(5, C))/(\mathbf{Z}_2 \times \mathbf{Z}_5)$
$E_{6(6)}$	$(Sp(1, \mathbf{R}) \times SL(6, \mathbf{R}))/\mathbf{Z}_2 \times 2$	$(Sp(1, \mathbf{R}) \times \mathbf{R}^+ \times SL(5, \mathbf{R})) \times 2$
E_6^C	$(C^* \times Spin(10, C))/\mathbf{Z}_4$	$(C^* \times C^* \times Spin(8, C))/(\mathbf{Z}_2 \times \mathbf{Z}_4)$
$E_{6(6)}$	$(\mathbf{R}^+ \times spin(5, 5)) \times 2$	$(\mathbf{R}^+ \times \mathbf{R}^+ \times spin(4, 4)) \times 2^2$
$E_{6(2)}$	$(U(1) \times spin(6, 4))/\mathbf{Z}_4$	$(U(1) \times \mathbf{R}^+ \times spin(5, 3))/\mathbf{Z}_2 \times 2$
$E_{6(-14)}$	$(U(1) \times spin(2, 8))/\mathbf{Z}_4$	$(U(1) \times \mathbf{R}^+ \times Spin(1, 7))/\mathbf{Z}_2 \times 2$
$E_{6(-26)}$	$\mathbf{R}^+ \times Spin(9, 1)$	$(\mathbf{R}^+ \times \mathbf{R}^+ \times Spin(8)) \times 2^2$

0.1. Introduction

Let G be a simple Lie group, $\mathfrak{g}$ the Lie algebra of G and G^C , $\mathfrak{g}^C$ their complexifications, respectively. In a 2-graded decomposition $\mathfrak{g} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2$ of $\mathfrak{g}$, we know that there exists $Z \in \mathfrak{g}_0$, called a characteristic element, such that each $\mathfrak{g}_k$ is the k -eigenspace of $\text{ad } Z : \mathfrak{g} \rightarrow \mathfrak{g}$. In $\mathfrak{g}^C$, let

$$\tilde{\gamma} = \exp \frac{2\pi i}{2} \text{ad } Z, \quad \tilde{\gamma}_3 = \exp \frac{2\pi i}{3} \text{ad } Z.$$

Then $\tilde{\gamma}$, $\tilde{\gamma}_3$ are automorphisms of order 2, 3 of $\mathfrak{g}^C$, respectively. In view of the above consideration, we explain our methods. Let $\tilde{\gamma}_3$ be an automorphism of order 3 of $\mathfrak{g}^C$ (all automorphisms of order 3 of $\mathfrak{g}^C$ are classified [3]) and $(\mathfrak{g}^C)_0$ be the 1-eigenspace of $\tilde{\gamma}_3$:

$$(\mathfrak{g}^C)_0 = (\mathfrak{g}^C)^{\tilde{\gamma}_3} = \{X \in \mathfrak{g}^C \mid \tilde{\gamma}_3 X = X\}.$$

If $(\mathfrak{g}^C)_0$ contains C as a direct summand, then a generator Z of C is a characteristic element of $\mathfrak{g}^C$. Using $\text{ad } Z$, we decompose $\mathfrak{g}^C$ into the eigenspaces,

$$\mathfrak{g}^C = (\mathfrak{g}^C)_{-2} \oplus (\mathfrak{g}^C)_{-1} \oplus (\mathfrak{g}^C)_0 \oplus (\mathfrak{g}^C)_1 \oplus (\mathfrak{g}^C)_2.$$

Next, let

$$\mathfrak{g}_k = (\mathfrak{g}^C)_k \cap \mathfrak{g}.$$

Then obviously we have

$$\mathfrak{g} \supset \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2, \quad [\mathfrak{g}_k, \mathfrak{g}_l] \subset \mathfrak{g}_{k+l}.$$

However, the above “ $\supset$ ” is not necessarily “ $=$ ”. So we need to find (if possible) a real form $(\mathfrak{g}^C)^{\tilde{\iota}}$ of $\mathfrak{g}^C$ (where $\tilde{\iota}$ is a complex-conjugate involutive automorphism of $\mathfrak{g}^C$) such that $(\mathfrak{g}^C)^{\tilde{\iota}} \cong \mathfrak{g}$ satisfying $((\mathfrak{g}^C)_k)^{\tilde{\iota}} \subset (\mathfrak{g}^C)_k$. Then we get a 2-graded decomposition $\mathfrak{g} \cong (\mathfrak{g}^C)^{\tilde{\iota}} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2$, where $\mathfrak{g}_k = (\mathfrak{g}^C)_k \cap (\mathfrak{g}^C)^{\tilde{\iota}}$. In this paper, for the simply connected exceptional complex simple Lie group G^C of type G_2 , F_4 and E_6 , we determine the group structures of

$$\begin{aligned} (G^C)_{\text{ev}} &= (G^C)^{\tilde{\gamma}}, & (G^C)_0 &= (G^C)^{\tilde{\gamma}_3}, \\ G_{\text{ev}} &= (G^C)^{\tilde{\gamma}} \cap (G^C)^{\tilde{\iota}}, & G_0 &= (G^C)^{\tilde{\gamma}_3} \cap (G^C)^{\tilde{\iota}} \end{aligned}$$

where $\tilde{\gamma}$, $\tilde{\gamma}_3$, $\tilde{\iota}$ are automorphisms of G^C associated with automorphisms $\tilde{\gamma}$, $\tilde{\gamma}_3$, $\tilde{\iota}$ of $\mathfrak{g}^C$.

0.2. Notations and preliminaries

$\mathbf{R}$, $\mathbf{C} = \{1, e_1\}$ ($e_1^2 = -1$), $\mathbf{H} = \mathbf{C} \oplus \mathbf{C}e_2 = \{1, e_1, e_2, e_3\}$ denote the fields of real, complex, quaternion numbers and $\mathbf{C}' = \{1, e_1'\}$ ($e_1'^2 = 1$), $\mathbf{H}' = \mathbf{C}' \oplus \mathbf{C}'e_2$ denote the $\mathbf{R}$ -algebras of split complex, split quaternion numbers, respectively. Let $\mathbf{R}^* = \mathbf{R} - \{0\}$, $\mathbf{R}^+ = \{x \in \mathbf{R} \mid x > 0\}$.

For an $\mathbf{R}$ -vector space V , its complexification $\{u + iv \mid u, v \in V\}$ is denoted by V^C . The complex conjugation in V^C is denoted by τ :

$$\tau(u + iv) = u - iv.$$

$\mathbf{R}^C$ is briefly denoted by C and let $C^* = C - \{0\}$.

For a K -vector space V ($K = \mathbf{R}, C$), $\text{Iso}_K(V)$ denotes all of K -linear isomorphisms of V . For a K -linear mapping f of V , V_f (or V^f) denotes $\{v \in V \mid f(v) = v\}$.

For a topological group G , G^0 denotes the connected component containing the identity of G and $G = G^0 \times r = G^0 \times \{1, g_1, \dots, g_{r-1}\}$ means that G has r connected components such that $G = G^0 \cup g_1 G^0 \cup \dots \cup g_{r-1} G^0$. When G is a transformation group of a space X , for $x_1, \dots, x_m \in X$, $G_{x_1, \dots, x_m}$ is $\{g \in G \mid gx_k = x_k, k = 1, \dots, m\}$.

Let G be a group and σ an automorphism of G , then G^σ denotes $\{g \in G \mid \sigma(g) = g\}$. For $s \in G$, G^s denotes $\{g \in G \mid sgs^{-1} = g\}$.

$$E = \text{diag}(1, \dots, 1) \in M(n, \mathbf{R}).$$

$$I_1 = \text{diag}(-1, 1, \dots, 1), I_2 = \text{diag}(-1, -1, 1, \dots, 1), \dots \in M(n, \mathbf{R}).$$

$$I = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \text{ moreover } I = \text{diag}(I, \dots, I), J = \text{diag}(J, \dots, J) \in M(2n, \mathbf{R}).$$

$$GL(n, K) = \{A \in M(n, K) \mid \det A \neq 0\}, K = \mathbf{R}, C,$$

$$SL(n, K) = \{A \in M(n, K) \mid \det A = 1\}, K = \mathbf{R}, C,$$

$$SO(n, K) = \{A \in M(n, K) \mid {}^tAA = E, \det A = 1\}, K = \mathbf{R}, C, SO(n) = SO(n, \mathbf{R}),$$

$$O(m, n-m) = \{A \in M(n, \mathbf{R}) \mid {}^tAI_mA = I_m\},$$

$$U(n, K) = \{A \in M(n, K) \mid A^*A = E\}, K = C, C', C^C, U(n) = U(n, C),$$

$$SU(m, n-m) = \{A \in M(n, C) \mid {}^t(\tau A)I_mA = I_m, \det A = 1\},$$

$$SU^*(2n, C^C) = \{A \in (2n, C^C) \mid JA = \bar{A}J, \det A = 1\},$$

$$Sp(n, H^C) = \{A \in M(n, H^C) \mid A^*A = E\},$$

$$Sp(n, K) = \{A \in M(n, K) \mid {}^tAJA = J\}, K = \mathbf{R}, C.$$

Then we have the following isomorphism ([5]).

$$U(n, C^C) \cong GL(n, C), U(n, C') \cong GL(n, \mathbf{R}) \text{ (in particular, } U(1, C^C) \cong C^*, U(1, C') \cong \mathbf{R}^*),$$

$$SU^*(2n, C^C) \cong SL(2n, C), Sp(n, H^C) \cong Sp(n, C), Sp(n, H') \cong Sp(n, \mathbf{R}).$$

The Lie algebra of a Lie group G is denoted by the corresponding Germain small letter $\mathfrak{g}$.

In the following arguments, the following two Lemmas are useful.

LEMMA 0.1. *Let G be a simply connected Lie group and σ an automorphism of G of finite order, then G^σ is connected.*

LEMMA 0.2. *Let $\varphi : G \rightarrow G'$ be a homomorphism of Lie groups satisfying*

- (1) G' connected (2) $\text{Ker } \varphi$ is discrete (3) $\dim \mathfrak{g} = \dim \mathfrak{g}'$.

Then φ is onto. (Thus we have an isomorphism $G' \cong G / \text{Ker } \varphi$).

Group G_2

1.1. Lie groups of type G_2 and their Lie algebras

Let $\mathfrak{C} = \mathbf{H} \oplus \mathbf{H}e$ (resp. $\mathfrak{C}' = \mathbf{H} \oplus \mathbf{H}e'$) be the Cayley algebra (resp. the split Cayley algebra) over $\mathbf{R}$ with the multiplication

$$(m_1 + n_1e)(m_2 + n_2e) = (m_1m_2 - \bar{n}_2n_1) + (n_1\bar{m}_2 + n_2m_1)e,$$

$$\text{(resp. } (m_1 + n_1e')(m_2 + n_2e') = (m_1m_2 + \bar{n}_2n_1) + (n_1\bar{m}_2 + n_2m_1)e'),$$

the conjugation $\overline{m + ne} = \overline{m} - ne$ (resp. $\overline{m + ne'} = \overline{m} - ne'$) and the inner product $(x, y) = \frac{1}{2}(x\bar{y} + y\bar{x})$.

The connected linear Lie groups of type G_2 are given by

$$\begin{aligned} G_2^C &= \{\alpha \in \text{Iso}_C(\mathfrak{C}^C) \mid \alpha(xy) = (\alpha x)(\alpha y)\}, \\ G_2 &= \{\alpha \in \text{Iso}_R(\mathfrak{C}) \mid \alpha(xy) = (\alpha x)(\alpha y)\}, \\ G_{2(2)} &= \{\alpha \in \text{Iso}_R(\mathfrak{C}') \mid \alpha(xy) = (\alpha x)(\alpha y)\}. \end{aligned}$$

G_2^C and G_2 are simply connected. We define $\mathbf{R}$ -linear transformations $\gamma, \gamma_1, \gamma_2$ and γ_3 of $\mathfrak{C}$ by

$$\begin{aligned} \gamma(m + ne) &= m - ne, & \gamma_1(m + ne) &= \gamma_1 m + (\gamma_1 n)e, \\ \gamma_2(m + ne) &= \gamma_2 m + (\gamma_2 n)e, & \gamma_3(m + ne) &= m + (\omega_1 n)e, \end{aligned}$$

$m + ne \in \mathbf{H} \oplus \mathbf{H}e = \mathfrak{C}$, where $\gamma_1, \gamma_2 : \mathbf{H} \rightarrow \mathbf{H}$ are defined as

$$\gamma_1(x + ye_2) = \bar{x} + \bar{y}e_2, \quad \gamma_2(x + ye_2) = x - ye_2, \quad x + ye_2 \in \mathbf{C} \oplus \mathbf{C}e_2 = \mathbf{H},$$

respectively, and $\omega_1 = -\frac{1}{2} + \frac{\sqrt{3}}{2}e_1 \in \mathbf{C} \subset \mathbf{H} \subset \mathfrak{C}$. Then $\gamma, \gamma_1, \gamma_2, \gamma_3 \in G_2 \subset G_2^C$ and $\gamma^2 = \gamma_1^2 = \gamma_2^2 = 1, \gamma_3^3 = 1$. (γ_1, γ_2 are denoted by γ_C, γ_H , respectively, in [5]).

Since $\mathfrak{C} = (\mathfrak{C}^C)_\tau, \mathfrak{C}' = (\mathfrak{C}^C)_{\tau\gamma}$ we have

PROPOSITION 1.1. $G_2 = (G_2^C)^\tau, G_{2(2)} \cong (G_2^C)^{\tau\gamma}$.
 $((G_2^C)^\tau$ is defined by $(G_2^C)^\tau = \{\alpha \in G_2^C \mid \tau\alpha\tau = \alpha\}$ etc.).

Let $\mathfrak{so}(8) = \mathfrak{so}(\mathfrak{C})$ be the Lie algebra of the group $SO(8) = SO(\mathfrak{C}) = \{\alpha \in \text{Iso}_R(\mathfrak{C}) \mid (\alpha x, \alpha y) = (x, y), \det \alpha = 1\}$:

$$\begin{aligned} \mathfrak{so}(8) &= \{D \in \text{Hom}_R(\mathfrak{C}, \mathfrak{C}) \mid \\ &\quad (Dx, y) + (x, Dy) = 0\}. \end{aligned}$$

Let $\{e_0 = 1, e_1, e_2, e_3, e_4 = e, e_5, e_6, e_7\}$ be a canonical $\mathbf{R}$ -basis of $\mathfrak{C}$ (cf. the right figure) and define an $\mathbf{R}$ -linear mapping G_{kl} of $\mathfrak{C}$ satisfying

$$\begin{aligned} G_{kl}e_k &= -e_l, \quad G_{kl}e_l = e_k, \\ G_{kl}e_m &= 0, \text{ otherwise.} \end{aligned}$$

Then $G_{kl} \in \mathfrak{so}(8)$ and $\{G_{kl} \mid 0 \leq k < l \leq 7\}$ is an $\mathbf{R}$ -basis of $\mathfrak{so}(8)$.

LEMMA 1.2. *The Lie bracket in $\mathfrak{so}(8)$ is given by*

$$[G_{kl}, G_{lm}] = G_{km}, \quad [G_{kl}, G_{mn}] = 0, \quad k, l, m, n \text{ are distinct.}$$

The Lie algebra $\mathfrak{g}_2$ of the group G_2 :

$$\mathfrak{g}_2 = \{D \in \text{Hom}_R(\mathfrak{C}, \mathfrak{C}) \mid D(xy) = (Dx)y + x(Dy)\}$$

is a subalgebra of $\mathfrak{so}(8)$.

PROPOSITION 1.3 ([1]). *The following 7 elements are additive generators of the Lie algebra $\mathfrak{g}_2$.*

$$\begin{aligned} \lambda G_{23} + \mu G_{45} + \nu G_{67}, & \quad -\lambda G_{13} - \mu G_{46} + \nu G_{57}, \\ \lambda G_{12} + \mu G_{47} + \nu G_{56}, & \quad -\lambda G_{15} + \mu G_{26} - \nu G_{37}, \quad \lambda, \mu, \nu \in \mathbf{R}, \\ \lambda G_{14} - \mu G_{27} - \nu G_{36}, & \quad -\lambda G_{17} - \mu G_{24} + \nu G_{35}, \quad \lambda + \mu + \nu = 0. \\ \lambda G_{16} + \mu G_{25} + \nu G_{34}, & \end{aligned}$$

1.2. Subgroups of type $C_1^C \oplus C_1^C$ and $C \oplus C_1^C$ of G_2^C

In the complexified Lie algebra $\mathfrak{g}_2^C$ of $\mathfrak{g}_2$, let $Z = i(G_{45} - G_{67})$. Then we can easily verify that

$$\gamma = \exp \frac{2\pi i}{2} Z, \quad \gamma_3 = \exp \frac{2\pi i}{3} Z.$$

Since γ, γ_1 are conjugate in G_2 ([5] Proposition 1.2.3), we have

PROPOSITION 1.4. $G_{2(2)} = (G_2^C)^{\tau\gamma} \cong (G_2^C)^{\tau\gamma_1}$.

LEMMA 1.5. *In the Lie algebra $\mathfrak{so}(8, C)$, we have*

$$\begin{cases} \tau G_{kl} \tau = G_{kl} \\ \tau(iG_{kl})\tau = -iG_{kl}. \end{cases} \quad \gamma_1 G_{kl} \gamma_1 = \begin{cases} G_{kl}, & k+l \text{ is even} \\ -G_{kl}, & k+l \text{ is odd.} \end{cases}$$

THEOREM 1.6. *The 2-graded decomposition of $\mathfrak{g}_{2(2)} = (\mathfrak{g}_2^C)^{\tau\gamma_1}$ (or $\mathfrak{g}_2^C$),*

$$\mathfrak{g}_{2(2)} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2$$

with respect to $\text{ad } Z$, $Z = i(G_{45} - G_{67})$, is given by

$$\mathfrak{g}_0 = \left\{ i(G_{45} - G_{67}), i(-2G_{23} + G_{45} + G_{67}), 2G_{13} - G_{46} + G_{57}, \right. \\ \left. i(-2G_{12} + G_{47} + G_{56}) \right\} 4$$

$$\begin{aligned} \mathfrak{g}_{-1} &= \left\{ \begin{aligned} &(G_{26} + G_{37}) - i(G_{27} - G_{36}), \quad (G_{24} + G_{35}) + i(G_{25} - G_{34}), \\ &(-2G_{15} - G_{26} + G_{37}) + i(2G_{14} + G_{27} + G_{36}), \\ &(2G_{17} - G_{24} + G_{35}) + i(2G_{16} - G_{25} - G_{34}) \end{aligned} \right\} 4 \\ \mathfrak{g}_{-2} &= \{(G_{46} + G_{57}) - i(G_{47} - G_{56})\} 1 \\ \mathfrak{g}_1 &= \tau(\mathfrak{g}_{-1})\tau, \quad \mathfrak{g}_2 = \tau(\mathfrak{g}_{-2})\tau. \end{aligned}$$

PROOF. We can verify that $\mathfrak{g}_k$ is the k -eigenspace of $\text{ad } Z : \mathfrak{g}_2^C \rightarrow \mathfrak{g}_2^C$ (Lemma 1.2). Moreover each element X of $\mathfrak{g}_2^C$ is invariant under $\text{Ad}(\tau\gamma_1)$, that is,

$$\gamma_1 \tau X \tau \gamma_1 = X \quad (\text{Lemma 1.5}).$$

Hence the 2-graded decomposition of $\mathfrak{g}_2^C$ over C induces also a 2-graded decomposition of $\mathfrak{g}_{2(2)} = (\mathfrak{g}_2^C)^{\tau\gamma_1}$ over $\mathbf{R}$. (Note. $\{X_1, \dots, X_m\} = \mathfrak{g}_k$ is a basis of $\mathfrak{g}_k$ and the number of the right side is the dimension of $\mathfrak{g}_k$).

Now, we shall determine the group structures of

$$(G_2^C)_{\text{ev}} = (G_2^C)^\gamma, \quad (G_2^C)_0 = (G_2^C)^{\gamma_3}.$$

THEOREM 1.7. (1) $(G_2^C)_{\text{ev}} \cong (Sp(1, C) \times Sp(1, C))/\mathbf{Z}_2$, $\mathbf{Z}_2 = \{(1, 1), (-1, -1)\}$.

(2) $(G_2^C)_0 \cong (C^* \times Sp(1, C))/\mathbf{Z}_2$, $\mathbf{Z}_2 = \{(1, 1), (-1, -1)\}$.

PROOF. (1) We define $\varphi : Sp(1, \mathbf{H}^C) \times Sp(1, \mathbf{H}^C) \rightarrow (G_2^C)^\gamma = (G_2^C)_{\text{ev}}$ by

$$\varphi(p, q)(m + ne) = qm\bar{q} + (pn\bar{q})e, \quad m + ne \in \mathbf{H}^C \oplus \mathbf{H}^C e = \mathfrak{C}^C.$$

Then φ is well-defined, is a homomorphism and $\text{Ker } \varphi = \mathbf{Z}_2$. Since $(G_2^C)^\gamma$ is connected and $\dim_C(\mathfrak{sp}(1, \mathbf{H}^C) \oplus \mathfrak{sp}(1, \mathbf{H}^C)) = 3 + 3 = 6 = 4 + 2 = \dim_C((\mathfrak{g}_2^C)_{\text{ev}})$ (Theorem 1.6), φ is onto. Therefore $(G_2^C)_{\text{ev}} \cong (Sp(1, \mathbf{H}^C) \times Sp(1, \mathbf{H}^C))/\mathbf{Z}_2 \cong (Sp(1, C) \times Sp(1, C))/\mathbf{Z}_2$.

(2) The restriction mapping $\varphi : U(1, \mathbf{C}^C) \times Sp(1, \mathbf{H}^C) \rightarrow (G_2^C)^{\gamma_3} = (G_2^C)_0$ of φ above (1) is well-defined (because $\gamma_3 = \varphi(\omega_1, 1)$) and $\text{Ker } \varphi = \mathbf{Z}_2$. Since $(G_2^C)^{\gamma_3}$ is connected and $\dim_C(\mathfrak{u}(1, \mathbf{C}^C) \oplus \mathfrak{sp}(1, \mathbf{H}^C)) = 1 + 3 = 4 = \dim_C((\mathfrak{g}_2^C)_0)$ (Theorem 1.6), φ is onto. Therefore $(G_2^C)_0 \cong (U(1, \mathbf{C}^C) \times Sp(1, \mathbf{H}^C))/\mathbf{Z}_2 \cong (C^* \times Sp(1, C))/\mathbf{Z}_2$.

1.2.1. Subgroups of type $C_{1(1)} \oplus C_{1(1)}$ and $\mathbf{R} \oplus C_{1(1)}$ of $G_{2(2)}$

We shall determine the group structures of

$$(G_{2(2)})_{\text{ev}} = (G_2^C)^\gamma \cap (G_2^C)^{\tau\gamma_1}, \quad (G_{2(2)})_0 = (G_2^C)^{\gamma_3} \cap (G_2^C)^{\tau\gamma_1}.$$

LEMMA 1.8. $\varphi : Sp(1, \mathbf{H}^C) \times Sp(1, \mathbf{H}^C) \rightarrow G_2^C$ of Theorem 1.7.(1) satisfies

- (1) $\gamma = \varphi(-1, 1), \quad \gamma_1 = \varphi(e_2, e_2), \quad \gamma_2 = \varphi(e_1, e_1).$
- (2) $\tau\varphi(p, q)\tau = \varphi(\tau p, \tau q), \quad \gamma_1\varphi(p, q)\gamma_1 = \varphi(\gamma_1 p, \gamma_1 q).$

THEOREM 1.9. (1) $(G_{2(2)})_{\text{ev}} \cong (Sp(1, \mathbf{R}) \times Sp(1, \mathbf{R}))/\mathbf{Z}_2 \times \{1, \gamma_2\}, \mathbf{Z}_2 = \{(1, 1), (-1, -1)\}.$

- (2) $(G_{2(2)})_0 \cong (\mathbf{R}^+ \times Sp(1, \mathbf{R})) \times \{1, \gamma_2\}.$

PROOF. (1) For $\alpha \in (G_{2(2)})_{\text{ev}} \subset (G_2^C)^\gamma$, there exist $p, q \in Sp(1, \mathbf{H}^C)$ such that $\alpha = \varphi(p, q)$ (Theorem 1.7.(1)). From $\gamma_1\tau\alpha\tau\gamma_1 = \alpha$, we have $\varphi(\gamma_1\tau p, \gamma_1\tau q) = \varphi(p, q)$ (Lemma 1.8.(2)). Hence

$$\gamma_1\tau p = p, \gamma_1\tau q = q \quad \text{or} \quad \gamma_1\tau p = -p, \gamma_1\tau q = -q.$$

In the former case, $p, q \in Sp(1, \mathbf{H}')$. Hence the group of the former case is $(Sp(1, \mathbf{H}') \times Sp(1, \mathbf{H}'))/\mathbf{Z}_2 \cong (Sp(1, \mathbf{R}) \times Sp(1, \mathbf{R}))/\mathbf{Z}_2$. In the latter case, $p = q = e_1$ satisfy these conditions and $\varphi(e_1, e_1) = \gamma_2$ (Lemma 1.8.(1)). Therefore $(G_{2(2)})_{\text{ev}} \cong (Sp(1, \mathbf{R}) \times Sp(1, \mathbf{R}))/\mathbf{Z}_2 \times \{1, \gamma_2\}.$

(2) For $\alpha \in (G_{2(2)})_0 \subset (G_2^C)^{\gamma_3}$, there exist $a \in U(1, \mathbf{C}^C)$ and $q \in Sp(1, \mathbf{H}^C)$ such that $\alpha = \varphi(a, q)$ (Theorem 1.7.(2)). From $\gamma_1\tau\alpha\tau\gamma_1 = \alpha$, we have $\varphi(\gamma_1\tau a, \gamma_1\tau q) = \varphi(a, q)$ (Lemma 1.8.(2)). Hence

$$\gamma_1\tau a = a, \gamma_1\tau q = q \quad \text{or} \quad \gamma_1\tau a = -a, \gamma_1\tau q = -q.$$

In the former case, $a \in U(1, \mathbf{C}')$ and $q \in Sp(1, \mathbf{H}')$. Hence the group of the former case is $(U(1, \mathbf{C}') \times Sp(1, \mathbf{H}'))/\mathbf{Z}_2 \cong (\mathbf{R}^* \times Sp(1, \mathbf{R}))/\mathbf{Z}_2$ ($\mathbf{Z}_2 = \{(1, 1), (-1, -1)\} \cong \mathbf{R}^+ \times Sp(1, \mathbf{R})$). In the latter case, $a = q = e_1$ satisfy these conditions and $\varphi(e_1, e_1) = \gamma_2$ (Lemma 1.8.(1)). Therefore $(G_{2(2)})_0 \cong (\mathbf{R}^+ \times Sp(1, \mathbf{R})) \times \{1, \gamma_2\}.$

Group F_4

2.1. Lie groups of type F_4 and their Lie algebras

Let

$$\begin{aligned} \mathfrak{J}(3, K) &= \{X \in M(3, K) \mid X^* = X\}, \\ \mathfrak{J}(1, 2, K) &= \{X \in M(3, K) \mid I_1 X^* I_1 = X\}, \end{aligned} \quad K = \mathfrak{C}, \mathfrak{C}', \mathfrak{C}^C \text{ (or } \mathbf{H})$$

be the Jordan algebras with the Jordan multiplication $X \circ Y$, the inner product (X, Y) and the Freudenthal multiplication $X \times Y$,

$$\begin{aligned} X \circ Y &= \frac{1}{2}(XY + YX), \quad (X, Y) = \text{tr}(X \circ Y), \\ X \times Y &= \frac{1}{2}(2X \circ Y - \text{tr}(X)Y - \text{tr}(Y)X + (\text{tr}(X)\text{tr}(Y) - (X, Y))E), \end{aligned}$$

respectively. From now on, we briefly denote $\mathfrak{J} = \mathfrak{J}(3, \mathfrak{C})$, $\mathfrak{J}' = \mathfrak{J}(3, \mathfrak{C}')$, $\mathfrak{J}^C = \mathfrak{J}(3, \mathfrak{C}^C)$. In $\mathfrak{J}$, $\mathfrak{J}'$ or $\mathfrak{J}^C$, we use the following notations.

$$E_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad E_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad E_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$F_1(x) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & x \\ 0 & \bar{x} & 0 \end{pmatrix}, \quad F_2(x) = \begin{pmatrix} 0 & 0 & \bar{x} \\ 0 & 0 & 0 \\ x & 0 & 0 \end{pmatrix}, \quad F_3(x) = \begin{pmatrix} 0 & x & 0 \\ \bar{x} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The tables of the Jordan and the Freudenthal multiplications among them are given as follows.

$$\left\{ \begin{array}{l} E_k \circ E_k = E_k \\ E_k \circ F_k(x) = 0 \\ F_k(x) \circ F_k(y) = (x, y)(E_{k+1} + E_{k+2}), \end{array} \right. \quad \left\{ \begin{array}{l} E_k \circ E_l = 0, \quad k \neq l \\ E_k \circ F_l(x) = \frac{1}{2}F_l(x), \quad k \neq l \\ F_k(x) \circ F_{k+1}(y) = \frac{1}{2}F_{k+2}(\overline{xy}), \end{array} \right.$$

$$\left\{ \begin{array}{l} E_k \times E_k = 0 \\ E_k \times F_k(x) = -\frac{1}{2}F_k(x) \\ F_k(x) \times F_k(y) = -(x, y)E_k, \end{array} \right. \quad \left\{ \begin{array}{l} E_k \times E_{k+1} = \frac{1}{2}E_{k+2} \\ E_k \times F_l(x) = 0, \quad k \neq l \\ F_k(x) \times F_{k+1}(y) = \frac{1}{2}F_{k+2}(\overline{xy}) \end{array} \right.$$

where the indices are considered as mod 3.

The connected linear Lie groups of type F_4 are given by

$$F_4^C = \{\alpha \in \text{Iso}_C(\mathfrak{J}(3, \mathfrak{C}^C)) \mid \alpha(X \circ Y) = \alpha X \circ \alpha Y\},$$

$$F_4 = \{\alpha \in \text{Iso}_R(\mathfrak{J}(3, \mathfrak{C})) \mid \alpha(X \circ Y) = \alpha X \circ \alpha Y\},$$

$$F_{4(4)} = \{\alpha \in \text{Iso}_R(\mathfrak{J}(3, \mathfrak{C}')) \mid \alpha(X \circ Y) = \alpha X \circ \alpha Y\},$$

$$F_{4(-20)} = \{\alpha \in \text{Iso}_R(\mathfrak{J}(1, 2, \mathfrak{C})) \mid \alpha(X \circ Y) = \alpha X \circ \alpha Y\}.$$

F_4^C , F_4 and $F_{4(-20)}$ are simply connected. In the definitions of the groups above, the condition $\alpha(X \circ Y) = \alpha X \circ \alpha Y$ is equivalent to

$$\alpha(X \times Y) = \alpha X \times \alpha Y.$$

Note that $\alpha \in F_4^C$ satisfies

$$(\alpha X, \alpha Y) = (X, Y), \quad \text{tr}(\alpha X) = \text{tr}(X).$$

The group F_4^C contains G_2^C as a subgroup by the following way. For $\alpha \in G_2^C$, we define $\tilde{\alpha} : \mathfrak{J}^C \rightarrow \mathfrak{J}^C$ by

$$\tilde{\alpha} \begin{pmatrix} \xi_1 & x_3 & \overline{x_2} \\ \overline{x_3} & \xi_2 & x_1 \\ x_2 & \overline{x_1} & \xi_3 \end{pmatrix} = \begin{pmatrix} \xi_1 & \alpha x_3 & \overline{\alpha x_2} \\ \overline{\alpha x_3} & \xi_2 & \alpha x_1 \\ \alpha x_2 & \overline{\alpha x_1} & \xi_3 \end{pmatrix},$$

then $\tilde{\alpha} \in F_4^C$ and $G_2^C \cong \{\tilde{\alpha} \mid \alpha \in G_2^C\} \subset F_4^C$. Similarly we have $G_2 \subset F_4$. In particular, $\gamma, \gamma_1, \gamma_2, \gamma_3 \in G_2 \subset F_4 \subset F_4^C$.

We define $\mathbf{R}$ -linear transformations σ, σ' and σ_3 of $\mathfrak{J}$ by

$$\sigma X = \sigma \begin{pmatrix} \xi_1 & x_3 & \overline{x_2} \\ \overline{x_3} & \xi_2 & x_1 \\ x_2 & \overline{x_1} & \xi_3 \end{pmatrix} = \begin{pmatrix} \xi_1 & -x_3 & -\overline{x_2} \\ -\overline{x_3} & \xi_2 & x_1 \\ -x_2 & \overline{x_1} & \xi_3 \end{pmatrix}, \quad \sigma' X = \begin{pmatrix} \xi_1 & x_3 & -\overline{x_2} \\ \overline{x_3} & \xi_2 & -x_1 \\ -x_2 & -\overline{x_1} & \xi_3 \end{pmatrix},$$

$$\sigma_3 X = \begin{pmatrix} \xi_1 & x_3 \omega_1 & \overline{\omega_1 x_2} \\ \overline{x_3 \omega_1} & \xi_2 & \overline{\omega_1 x_1 \omega_1} \\ \omega_1 x_2 & \omega_1 \overline{x_1} \omega_1 & \xi_3 \end{pmatrix}$$

($\omega_1 = -\frac{1}{2} + \frac{\sqrt{3}}{2}e_1$), respectively. Then $\sigma, \sigma', \sigma_3 \in F_4 \subset F_4^C$ and $\sigma^2 = \sigma'^2 = 1$, $\sigma_3^3 = 1$.

Since $\mathfrak{J}(3, \mathfrak{C}) = (\mathfrak{J}^C)_\tau$, $\mathfrak{J}(3, \mathfrak{C}') = (\mathfrak{J}^C)_{\tau\gamma}$, $\mathfrak{J}(1, 2, \mathfrak{C}) \cong (\mathfrak{J}^C)_{\tau\sigma}$, we have

PROPOSITION 2.1. $F_4 = (F_4^C)^\tau$, $F_{4(4)} = (F_4^C)^{\tau\gamma} \cong (F_4^C)^{\tau\gamma_1}$, $F_{4(-20)} = (F_4^C)^{\tau\sigma}$.

The Lie algebra $\mathfrak{f}_4$ of the group F_4 is given by

$$\mathfrak{f}_4 = \{\delta \in \text{Hom}_R(\mathfrak{J}, \mathfrak{J}) \mid \delta(X \circ Y) = \delta X \circ Y + X \circ \delta Y\}.$$

The subalgebra $\mathfrak{d}_4$ of $\mathfrak{f}_4$:

$$\mathfrak{d}_4 = \{\delta \in \mathfrak{f}_4 \mid \delta E_k = 0, k = 1, 2, 3\}$$

is isomorphic to the Lie algebra $\mathfrak{so}(8) = \mathfrak{so}(\mathfrak{C})$ by the correspondence

$$D_1 \in \mathfrak{so}(8) \rightarrow \delta = \delta(D_1, D_2, D_3) \in \mathfrak{d}_4,$$

$$\delta \begin{pmatrix} \xi_1 & x_3 & \overline{x_2} \\ \overline{x_3} & \xi_2 & x_1 \\ x_2 & \overline{x_1} & \xi_3 \end{pmatrix} = \begin{pmatrix} 0 & D_3 x_3 & \overline{D_2 x_2} \\ \overline{D_3 x_3} & 0 & D_1 x_1 \\ D_2 x_2 & \overline{D_1 x_1} & 0 \end{pmatrix}$$

where D_2, D_3 are elements of $\mathfrak{so}(8)$ determined by the principle of triality

$$(D_1 x)y + x(D_2 y) = \overline{D_3(\overline{xy})}, \quad x, y \in \mathfrak{C}.$$

From now on, we identify $D_1 \in \mathfrak{so}(8)$ with $\delta(D_1, D_2, D_3) \in \mathfrak{d}_4 \subset \mathfrak{f}_4$.

Let $(\mathfrak{M}^-)_0 = \{A \in M(3, \mathfrak{C}) \mid A^* = -A, \text{tr}(A) = 0\}$. For $A \in (\mathfrak{M}^-)_0$, we define an $\mathbf{R}$ -linear mapping $\tilde{A}$ of $\mathfrak{J}$ by

$$\tilde{A}X = \frac{1}{2}[A, X] = \frac{1}{2}(AX - XA).$$

Then we can verify that $\tilde{A} \in \mathfrak{f}_4$. In $(\mathfrak{M}^-)_0$, we use the following notations.

$$A_1(a) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & a \\ 0 & -\bar{a} & 0 \end{pmatrix}, \quad A_2(a) = \begin{pmatrix} 0 & 0 & -\bar{a} \\ 0 & 0 & 0 \\ a & 0 & 0 \end{pmatrix}, \quad A_3(a) = \begin{pmatrix} 0 & a & 0 \\ -\bar{a} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The action of $\tilde{A}_k(a) \in \mathfrak{f}_4$ on $\mathfrak{J}$ is given as follows.

$$\begin{cases} \tilde{A}_k(a)E_k = 0 \\ \tilde{A}_k(a)E_{k+1} = -\frac{1}{2}F_k(a) \\ \tilde{A}_k(a)E_{k+2} = \frac{1}{2}F_k(a), \end{cases} \quad \begin{cases} \tilde{A}_k(a)F_k(x) = (a, x)(E_{k+1} - E_{k+2}) \\ \tilde{A}_k(a)F_{k+1}(x) = \frac{1}{2}F_{k+2}(\bar{a}x) \\ \tilde{A}_k(a)F_{k+2}(x) = -\frac{1}{2}F_{k+1}(\bar{x}a). \end{cases}$$

PROPOSITION 2.2 ([1]). *Any element δ of the Lie algebra $\mathfrak{f}_4$ can be uniquely expressed as*

$$\delta = D + \tilde{A}_1(a_1) + \tilde{A}_2(a_2) + \tilde{A}_3(a_3), \quad D \in \mathfrak{so}(8), \quad a_k \in \mathfrak{C}.$$

2.2. Subgroups of type $C_1^C \oplus C_3^C$ and $C \oplus C_3^C$ of F_4^C

In the complexified Lie algebra $\mathfrak{f}_4^C$ of $\mathfrak{f}_4$, let $Z = i(G_{45} - G_{67})$. Then, as is shown in $\mathfrak{g}_2^C$, we have

$$\gamma = \exp \frac{2\pi i}{2} Z, \quad \gamma_3 = \exp \frac{2\pi i}{3} Z.$$

LEMMA 2.3. *In the Lie algebra $\mathfrak{f}_4^C$, we have*

(1) $[D_1, \tilde{A}_1(a)] = \tilde{A}_1(D_1 a)$, $[D_1, \tilde{A}_2(a)] = \tilde{A}_2(D_2 a)$, $[D_1, \tilde{A}_3(a)] = \tilde{A}_3(D_3 a)$
where $D_1 = \delta(D_1, D_2, D_3) \in \mathfrak{so}(8) \subset \mathfrak{f}_4^C$.

(2) $\tau \tilde{A}_k(a) \tau = \tilde{A}_k(\tau a)$, $\gamma_1 \tilde{A}_k(a) \gamma_1 = \tilde{A}_k(\gamma_1 a)$.

THEOREM 2.4. *The 2-graded decomposition of $\mathfrak{f}_{4(4)} = (\mathfrak{f}_4^C)^{\tau\gamma_1}$ (or $\mathfrak{f}_4^C$),*

$$\mathfrak{f}_{4(4)} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2$$

with respect to $\text{ad } Z$, $Z = i(G_{45} - G_{67})$, is given by

$$\mathfrak{g}_0 = \left\{ \begin{array}{ccccc} G_{46} - G_{57}, & i(G_{47} + G_{56}), & \tilde{A}_1(1), & \tilde{A}_2(1), & \tilde{A}_3(1), \\ iG_{01}, & G_{02}, & iG_{03}, & i\tilde{A}_1(e_1), & i\tilde{A}_2(e_1), & i\tilde{A}_3(e_1), \\ iG_{12}, & G_{13}, & iG_{23}, & \tilde{A}_1(e_2), & \tilde{A}_2(e_2), & \tilde{A}_3(e_2), \\ iG_{45}, & iG_{67}, & i\tilde{A}_1(e_3), & i\tilde{A}_2(e_3), & i\tilde{A}_3(e_3) \end{array} \right\} \quad 22$$

$$\mathfrak{g}_{-1} = \left\{ \begin{array}{l} G_{04} + iG_{05}, G_{06} - iG_{07}, iG_{14} - G_{15}, iG_{16} + G_{17}, \\ G_{24} + iG_{25}, G_{26} - iG_{27}, iG_{34} - G_{35}, iG_{36} + G_{37}, \\ \tilde{A}_1(e_4 + ie_5), \tilde{A}_2(e_4 + ie_5), \tilde{A}_3(e_4 + ie_5), \\ \tilde{A}_1(e_6 - ie_7), \tilde{A}_2(e_6 - ie_7), \tilde{A}_3(e_6 - ie_7) \end{array} \right\} 14$$

$$\mathfrak{g}_{-2} = \{(G_{46} + G_{57}) - i(G_{47} - G_{56})\} 1$$

$$\mathfrak{g}_1 = \tau(\mathfrak{g}_{-1})\tau, \quad \mathfrak{g}_2 = \tau(\mathfrak{g}_{-2})\tau.$$

PROOF. Note that for $D_1 = G_{45} - G_{67} \in \mathfrak{so}(8)$ we have $D_2 = D_3 = G_{45} - G_{67}$. Then we can prove this theorem in a way similar to Theorem 1.6, using Lemmas 1.5 and 2.3.

Now, we shall determine the group structures of

$$(F_4^C)_{\text{ev}} = (F_4^C)^\gamma, \quad (F_4^C)_0 = (F_4^C)^{\gamma_3}.$$

To an element $X = \begin{pmatrix} \xi_1 & x_3 & \overline{x_2} \\ \overline{x_3} & \xi_2 & x_1 \\ x_2 & \overline{x_1} & \xi_3 \end{pmatrix} \in \mathfrak{J}(3, \mathfrak{C})$, we associate an element $M + n \in \mathfrak{J}(2, \mathbf{H}) \oplus \mathbf{H}^3$ such that

$$\begin{pmatrix} \xi_1 & m_3 & \overline{m_2} \\ \overline{m_3} & \xi_2 & m_1 \\ m_2 & \overline{m_1} & \xi_3 \end{pmatrix} + (n_1, n_2, n_3), \quad x_k = m_k + n_k e \in \mathbf{H} \oplus \mathbf{H}e = \mathfrak{C}.$$

Then $\mathfrak{J}(3, \mathbf{H}) \oplus \mathbf{H}^3$ has a multiplication

$$(M_1 + n_1) \times (M_2 + n_2) = \left(M_1 \times M_2 - \frac{1}{2}(n_1^* n_2 + n_2^* n_1) \right) - \frac{1}{2}(n_1 M_2 + n_2 M_1)$$

corresponding to the multiplication $X \times Y$ of $\mathfrak{J}(3, \mathfrak{C})$.

THEOREM 2.5. (1) $(F_4^C)_{\text{ev}} \cong (Sp(1, C) \times Sp(3, C))/Z_2$, $Z_2 = \{(1, E), (-1, -E)\}$.

(2) $(F_4^C)_0 \cong (C^* \times Sp(3, C))/Z_2$, $Z_2 = \{(1, E), (-1, -E)\}$.

PROOF. (1) We define $\varphi : Sp(1, \mathbf{H}^C) \times Sp(3, \mathbf{H}^C) \rightarrow (F_4^C)^\gamma = (F_4^C)_{\text{ev}}$ by

$$\varphi(p, A)(M + n) = AMA^* + p n A^*, \quad M + n \in \mathfrak{J}(3, \mathbf{H}^C) \oplus (\mathbf{H}^C)^3 = \mathfrak{J}(3, \mathfrak{C}^C).$$

Then φ is well-defined, is a homomorphism and $\text{Ker } \varphi = Z_2$. Since $(F_4^C)^\gamma$ is connected and $\dim_C(\mathfrak{sp}(1, \mathbf{H}^C) \oplus \mathfrak{sp}(3, \mathbf{H}^C)) = 3 + 21 = 24 = 22 + 2 \times 1 = \dim_C((F_4^C)_{\text{ev}})$ (Theorem 2.4), φ is onto. Therefore $(F_4^C)_{\text{ev}} \cong (Sp(1, \mathbf{H}^C) \times Sp(3, \mathbf{H}^C))/Z_2 \cong (Sp(1, C) \times Sp(3, C))/Z_2$.

(2) The restriction mapping $\varphi : U(1, \mathbf{C}^C) \times Sp(3, \mathbf{H}^C) \rightarrow (F_4^C)^{\gamma_3} = (F_4^C)_0$ of φ above (1) is well-defined (because $\gamma_3 = \varphi(\omega_1, E)$) and $\text{Ker } \varphi = \mathbf{Z}_2$. Since $(F_4^C)^{\gamma_3}$ is connected and $\dim_C(\mathfrak{u}(1, \mathbf{C}^C) \oplus \mathfrak{sp}(3, \mathbf{H}^C)) = 1+21 = 22 = \dim_C((F_4^C)_0)$ (Theorem 2.4), φ is onto. Therefore $(F_4^C)_0 \cong (U(1, \mathbf{C}^C) \times Sp(3, \mathbf{H}^C))/\mathbf{Z}_2 \cong (C^* \times Sp(3, C))/\mathbf{Z}_2$.

2.2.1. Subgroups of type $C_{1(1)} \oplus C_{3(3)}$ and $R \oplus C_{3(3)}$ of $F_{4(4)}$

We shall determine the group structures of

$$(F_{4(4)})_{\text{ev}} = (F_4^C)^\gamma \cap (F_4^C)^{\tau\gamma_1}, \quad (F_{4(4)})_0 = (F_4^C)^{\gamma_3} \cap (F_4^C)^{\tau\gamma_1}.$$

LEMMA 2.6. $\varphi : Sp(1, \mathbf{H}^C) \times Sp(3, \mathbf{H}^C) \rightarrow F_4^C$ of Theorem 2.5.(1) satisfies

$$(1) \quad \gamma = \varphi(-1, E), \quad \gamma_1 = \varphi(e_2, e_2 E), \quad \gamma_2 = \varphi(e_1, e_1 E).$$

$$(2) \quad \tau\varphi(p, A)\tau = \varphi(\tau p, \tau A), \quad \gamma_1\varphi(p, A)\gamma_1 = \varphi(\gamma_1 p, \gamma_1 A).$$

THEOREM 2.7. (1) $(F_{4(4)})_{\text{ev}} \cong (Sp(1, \mathbf{R}) \times Sp(3, \mathbf{R}))/\mathbf{Z}_2 \times \{1, \gamma_2\}$, $\mathbf{Z}_2 = \{(1, E), (-1, -E)\}$.

$$(2) \quad (F_{4(4)})_0 \cong (\mathbf{R}^+ \times Sp(3, \mathbf{R})) \times \{1, \gamma_2\}.$$

PROOF. We can prove this theorem in a way similar to Theorem 1.9, using Lemma 2.6.

2.3. Subgroups of type B_4^C and $C \oplus B_3^C$ of F_4^C

In the Lie algebra $\mathfrak{f}_4^C$, let $Z = -2iG_{01}$. Then we can easily verify that

$$\sigma = \exp \frac{2\pi i}{2} Z, \quad \sigma_3 = \exp \frac{2\pi i}{3} Z.$$

THEOREM 2.8. The 2-graded decomposition of $\mathfrak{f}_{4(4)} = (\mathfrak{f}_4^C)^{\tau\gamma_1}$ (or $\mathfrak{f}_4^C$),

$$\mathfrak{f}_{4(4)} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2$$

with respect to $\text{ad } Z$, $Z = -2iG_{01}$, is given by

$$\mathfrak{g}_0 = \left\{ \begin{array}{cccccccc} iG_{01}, & iG_{23}, & G_{24}, & iG_{25}, & G_{26}, & iG_{27}, & \tilde{A}_1(e_2), & i\tilde{A}_1(e_3), \\ & & iG_{34}, & G_{35}, & iG_{36}, & G_{37}, & \tilde{A}_1(e_4), & i\tilde{A}_1(e_5), \\ & & & iG_{45}, & G_{46}, & iG_{47}, & \tilde{A}_1(e_6), & i\tilde{A}_2(e_7) \\ & & & & iG_{56}, & G_{57}, & & \\ & & & & & iG_{67}, & & \end{array} \right\} 22$$

$$\mathfrak{g}_{-1} = \left\{ \begin{array}{l} \tilde{A}_2(1 + ie_1), \tilde{A}_2(e_2 + ie_3), \tilde{A}_2(e_4 + ie_5), \tilde{A}_2(e_6 + ie_7), \\ \tilde{A}_3(1 + ie_1), \tilde{A}_3(e_2 - ie_3), \tilde{A}_3(e_4 - ie_5), \tilde{A}_3(e_6 - ie_7) \end{array} \right\} 8$$

$$\mathfrak{g}_{-2} = \left\{ \begin{array}{l} G_{02} - iG_{12}, iG_{03} + G_{13}, G_{04} - iG_{14}, \tilde{A}_1(1 - ie_1) \\ iG_{05} + G_{15}, G_{06} - iG_{16}, iG_{07} + G_{17}, \end{array} \right\} 7$$

$$\mathfrak{g}_1 = \tau(\mathfrak{g}_{-1})\tau, \quad \mathfrak{g}_2 = \tau(\mathfrak{g}_{-2})\tau.$$

PROOF. Note that for $D_1 = -2G_{01} \in \mathfrak{so}(8)$ we have

$$D_2 = G_{01} + G_{23} + G_{45} + G_{67}, \quad D_3 = G_{01} - G_{23} - G_{45} - G_{67}.$$

Then we can prove this theorem in a way simialr to Theorem 2.4, using Lemmas 1.5 and 2.3.

Now, we shall determine the group structures of

$$(F_4^C)_{\text{ev}} = (F_4^C)^\sigma, \quad (F_4^C)_0 = (F_4^C)^{\sigma^3}.$$

PROPOSITION 2.9. (1) $(F_4^C)_{E_1, F_1(1), F_1(e_1)} \cong Spin(7, C)$.

(2) $(F_4^C)_{E_1, F_1(1)} \cong Spin(8, C)$.

(3) $(F_4^C)_{E_1} \cong Spin(9, C)$.

PROOF. (1) The group $(F_4^C)_{E_1, F_1(1), F_1(e_1)}$ acts on the C -vector space

$$(V^C)^7 = \{X \in \mathfrak{J}^C \mid E_1 \circ X = 0, \text{tr}(X) = 0, (F_1(1), X) = (F_1(e_1), X) = 0\}$$

$$= \left\{ X = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \xi & x \\ 0 & \bar{x} & -\xi \end{pmatrix} \mid \xi \in C, x \in \mathfrak{C}^C, (1, x) = (e_1, x) = 0 \right\}$$

with the norm $(X, X)/2 = \xi^2 + x\bar{x}$. The connectedness of $(F_4^C)_{E_1, F_1(1), F_1(e_1)}$ follows from

$$(F_4^C)_{E_1, F_1(1), F_1(e_1)} / (F_4^C)_{E_1, E_2 - E_3, F_1(1), F_1(e_1)} \\ \simeq (S^C)^6 = \{X \in (V^C)^7 \mid (X, X) = 2\},$$

$$(F_4^C)_{E_1, E_2 - E_3, F_1(1), F_1(e_1)} = (F_4^C)_{E_1, E_2, E_3, F_1(1), F_1(e_1)} \cong Spin(6, C) \cong SU(4, C^C).$$

Hence we can define a homomorphism $\pi : (F_4^C)_{E_1, F_1(1), F_1(e_1)} \rightarrow SO(7, C) = SO((V^C)^7)$ by $\pi(\alpha) = \alpha|(V^C)^7$. $\text{Ker } \pi = \{1, \sigma\} = \mathbf{Z}_2$. Since

$$\dim_C((F_4^C)_{E_1, F_1(1), F_1(e_1)}) = \dim_C(Spin(6, C)) + \dim_C((S^C)^6) \\ = 15 + 6 = 21 = \dim_C(SO(7, C)),$$

π is onto. Therefore $(F_4^C)_{E_1, F_1(1), F_1(e_1)}$ is $Spin(7, C)$ as a double covering group of $SO(7, C)$.

(2) The group $(F_4^C)_{E_1, F_1(1)}$ acts on the C -vector space

$$\begin{aligned}(V^C)^8 &= \{X \in \mathfrak{J}^C \mid E_1 \circ X = 0, \operatorname{tr}(X) = 0, (F_1(1), X) = 0\} \\ &= \{X = \xi E_2 - \xi E_3 + F_1(x) \mid \xi \in C, x \in \mathfrak{C}^C, (1, x) = 0\}\end{aligned}$$

with the norm $(X, X)/2 = \xi^2 + x\bar{x}$. The connectedness of $(F_4^C)_{E_1, F_1(1)}$ follows from

$$(F_4^C)_{E_1, F_1(1)} / (F_4^C)_{E_1, F_1(1), F_1(e_1)} \simeq (S^C)^7 = \{X \in (V^C)^8 \mid (X, X) = 2\}.$$

Therefore, in a similar way as in (1), $(F_4^C)_{E_1, F_1(1)}$ is $Spin(8, C)$ as a covering group of $SO(8, C) = SO((V^C)^8)$.

(3) The group $(F_4^C)_{E_1}$ acts on the C -vector space

$$\begin{aligned}(V^C)^9 &= \{X \in \mathfrak{J}^C \mid E_1 \circ X = 0, \operatorname{tr}(X) = 0\} \\ &= \{X = \xi E_2 - \xi E_3 + F_1(x) \mid \xi \in C, x \in \mathfrak{C}^C\}\end{aligned}$$

with the norm $(X, X)/2 = \xi^2 + x\bar{x}$. The connectedness of $(F_4^C)_{E_1}$ follows from

$$(F_4^C)_{E_1} / (F_4^C)_{E_1, F_1(1)} \simeq (S^C)^8 = \{X \in (V^C)^9 \mid (X, X) = 2\}.$$

Therefore, in a similar way as in (1), $(F_4^C)_{E_1}$ is $Spin(9, C)$ as a covering group of $SO(9, C) = SO((V^C)^9)$.

We define a mapping $D : U(1, C^C) \rightarrow F_4^C$ by

$$D(a) \begin{pmatrix} \xi_1 & x_3 & \bar{x}_2 \\ \bar{x}_3 & \xi_2 & x_1 \\ x_2 & \bar{x}_1 & \xi_3 \end{pmatrix} = \begin{pmatrix} \xi_1 & x_3 a & \bar{a} \bar{x}_2 \\ \bar{x}_3 a & \xi_2 & \bar{a} x_1 \bar{a} \\ a x_2 & a \bar{x}_1 a & \xi_3 \end{pmatrix}.$$

Then $\sigma = D(-1)$ and $\sigma_3 = D(\omega_1)$.

THEOREM 2.10. (1) $(F_4^C)_{\text{ev}} \cong Spin(9, C)$.

(2) $(F_4^C)_0 \cong (C^* \times Spin(7, C))/Z_2$, $Z_2 = \{(1, 1), (-1, \sigma)\}$.

PROOF. (1) Since $(f_4^C)_{\text{ev}} = (f_4^C)_{E_1} = \{\delta \in f_4^C \mid \delta E_1 = 0\}$ (Theorem 2.8) and $(F_4^C)^\sigma$, $(F_4^C)_{E_1}$ are connected (Proposition 2.9.(3)), we have $(F_4^C)^\sigma = (F_4^C)_{\text{ev}} = (F_4^C)_{E_1} \cong Spin(9, C)$ (Proposition 2.9.(3)). (Another proof is found in [5] Theorem 2.4.3).

(2) Let $Spin(7, C) = (F_4^C)_{E_1, F_1(1), F_1(e_1)}$ (Proposition 2.9.(1)). We define $\varphi : U(1, C^C) \times Spin(7, C) \rightarrow (F_4^C)^{\sigma_3}$ by

$$\varphi(a, \beta) = D(a)\beta.$$

Since iG_{01} is a center of $(\mathfrak{f}_4^C)_0$ (Theorem 2.8), $D(a) \in U(1, \mathbf{C}^C)$ and $\beta \in Spin(7, C)$ commute. Hence φ is well-defined and is a homomorphism. $\text{Ker } \varphi = \mathbf{Z}_2$. Since $(F_4^C)^{\sigma_3}$ is connected and $\dim_C(\mathfrak{u}(1, \mathbf{C}^C) \oplus \mathfrak{spin}(7, C)) = 1 + 21 = 22 = \dim_C((\mathfrak{f}_4^C)_0)$ (Theorem 2.8), φ is onto. Therefore $(F_4^C)_0 \cong (U(1, \mathbf{C}^C) \times Spin(7, C))/\mathbf{Z}_2 \cong (C^* \times Spin(7, C))/\mathbf{Z}_2$.

2.3.1. Subgroups of type $B_{4(4)}$ and $R \oplus B_{3(3)}$ of $F_{4(4)}$

We shall determine the group structures of

$$(F_{4(4)})_{\text{ev}} = (F_4^C)^\sigma \cap (F_4^C)^{\tau\gamma_1}, \quad (F_{4(4)})_0 = (F_4^C)^{\sigma_3} \cap (F_4^C)^{\tau\gamma_1}.$$

LEMMA 2.11. *In the group F_4^C , we have*

$$\tau D(a)\tau = D(\tau a), \quad \gamma_1 D(a)\gamma_1 = D(\bar{a}).$$

THEOREM 2.12. (1) $(F_{4(4)})_{\text{ev}} \cong spin(4, 5)$.

(2) $(F_{4(4)})_0 \cong (\mathbf{R}^* \times spin(3, 4)) \times \{1, \sigma'\}$.

PROOF. (1) The group $(F_{4(4)})_{\text{ev}} = (F_4^C)^\sigma \cap (F_4^C)^{\tau\gamma_1} = (F_4^C)_{E_1} \cap (F_4^C)^{\tau\gamma_1}$ (Theorem 2.10.(1)) = $((F_4^C)^{\tau\gamma_1})_{E_1}$ acts on the $\mathbf{R}$ -vector space

$$\begin{aligned} V^{4,5} &= \{X \in (\mathfrak{J}^C)_{\tau\gamma_1} \mid E_1 \circ X = 0, \text{tr}(X) = 0\} \\ &= \{X = \xi E_2 - \xi E_3 + F_1(x) \mid \xi \in \mathbf{R}, x \in (\mathfrak{C}^C)_{\tau\gamma_1} = \mathfrak{C}'\} \end{aligned}$$

with the norm $(X, X)/2 = \xi^2 + x\bar{x}$. Since $(F_{4(4)})_{\text{ev}} = ((F_4^C)^\sigma)^{\tau\gamma_1}$ (because σ and $\tau\gamma_1$ commute) = $(Spin(9, C))^{\tau\gamma_1}$ (Theorem 2.10.(1)), $(F_{4(4)})_{\text{ev}}$ is connected. Hence we can define a homomorphism $\pi : (F_{4(4)})_{\text{ev}} \rightarrow O(4, 5)^0 = O(V^{4,5})^0$ by $\pi(\alpha) = \alpha|V^{4,5}$. $\text{Ker } \pi = \{1, \sigma\} = \mathbf{Z}_2$. Since $\dim((\mathfrak{f}_{4(4)})_{\text{ev}}) = 22 + 2 \times 7$ (Theorem 2.8) = $36 = \dim(\mathfrak{so}(4, 5))$, π is onto. Therefore $(F_{4(4)})_{\text{ev}}$ is $spin(4, 5)$ (not simply connected) as a double covering group of $O(4, 5)^0$.

(2) For $\alpha \in (F_{4(4)})_0 \subset (F_4^C)^{\sigma_3}$, there exist $a \in U(1, \mathbf{C}^C)$ and $\beta \in Spin(7, C)$ such that $\alpha = \varphi(a, \beta) = D(a)\beta$ (Theorem 2.10.(2)). From $\gamma_1\tau\alpha\tau\gamma_1 = \alpha$, we have

$$\begin{cases} \gamma_1\tau D(a)\tau\gamma_1 = D(a) \\ \gamma_1\tau\beta\tau\gamma_1 = \beta \end{cases} \quad \text{or} \quad \begin{cases} \gamma_1\tau D(a)\tau\gamma_1 = D(-a) \\ \gamma_1\tau\beta\tau\gamma_1 = \sigma\beta. \end{cases}$$

In the former case, from $D(\tau\bar{a}) = D(a)$ (Theorem 2.11), we have $\tau\bar{a} = a$, hence $a \in U(1, \mathbf{C}') \cong \mathbf{R}^*$. The group $(Spin(7, C))^{\tau\gamma_1} = ((F_4^C)_{E_1, F_1(1), F_1(e_1)})^{\tau\gamma_1}$ (Proposition 2.9.(1)) = $((F_4^C)^{\tau\gamma_1})_{E_1, F_1(1), F_1(e_1)}$ acts on the $\mathbf{R}$ -vector space

$$\begin{aligned} V^{3,4} &= \{X \in (\mathfrak{J}^C)_{\tau\gamma_1} \mid E_1 \circ X = 0, \text{tr}(X) = 0, (F_1(1), X) = (F_1(e_1), X) = 0\} \\ &= \{X = \xi E_2 - \xi E_3 + F_1(x) \mid \xi \in \mathbf{R}, x \in (\mathfrak{C}^C)_{\tau\gamma_1} = \mathfrak{C}', (1, x) = (e_1, x) = 0\} \end{aligned}$$

with the norm $(X, X)/2 = \xi^2 + x\bar{x}$. Since $(Spin(7, C))^{\tau\gamma_1}$ is connected, in a similar way as in (1), $(Spin(7, C))^{\tau\gamma_1}$ is $spin(3, 4)$ (not simply connected) as a double covering group of $O(3, 4)^0 = O(V^{3,4})^0$. Hence the group of the former case is $(\mathbf{R}^* \times spin(3, 4))/\mathbf{Z}_2$ ($\mathbf{Z}_2 = \{(1, 1), (-1, \sigma)\} \cong \mathbf{R}^+ \times spin(3, 4)$). In the latter case, $a = e_1$ and $\beta = \sigma'D(-e_1)$ satisfy these conditions and $\varphi(e_1, \sigma'D(-e_1)) = \sigma'$. Therefore $(F_{4(4)})_0 \cong (\mathbf{R}^* \times spin(3, 4)) \times \{1, \sigma'\}$.

2.3.2. Subgroups of type $B_{4(-20)}$ and $R \oplus B_{3(-21)}$ of $F_{4(-20)}$

We define $\delta \in F_4$ by $\delta = \exp \frac{\pi}{2} \tilde{A}_1(e_1)$. This δ is an $\mathbf{R}$ -linear transformation of $\mathfrak{J}$ satisfying

$$\delta : \begin{cases} E_1 \rightarrow E_1 \\ E_2 + E_3 \rightarrow E_2 + E_3 \\ F_1(e_k) \rightarrow F_1(e_k), \quad k \neq 1, \end{cases} \quad \begin{cases} E_2 - E_3 \rightarrow -F_1(e_1) \\ F_1(e_1) \rightarrow E_2 - E_3 \\ F_2(x) \rightarrow \frac{1}{\sqrt{2}}(F_2(x) + F_3(\overline{e_1 x})) \\ F_3(x) \rightarrow \frac{1}{\sqrt{2}}(-F_2(\overline{x e_1}) + F_3(x)). \end{cases}$$

And we define an $\mathbf{R}$ -linear transformation σ_1 of $\mathfrak{J}$ by $\sigma_1 = \delta^{-1} \sigma' \delta$, that is,

$$\sigma_1 \begin{pmatrix} \xi_1 & x_3 & \overline{x_2} \\ \overline{x_3} & \xi_2 & x_1 \\ x_2 & \overline{x_1} & \xi_3 \end{pmatrix} = \begin{pmatrix} \xi_1 & \overline{e_1 x_2} & x_3 e_1 \\ e_1 x_2 & \xi_3 & \overline{e_1 x_1 e_1} \\ \overline{x_3 e_1} & e_1 x_1 e_1 & \xi_2 \end{pmatrix}.$$

Obviously $\sigma_1 \in F_4 \subset F_4^C$ and $\sigma_1^2 = 1$. Since σ, σ' and σ_1 are conjugate with each other in F_4 ([5] Proposition 2.2.3), we have

PROPOSITION 2.13. $F_{4(-20)} = (F_4^C)^{\tau\sigma} \cong (F_4^C)^{\tau\sigma'} \cong (F_4^C)^{\tau\sigma_1}$.

LEMMA 2.14. In the Lie algebra $\mathfrak{f}_4^C$, we have

- (1) $\sigma_1 G_{kl} \sigma_1 = -G_{kl}$, $k = 1$ or $l = 1$, $\sigma_1 G_{kl} \sigma_1 = G_{kl}$, otherwise,
- (2) $\sigma_1 \tilde{A}_1(a) \sigma_1 = -\tilde{A}_1(\overline{e_1 a e_1})$, $\sigma_1 \tilde{A}_2(a) \sigma_1 = -\tilde{A}_3(\overline{e_1 a})$,
 $\sigma_1 \tilde{A}_3(a) \sigma_1 = -\tilde{A}_2(\overline{a e_1})$.

THEOREM 2.15. The 2-graded decomposition of $\mathfrak{f}_{4(-20)} = (\mathfrak{f}_4^C)^{\tau\sigma_1}$ (or $\mathfrak{f}_4^C$),

$$\mathfrak{f}_{4(-20)} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2$$

with respect to $\text{ad } Z$, $Z = -2iG_{01}$, is given by

$$\mathfrak{g}_0 = \left\{ \begin{array}{cccccccc} iG_{01}, & G_{23}, & G_{24}, & G_{25}, & G_{26}, & G_{27}, & \tilde{A}_1(e_2), & \tilde{A}_1(e_3), \\ & & G_{34}, & G_{35}, & G_{36}, & G_{37}, & \tilde{A}_1(e_4), & \tilde{A}_1(e_5), \\ & & & G_{45}, & G_{46}, & G_{47}, & \tilde{A}_1(e_6), & \tilde{A}_2(e_7) \\ & & & & G_{56}, & G_{57}, & & \\ & & & & & G_{67}, & & \end{array} \right\} \quad 22$$

$$\mathfrak{g}_{-1} = \left\{ \begin{array}{ll} \tilde{A}_2(1 + ie_1) - i\tilde{A}_3(1 + ie_1), & i\tilde{A}_2(1 + ie_1) - \tilde{A}_3(1 + ie_1), \\ \tilde{A}_2(e_2 + ie_3) + i\tilde{A}_3(e_2 - ie_3), & i\tilde{A}_2(e_2 + ie_3) + \tilde{A}_3(e_2 - ie_3), \\ \tilde{A}_2(e_4 + ie_5) + i\tilde{A}_3(e_4 - ie_5), & i\tilde{A}_2(e_4 + ie_5) + \tilde{A}_3(e_4 - ie_5), \\ \tilde{A}_2(e_6 + ie_7) + i\tilde{A}_3(e_6 - ie_7), & i\tilde{A}_2(e_6 + ie_7) + \tilde{A}_3(e_6 - ie_7) \end{array} \right\} 8$$

$$\mathfrak{g}_{-2} = \left\{ \begin{array}{l} G_{02} - iG_{12}, \quad G_{03} - iG_{13}, \quad G_{04} - iG_{14}, \quad \tilde{A}_1(1 - ie_1) \\ G_{05} - iG_{15}, \quad G_{06} - iG_{16}, \quad G_{07} - iG_{17}, \end{array} \right\} 7$$

$$\mathfrak{g}_1 = \tau(\mathfrak{g}_{-1})\tau, \quad \mathfrak{g}_2 = \tau(\mathfrak{g}_{-2})\tau.$$

PROOF. We can prove this theorem in a way similar to Theorem 2.8, using Lemma 2.14.

We shall determine the group structures of

$$(F_{4(-20)})_{\text{ev}} = (F_4^C)^\sigma \cap (F_4^C)^{\tau\sigma_1}, \quad (F_{4(-20)})_0 = (F_4^C)^{\sigma_3} \cap (F_4^C)^{\tau\sigma_1}.$$

LEMMA 2.16. *In the group F_4^C , we have*

$$\sigma_1 D(a) \sigma_1 = D(\bar{a}).$$

THEOREM 2.17. (1) $(F_{4(-20)})_{\text{ev}} \cong Spin(8, 1)$.

(2) $(F_{4(-20)})_0 \cong (\mathbf{R}^+ \times Spin(7)) \times \{1, \sigma'\}$.

PROOF. (1) The group $(F_{4(-20)})_{\text{ev}} = (F_4^C)^\sigma \cap (F_4^C)^{\tau\sigma_1} = (F_4^C)_{E_1} \cap (F_4^C)^{\tau\sigma_1}$ (Theorem 2.10.(1)) $= ((F_4^C)^{\tau\sigma_1})_{E_1}$ acts on the $\mathbf{R}$ -vector space

$$V^{8,1} = \{X \in (\mathfrak{J}^C)_{\tau\sigma_1} \mid E_1 \circ X = 0, \text{tr}(X) = 0\}$$

$$= \left\{ X = i\xi E_2 - i\xi E_3 + F_1(x) \mid \begin{array}{l} \xi \in \mathbf{R}, x = x_1 e_1 + iy, \\ x_1 \in \mathbf{R}, y \in \mathfrak{C}, (e_1, y) = 0 \end{array} \right\}$$

with the norm $(X, X)/2 = -\xi^2 + x\bar{x} = -\xi^2 + x_1^2 - y\bar{y}$. Since $(F_{4(-20)})_{\text{ev}} = (F_4^C)^\sigma \cap (F_4^C)^{\tau\sigma_1} = ((F_4^C)^\sigma)^{\tau\sigma_1}$ (because σ and $\tau\sigma_1$ commute) $= (Spin(9, C))^{\tau\sigma_1}$ (Theorem 2.10.(1)), $(F_{4(-20)})_{\text{ev}}$ is connected. Hence, in a similar way as in Theorem 2.12.(1), $(F_{4(-20)})_{\text{ev}}$ is $Spin(8, 1)$ as a double covering group of $O(8, 1)^0 = O(V^{8,1})^0$.

(2) For $\alpha \in (F_{4(-20)})_0 \subset (F_4^C)^{\sigma_3}$, there exist $a \in U(1, \mathbf{C}^C)$ and $\beta \in Spin(7, C)$ such that $\alpha = \varphi(a, \beta) = D(a)\beta$ (Theorem 2.10.(2)). From $\sigma_1 \tau \alpha \tau \sigma_1 = \alpha$, we have

$$\left\{ \begin{array}{l} \sigma_1 \tau D(a) \tau \sigma_1 = D(a) \\ \sigma_1 \tau \beta \tau \sigma_1 = \beta \end{array} \right\} \quad \text{or} \quad \left\{ \begin{array}{l} \sigma_1 \tau D(a) \tau \sigma_1 = D(-a) \\ \sigma_1 \tau \beta \tau \sigma_1 = \sigma \beta. \end{array} \right.$$

In the former case, from $D(\tau\bar{a}) = D(a)$ (Lemma 2.16), we have $\tau\bar{a} = a$, hence $a \in U(1, \mathbf{C}') \cong \mathbf{R}^*$. The group $(Spin(7, C))^{\tau\sigma_1} = ((F_4^C)^{\tau\sigma_1})_{E_1, F_1(1), F_1(e_1)}$ (Proposition

2.9.(1)) acts on the $\mathbf{R}$ -vector space

$$\begin{aligned} V^7 &= \{X \in (\mathfrak{J}^C)_{\tau\sigma_1} \mid E_1 \circ X = 0, \operatorname{tr}(X) = 0, (F_1(1), X) = (F_1(e_1), X) = 0\} \\ &= \{X = i\xi E_2 - i\xi E_3 + iF_1(y) \mid \xi \in \mathbf{R}, y \in \mathfrak{C}, (1, y) = (e_1, y) = 0\} \end{aligned}$$

with the norm $-(X, X)/2 = \xi^2 + y\bar{y}$. Hence, in a way as in Theorem 2.12.(2), $(Spin(7, C))^{\tau\sigma_1}$ is $Spin(7)$ as a double covering group of $SO(7) = SO(V^7)$. Hence the group of the former case is $(\mathbf{R}^* \times Spin(7))/\mathbf{Z}_2$ ($\mathbf{Z}_2 = \{(1, 1), (-1, \sigma)\} \cong \mathbf{R}^+ \times Spin(7)$). In the latter case, $a = e_1$ and $\beta = \sigma'D(-e_1)$ satisfy these conditions and $\varphi(e_1, \sigma'D(-e_1)) = \sigma'$. Therefore $(F_{4(-20)})_0 \cong (\mathbf{R}^* \times Spin(7)) \times \{1, \sigma'\}$.

Group E_6

3.1. Lie groups of type E_6 and their Lie algebras

In $\mathfrak{J}$, $\mathfrak{J}'$ or $\mathfrak{J}^C$, we define a trilinear form (X, Y, Z) and the determinant $\det X$ by

$$(X, Y, Z) = (X, Y \times Z), \quad \det X = \frac{1}{3}(X, X, X)$$

and in $\mathfrak{J}^C$, hermitian inner products $\langle X, Y \rangle$, $\langle X, Y \rangle_\gamma$ and $\langle X, Y \rangle_\sigma$ by

$$\langle X, Y \rangle = (\tau X, Y), \quad \langle X, Y \rangle_\gamma = (\gamma\tau X, Y), \quad \langle X, Y \rangle_\sigma = (\sigma\tau X, Y),$$

respectively.

The connected universal linear Lie groups of type E_6 are given by

$$\begin{aligned} E_6^C &= \{\alpha \in \operatorname{Iso}_C(\mathfrak{J}^C) \mid \det \alpha X = \det X\}, \\ E_6 &= \{\alpha \in \operatorname{Iso}_C(\mathfrak{J}^C) \mid \det \alpha X = \det X, \langle \alpha X, \alpha Y \rangle = \langle X, Y \rangle\}, \\ E_{6(6)} &= \{\alpha \in \operatorname{Iso}_R(\mathfrak{J}') \mid \det \alpha X = \det X\}, \\ E_{6(2)} &= \{\alpha \in \operatorname{Iso}_C(\mathfrak{J}^C) \mid \det \alpha X = \det X, \langle \alpha X, \alpha Y \rangle_\gamma = \langle X, Y \rangle_\gamma\}, \\ E_{6(-14)} &= \{\alpha \in \operatorname{Iso}_C(\mathfrak{J}^C) \mid \det \alpha X = \det X, \langle \alpha X, \alpha Y \rangle_\sigma = \langle X, Y \rangle_\sigma\}, \\ E_{6(-26)} &= \{\alpha \in \operatorname{Iso}_R(\mathfrak{J}) \mid \det \alpha X = \det X\}. \end{aligned}$$

E_6^C , E_6 and $E_{6(-26)}$ are simply connected. For $\alpha \in \operatorname{Iso}_C(\mathfrak{J}^C)$, ${}^t\alpha$ denotes the transpose of α : $({}^t\alpha X, Y) = (X, \alpha Y)$. Then, in the definitions of the groups above, the condition $\det \alpha X = \det X$ is equivalent to

$$(\alpha X, \alpha Y, \alpha Z) = (X, Y, Z) \quad \text{or} \quad {}^t\alpha^{-1}(X \times Y) = \alpha X \times \alpha Y.$$

Since for $\alpha \in E_6^C$ we have ${}^t\alpha^{-1} \in E_6^C$ ([5] Lemma 3.2.1), we define an automorphism λ of E_6^C by

$$\lambda(\alpha) = {}^t\alpha^{-1}, \quad \alpha \in E_6^C.$$

The group E_6^C contains F_4^C as a subgroup by

$$\begin{aligned} F_4^C &= (E_6^C)^\lambda = \{\alpha \in E_6^C \mid (\alpha X, \alpha Y) = (X, Y)\} \\ &= \{\alpha \in E_6^C \mid \alpha E = E\}. \end{aligned}$$

Similarly we have $F_4 \subset E_6$. In particular, $\gamma, \gamma_1, \gamma_2, \gamma_3, \sigma, \sigma', \sigma_1, \sigma_3 \in F_4 \subset E_6 \subset E_6^C$.

From the definitions of the groups above, we have

PROPOSITION 3.1. $E_6 = (E_6^C)^{\lambda\tau}$, $E_{6(6)} = (E_6^C)^{\tau\gamma} \cong (E_6^C)^{\tau\gamma_1}$, $E_{6(2)} = (E_6^C)^{\lambda\tau\gamma} \cong (E_6^C)^{\lambda\tau\gamma_1}$, $E_{6(-14)} = (E_6^C)^{\lambda\tau\sigma} \cong (E_6^C)^{\lambda\tau\sigma_1}$, $E_{6(-26)} = (E_6^C)^\tau$. $((E_6^C)^{\lambda\tau}$ is defined by $(E_6^C)^{\lambda\tau} = \{\alpha \in E_6^C \mid \tau^t \alpha^{-1} \tau = \alpha\}$ etc.).

The Lie algebra $\mathfrak{e}_6^C$ of the group E_6^C is given by

$$\mathfrak{e}_6^C = \{\phi \in \text{Hom}_C(\mathfrak{J}^C, \mathfrak{J}^C) \mid (\phi X, X, X) = 0\}.$$

Let $\mathfrak{J}_0^C = \{T \in \mathfrak{J}^C \mid \text{tr}(T) = 0\}$. For $T \in \mathfrak{J}_0^C$, we define a C -linear mapping $\tilde{T}$ of $\mathfrak{J}^C$ by

$$\tilde{T}X = T \circ X = \frac{1}{2}(TX + XT).$$

Then we can verified that $\tilde{T} \in \mathfrak{e}_6^C$.

PROPOSITION 3.2 ([1]). Any element ϕ of the Lie algebra $\mathfrak{e}_6^C$ can be uniquely expressed as

$$\phi = \delta + \tilde{T}, \quad \delta \in \mathfrak{f}_4^C, \quad T \in \mathfrak{J}_0^C.$$

3.2. Subgroups of type $C_1^C \oplus A_5^C$ and $C \oplus A_5^C$ of E_6^C

In the Lie algebra $\mathfrak{e}_6^C$, let $Z = i(G_{45} - G_{67})$. Then, as is shown in $\mathfrak{g}_2^C$ or $\mathfrak{f}_4^C$, we have

$$\gamma = \exp \frac{2\pi i}{2} Z, \quad \gamma_3 = \exp \frac{2\pi i}{3} Z.$$

LEMMA 3.3. In the Lie algebra $\mathfrak{e}_6^C$, we have

$$(1) \quad [D_1, \tilde{F}_1(a)] = \tilde{F}_1(D_1 a), \quad [D_1, \tilde{F}_2(a)] = \tilde{F}_2(D_2 a), \quad [D_1, \tilde{F}_3(a)] = \tilde{F}_3(D_3 a), \\ [D_1, (E_k - E_{k+1})^\sim] = 0$$

where $D_1 = \delta(D_1, D_2, D_3) \in \mathfrak{so}(8) \subset \mathfrak{f}_4^C \subset \mathfrak{e}_6^C$.

$$(2) \quad \tau \tilde{F}_k(a) \tau = \tilde{F}_k(\tau a), \quad \tau(E_k - E_{k+1})^\sim \tau = (E_k - E_{k+1})^\sim, \\ \gamma_1 \tilde{F}_k(a) \gamma_1 = \tilde{F}_k(\gamma_1 a), \quad \gamma_1(E_k - E_{k+1})^\sim \gamma_1 = (E_k - E_{k+1})^\sim.$$

THEOREM 3.4. The 2-graded decomposition of $\mathfrak{e}_{6(6)} = (\mathfrak{e}_6^C)^{\tau\gamma_1}$ (or $\mathfrak{e}_6^C$),

$$\mathfrak{e}_{6(6)} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2$$

with respect to $\text{ad } Z$, $Z = i(G_{45} - G_{67})$, is given by

$$\begin{aligned} \mathfrak{g}_0 &= \left\{ \begin{array}{cccccc} iG_{45}, iG_{67}, & \tilde{A}_1(1), & i\tilde{A}_1(e_1), & \tilde{A}_1(e_2), & i\tilde{A}_1(e_3), \\ G_{46} - G_{57}, i(G_{47} + G_{56}), & \tilde{A}_2(1), & i\tilde{A}_2(e_1), & \tilde{A}_2(e_2), & i\tilde{A}_2(e_3), \\ iG_{01}, G_{02}, iG_{03}, & \tilde{A}_3(1), & i\tilde{A}_3(e_1), & \tilde{A}_3(e_2), & i\tilde{A}_3(e_3), \\ iG_{12}, G_{13}, & \tilde{F}_1(1), & i\tilde{F}_1(e_1), & \tilde{F}_1(e_2), & i\tilde{F}_1(e_3), \\ iG_{23}, & \tilde{F}_2(1), & i\tilde{F}_2(e_1), & \tilde{F}_2(e_2), & i\tilde{F}_2(e_3), \\ (E_1 - E_2)^\sim, (E_2 - E_3)^\sim, & \tilde{F}_3(1), & i\tilde{F}_3(e_1), & \tilde{F}_3(e_2), & i\tilde{F}_3(e_3) \end{array} \right\} 36 \\ \mathfrak{g}_{-1} &= \left\{ \begin{array}{cccccc} G_{04} + iG_{05}, G_{06} - iG_{07}, & \tilde{A}_1(e_4 + ie_5), & \tilde{A}_2(e_4 + ie_5), & \tilde{A}_3(e_4 + ie_5), \\ iG_{14} - G_{15}, iG_{16} + iG_{17}, & \tilde{A}_1(e_6 - ie_7), & \tilde{A}_2(e_6 - ie_7), & \tilde{A}_3(e_6 - ie_7), \\ G_{24} + iG_{25}, G_{26} - iG_{27}, & \tilde{F}_1(e_4 + ie_5), & \tilde{F}_2(e_4 + ie_5), & \tilde{F}_3(e_4 + ie_5), \\ iG_{34} - G_{35}, iG_{36} + G_{37}, & \tilde{F}_1(e_6 - ie_7), & \tilde{F}_2(e_6 - ie_7), & \tilde{F}_3(e_6 - ie_7) \end{array} \right\} 20 \\ \mathfrak{g}_{-2} &= \{(G_{46} + G_{57}) - i(G_{47} - G_{56})\} 1 \\ \mathfrak{g}_1 &= \tau(\mathfrak{g}_{-1})\tau, \quad \mathfrak{g}_2 = \tau(\mathfrak{g}_{-2})\tau. \end{aligned}$$

PROOF. We can prove this theorem in a way similar to Theorem 2.4, using Lemma 3.3.

Now, we shall determine the group structures of

$$(E_6^C)_{\text{ev}} = (E_6^C)^\gamma, \quad (E_6^C)_0 = (E_6^C)^{\gamma_3}.$$

Let $k : M(3, \mathbf{H}^C) \rightarrow MJ(6, \mathbf{C}^C) = \{P \in M(6, \mathbf{C}^C) \mid JP = \bar{P}J\}$ (resp. $k : (\mathbf{H}^C)^3 \rightarrow MJ(2, 6, \mathbf{C}^C) = \{P \in M(2, 6, \mathbf{C}^C) \mid JP = \bar{P}J\}$) be a $\mathbf{C}$ -linear isomorphism defined by

$$k\left((a + be_2)\right) = \left(\begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix}\right), \quad a, b \in \mathbf{C}^C,$$

and we denote the inverse k^{-1} of k by h .

We define a mapping $\psi : Sp(1, \mathbf{H}^C) \times SU^*(6, \mathbf{C}^C) \rightarrow (E_6^C)^\gamma$ by

$$\psi(p, A)(M + \mathbf{n}) = (hA)M(hA)^* + p\mathbf{n}(hA)^{-1}, \quad M + \mathbf{n} \in \mathfrak{J}(3, \mathbf{H}^C) \oplus (\mathbf{H}^C)^3 = \mathfrak{J}^C.$$

Then ψ is well-defined ([5] Proposition 3.5.4), is a homomorphism and $\text{Ker } \psi = \{(1, E), (-1 - E)\} = \mathbf{Z}_2$. Since $(E_6^C)^\gamma$ is connected and $\dim_{\mathbf{C}}(\mathfrak{sp}(1, \mathbf{H}^C) \oplus$

$\mathfrak{su}^*(6, \mathbf{C}^C) = 3 + 35 = 38 = 36 + 2 = \dim_C((\mathfrak{e}_6^C)_{\text{ev}})$ (Theorem 3.4), ψ is onto. Therefore $(E_6^C)^\gamma \cong (Sp(1, \mathbf{H}^C) \times SU^*(6, \mathbf{C}^C))/\mathbf{Z}_2$.

THEOREM 3.5. (1) $(E_6^C)_{\text{ev}} \cong (Sp(1, C) \times SL(6, C))/\mathbf{Z}_2$, $\mathbf{Z}_2 = \{(1, E), (-1, -E)\}$.

(2) $(E_6^C)_0 \cong (C^* \times SL(6, C))/\mathbf{Z}_2$, $\mathbf{Z}_2 = \{(1, E), (-1, -E)\}$.

PROOF. (1) Note that $f : SL(6, C) \rightarrow SU^*(6, \mathbf{C}^C)$ given by $f(A) = \varepsilon A - \bar{\varepsilon} JAJ$, where $\varepsilon = \frac{1}{2}(1 + ie_1)$, is an isomorphism, and we define $\varphi : Sp(1, \mathbf{H}^C) \times SL(6, C) \rightarrow (E_6^C)^\gamma = (E_6^C)_{\text{ev}}$ by

$$\varphi(p, A) = \psi(p, f(A)).$$

Then φ is also a homomorphism, onto and $\text{Ker } \varphi = \mathbf{Z}_2$. Therefore $(E_6^C)_{\text{ev}} \cong (Sp(1, \mathbf{H}^C) \times SL(6, C))/\mathbf{Z}_2 \cong (Sp(1, C) \times SL(6, C))/\mathbf{Z}_2$.

(2) The restriction mapping $\varphi : U(1, \mathbf{C}^C) \times SL(6, C) \rightarrow (E_6^C)^{\gamma_3} = (E_6^C)_0$ of φ above (1) is well-defined and $\text{Ker } \varphi = \mathbf{Z}_2$. Since $(E_6^C)^{\gamma_3}$ is connected and $\dim_C(\mathfrak{u}(1, \mathbf{C}^C) \oplus \mathfrak{sl}(6, C)) = 1 + 35 = 36 = \dim_C((\mathfrak{e}_6^C)_0)$ (Theorem 3.4), φ is onto. Therefore $(E_6^C)_0 \cong (U(1, \mathbf{C}^C) \times SL(6, C))/\mathbf{Z}_2 \cong (C^* \times SL(6, C))/\mathbf{Z}_2$.

3.2.1. Subgroups of type $C_{1(1)} \oplus A_{5(5)}$ and $R \oplus A_{5(5)}$ of $E_{6(6)}$

We shall determine the group structures of

$$(E_{6(6)})_{\text{ev}} = (E_6^C)^\gamma \cap (E_6^C)^{\tau\gamma_1}, \quad (E_{6(6)})_0 = (E_6^C)^{\gamma_3} \cap (E_6^C)^{\tau\gamma_1}.$$

LEMMA 3.6. $\varphi : Sp(1, \mathbf{H}^C) \times SL(6, C) \rightarrow E_6^C$ of Theorem 3.5.(1) satisfies

- (1) $\gamma = \varphi(-1, E)$, $\gamma_1 = \varphi(e_2, J)$, $\gamma_2 = \varphi(e_1, e_1 I)$.
- (2) $\tau\varphi(p, A)\tau = \varphi(\tau p, -J(\tau A)J)$, $\gamma_1\varphi(p, A)\gamma_1 = \varphi(\gamma_1 p, -JAJ)$.

THEOREM 3.7. (1) $(E_{6(6)})_{\text{ev}} \cong (Sp(1, \mathbf{R}) \times SL(6, \mathbf{R}))/\mathbf{Z}_2 \times \{1, \gamma_2\}$, $\mathbf{Z}_2 = \{(1, E), (-1, -E)\}$.

(2) $(E_{6(6)})_0 \cong (\mathbf{R}^+ \times SL(6, \mathbf{R}) \times \{1, \gamma_2\})$.

PROOF. We can prove this theorem in a way similar to Theorem 1.9 or Theorem 2.7, using Lemma 3.6.

3.2.2. Subgroups of type $C_{1(1)} \oplus A_{5(1)}$ and $R \oplus A_{5(1)}$ of $E_{6(2)}$

LEMMA 3.8. In the Lie algebra $\mathfrak{e}_6^C$, we have

$$\begin{cases} \lambda(G_{kl}) = G_{kl} \\ \lambda(\tilde{A}_k(a)) = \tilde{A}_k(a), \end{cases} \quad \begin{cases} \lambda(\tilde{F}_k(a)) = -\tilde{F}_k(a) \\ \lambda((E_k - E_{k+1})^\sim) = -(E_k - E_{k+1})^\sim. \end{cases}$$

THEOREM 3.9. The 2-graded decomposition of $\mathfrak{e}_{6(2)} = (\mathfrak{e}_6^C)^{\lambda\tau\gamma_1}$ (or $\mathfrak{e}_6^C$),

$$\mathfrak{e}_{6(2)} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2$$

with respect to $\text{ad } Z$, $Z = i(G_{45} - G_{67})$, is given by

$$\mathfrak{g}_0 = \left\{ \begin{array}{cccccc} iG_{45}, iG_{67}, & \tilde{A}_1(1), & i\tilde{A}_1(e_1), & \tilde{A}_1(e_2), & i\tilde{A}_1(e_3), \\ G_{46} - G_{57}, i(G_{47} + G_{56}), & \tilde{A}_2(1), & i\tilde{A}_2(e_1), & \tilde{A}_2(e_2), & i\tilde{A}_2(e_3), \\ iG_{01}, G_{02}, iG_{03}, & \tilde{A}_3(1), & i\tilde{A}_3(e_1), & \tilde{A}_3(e_2), & i\tilde{A}_3(e_3), \\ iG_{12}, G_{13}, & i\tilde{F}_1(1), & \tilde{F}_1(e_1), & i\tilde{F}_1(e_2), & \tilde{F}_1(e_3), \\ iG_{23}, & i\tilde{F}_2(1), & \tilde{F}_2(e_1), & i\tilde{F}_2(e_2), & \tilde{F}_2(e_3), \\ i(E_1 - E_2)^\sim, i(E_2 - E_3)^\sim, & i\tilde{F}_3(1), & \tilde{F}_3(e_1), & i\tilde{F}_3(e_2), & \tilde{F}_3(e_3) \end{array} \right\} \quad 36$$

$$\mathfrak{g}_{-1} = \left\{ \begin{array}{cccc} G_{04} + iG_{05}, & G_{06} - iG_{07}, & iG_{14} - G_{15}, & iG_{16} + G_{17}, \\ G_{24} + iG_{25}, & G_{26} - iG_{27}, & iG_{34} - G_{35}, & iG_{36} + G_{37}, \\ \tilde{A}_1(e_4 + ie_5), & \tilde{A}_2(e_4 + ie_5), & \tilde{A}_3(e_4 + ie_5), & \\ \tilde{A}_1(e_6 - ie_7), & \tilde{A}_2(e_6 - ie_7), & \tilde{A}_3(e_6 - ie_7), & \\ i\tilde{F}_1(e_4 + ie_5), & i\tilde{F}_2(e_4 + ie_5), & i\tilde{F}_3(e_4 + ie_5), & \\ i\tilde{F}_1(e_6 - ie_7), & i\tilde{F}_2(e_6 - ie_7), & i\tilde{F}_3(e_6 - ie_7), & \end{array} \right\} \quad 20$$

$$\mathfrak{g}_{-2} = \{(G_{46} + G_{57}) - i(G_{47} - G_{56})\} \cdot 1$$

$$\mathfrak{g}_1 = \tau(\mathfrak{g}_{-1})\tau, \quad \mathfrak{g}_2 = \tau(\mathfrak{g}_{-2})\tau.$$

PROOF. We can prove this theorem in a way similar to Theorem 3.4, using Lemma 3.8.

We shall determine the group structures of

$$(E_{6(2)})_{\text{ev}} = (E_6^C)^\gamma \cap (E_6^C)^{\lambda\tau\gamma_1}, \quad (E_{6(2)})_0 = (E_6^C)^{\gamma_3} \cap (E_6^C)^{\lambda\tau\gamma_1}.$$

LEMMA 3.10. $\varphi : Sp(1, \mathbf{H}^C) \times SL(6, C) \rightarrow E_6^C$ of Theorem 3.5.(1) satisfies

$${}^t\varphi(p, A)^{-1} = \varphi(p, -J^t A^{-1} J).$$

THEOREM 3.11. (1) $(E_{6(2)})_{\text{ev}} \cong (Sp(1, \mathbf{R}) \times SU(3, 3))/\mathbf{Z}_2 \times \{1, \gamma_2\}$, $\mathbf{Z}_2 = \{(1, E), (-1, -E)\}$.

(2) $(E_{6(2)})_0 \cong (\mathbf{R}^+ \times SU(3, 3))/\mathbf{Z}_2 \times \{1, \gamma_2\}$.

PROOF. (1) For $\alpha \in (E_{6(2)})_{\text{ev}} \subset (E_6^C)^\gamma$, there exist $p \in Sp(1, \mathbf{H}^C)$ and $A \in SL(6, C)$ such that $\alpha = \varphi(p, A)$ (Theorem 3.5.(1)). From $\gamma_1 \tau^t \alpha^{-1} \tau \gamma_1 = \alpha$, we have $\varphi(\gamma_1 \tau p, -J^t(\tau A)^{-1} J) = \varphi(p, A)$ (Lemmas 3.6.(2), 3.10). Hence

$$\gamma_1 \tau p = p, \quad -J^t(\tau A)^{-1} J = A \quad \text{or} \quad \gamma_1 \tau p = -p, \quad -J^t(\tau A)^{-1} J = -A.$$

In the former case, $p \in Sp(1, \mathbf{H}') \cong Sp(1, \mathbf{R})$ and the group $\{A \in SL(6, C) \mid -J^t(\tau A)^{-1} J = A\}$ is $\{A \in SL(6, C) \mid {}^t(\tau A)JA = J\} \cong \{A \in SL(6, C) \mid$

${}^t(\tau A)IA = I\} \cong SU(3, 3)$. Hence the group of the former case is $(Sp(1, \mathbf{R}) \times SU(3, 3))/\mathbf{Z}_2$. In the latter case, $p = e_1$ and $A = e_1 I$ satisfy these conditions and $\varphi(e_1, e_1 I) = \gamma_2$ (Lemma 3.6.(1)). Therefore $(E_{6(2)})_{\text{ev}} \cong (Sp(1, \mathbf{R}) \times SU(3, 3))/\mathbf{Z}_2 \times \{1, \gamma_2\}$.

(2) The restriction mapping $\varphi : U(1, \mathbf{C}^C) \times SL(6, C) \rightarrow (E_6^C)^{\gamma_3}$ of φ above (1) induces, in a similar way as in Theorems 1.9.(2) or 2.7.(2), an isomorphism $(E_{6(2)})_0 \cong (\mathbf{R}^+ \times SU(3, 3)) \times \{1, \gamma_2\}$.

3.2.3. Subgroups of type $\mathbf{C}_{1(1)} \oplus \mathbf{A}_{5(-15)}$ and $\mathbf{R} \oplus \mathbf{A}_{5(-15)}$ of $\mathbf{E}_{6(-14)}$

We define $\delta' \in F_4 \subset E_6$, $\delta_1' \in E_6$ by $\delta' = \exp \frac{\pi}{2} \tilde{A}_1(e_4)$, $\delta_1' = \exp \frac{\pi}{2} i \tilde{F}_1(e_6)$, respectively. δ' is an $\mathbf{R}$ -linear transformation of $\mathfrak{J}$ defined as similar to $\delta \in F_4$ of Section 2.3.2 and δ_1' is a C -linear transformation of $\mathfrak{J}^C$ satisfying

$$\delta_1' : \begin{cases} E_1 \rightarrow E_1 \\ E_2 - E_3 \rightarrow E_2 - E_3 \\ F_1(e_k) \rightarrow F_1(e_k), \quad k \neq 6, \end{cases} \quad \begin{cases} E_2 + E_3 \rightarrow iF_1(e_6) \\ F_1(e_6) \rightarrow i(E_2 + E_3) \\ F_2(x) \rightarrow \frac{1}{\sqrt{2}}(F_2(x) + iF_3(\overline{e_6 x})) \\ F_3(x) \rightarrow \frac{1}{\sqrt{2}}(iF_2(\overline{x e_6}) + F_3(x)). \end{cases}$$

And we define a C -linear transformation σ_2 of $\mathfrak{J}^C$ by $\sigma_2 = \delta_1'^{-1} \delta'^{-1} \sigma' \delta' \delta_1'$, that is,

$$\sigma_2 \begin{pmatrix} \xi_1 & x_3 & \overline{x_2} \\ \overline{x_3} & \xi_2 & x_1 \\ x_2 & \overline{x_1} & \xi_3 \end{pmatrix} = \begin{pmatrix} \xi_1 & i(x_3 e_4) e_6 & \overline{i e_6(e_4 x_2)} \\ i \overline{(x_3 e_4) e_6} & -\xi_2 & -e_6(e_4 x_1 e_4) e_6 \\ i e_6(e_4 x_2) & -e_6(e_4 x_1 e_4) e_6 & -\xi_3 \end{pmatrix}.$$

Then $\sigma_2 \in E_6$ and $\sigma_2^2 = 1$. Since σ , σ' and σ_2 are conjugate with each other in E_6 , we have

PROPOSITION 3.12. $E_{6(-14)} = (E_6^C)^{\lambda \tau \sigma} \cong (E_6^C)^{\lambda \tau \sigma_2}$.

We shall determine the group structures of

$$(E_{6(-14)})_{\text{ev}} = (E_6^C)^\gamma \cap (E_6^C)^{\lambda \tau \sigma_2}, \quad (E_{6(-14)})_0 = (E_6^C)^{\gamma_3} \cap (E_6^C)^{\lambda \tau \sigma_2}.$$

LEMMA 3.13. In the Lie algebra $\mathfrak{e}_6^C$, we have

$$(1) \quad \begin{cases} \sigma_2 G_{k4} \sigma_2 = -G_{k4}, \quad k \neq 6 \\ \sigma_2 G_{k6} \sigma_2 = -G_{k6}, \quad k \neq 4, \end{cases} \quad \sigma_2 G_{kl} \sigma_2 = G_{kl}, \quad \text{otherwise,}$$

$$(2) \quad \begin{cases} \sigma_2 \tilde{A}_1(a) \sigma_2 = \tilde{A}_1(e_6(e_4 a e_4) e_6) \\ \sigma_2 \tilde{A}_2(a) \sigma_2 = i \tilde{F}_2(e_6(e_4 a)) \\ \sigma_2 \tilde{A}_3(a) \sigma_2 = -i \tilde{F}_3((a e_4) e_6), \end{cases} \quad \begin{cases} \sigma_2 \tilde{F}_1(a) \sigma_2 = \tilde{F}_1(e_6(e_4 a e_4) e_6) \\ \sigma_2 \tilde{F}_2(a) \sigma_2 = i \tilde{A}_2(e_6(e_4 a)) \\ \sigma_2 \tilde{F}_3(a) \sigma_2 = -i \tilde{A}_3((a e_4) e_6) \\ \sigma_2 (E_k - E_{k+1}) \sim \sigma_2 = (E_k - E_{k+1})^\sim. \end{cases}$$

THEOREM 3.14. The 2-graded decomposition of $\mathfrak{e}_{6(-14)} = (\mathfrak{e}_6^C)^{\lambda\tau\sigma_2}$ (or $\mathfrak{e}_6^C$),

$$\mathfrak{e}_{6(-14)} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2$$

with respect to $\text{ad } Z$, $Z = i(G_{45} - G_{67})$, is given by

$$\mathfrak{g}_0 = \left\{ \begin{array}{l} G_{46} - G_{57}, \ i(G_{47} + G_{56}), \ iG_{45}, \ iG_{67}, \ i(E_1 - E_2)^\sim, \\ G_{01}, \ G_{02}, \ G_{03}, \ G_{12}, \ G_{13}, \ G_{23}, \ i(E_2 - E_3)^\sim, \\ \tilde{A}_1(1), \ \tilde{A}_1(e_1), \ \tilde{A}_1(e_2), \ \tilde{A}_1(e_3), \ i\tilde{F}_1(1), \ i\tilde{F}_1(e_1), \ i\tilde{F}_1(e_2), \ i\tilde{F}_1(e_3), \\ \tilde{A}_2(1) + i\tilde{F}_2(e_2), \ i\tilde{A}_2(1) + \tilde{F}_2(e_2), \ \tilde{A}_2(e_2) - i\tilde{F}_2(1), \ i\tilde{A}_2(e_2) - \tilde{F}_2(1), \\ \tilde{A}_2(e_1) + i\tilde{F}_2(e_3), \ i\tilde{A}_2(e_1) + \tilde{F}_2(e_3), \ \tilde{A}_2(e_3) - i\tilde{F}_2(e_1), \\ i\tilde{A}_2(e_3) - \tilde{F}_2(e_1), \ \tilde{A}_3(1) + i\tilde{F}_3(e_2), \ i\tilde{A}_3(1) + \tilde{F}_3(e_2), \\ \tilde{A}_3(e_2) - i\tilde{F}_3(1), \ i\tilde{A}_3(e_2) - \tilde{F}_3(1), \ \tilde{A}_3(e_1) - i\tilde{F}_3(e_3), \\ i\tilde{A}_3(e_1) - \tilde{F}_3(e_3), \ \tilde{A}_3(e_3) + i\tilde{F}_3(e_1), \ i\tilde{A}_3(e_3) + \tilde{F}_3(e_1) \end{array} \right\} \quad 36$$

$$\mathfrak{g}_{-1} = \left\{ \begin{array}{l} iG_{04} - G_{05}, \ iG_{06} + G_{07}, \ iG_{14} - G_{15}, \ iG_{16} + G_{17}, \\ iG_{24} - G_{25}, \ iG_{26} + G_{27}, \ iG_{34} - G_{35}, \ iG_{36} + G_{37}, \\ i\tilde{A}_1(e_4 + ie_5), \ i\tilde{A}_1(e_6 - ie_7), \ \tilde{F}_1(e_4 + ie_5), \ \tilde{F}_1(e_6 - ie_7), \\ \tilde{A}_2(e_4 + ie_5) - i\tilde{F}_2(e_6 - ie_7), \ i\tilde{A}_2(e_4 + ie_5) - \tilde{F}_2(e_6 - ie_7), \\ \tilde{A}_2(e_6 - ie_7) + i\tilde{F}_2(e_4 + ie_5), \ i\tilde{A}_2(e_6 - ie_7) + \tilde{F}_2(e_4 + ie_5), \\ \tilde{A}_3(e_4 + ie_5) + i\tilde{F}_3(e_6 - ie_7), \ i\tilde{A}_3(e_4 + ie_5) + \tilde{F}_2(e_6 - ie_7), \\ \tilde{A}_3(e_6 - ie_7) - i\tilde{F}_3(e_4 + ie_5), \ i\tilde{A}_3(e_6 - ie_7) - \tilde{F}_3(e_4 + ie_5) \end{array} \right\} \quad 20$$

$$\mathfrak{g}_{-2} = \{(G_{46} + G_{57}) - i(G_{47} - G_{56})\} \cdot 1$$

$$\mathfrak{g}_1 = \tau(\lambda(\mathfrak{g}_{-1}))\tau, \quad \mathfrak{g}_2 = \tau(\lambda(\mathfrak{g}_{-2}))\tau.$$

PROOF. We can prove this theorem in a way similar to Theorem 3.9, using Lemmas 3.8 and 3.13.

In the identification $\mathfrak{J}^C = \mathfrak{J}(3, \mathbf{H}^C) \oplus (\mathbf{H}^C)^3$ in Section 2.2, we have

$$\begin{aligned} \sigma_2(M + \mathbf{n}) &= \sigma_2 X = \sigma_2 \begin{pmatrix} \xi_1 & x_3 & \overline{x_2} \\ \overline{x_3} & \xi_2 & x_1 \\ x_2 & \overline{x_1} & \xi_3 \end{pmatrix} \\ &= \begin{pmatrix} \xi_1 & i(x_3 e_4) e_6 & i\overline{e_6(e_4 x_2)} \\ i(x_3 e_4) e_6 & -\xi_2 & -e_6(e_4 x_1 e_4) e_6 \\ ie_6(e_4 x_2) & -\overline{e_6(e_4 x_1 e_4) e_6} & -\xi_3 \end{pmatrix} \\ &= \begin{pmatrix} \xi_1 & -ie_2 m_3 & i\overline{m_2 e_2} \\ -ie_2 m_3 & -\xi_2 & -m_1 \\ im_2 e_2 & -\overline{m_1} & -\xi_3 \end{pmatrix} + (\gamma_1 n_1, i(\gamma_1 n_2) e_2, i(\gamma_1 n_3) e_2) \end{aligned}$$

$$\begin{aligned}
&= \begin{pmatrix} e_2 & 0 & 0 \\ 0 & -i & 0 \\ 0 & 0 & -i \end{pmatrix} \begin{pmatrix} \xi_1 & m_3 & \overline{m_2} \\ \overline{m_3} & \xi_2 & m_1 \\ m_2 & \overline{m_1} & \xi_3 \end{pmatrix} \begin{pmatrix} -e_2 & 0 & 0 \\ 0 & -i & 0 \\ 0 & 0 & -i \end{pmatrix} \\
&\quad + (\gamma_1 n_1, \gamma_1 n_2, \gamma_1 n_3) \begin{pmatrix} 1 & 0 & 0 \\ 0 & ie_2 & 0 \\ 0 & 0 & ie_2 \end{pmatrix} \\
&= P_1 M P_1^* - (\gamma_1 \mathbf{n}) e_2 P_1, \quad P_1 = \text{diag}(e_2, -i, -i) \in M(3, \mathbf{H}^C),
\end{aligned}$$

where $x_k = m_k + n_k e \in \mathbf{H}^C \oplus \mathbf{H}^C e = \mathfrak{C}^C$.

LEMMA 3.15. $\varphi : Sp(1, \mathbf{H}^C) \times SL(6, C) \rightarrow E_6^C$ of Theorem 3.5.(1) satisfies

$$\sigma_2 \varphi(p, A) \sigma_2 = \varphi(\gamma_1 p, -PAP), \quad P = \text{diag}(J, i, i, i, i) \in M(6, C).$$

THEOREM 3.16. (1) $(E_{6(-14)})_{\text{ev}} \cong (Sp(1, \mathbf{R}) \times SU(1, 5))/\mathbf{Z}_2 \times \{1, \gamma_2\}$, $\mathbf{Z}_2 = \{(1, E), (-1, -E)\}$.

(2) $(E_{6(-14)})_0 \cong (\mathbf{R}^+ \times SU(1, 5)) \times \{1, \gamma_2\}$.

PROOF. (1) For $\alpha \in (E_{6(-14)})_{\text{ev}} \subset (E_6^C)^\gamma$, there exist $p \in Sp(1, \mathbf{H}^C)$ and $A \in SL(6, C)$ such that $\alpha = \varphi(p, A)$ (Theorem 3.5.(1)). From $\sigma_2 \tau^t \alpha^{-1} \tau \sigma_2 = \alpha$, we have $\varphi(\gamma_1 \tau p, -P^t(\tau A)^{-1} P) = \varphi(p, A)$ (Lemmas 3.6.(2), 3.10, 3.15). Hence

$$\gamma_1 \tau p = p, \quad -P^t(\tau A)^{-1} P = A \quad \text{or} \quad \gamma_1 \tau p = -p, \quad -P^t(\tau A)^{-1} P = -A.$$

In the former case, $p \in Sp(1, \mathbf{H}') \cong Sp(1, \mathbf{R})$ and the group $\{A \in SL(6, C) \mid -P^t(\tau A)^{-1} P = A\}$ is $\{A \in SL(6, C) \mid {}^t(\tau A)PA = P\} \cong \{A \in SL(6, C) \mid {}^t(\tau A)I_1 A = I_1\} \cong SU(1, 5)$. Hence the group of the former case is $(Sp(1, \mathbf{R}) \times SU(1, 5))/\mathbf{Z}_2$. In the latter case, $p = e_1$ and $A = e_1 I$ satisfy these conditions and $\varphi(e_1, e_1 I) = \gamma_2$ (Lemma 3.6.(1)). Therefore $(E_{6(-14)})_{\text{ev}} \cong (Sp(1, \mathbf{R}) \times SU(1, 5))/\mathbf{Z}_2 \times \{1, \gamma_2\}$.

(2) The restriction mapping $\varphi : U(1, \mathbf{C}^C) \times SL(6, C) \rightarrow (E_6^C)^{\gamma_3}$ of φ above (1) induces, in a similar way as in Theorems 1.9.(1), 2.7.(2) or 3.11.(2), an isomorphism $(E_{6(-14)})_0 \cong (\mathbf{R}^* \times SU(1, 5)) \times \{1, \gamma_2\}$.

3.3. Subgroups of type $C_1^C \oplus A_5^C$ and $C_1^C \oplus C \oplus A_4^C$ of E_6^C

Let $\omega = \exp \frac{2\pi i}{3}$, $\mu = \exp \frac{2\pi i}{6}$, $\nu = \exp \frac{2\pi i}{9} \in C$ and let $A_\mu = \text{diag}(\mu^5, \mu^{-1}, \dots, \mu^{-1}) = \mu^{-1} E$, $A_\nu = \text{diag}(\nu^5, \nu^{-1}, \dots, \nu^{-1}) \in SL(6, C)$. We define $\mu_2, \mu_3 \in E_6 \subset E_6^C$ by

$$\mu_2 = \varphi(1, A_\mu), \quad \mu_3 = \varphi(1, A_\nu),$$

respectively, where $\varphi : Sp(1, \mathbf{H}^C) \times SL(6, C) \rightarrow E_6^C$ is the mapping defined in Theorem 3.5.(1). Now, in the Lie algebra $\mathfrak{e}_6^C$, let $Z = i(G_{45} + G_{67}) + \frac{2}{3}(2E_1 - E_2 - E_3)^\sim$. Then we can verify that

$$\omega^2\gamma = \mu_2 = \exp \frac{2\pi i}{2}Z, \quad \mu_3 = \exp \frac{2\pi i}{3}Z.$$

LEMMA 3.17. *In the Lie algebra $\mathfrak{e}_6^C$, we have*

$$\begin{aligned} [(\xi_1 E_1 + \xi_2 E_2 + \xi_3 E_3)^\sim, \tilde{A}_k(a)] &= \frac{1}{2}(\xi_{k+1} - \xi_{k+2})\tilde{F}_k(a), \\ [(\xi_1 E_1 + \xi_2 E_2 + \xi_3 E_3)^\sim, \tilde{F}_k(a)] &= \frac{1}{2}(\xi_{k+1} - \xi_{k+2})\tilde{A}_k(a), \end{aligned} \quad \xi_1 + \xi_2 + \xi_3 = 0.$$

THEOREM 3.18. *The 2-graded decomposition of $\mathfrak{e}_{6(6)} = (\mathfrak{e}_6^C)^{\tau\gamma_1}$ (or $\mathfrak{e}_6^C$),*

$$\mathfrak{e}_{6(6)} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2$$

with respect to $\text{ad } Z$, $Z = i(G_{45} + G_{67}) + \frac{2}{3}(2E_1 - E_2 - E_3)^\sim$, is given by

$$\begin{aligned} \mathfrak{g}_0 &= \left\{ \begin{array}{l} G_{46} + G_{57}, i(G_{47} - G_{56}), iG_{45}, iG_{67}, (E_2 - E_3)^\sim, \\ iG_{01}, G_{02}, iG_{03}, iG_{12}, G_{13}, iG_{23}, (2E_1 - E_2 - E_3)^\sim, \\ \tilde{A}_1(1), i\tilde{A}_1(e_1), \tilde{A}_1(e_2), i\tilde{A}_1(e_3), \tilde{F}_1(1), i\tilde{F}_1(e_1), \tilde{F}_1(e_2), i\tilde{F}_1(e_3), \\ \tilde{A}_2(1 + ie_1) - \tilde{F}_2(1 + ie_1), \quad \tilde{A}_2(1 - ie_1) + \tilde{F}_2(1 - ie_1), \\ \tilde{A}_2(e_2 + ie_3) + \tilde{F}_2(e_2 + ie_3), \quad \tilde{A}_2(e_2 - ie_3) - \tilde{F}_2(e_2 - ie_3), \\ \tilde{A}_3(1 + ie_1) - \tilde{F}_3(1 + ie_1), \quad \tilde{A}_3(1 - ie_1) + \tilde{F}_3(1 - ie_1), \\ \tilde{A}_3(e_2 + ie_3) - \tilde{F}_3(e_2 + ie_3), \quad \tilde{A}_3(e_2 - ie_3) + \tilde{F}_3(e_2 - ie_3) \end{array} \right\} 28 \\ \mathfrak{g}_{-1} &= \left\{ \begin{array}{l} G_{04} + iG_{05}, G_{06} + iG_{07}, iG_{14} - G_{15}, iG_{16} - G_{17}, \\ G_{24} + iG_{25}, G_{26} + iG_{27}, iG_{34} - G_{35}, iG_{36} - G_{37}, \\ \tilde{A}_1(e_4 + ie_5), \tilde{A}_1(e_6 + ie_7), \tilde{F}_1(e_4 + ie_5), \tilde{F}_1(e_6 + ie_7), \\ \tilde{A}_2(e_4) + \tilde{F}_2(e_4), \quad i\tilde{A}_2(e_5) + i\tilde{F}_2(e_5), \tilde{A}_2(e_6) + \tilde{F}_2(e_6), \\ i\tilde{A}_2(e_7) + i\tilde{F}_2(e_7), \quad \tilde{A}_3(e_4) - \tilde{F}_3(e_4), \quad i\tilde{A}_3(e_5) - i\tilde{F}_3(e_5), \\ \tilde{A}_3(e_6) - \tilde{F}_3(e_6), \quad i\tilde{A}_2(e_7) - i\tilde{F}_3(e_7) \end{array} \right\} 20 \\ \mathfrak{g}_{-2} &= \left\{ \begin{array}{l} (G_{46} - G_{57}) + i(G_{47} + G_{56}), \\ \tilde{A}_2(1 + ie_1) + \tilde{F}_2(1 + ie_1), \quad \tilde{A}_2(e_2 - ie_3) + \tilde{F}_2(e_2 - ie_3), \\ \tilde{A}_3(1 - ie_1) - \tilde{F}_3(1 - ie_1), \quad \tilde{A}_3(e_2 - ie_3) - \tilde{F}_3(e_2 - ie_3) \end{array} \right\} 5 \\ \mathfrak{g}_1 &= \tau(\lambda(\mathfrak{g}_{-1}))\tau, \quad \mathfrak{g}_2 = \tau(\lambda(\mathfrak{g}_{-2}))\tau. \end{aligned}$$

PROOF. Note that for $D_1 = G_{45} + G_{67} \in \mathfrak{so}(8)$ we have $D_2 = G_{01} - G_{23}$, $D_3 = -G_{01} - G_{23}$. Then we can prove this theorem in a way similar to Theorem 3.4, using Lemma 3.17.

Now, we shall determine the group structures of

$$(E_6^C)_{\text{ev}} = (E_6^C)^{\mu_2}, \quad (E_6^C)_0 = (E_6^C)^{\mu_3}.$$

THEOREM 3.19. (1) $(E_6^C)_{\text{ev}} \cong (Sp(1, C) \times SL(6, C))/Z_2$, $Z_2 = \{(1, E), (-1, -E)\}$.

(2) $(E_6^C)_0 \cong (Sp(1, C) \times C^* \times SL(5, C))/(Z_2 \times Z_5)$, $Z_2 = \{(1, 1, E), (-1, -1, E)\}$, $Z_5 = \{(1, \kappa^k, \kappa^k E), k = 0, 1, \dots, 4\}$ ($\kappa = \exp \frac{2\pi i}{5}$).

PROOF. (1) Since $\mu_2 = \omega^2 \gamma$ and $\omega^2 1$ is a center of E_6^C , $(E_6^C)_{\text{ev}} = (E_6^C)^{\mu_2} = (E_6^C)^\gamma \cong (Sp(1, C) \times SL(6, C))/Z_2$ (Theorem 3.5.(1)).

(2) The restriction mapping $\varphi : Sp(1, H^C) \times S(GL(1, C) \times GL(5, C)) \rightarrow (E_6^C)^{\mu_3}$ of φ of Theorem 3.5.(2) is well-defined and $\text{Ker } \varphi = \{(1, E), (-1, -E)\} = Z_2$. Since $(E_6^C)^{\mu_3}$ is connected and $\dim_C(\mathfrak{sp}(1, C) \oplus \mathfrak{s}(\mathfrak{gl}(1, C) \oplus \mathfrak{gl}(5, C))) = 3 + 1 + 24 = 28 = \dim_C((E_6^C)_0)$ (Theorem 3.18), φ is onto. Hence $(E_6^C)^{\mu_3} \cong (Sp(1, H^C) \times S(GL(1, C) \times GL(5, C)))/Z_2$. Since $h : C^* \times SL(5, C) \rightarrow S(GL(1, C) \times GL(5, C))$, $h(z, B) = \begin{pmatrix} z^5 & 0 \\ 0 & z^{-1}B \end{pmatrix}$ induces an isomorphism $(C^* \times SL(5, C))/Z_5$ ($Z_5 = \{(\kappa^k, \kappa^k E), k = 0, 1, \dots, 4\} \cong S(GL(1, C) \times GL(5, C))$), we have $(E_6^C)^{\mu_3} \cong (Sp(1, C) \times C^* \times SL(5, C))/(Z_2 \times Z_5)$.

3.3.1. Subgroups of type $C_{1(1)} \oplus A_{5(5)}$ and $C_{1(1)} \oplus R \oplus A_{4(4)}$ of $E_{6(6)}$

We shall determine the group structures of

$$(E_{6(6)})_{\text{ev}} = (E_6^C)^{\mu_2} \cap (E_6^C)^{\tau\gamma_1}, \quad (E_{6(6)})_0 = (E_6^C)^{\mu_3} \cap (E_6^C)^{\tau\gamma_1}.$$

THEOREM 3.20. (1) $(E_{6(6)})_{\text{ev}} \cong (Sp(1, R) \times SL(6, R))/Z_2 \times \{1, \gamma_2\}$, $Z_2 = \{(1, E), (-1, -E)\}$.

(2) $(E_{6(6)})_0 \cong (Sp(1, R) \times R^+ \times SL(5, R)) \times \{1, \gamma_2\}$.

PROOF. (1) $(E_{6(6)})_{\text{ev}} = (E_6^C)^{\mu_2} \cap (E_6^C)^{\tau\gamma_1} = (E_6^C)^\gamma \cap (E_6^C)^{\tau\gamma_1} \cong (Sp(1, R) \times SL(6, C))/Z_2 \times \{1, \gamma_2\}$ (Theorem 3.7.(1)).

(2) For $\alpha \in (E_{6(0)})_0 \subset (E_6^C)^{\mu_3}$, there exist $p \in Sp(1, H^C)$ and $A \in S(GL(1, C) \times GL(5, C))$ such that $\alpha = \varphi(p, A)$ (Theorem 3.19.(2)). From $\gamma_1 \tau \alpha \tau \gamma_1 = \alpha$, we have $\varphi(\gamma_1 \tau p, \gamma_1 \tau A) = \varphi(p, A)$ (Lemma 3.6.(2)). Hence

$$\gamma_1 \tau p = p, \quad \gamma_1 \tau A = A \quad \text{or} \quad \gamma_1 \tau p = -p, \quad \gamma_1 \tau A = -A.$$

The group of the former case is $(Sp(1, R) \times S(GL(1, R) \times GL(5, R)))/Z_2$ ($Z_2 = \{(1, E), (-1, -E)\} \cong (SL(1, R) \times R^+ \times SL(5, R))/Z_2$ ($Z_2 = \{(1, 1, E), (-1, -1, -E)\} \cong Sp(1, R) \times R^+ \times SL(5, R)$). $p = e_1$ and $A = e_1 I$ satisfy the latter case and $\varphi(e_1, e_1 I) = \gamma_2$ (Lemma 3.6.(1)). Therefore $(E_{6(6)})_0 \cong (Sp(1, R) \times R^+ \times SL(5, R)) \times \{1, \gamma_2\}$.

3.4. Subgroups of type $C \oplus D_5^C$ and $C \oplus C \oplus D_4^C$ of E_6^C

In the Lie algebra $\mathfrak{e}_6^C$, let $Z = -2iG_{01}$. Then, as is shown in $\mathfrak{f}_4^C$, we have

$$\sigma = \exp \frac{2\pi i}{2} Z, \quad \sigma_3 = \exp \frac{2\pi i}{3} Z.$$

THEOREM 3.21. *The 2-graded decomposition of $\mathfrak{e}_{6(6)} = (\mathfrak{e}_6^C)^{\tau\gamma_1}$ (or $\mathfrak{e}_6^C$),*

$$\mathfrak{e}_{6(6)} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2$$

with respect to $\text{ad } Z$, $Z = -2iG_{01}$, is given by

$$\begin{aligned} \mathfrak{g}_0 &= \left\{ \begin{array}{cccccc} iG_{01}, iG_{23}, G_{24}, iG_{25}, G_{26}, iG_{27}, & \tilde{A}_1(e_2), & \tilde{F}_1(e_2), \\ & i\tilde{A}_1(e_3), & i\tilde{F}_1(e_3), \\ & i\tilde{A}_1(e_4), & \tilde{F}_1(e_4), \\ & i\tilde{A}_1(e_5), & i\tilde{F}_1(e_5), \\ & i\tilde{A}_1(e_6), & \tilde{F}_1(e_6), \\ (2E_1 - E_2 - E_3)^\sim, (E_2 - E_3)^\sim, & i\tilde{A}_1(e_7), & i\tilde{F}_1(e_7) \end{array} \right\} 30 \\ \mathfrak{g}_{-1} &= \left\{ \begin{array}{cccc} \tilde{A}_2(1 + ie_1), & \tilde{A}_2(e_2 + ie_3), & \tilde{A}_2(e_4 + ie_5), & \tilde{A}_2(e_6 + ie_7), \\ \tilde{A}_3(1 + ie_1), & \tilde{A}_3(e_2 - ie_3), & \tilde{A}_3(e_4 - ie_5), & \tilde{A}_3(e_6 - ie_7), \\ \tilde{F}_2(1 + ie_1), & \tilde{F}_2(e_2 + ie_3), & \tilde{F}_2(e_4 + ie_5), & \tilde{F}_2(e_6 + ie_7), \\ \tilde{F}_3(1 + ie_1), & \tilde{F}_3(e_2 - ie_3), & \tilde{F}_3(e_4 - ie_5), & \tilde{F}_3(e_6 - ie_7) \end{array} \right\} 16 \\ \mathfrak{g}_{-2} &= \left\{ \begin{array}{cc} G_{02} - iG_{12}, iG_{03} + G_{13}, G_{04} - iG_{14} & \tilde{A}_1(1 - ie_1), \\ iG_{05} + G_{15}, G_{06} - iG_{16}, iG_{07} + iG_{17} & \tilde{F}_1(1 - ie_1) \end{array} \right\} 8 \\ \mathfrak{g}_1 &= \tau(\mathfrak{g}_{-1})\tau, \quad \mathfrak{g}_2 = \tau(\mathfrak{g}_{-2})\tau. \end{aligned}$$

PROOF. We can prove this theorem in a way similar to Theorem 2.8, using Lemma 3.3.

We shall determine the group structures of

$$(E_6^C)_{\text{ev}} = (E_6^C)^\sigma, \quad (E_6^C)_0 = (E_6^C)^{\sigma_3}.$$

PROPOSITION 3.22. (1) $(E_6^C)_{E_1, F_1(1), F_1(e_1)} \cong \text{Spin}(8, C)$.
 (2) $(E_6^C)_{E_1} \cong \text{Spin}(10, C)$.

PROOF. (1) The group $(E_6^C)_{E_1, F_1(1), F_1(e_1)}$ acts on the C -vector space

$$\begin{aligned} (V^C)^8 &= \{X \in \mathfrak{J}^C \mid 4E_1 \times (E_1 \times X) = X, (E_1, F_1(1), X) = (E_1, F_1(e_1), X) = 0\} \\ &= \{X = \xi E_2 + \eta E_3 + F_1(x) \mid \xi, \eta \in C, x \in \mathfrak{C}^C, (1, x) = (e_1, x) = 0\} \end{aligned}$$

with the norm $(E_1, X, X) = \xi\eta - x\bar{x}$. The connectedness of $(E_6^C)_{E_1, F_1(1), F_1(e_1)}$ follows from

$$(E_6^C)_{E_1, F_1(1), F_1(e_1)} / (F_4^C)_{E_1, F_1(1), F_1(e_1)} \simeq (S^C)^7 = \{X \in (V^C)^8 \mid (E_1, X, X) = 1\}$$

(Proposition 2.9.(1)). Hence we can define a homomorphism π :

$$(E_6^C)_{E_1, F_1(1), F_1(e_1)} \rightarrow SO(8, C) = SO((V^C)^8)$$

by $\pi(\alpha) = \alpha|(V^C)^8$. $\text{Ker } \pi = \{1, \sigma\} = \mathbf{Z}_2$. Since $\dim_C((E_6^C)_{E_1, F_1(1), F_1(e_1)}) = \dim_C((F_4^C)_{E_1, F_1(1), F_1(e_1)}) + \dim_C((S^C)^7) = 21 + 7 = 28 = \dim_C(SO(8, C))$, π is onto. Therefore $(E_6^C)_{E_1, F_1(1), F_1(e_1)}$ is $Spin(8, C)$ as a double covering group of $SO(8, C)$.

(2) The group $(E_6^C)_{E_1}$ acts on the C -vector space

$$\begin{aligned} (V^C)^{10} &= \{X \in \mathfrak{J}^C \mid 4E_1 \times (E_1 \times X) = X\} \\ &= \{X = \xi E_2 + \eta E_3 + F_1(x) \mid \xi, \eta \in C, x \in \mathfrak{C}^C\} \end{aligned}$$

with the norm $(E_1, X, X) = \xi\eta - x\bar{x}$. The connectedness of $(E_6^C)_{E_1}$ follows from

$$(E_6^C)_{E_1} / (F_4^C)_{E_1} \simeq (S^C)^9 = \{X \in (V^C)^{10} \mid (E_1, X, X) = 1\}$$

(Proposition 2.9.(3)). Hence we can define a homomorphism $\pi : (E_6^C)_{E_1} \rightarrow SO(10, C) = SO((V^C)^{10})$ by $\pi(\alpha) = \alpha|(V^C)^{10}$. $\text{Ker } \pi = \{1, \sigma\} = \mathbf{Z}_2$. Since $\dim_C((E_6^C)_{E_1}) = \dim_C((F_4^C)_{E_1}) + \dim_C((S^C)^7) = 36 + 9 = 45 = \dim_C(SO(10, C))$, π is onto. Therefore $(E_6^C)_{E_1}$ is $Spin(10, C)$ as a double covering group of $SO(10, C)$.

We define a mapping $\phi : C^* \rightarrow E_6^C$ by

$$\phi(\theta) \begin{pmatrix} \xi_1 & x_3 & \bar{x}_2 \\ \bar{x}_3 & \xi_2 & x_1 \\ x_2 & \bar{x}_1 & \xi_3 \end{pmatrix} = \begin{pmatrix} \theta^4 \xi_1 & \theta x_3 & \theta \bar{x}_2 \\ \theta \bar{x}_3 & \theta^{-2} \xi_2 & \theta^{-2} x_1 \\ \theta x_2 & \theta^{-2} \bar{x}_1 & \theta^{-2} \xi_3 \end{pmatrix}.$$

THEOREM 3.23. (1) $(E_6^C)_{\text{ev}} \cong C^* \times Spin(10, C) / \mathbf{Z}_4$, $\mathbf{Z}_4 = \{(1, 1), (-1, \sigma), (i, \phi(-i)), (-i, \phi(i))\}$.

(2) $(E_6^C)_0 \cong (C^* \times C^* \times Spin(8, C)) / (\mathbf{Z}_4 \times \mathbf{Z}_2)$, $\mathbf{Z}_4 = \{(1, 1, 1), (-1, -1, 1), (i, e_1, \phi(-i)D(-e_1)), (-i, -e_1, \phi(i)D(e_1))\}$, $\mathbf{Z}_2 = \{(1, 1, 1), (1, -1, \sigma)\}$.

PROOF. (1) Let $Spin(10, C) = (E_6^C)_{E_1}$ (Proposition 3.22.(2)). We define $\varphi : C^* \times Spin(10, C) \rightarrow (E_6^C)^\sigma$ by

$$\varphi(\theta, \beta) = \phi(\theta)\beta.$$

φ is well-defined. In fact, $(E_6^C)_{E_1} \subset (E_6^C)^\sigma$ (because $(\mathfrak{e}_6^C)_{E_1} \subset (\mathfrak{e}_6^C)_{\text{ev}}$ (Theorem 3.21)) and $\phi(\theta) \in (E_6^C)^\sigma$. Since $\phi(\theta)$ and $\beta \in \text{Spin}(10, C)$ commute, ϕ is a homomorphism. $\text{Ker } \varphi = \mathbf{Z}_4$. Since $(E_6^C)^\sigma$ is connected and $\dim_C(C \oplus \mathfrak{spin}(10, C)) = 1 + 45 = 46 = 30 + 2 \times 8 = \dim_C((\mathfrak{e}_6^C)_{\text{ev}})$ (Theorem 3.21), φ is onto. Therefore $(E_6^C)_{\text{ev}} \cong (C^* \times \text{Spin}(10, C))/\mathbf{Z}_4$.

(2) Let $\text{Spin}(8, C) = (E_6^C)_{E_1, F_1(1), F_1(e_1)}$ (Proposition 3.22.(1)). We define $\varphi : C^* \times U(1, C^C) \times \text{Spin}(8, C) \rightarrow (E_6^C)^{\sigma_3}$ by

$$\varphi(\theta, a, \beta) = \phi(\theta)D(a)\beta.$$

φ is well-defined (because $\sigma_3 = \varphi(1, \omega_1, 1)$). Since $D(a)$ commute with $\phi(\theta)$ and β (because iG_{01} is a center of $(\mathfrak{e}_6^C)_0$ (Theorem 3.21)), φ is a homomorphism. $\text{Ker } \varphi$ is $\{(1, 1, 1), (-1, -1, 1), (1, -1, \sigma), (-1, 1, \sigma), (i, e_1, \varphi(-i)D(-e_1)), (i, -e_1, \phi(-i)D(e_1)), (-i, -e_1, \phi(i)D(e_1)), (-i, e_1, \phi(i)D(-e_1))\} = \mathbf{Z}_4 \times \mathbf{Z}_2$. Since $(E_6^C)^{\sigma_3}$ is connected and $\dim_C(C \oplus \mathfrak{u}(1, C^C) \oplus \mathfrak{spin}(8, C)) = 1 + 1 + 28 = 30 = \dim_C((\mathfrak{e}_6^C)_0)$ (Theorem 3.21), φ is onto. Therefore $(E_6^C)_0 \cong (C^* \times U(1, C^C) \times \text{Spin}(8, C))/(\mathbf{Z}_4 \times \mathbf{Z}_2) \cong (C^* \times C^* \times \text{Spin}(8, C))/(\mathbf{Z}_4 \times \mathbf{Z}_2)$.

3.4.1. Subgroups of type $R \oplus D_{5(5)}$ and $R \oplus R \oplus D_{4(4)}$ of $E_{6(6)}$

We shall determine the group structures of

$$(E_{6(6)})_{\text{ev}} = (E_6^C)^\sigma \cap (E_6^C)^{\tau\gamma_1}, \quad (E_{6(6)})_0 = (E_6^C)^{\sigma_3} \cap (E_6^C)^{\tau\gamma_1}.$$

We define $\rho \in E_6 \subset E_6^C$ by $\rho = \varphi(1, I_2')$ where $I_2' = \text{diag}(1, -1, 1, -1, 1, 1)$ and $\varphi : Sp(1, \mathbf{H}^C) \times SL(6, C) \rightarrow E_6^C$ is the mapping defined in Theorem 3.5.(1). The explicit form of ρ is

$$\rho \begin{pmatrix} \xi_1 & x_3 & \overline{x_2} \\ \overline{x_3} & \xi_2 & x_1 \\ x_2 & \overline{x_1} & \xi_3 \end{pmatrix} = \begin{pmatrix} -\xi_1 & e_1 x_3 e_1 & -i \overline{x_2} \overline{e_1} \\ \overline{e_1} x_3 e_1 & -\xi_2 & i e_1 x_1 \\ -i x_2 e_1 & i \overline{e_1} \overline{x_1} & \xi_3 \end{pmatrix}.$$

LEMMA 3.24. In the group E_6^C , we have

- (1) $\tau\phi(\theta)\tau = \phi(\tau\theta), \quad \gamma_1\phi(\theta)\gamma_1 = \phi(\theta).$
- (2) $\tau\rho\tau = \sigma'\rho, \quad \gamma_1\rho\gamma_1 = \sigma'\rho.$

THEOREM 3.25. (1) $(E_{6(6)})_{\text{ev}} \cong (\mathbf{R}^+ \times \text{spin}(5, 5)) \times \{1, \rho\}.$

(2) $(E_{6(6)})_0 \cong (\mathbf{R}^+ \times \mathbf{R}^+ \times \text{spin}(4, 4)) \times (\{1, \sigma'\} \times \{1, \rho\}).$

PROOF. (1) For $\alpha \in (E_{6(6)})_{\text{ev}} \subset (E_6^C)^\sigma$, there exist $\theta \in C^*$ and $\beta \in \text{Spin}(10, C)$ such that $\alpha = \varphi(\theta, \beta) = \phi(\theta)\beta$ (Theorem 3.23.(1)). From $\gamma_1\tau\alpha\tau\gamma_1 = \alpha$, we have

$$(i) \begin{cases} \gamma_1\tau\phi(\theta)\tau\gamma_1 = \phi(\theta) \\ \gamma_1\tau\beta\tau\gamma_1 = \beta \end{cases} \quad (ii) \begin{cases} \phi(-\theta) \\ \sigma\beta \end{cases} \quad (iii) \begin{cases} \phi(i\theta) \\ \phi(-i)\beta \end{cases} \quad (iv) \begin{cases} \phi(-i\theta) \\ \phi(i)\beta \end{cases}$$

(i) From $\phi(\tau\theta) = \phi(\theta)$ (Lemma 3.24.(1)), we have $\tau\theta = \theta$, hence $\theta \in \mathbf{R}^*$. The group $(Spin(10, C))^{\tau\gamma_1} = ((E_6^C)_{E_1})^{\tau\gamma_1}$ (Proposition 3.22.(2)) $= ((E_6^C)^{\tau\gamma_1})_{E_1}$ acts on the $\mathbf{R}$ -vector space

$$\begin{aligned} V^{5,5} &= \{X \in (\mathfrak{J}^C)_{\tau\gamma_1} \mid 4E_1 \times (E_1 \times X) = X\} \\ &= \{X = \xi E_2 + \eta E_3 + F_1(x) \mid \xi, \eta \in \mathbf{R}, x \in (\mathfrak{C}^C)_{\tau\gamma_1} = \mathfrak{C}'\} \end{aligned}$$

with the norm $(E_1, X, X) = \xi\eta - x\bar{x}$. Hence $(Spin(10, C))^{\tau\gamma_1}$ is $spin(5, 5)$, in a similar way as in Theorem 2.12.(2), as a double covering group of $O(5, 5)^0 = O(V^{5,5})^0$. Therefore the group of the case (i) is $(\mathbf{R}^* \times spin(5, 5))/\mathbf{Z}_2$ ($\mathbf{Z}_2 = \{(1, 1), (-1, \sigma)\} \cong \mathbf{R}^+ \times spin(5, 5)$).

(ii) $\tau\theta = -\theta$, $\gamma_1\tau\beta\tau\gamma_1 = \sigma\beta$. Then $\theta = i$ and $\beta = \phi(-i)$ satisfy these conditions and $\varphi(i, \phi(-i)) = 1$.

$$(iii) \quad \varphi\left(\frac{1-i}{\sqrt{2}}, \rho\phi\left(\frac{1+i}{\sqrt{2}}\right)\right) = \rho. \quad (iv) \quad \varphi\left(\frac{1+i}{\sqrt{2}}, \rho\phi\left(\frac{1-i}{\sqrt{2}}\right)\right) = \rho.$$

Therefore $(E_{6(6)})_{ev} \cong (\mathbf{R}^+ \times spin(5, 5)) \times \{1, \rho\}$.

(2) For $\alpha \in (E_{6(6)})_{ev} \subset (E_6^C)^{\sigma_3}$, there exist $\theta \in C^*$, $a \in U(1, C^C)$ and $\beta \in Spin(8, C)$ such that $\alpha = \varphi(\theta, a, \beta) = \phi(\theta)D(a)\beta$ (Theorem 3.23.(2)). From $\gamma_1\tau\alpha\tau\gamma_1 = \alpha$, we have

$$\begin{aligned} (i) \quad & \begin{cases} \gamma_1\tau\phi(\theta)\tau\gamma_1 = \phi(\theta) \\ \gamma_1\tau D(a)\tau\gamma_1 = D(a) \\ \gamma_1\tau\beta\tau\gamma_1 = \beta \end{cases} & (ii) \quad & \begin{cases} \phi(-\theta) \\ D(-a) \\ \beta \end{cases} & (iii) \quad & \begin{cases} \phi(\theta) \\ D(-a) \\ \sigma\beta \end{cases} \\ (iv) \quad & \begin{cases} \phi(-\theta) \\ D(a) \\ \sigma\beta \end{cases} & (v) \quad & \begin{cases} \phi(-i\theta) \\ D(-e_1a) \\ \phi(i)D(e_1)\beta \end{cases} & (vi) \quad & \begin{cases} \phi(-i\theta) \\ D(e_1a) \\ \phi(i)D(-e_1)\beta \end{cases} \\ (vii) \quad & \begin{cases} \phi(i\theta) \\ D(e_1a) \\ \phi(-i)D(-e_1)\beta \end{cases} & (viii) \quad & \begin{cases} \phi(i\theta) \\ D(-e_1a) \\ \phi(-i)D(e_1)\beta \end{cases} \end{aligned}$$

(i) From $\phi(\tau\theta) = \phi(\theta)$, $D(\tau\bar{a}) = D(a)$ (Lemma 3.24.(1)), we have $\tau\theta = \theta$, $\tau\bar{a} = a$, hence $\theta \in \mathbf{R}^*$, $a \in U(1, C') \cong \mathbf{R}^*$, respectively. The group $(Spin(8, C))^{\tau\gamma_1} = ((E_6^C)^{\tau\gamma_1})_{E_1, F_1(1), F_1(e_1)}$ (Proposition 3.22.(1)) acts on the $\mathbf{R}$ -vector space

$$\begin{aligned} V^{4,4} &= \{X \in (\mathfrak{J}^C)_{\tau\gamma_1} \mid 4E_1 \times (E_1 \times X) = X, (E_1, F_1(1), X) = (E_1, F_1(e_1), X) = 0\} \\ &= \{X = \xi E_2 + \eta E_3 + F_1(x) \mid \xi, \eta \in \mathbf{R}, x \in (\mathfrak{C}^C)_{\tau\gamma_1} = \mathfrak{C}', (1, x) = (e_1, x) = 0\} \end{aligned}$$

with the norm $(E_1, X, X) = \xi\eta - x\bar{x}$. Hence $(Spin(8, C))^{\tau\gamma_1}$ is $spin(4, 4)$, in a similar way as in (1). Therefore the group of the case (i) is $(\mathbf{R}^* \times \mathbf{R}^* \times spin(4, 4))/(\mathbf{Z}_2 \times \mathbf{Z}_2)$ ($\mathbf{Z}_2 \times \mathbf{Z}_2 = \{(1, 1, 1), (-1, -1, 1)\} \times \{(1, 1, 1), (1, -1, \sigma)\} \cong \mathbf{R}^+ \times \mathbf{R}^+ \times spin(4, 4)$).

(ii) $\theta = i$, $a = e_1$ and $\beta = \phi(-i)D(-e_1)$ satisfy these conditions and $\varphi(i, e_1, \phi(-i)D(-e_1)) = 1$.

Similarly, using Lemma 3.24, we have

$$(iii) \quad \varphi(1, e_1, \sigma' D(-e_1)) = \sigma'. \quad (iv) \quad \varphi(i, 1, \sigma' \phi(-i)) = \sigma'.$$

$$(v) \quad \varphi\left(\frac{1-i}{\sqrt{2}}, \frac{1+e_1}{\sqrt{2}}, \rho\phi\left(\frac{1+i}{\sqrt{2}}\right)D\left(\frac{1-e_1}{\sqrt{2}}\right)\right) = \rho.$$

$$(vi) \quad \varphi\left(\frac{1+i}{\sqrt{2}}, \frac{1-e_1}{\sqrt{2}}, \rho\phi\left(\frac{1-i}{\sqrt{2}}\right)D\left(\frac{1+e_1}{\sqrt{2}}\right)\right) = \rho.$$

$$(vii) \quad \varphi\left(\frac{1-i}{\sqrt{2}}, \frac{1-e_1}{\sqrt{2}}, \sigma'\rho\phi\left(\frac{1+i}{\sqrt{2}}\right)D\left(\frac{1+e_1}{\sqrt{2}}\right)\right) = \sigma'\rho.$$

$$(viii) \quad \varphi\left(\frac{1+i}{\sqrt{2}}, \frac{1+e_1}{\sqrt{2}}, \sigma'\rho\phi\left(\frac{1-i}{\sqrt{2}}\right)D\left(\frac{1-e_1}{\sqrt{2}}\right)\right) = \sigma'\rho.$$

Therefore $(E_{6(6)})_0 \cong (\mathbf{R}^+ \times \mathbf{R}^+ \times \text{spin}(4, 4)) \times \{1, \sigma', \rho, \sigma'\rho\}$.

3.4.2. Subgroups of type $iR \oplus D_{5(3)}$ and $iR \oplus R \oplus D_{4(2)}$ of $E_{6(2)}$

THEOREM 3.26. The 2-graded decomposition of $\mathfrak{e}_{6(2)} = (\mathfrak{e}_6^C)^{\lambda\tau\gamma_1}$ (or $\mathfrak{e}_6^C$),

$$\mathfrak{e}_{6(2)} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2$$

with respect to $\text{ad } Z$, $Z = -2iG_{01}$, is given by

$$\begin{aligned} \mathfrak{g}_0 &= \left\{ \begin{array}{llll} iG_{01}, & iG_{23}, & G_{24}, & iG_{25}, & G_{26}, & iG_{27}, & \tilde{A}_1(e_2), & i\tilde{F}_1(e_2), \\ & iG_{34}, & G_{35}, & iG_{36}, & G_{37}, & i\tilde{A}_1(e_3), & \tilde{F}_1(e_3), \\ & iG_{45}, & G_{46}, & iG_{47}, & \tilde{A}_1(e_4), & i\tilde{F}_1(e_4), \\ & iG_{56}, & G_{57}, & i\tilde{A}_1(e_5), & \tilde{F}_1(e_5), \\ & iG_{67}, & \tilde{A}_1(e_6), & i\tilde{F}_1(e_6), \\ i(2E_1 - E_2 - E_3)^\sim, & i(E_2 - E_3)^\sim, & i\tilde{A}_1(e_7), & \tilde{F}_1(e_7) \end{array} \right\} 30 \\ \mathfrak{g}_{-1} &= \left\{ \begin{array}{llll} \tilde{A}_2(1 + ie_1), & \tilde{A}_2(e_2 + ie_3), & \tilde{A}_2(e_4 + ie_5), & \tilde{A}_2(e_6 + ie_7), \\ \tilde{A}_3(1 + ie_1), & \tilde{A}_3(e_2 - ie_3), & \tilde{A}_3(e_4 - ie_5), & \tilde{A}_3(e_6 - ie_7), \\ i\tilde{F}_2(1 + ie_1), & i\tilde{F}_2(e_2 + ie_3), & i\tilde{F}_2(e_4 + ie_5), & i\tilde{F}_2(e_6 + ie_7), \\ i\tilde{F}_3(1 + ie_1), & i\tilde{F}_3(e_2 - ie_3), & i\tilde{F}_3(e_4 - ie_5), & i\tilde{F}_3(e_6 - ie_7) \end{array} \right\} 16 \\ \mathfrak{g}_{-2} &= \left\{ \begin{array}{ll} G_{02} - iG_{12}, & iG_{03} + G_{13}, & G_{04} - iG_{14} & \tilde{A}_1(1 - ie_1), \\ iG_{05} + G_{15}, & G_{06} - iG_{16}, & G_{07} + iG_{17} & i\tilde{F}_1(1 - ie_1) \end{array} \right\} 8 \\ \mathfrak{g}_1 &= \tau(\mathfrak{g}_{-1})\tau, \quad \mathfrak{g}_2 = \tau(\mathfrak{g}_{-2})\tau. \end{aligned}$$

PROOF. We can prove this theorem in a way similar to Theorem 3.21, using Lemma 3.8.

We shall determine the group structures of

$$(E_{6(2)})_{\text{ev}} = (E_6^C)^\sigma \cap (E_6^C)^{\lambda\tau\gamma_1}, \quad (E_{6(2)})_0 = (E_6^C)^{\sigma_3} \cap (E_6^C)^{\lambda\tau\gamma_1}.$$

LEMMA 3.27. *In the group E_6^C , we have*

$${}^t\phi(\theta)^{-1} = \phi(\theta^{-1}).$$

THEOREM 3.28. (1) $(E_{6(2)})_{\text{ev}} \cong (U(1) \times \text{spin}(6, 4))/\mathbf{Z}_4$, $\mathbf{Z}_4 = \{(1, 1), (-1, \sigma), (i, \phi(-i)), (-i, \phi(i))\}$.
 (2) $(E_{6(2)})_0 \cong (U(1) \times \mathbf{R}^+ \times \text{spin}(5, 3))/\mathbf{Z}_2 \times \{1, \sigma'\}$, $\mathbf{Z}_2 = \{(1, 1, 1), (-1, 1, \sigma)\}$.

PROOF. (1) For $\alpha \in (E_{6(2)})_{\text{ev}} \subset (E_6^C)^\sigma$, there exist $\theta \in C^*$ and $\beta \in \text{Spin}(10, C)$ such that $\alpha = \phi(\theta)\beta$ (Theorem 3.23.(1)). From $\gamma_1\tau^t\alpha^{-1}\tau\gamma_1 = \alpha$, we have

$$\gamma_1\tau^t\phi(\theta)^{-1}\tau\gamma_1 = \phi(\theta), \quad \phi(-\theta), \quad \phi(i\theta) \quad \text{or} \quad \phi(-i\theta).$$

Hence $\tau\theta^{-1} = \theta, -\theta, i\theta$ or $-i\theta$ (Lemmas 3.24.(1), 3.27), but the last three cases are impossible. Hence $\theta \in U(1) = \{\theta \in C \mid \theta(\tau\theta) = 1\}$. The group $(\text{Spin}(10, C))^{\lambda\tau\gamma_1} = ((E_6^C)^{\lambda\tau\gamma_1})_{E_1}$ (Proposition 3.22.(2)) acts on the $\mathbf{R}$ -vector space

$$\begin{aligned} V^{6,4} &= \{X \in \mathfrak{J}^C \mid 2E_1 \times X = -\tau\gamma_1 X\} \\ &= \{X = \xi E_2 - \tau\xi E_3 + F_1(x) \mid \xi \in C, x \in (\mathfrak{C}^C)_{\tau\gamma_1} = \mathfrak{C}'\} \end{aligned}$$

with the norm $(E_1, X, X) = -\xi(\tau\xi) - x\bar{x}$. Hence $(\text{Spin}(10, C))^{\lambda\tau\gamma_1}$ is $\text{spin}(6, 4)$, in a similar way as in Theorem 2.12.(1). Therefore $(E_{6(2)})_{\text{ev}} \cong (U(1) \times \text{spin}(6, 4))/\mathbf{Z}_4$.

(2) For $\alpha \in (E_{6(2)})_0 \subset (E_6^C)^{\sigma_3}$, there exist $\theta \in C^*$, $a \in U(1, \mathbf{C}^C)$ and $\beta \in \text{Spin}(8, C)$ such that $\alpha = \varphi(\theta, a, \beta) = \phi(\theta)D(a)\beta$ (Theorem 3.23.(2)). From $\gamma_1\tau^t\alpha^{-1}\tau\gamma_1 = \alpha$, in a similar way as in (1), we have $\theta \in U(1)$ and

$$\begin{cases} \gamma_1\tau^t D(a)^{-1}\tau\gamma_1 = D(a) \\ \gamma_1\tau^t \beta^{-1}\tau\gamma_1 = \beta \end{cases} \quad \text{or} \quad \begin{cases} \gamma_1\tau^t D(a)^{-1}\tau\gamma_1 = D(-a) \\ \gamma_1\tau^t \beta^{-1}\tau\gamma_1 = \sigma\beta. \end{cases}$$

In the former case, from $D(\tau\bar{a}) = D(a)$ (Lemma 2.11), we have $\tau a = \bar{a}$, hence $a \in U(1, \mathbf{C}') \cong \mathbf{R}^*$. The group $(\text{Spin}(8, C))^{\lambda\tau\gamma_1} = ((E_6^C)^{\lambda\tau\gamma_1})_{E_1, F_1(1), F_1(e_1)}$ (Proposition 3.22.(1)) acts on the $\mathbf{R}$ -vector space

$$\begin{aligned} V^{5,3} &= \{X \in \mathfrak{J}^C \mid 2E_1 \times X = -\gamma_1\tau X, (E_1, F_1(1), X) = (E_1, F_1(e_1), X) = 0\} \\ &= \{X = \xi E_2 - \tau\xi E_3 + F_1(x) \mid \xi \in C, x \in (\mathfrak{C}^C)_{\tau\gamma_1} = \mathfrak{C}', (1, x) = (e_1, x) = 0\} \end{aligned}$$

with the norm $(E_1, X, X) = -\xi(\tau\xi) - x\bar{x}$. Hence $(\text{Spin}(8, C))^{\lambda\tau\gamma_1}$ is $\text{spin}(5, 3)$, in a similar way as in Theorem 2.12.(1). Therefore the group of the former case is

$(U(1) \times \mathbf{R}^* \times \text{spin}(5, 3))/(\mathbf{Z}_2 \times \mathbf{Z}_2)$ ($\mathbf{Z}_2 \times \mathbf{Z}_2 = \{(1, 1, 1), (-1, -1, 1)\} \times \{(1, 1, 1), (-1, 1, \sigma)\}$) $\cong (U(1) \times \mathbf{R}^+ \times \text{spin}(5, 3))/\mathbf{Z}_2$. In the latter case, $\theta = 1$, $a = e_1$ and $\beta = \sigma' D(-e_1)$ satisfy these conditions and $\varphi(1, e_1, \sigma' D(-e_1)) = \sigma'$. Therefore $(E_{6(2)})_0 \cong (U(1) \times \mathbf{R}^+ \times \text{spin}(5, 3))/\mathbf{Z}_2 \times \{1, \sigma'\}$.

3.4.3. Subgroups of type $iR \oplus D_{5(-13)}$ and $iR \oplus R \oplus D_{4(-14)}$ of $E_{6(-14)}$

LEMMA 3.29. In the Lie algebra $\mathfrak{e}_6^C$, we have

$$\begin{aligned} \sigma_1 \tilde{F}_1(a) \sigma_1 &= \tilde{F}_1(\overline{e_1 a e_1}), & \sigma_1 \tilde{F}_2(a) \sigma_1 &= \tilde{F}_3(\overline{e_1 a}), & \sigma_1 \tilde{F}_3(a) \sigma_1 &= \tilde{F}_2(\overline{a e_1}), \\ \sigma_1 (\xi_1 E_1 + \xi_2 E_2 + \xi_3 E_3)^\sim \sigma_1 &= (\xi_1 E_1 + \xi_2 E_2 + \xi_3 E_3)^\sim, & \xi_1 + \xi_2 + \xi_3 &= 0. \end{aligned}$$

THEOREM 3.30. The 2-graded decomposition of $\mathfrak{e}_{6(-14)} = (\mathfrak{e}_6^C)^{\lambda_{\tau\sigma_1}}$ (or $\mathfrak{e}_6^C$),

$$\mathfrak{e}_{6(-14)} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2$$

with respect to $\text{ad } Z$, $Z = -2iG_{01}$, is given by

$$\begin{aligned} \mathfrak{g}_0 &= \left\{ \begin{array}{cc} iG_{01}, G_{23}, G_{24}, G_{25}, G_{26}, G_{27}, & \tilde{A}_1(e_2), \tilde{F}_1(e_2), \\ & G_{34}, G_{35}, G_{36}, G_{37}, \tilde{A}_1(e_3), \tilde{F}_1(e_3), \\ & G_{45}, G_{46}, G_{47}, \tilde{A}_1(e_4), \tilde{F}_1(e_4), \\ & G_{56}, G_{57}, \tilde{A}_1(e_5), \tilde{F}_1(e_5), \\ & G_{67}, \tilde{A}_1(e_6), \tilde{F}_1(e_6), \\ i(2E_1 - E_2 - E_3)^\sim, (E_2 - E_3)^\sim, & \tilde{A}_1(e_7), \tilde{F}_1(e_7) \end{array} \right\} 30 \\ \mathfrak{g}_{-1} &= \left\{ \begin{array}{cc} \tilde{A}_2(1 + ie_1) - i\tilde{A}_3(1 + ie_2), & i\tilde{A}_2(1 + ie_1) - \tilde{A}_3(1 + ie_2), \\ \tilde{A}_2(e_2 + ie_3) + i\tilde{A}_3(e_2 - ie_3), & i\tilde{A}_2(e_2 + ie_3) + \tilde{A}_3(e_2 - ie_3), \\ \tilde{A}_2(e_4 + ie_5) + i\tilde{A}_3(e_4 - ie_5), & i\tilde{A}_2(e_4 + ie_5) + \tilde{A}_3(e_4 - ie_5), \\ \tilde{A}_2(e_6 + ie_7) + i\tilde{A}_3(e_6 - ie_7), & i\tilde{A}_2(e_6 + ie_7) + \tilde{A}_3(e_6 - ie_7), \\ \tilde{F}_2(1 + ie_1) - i\tilde{F}_3(1 + ie_2), & i\tilde{F}_2(1 + ie_1) - \tilde{F}_3(1 + ie_2), \\ \tilde{F}_2(e_2 + ie_3) + i\tilde{F}_3(e_2 - ie_3), & i\tilde{F}_2(e_2 + ie_3) + \tilde{F}_3(e_2 - ie_3), \\ \tilde{F}_2(e_4 + ie_5) + i\tilde{F}_3(e_4 - ie_5), & i\tilde{F}_2(e_4 + ie_5) + \tilde{F}_3(e_4 - ie_5), \\ \tilde{F}_2(e_6 + ie_7) + i\tilde{F}_3(e_6 - ie_7), & i\tilde{F}_2(e_6 + ie_7) + \tilde{F}_3(e_6 - ie_7) \end{array} \right\} 16 \\ \mathfrak{g}_{-2} &= \left\{ \begin{array}{cc} G_{02} - iG_{12}, & G_{03} - iG_{13}, & G_{04} - iG_{14}, & \tilde{A}_1(1 - ie_1), \\ G_{05} - G_{15}, & G_{06} - iG_{16}, & G_{07} - iG_{17}, & \tilde{F}_1(1 - ie_1) \end{array} \right\} 8 \\ \mathfrak{g}_1 &= \tau(\mathfrak{g}_{-1})\tau, & \mathfrak{g}_2 &= \tau(\mathfrak{g}_{-2})\tau. \end{aligned}$$

PROOF. We can prove this theorem in a way similar to Theorem 2.15, using Lemmas 2.14 and 3.29.

We shall determine the group structures of

$$(E_{6(-14)})_{\text{ev}} = (E_6^C)^\sigma \cap (E_6^C)^{\lambda\tau\sigma_1}, \quad (E_{6(-14)})_0 = (E_6^C)^{\sigma_3} \cap (E_6^C)^{\lambda\tau\sigma_1}.$$

LEMMA 3.31. *In the group E_6^C , we have*

$$\sigma_1\phi(\theta)\sigma_1 = \phi(\theta).$$

THEOREM 3.32. (1) $(E_{6(-14)})_{\text{ev}} \cong (U(1) \times \text{spin}(2, 8))/\mathbf{Z}_4$, $\mathbf{Z}_4 = \{(1, 1), (-1, \sigma), (i, \phi(-i)), (-i, \phi(i))\}$.

(2) $(E_{6(-14)})_0 \cong (U(1) \times \mathbf{R}^+ \times \text{Spin}(1, 7))/\mathbf{Z}_2 \times \{1, \sigma'\}$, $\mathbf{Z}_2 = \{(1, 1, 1), (-1, 1, \sigma)\}$.

PROOF. (1) For $\alpha \in (E_{6(-14)})_{\text{ev}} \subset (E_6^C)^\sigma$, there exist $\theta \in C^*$ and $\beta \in \text{Spin}(10, C)$ such that $\alpha = \phi(\theta)\beta$ (Theorem 3.23.(1)). From $\sigma_1\tau^t\alpha^{-1}\tau\sigma_1 = \alpha$, we have

$$\sigma_1\tau^t\phi(\theta)^{-1}\tau\sigma_1 = \phi(\theta), \quad \phi(-\theta), \quad \phi(i\theta) \quad \text{or} \quad \phi(-i\theta).$$

Hence $\tau\theta^{-1} = \theta, -\theta, i\theta$ or $-i\theta$ (Lemmas 3.24.(1), 3.27, 3.31), but the last three cases are impossible. Hence $\theta \in U(1) = \{\theta \in C \mid \theta(\tau\theta) = 1\}$. The group $(\text{Spin}(10, C))^{\lambda\tau\sigma_1} = ((E_6^C)^{\lambda\tau\sigma_1})_{E_1}$ acts on the $\mathbf{R}$ -vector space

$$\begin{aligned} V^{2,8} &= \{X \in \mathfrak{J}^C \mid 2E_1 \times X = -\tau\sigma_1 X\} \\ &= \left\{ X = i\xi E_2 + i\eta E_3 + F_1(x) \mid \begin{array}{l} \xi, \eta \in \mathbf{R}, x = x_1 e_1 + iy, \\ x_1 \in \mathbf{R}, y \in \mathfrak{C}, (e_1, y) = 0 \end{array} \right\} \end{aligned}$$

with the norm $(E_1, X, X) = -\xi\eta - x\bar{x} = -\xi\eta - x_1^2 + y\bar{y}$. Hence $(\text{Spin}(10, C))^{\lambda\tau\sigma_1}$ is $\text{spin}(2, 8)$, in a similar way as in Theorem 2.12.(1). Therefore $(E_{6(-14)})_{\text{ev}} \cong (U(1) \times \text{spin}(2, 8))/\mathbf{Z}_4$.

(2) For $\alpha \in (E_{6(-14)})_0 \subset (E_6^C)^{\sigma_3}$, there exist $\theta \in C^*$, $a \in U(1, C^C)$ and $\beta \in \text{Spin}(8, C)$ such that $\alpha = \varphi(\theta, a, \beta) = \phi(\theta)D(a)\beta$ (Theorem 3.23.(2)). From $\sigma_1\tau^t\alpha^{-1}\tau\sigma_1 = \alpha$, in a similar way as in (1), we have $\theta \in U(1)$ and

$$\left\{ \begin{array}{l} \sigma_1\tau^t D(a)^{-1}\tau\sigma_1 = D(a) \\ \sigma_1\tau^t \beta^{-1}\tau\sigma_1 = \beta \end{array} \right. \quad \text{or} \quad \left\{ \begin{array}{l} \sigma_1\tau^t D(a)^{-1}\tau\sigma_1 = D(-a) \\ \sigma_1\tau^t \beta^{-1}\tau\sigma_1 = \sigma\beta. \end{array} \right.$$

In the former case, from $D(\tau\bar{a}) = D(a)$, (Lemmas 2.11, 2.16), we have $\tau\bar{a} = a$, hence $a \in U(1, C') \cong \mathbf{R}^*$. The group $(\text{Spin}(8, C))^{\lambda\tau\sigma_1} = ((E_6^C)^{\lambda\tau\sigma_1})_{E_1, F_1(1), F_1(e_1)}$ (Proposition 3.22.(1)) acts on the $\mathbf{R}$ -vector space

$$\begin{aligned} V^{1,7} &= \{X \in \mathfrak{J}^C \mid 2E_1 \times X = -\tau\sigma_1 X, (E_1, F_1(1), X) = (E_1, F_1(e_1), X) = 0\} \\ &= \{X = i\xi E_2 + i\eta E_3 + F_1(iy) \mid \xi, \eta \in \mathbf{R}, y \in \mathfrak{C}, (1, y) = (e_1, y) = 0\} \end{aligned}$$

with the norm $(E_1, X, X) = -\xi\eta + y\bar{y}$. Hence $(Spin(8, C))^{\lambda\tau\sigma_1}$ is $Spin(1, 7)$, in a similar way as in Theorem 2.12. Therefore the group of the former case is $(U(1) \times \mathbf{R}^* \times Spin(1, 7))/(\mathbf{Z}_2 \times \mathbf{Z}_2) (\mathbf{Z}_2 \times \mathbf{Z}_2 = \{(1, 1, 1), (-1, -1, 1)\} \times \{(1, 1, 1), (-1, 1, \sigma)\}) \cong (U(1) \times \mathbf{R}^+ \times Spin(1, 7))/\mathbf{Z}_2$. In the latter case, $\theta = 1$, $a = e_1$ and $\beta = \sigma'D(-e_1)$ satisfy these conditions and $\varphi(1, e_1, \sigma'D(-e_1)) = \sigma'$. Therefore $(E_{6(-14)})_0 \cong (U(1) \times \mathbf{R}^+ \times Spin(1, 7))/\mathbf{Z}_2 \times \{1, \sigma'\}$.

3.4.4. Subgroups of type $R \oplus D_{5(-27)}$ and $R \oplus R \oplus D_{4(-28)}$ of $E_{6(-26)}$

Define $\delta_1 \in E_6 \subset E_6^C$ by $\delta_1 = \exp \frac{\pi}{2} i \tilde{F}_1(1)$ (which is similar to the one defined in Section 3.2.3). We define a complex-conjugate linear transformation τ_1 of $\mathfrak{J}^C$ by

$$\tau_1 X = \delta_1^{-1} \tau \delta_1 \begin{pmatrix} \xi_1 & x_3 & \bar{x}_2 \\ \bar{x}_3 & \xi_2 & x_1 \\ x_2 & \bar{x}_1 & \xi_3 \end{pmatrix} = \begin{pmatrix} \tau\xi_1 & -i\tau\bar{x}_2 & -i\tau x_3 \\ -i\tau x_2 & -\tau\xi_3 & -\tau\bar{x}_1 \\ -i\tau\bar{x}_3 & -\tau x_1 & -\tau\xi_2 \end{pmatrix}.$$

Since τ and τ_1 are related with $\tau_1 = \delta_1^{-1} \tau \delta_1$, we have

PROPOSITION 3.33. $E_{6(-26)} = (E_6^C)^\tau \cong (E_6^C)^{\tau_1}$.

LEMMA 3.34. In the Lie algebra $\mathfrak{e}_6^C$, we have

- (1) $\tau_1 G_{0l} \tau_1 = -G_{0l}$, $\tau_1 G_{kl} \tau_1 = G_l$, $kl \neq 0$,
- (2) $\tau_1 \tilde{A}_1(a) \tau_1 = -\tilde{A}_1(\tau\bar{a})$, $\tau_1 \tilde{A}_2(a) \tau_1 = -i\tilde{F}_3(\tau\bar{a})$, $\tau_1 \tilde{A}_3(a) \tau_1 = i\tilde{F}_2(\tau\bar{a})$,
 $\tau_1 \tilde{F}_1(a) \tau_1 = \tilde{F}_1(\tau\bar{a})$, $\tau_1 \tilde{F}_2(a) \tau_1 = i\tilde{A}_3(\tau\bar{a})$, $\tau_1 \tilde{F}_3(a) \tau_1 = -i\tilde{A}_2(\tau\bar{a})$,
- (3) $\tau_1(\xi_1 E_1 + \xi_2 E_2 + \xi_3 E_3)^\sim \tau_1 = ((\tau\xi_1)E_1 + (\tau\xi_3)E_2 + (\tau\xi_2)E_3)^\sim$, $\xi_1 + \xi_2 + \xi_3 = 0$.

THEOREM 3.35. The 2-graded decomposition of $\mathfrak{e}_{6(-26)} = (\mathfrak{e}_6^C)^{\tau_1}$ (or $\mathfrak{e}_6^C$),

$$\mathfrak{e}_{6(-26)} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2$$

with respect to $\text{ad } Z$, $Z = -2iG_{01}$, is given by

$$\mathfrak{g}_0 = \left\{ \begin{array}{ll} iG_{01}, G_{23}, G_{24}, G_{25}, G_{26}, G_{27}, & \tilde{A}_1(e_2), i\tilde{F}_1(e_2), \\ & G_{34}, G_{35}, G_{36}, G_{37}, \tilde{A}_1(e_3), i\tilde{F}_1(e_3), \\ & G_{45}, G_{46}, G_{47}, \tilde{A}_1(e_4), i\tilde{F}_1(e_4), \\ & G_{56}, G_{57}, \tilde{A}_1(e_5), i\tilde{F}_1(e_5), \\ & G_{67}, \tilde{A}_1(e_6), i\tilde{F}_1(e_6), \\ (2E_1 - E_2 - E_3)^\sim, i(E_2 - E_3)^\sim, & \tilde{A}_1(e_7), i\tilde{F}_1(e_7) \end{array} \right\} 30$$

$$\begin{aligned}
\mathfrak{g}_{-1} &= \left\{ \begin{array}{ll} \tilde{A}_2(1 + ie_1) - i\tilde{F}_3(1 + ie_2), & i\tilde{A}_2(1 + ie_1) - \tilde{F}_3(1 + ie_2), \\ \tilde{A}_2(e_2 + ie_3) + i\tilde{F}_3(e_2 - ie_3), & i\tilde{A}_2(e_2 + ie_3) + \tilde{F}_3(e_2 - ie_3), \\ \tilde{A}_2(e_4 + ie_5) + i\tilde{F}_3(e_4 - ie_5), & i\tilde{A}_2(e_4 + ie_5) + \tilde{F}_3(e_4 - ie_5), \\ \tilde{A}_2(e_6 + ie_7) + i\tilde{F}_3(e_6 - ie_7), & i\tilde{A}_2(e_6 + ie_7) + \tilde{F}_3(e_6 - ie_7), \\ \tilde{F}_2(1 + ie_1) + i\tilde{A}_3(1 + ie_2), & i\tilde{F}_2(1 + ie_1) + \tilde{A}_3(1 + ie_2), \\ \tilde{F}_2(e_2 + ie_3) - i\tilde{A}_3(e_2 - ie_3), & i\tilde{F}_2(e_2 + ie_3) - \tilde{A}_3(e_2 - ie_3), \\ \tilde{F}_2(e_4 + ie_5) - i\tilde{A}_3(e_4 - ie_5), & i\tilde{F}_2(e_4 + ie_5) - \tilde{A}_3(e_4 - ie_5), \\ \tilde{F}_2(e_6 + ie_7) - i\tilde{A}_3(e_6 - ie_7), & i\tilde{F}_2(e_6 + ie_7) - \tilde{A}_3(e_6 - ie_7) \end{array} \right\} 16 \\
\mathfrak{g}_{-2} &= \left\{ \begin{array}{ll} iG_{02} + G_{12}, & iG_{03} + G_{13}, & iG_{04} + G_{14}, & i\tilde{A}_1(1 - ie_1), \\ iG_{05} + G_{15}, & iG_{06} + G_{16}, & iG_{07} + iG_{17}, & \tilde{F}_1(1 - ie_1) \end{array} \right\} 8 \\
\mathfrak{g}_1 &= \tau(\lambda(\mathfrak{g}_{-1}))\tau, \quad \mathfrak{g}_2 = \tau(\lambda(\mathfrak{g}_{-2}))\tau.
\end{aligned}$$

PROOF. We can prove this theorem in a way similar to Theorem 3.21, using Lemma 3.34.

We shall determine the group structures of

$$(E_{6(-26)})_{\text{ev}} = (E_6^C)^\sigma \cap (E_6^C)^{\tau_1}, \quad (E_{6(-26)})_0 = (E_6^C)^{\sigma_3} \cap (E_6^C)^{\tau_1}.$$

LEMMA 3.36. In the group E_6^C , we have

$$\begin{aligned}
\tau_1 \phi(\theta) \tau_1 &= \phi(\tau\theta), & \tau_1 D(a) \tau_1 &= D(\tau\bar{a}), \\
\tau_1 \sigma' \tau_1 &= \sigma\sigma', & \tau_1 \rho \tau_1 &= \rho\phi(i)D(e_1).
\end{aligned}$$

THEOREM 3.37. (1) $(E_{6(-26)})_{\text{ev}} \cong \mathbf{R}^+ \times Spin(9, 1)$.

(2) $(E_{6(-26)})_0 \cong (\mathbf{R}^+ \times \mathbf{R}^+ \times Spin(8)) \times (\{1, \sigma'\} \times \{1, \rho\})$.

PROOF. (1) For $\alpha \in (E_{6(-26)})_{\text{ev}} \subset (E_6^C)^\sigma$, there exist $\theta \in C^*$ and $\beta \in Spin(10, C)$ such that $\alpha = \phi(\theta)\beta$ (Theorem 3.23.(1)). $(E_{6(-26)})_{\text{ev}} = (E_6^C)^\sigma \cap (E_6^C)^{\tau_1} = ((E_6^C)^{\tau_1})^\sigma$ (because σ and τ_1 commute) $\cong (E_{6(6)})^\sigma$ is connected. Hence, from $\tau_1 \alpha \tau_1 = \alpha$, we have only one case

$$\tau_1 \phi(\theta) \tau_1 = \phi(\theta), \quad \tau_1 \beta \tau_1 = \beta.$$

From $\phi(\tau\theta) = \phi(\theta)$ (Lemma 3.36), we have $\tau\theta = \theta$, hence $\theta \in \mathbf{R}^*$. The group $(Spin(10, C))^{\tau_1} = ((E_6^C)^{\tau_1})_{E_1}$ (Proposition 3.22.(2)) acts on the $\mathbf{R}$ -vector space

$$\begin{aligned}
V^{9,1} &= \{X \in (\mathfrak{J}^C)_{\tau_1} \mid 4E_1 \times (E_1 \times X) = X\} \\
&= \{X = \xi E_2 - \tau\xi E_3 + F_1(x) \mid \xi \in C, x = ix_1 + y, x_1 \in \mathbf{R}, y \in \mathfrak{C}, (1, y) = 0\}
\end{aligned}$$

with the norm $(E_1, X, X) = -\xi(\tau\xi) - x\bar{x} = -\xi(\tau\xi) + x_1^2 - y\bar{y}$. Hence $(Spin(10, C))^{\tau_1}$ is $Spin(9, 1)$, in a similar way as in Theorem 2.12. Therefore $(E_{6(-26)})_{\text{ev}} \cong (\mathbf{R}^* \times Spin(9, 1))/\mathbf{Z}_2$ ($\mathbf{Z}_2 = \{(1, 1), (-1, \sigma)\}$) $\cong \mathbf{R}^+ \times Spin(9, 1)$.

(2) For $\alpha \in (E_{6(-26)})_{\text{ev}} \subset (E_6^C)^{\sigma_3}$, there exist $\theta \in C^*$, $a \in U(1, C^C)$ and $\beta \in Spin(8, C)$ such that $\alpha = \varphi(\theta, a, \beta) = \phi(\theta)D(a)\beta$ (Theorem 3.23.(2)). From $\tau_1\alpha\tau_1 = \alpha$, we have

$$\begin{aligned} \text{(i)} \quad & \begin{cases} \tau_1\phi(\theta)\tau_1 = \phi(\theta) \\ \tau_1D(a)\tau_1 = D(a) \\ \tau_1\beta\tau_1 = \beta \end{cases} & \text{(ii)} \quad & \begin{cases} \phi(-\theta) \\ D(-a) \\ \beta \end{cases} & \text{(iii)} \quad & \begin{cases} \phi(\theta) \\ D(-a) \\ \sigma\beta \end{cases} \\ \text{(iv)} \quad & \begin{cases} \phi(-\theta) \\ D(a) \\ \sigma\beta \end{cases} & \text{(v)} \quad & \begin{cases} \phi(i\theta) \\ D(e_1a) \\ \phi(-i)D(-e_1)\beta \end{cases} & \text{(vi)} \quad & \begin{cases} \phi(-i\theta) \\ D(-e_1a) \\ \phi(i)D(e_1)\beta \end{cases} \\ \text{(vii)} \quad & \begin{cases} \phi(i\theta) \\ D(-e_1a) \\ \phi(-i)D(e_1)\beta \end{cases} & \text{(viii)} \quad & \begin{cases} \phi(-i\theta) \\ D(e_1a) \\ \phi(i)D(-e_1)\beta \end{cases} \end{aligned}$$

(i) From $\phi(\tau\theta) = \phi(\theta)$, $D(\tau\bar{a}) = D(a)$ (Lemma 3.36), we have $\tau\theta = \theta$, $\tau\bar{a} = a$, hence $\theta \in \mathbf{R}^*$, $a \in U(1, C') \cong \mathbf{R}^*$, respectively. The group $(Spin(8, C))^{\tau_1} = ((E_6^C)^{\tau_1})_{E_1, F_1(1), F_1(e_1)}$ acts (Proposition 3.22.(1)) on the $\mathbf{R}$ -vector space

$$\begin{aligned} V^8 &= \{X \in (\mathfrak{J}^C)_{\tau_1} \mid 4E_1 \times (E_1 \times X) = X, (E_1, F_1(1), X) = (E_1, F_1(e_1), X) = 0\} \\ &= \{X = \xi E_2 - \tau\xi E_3 + F_1(y) \mid \xi \in C, y \in \mathfrak{C}, (1, y) = (e_1, y) = 0\} \end{aligned}$$

with the norm $-(E_1, X, X) = \xi(\tau\xi) + y\bar{y}$. Hence $(Spin(8, C))^{\tau_1}$ is $Spin(8)$, in a similar way as in Theorem 2.12. Therefore the group of the case (i) is $(\mathbf{R}^* \times \mathbf{R}^* \times Spin(8))/(\mathbf{Z}_2 \times \mathbf{Z}_2)$ ($\mathbf{Z}_2 \times \mathbf{Z}_2 = \{(1, 1, 1), (-1, -1, 1)\} \times \{(1, 1, 1), (1, -1, \sigma)\}$) $\cong \mathbf{R}^+ \times \mathbf{R}^+ \times Spin(8)$.

$$\text{(ii)} \quad \varphi(i, e_1, \phi(-i)D(-e_1)) = 1.$$

$$\text{(iii)} \quad \varphi(1, e_1, \sigma'D(-e_1)) = \sigma' \text{ (Lemma 3.36).}$$

$$\text{(iv)} \quad \varphi(i, 1, \sigma'\phi(-i)) = \sigma'.$$

$$\text{(v)} \quad \varphi\left(\frac{1-i}{\sqrt{2}}, \frac{1-e_1}{\sqrt{2}}, \rho\phi\left(\frac{1+i}{\sqrt{2}}\right)D\left(\frac{1+e_1}{\sqrt{2}}\right)\right) = \rho \text{ (Lemma 3.36).}$$

$$\text{(vi)} \quad \varphi\left(\frac{1+i}{\sqrt{2}}, \frac{1+e_1}{\sqrt{2}}, \rho\phi\left(\frac{1-i}{\sqrt{2}}\right)D\left(\frac{1-e_1}{\sqrt{2}}\right)\right) = \rho.$$

$$\text{(vii)} \quad \varphi\left(\frac{1-i}{\sqrt{2}}, \frac{1+e_1}{\sqrt{2}}, \sigma'\rho\phi\left(\frac{1+i}{\sqrt{2}}\right)D\left(\frac{1-e_1}{\sqrt{2}}\right)\right) = \sigma'\rho \text{ (Lemma 3.36).}$$

$$\text{(viii)} \quad \varphi\left(\frac{1+i}{\sqrt{2}}, \frac{1-e_1}{\sqrt{2}}, \sigma'\rho\phi\left(\frac{1-i}{\sqrt{2}}\right)D\left(\frac{1+e_1}{\sqrt{2}}\right)\right) = \sigma'\rho.$$

Therefore $(E_{6(-26)})_0 \cong (\mathbf{R}^+ \times \mathbf{R}^+ \times Spin(8)) \times \{1, \sigma', \rho, \sigma'\rho\}$.

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