2-graded decompositions of exceptional Lie algebras $\mathfrak{g}$
and group realizations of $\mathfrak{g}_{ev}, \mathfrak{g}_0$
Part I, $G = G_2, F_4, E_6$

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The 2-graded decompositions of simple Lie algebras $\mathfrak{g}$,

$$\mathfrak{g} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2, \quad [\mathfrak{g}_k, \mathfrak{g}_l] \subseteq \mathfrak{g}_{k+l}$$

are classified and the types of subalgebras $\mathfrak{g}_{ev} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_2, \mathfrak{g}_0$ are determined. The following table is the results of $\mathfrak{g}_{ev}$, $\mathfrak{g}_0$ for exceptional Lie algebras $\mathfrak{g}$ of type $G_2, F_4$ and $E_6$ (Kaneyuki [2]). (The order is arranged for our uses).

<table>
<thead>
<tr>
<th>$\mathfrak{g}$</th>
<th>$\mathfrak{g}_{ev}$</th>
<th>$\mathfrak{g}_0$</th>
<th>dim $\mathfrak{g}_1$</th>
<th>dim $\mathfrak{g}_2$</th>
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<tbody>
<tr>
<td>$\mathfrak{g}_2$</td>
<td>$\mathfrak{sl}(2, C) \oplus \mathfrak{sl}(2, C)$</td>
<td>$\mathfrak{C} \oplus \mathfrak{sl}(2, C)$</td>
<td>4</td>
<td>1</td>
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<tr>
<td>$\mathfrak{g}_2(2)$</td>
<td>$\mathfrak{sl}(2, R) \oplus \mathfrak{sl}(2, R)$</td>
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<tr>
<td>$\mathfrak{f}_4$</td>
<td>$\mathfrak{sl}(2, C) \oplus \mathfrak{sp}(3, C)$</td>
<td>$\mathfrak{C} \oplus \mathfrak{sp}(3, C)$</td>
<td>14</td>
<td>1</td>
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<tr>
<td>$\mathfrak{f}_4$</td>
<td>$\mathfrak{so}(9, C)$</td>
<td>$\mathfrak{C} \oplus \mathfrak{so}(7, C)$</td>
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<td>7</td>
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<td>$\mathfrak{f}_4(4)$</td>
<td>$\mathfrak{so}(4, 5)$</td>
<td>$\mathfrak{R} \oplus \mathfrak{so}(3, 4)$</td>
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<td>7</td>
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<td>$\mathfrak{f}_4(-20)$</td>
<td>$\mathfrak{so}(8, 1)$</td>
<td>$\mathfrak{R} \oplus \mathfrak{so}(7)$</td>
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<td>7</td>
</tr>
<tr>
<td>$\mathfrak{e}_6$</td>
<td>$\mathfrak{sl}(2, C) \oplus \mathfrak{so}(6, C)$</td>
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<tr>
<td>$\mathfrak{e}_6(6)$</td>
<td>$\mathfrak{sl}(2, R) \oplus \mathfrak{so}(6, R)$</td>
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</tr>
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<td>$\mathfrak{e}_6(2)$</td>
<td>$\mathfrak{sl}(2, R) \oplus \mathfrak{su}(3, 3)$</td>
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<td>1</td>
</tr>
<tr>
<td>$\mathfrak{e}_6(-14)$</td>
<td>$\mathfrak{sl}(2, R) \oplus \mathfrak{su}(1, 5)$</td>
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Now, for the exceptional universal Lie groups $G$ of type $G_2$, $F_4$, and $E_6$, we realize the subgroups $G_{ev}, G_0$ of $G$ corresponding to the subalgebras $g_{ev}, g_0$ of $g = \text{Lie} G$. Our results are as follows.

\[
\begin{array}{lll}
G & G_{ev} & G_0 \\
G_2^C & (\text{Sp}(1, C) \times \text{Sp}(1, C))/\mathbb{Z}_2 & (\mathbb{C}^* \times \text{Sp}(1, C))/\mathbb{Z}_2 \\
G_{2(2)}^C & (\text{Sp}(1, \mathbb{R}) \times \text{Sp}(1, \mathbb{R}))/\mathbb{Z}_2 \times 2 & (\mathbb{R}^+ \times \text{Sp}(1, \mathbb{R})) \times 2 \\
F_4^C & (\text{Sp}(1, C) \times \text{Sp}(3, C))/\mathbb{Z}_2 & (\mathbb{C}^* \times \text{Sp}(3, C))/\mathbb{Z}_2 \\
F_{4(4)}^C & (\text{Sp}(1, \mathbb{R}) \times \text{Sp}(3, \mathbb{R}))/\mathbb{Z}_2 \times 2 & (\mathbb{R}^+ \times \text{Sp}(3, \mathbb{R})) \times 2 \\
F_4^C & \text{Spin}(9, C) & (\mathbb{C}^* \times \text{Spin}(7, C))/\mathbb{Z}_2 \\
F_{4(4)} & \text{Spin}(4, 5) & (\mathbb{R}^+ \times \text{Spin}(3, 4)) \times 2 \\
F_{4(10)} & \text{Spin}(8, 1) & (\mathbb{R}^+ \times \text{Spin}(7)) \times 2 \\
E_6^C & (\text{Sp}(1, C) \times \text{SL}(6, C))/\mathbb{Z}_2 & (\mathbb{C}^* \times \text{SL}(6, C))/\mathbb{Z}_2 \\
E_{6(6)}^C & (\text{Sp}(1, \mathbb{R}) \times \text{SL}(6, \mathbb{R}))/\mathbb{Z}_2 \times 2 & (\mathbb{R}^+ \times \text{SL}(6, \mathbb{R})) \times 2 \\
E_{6(2)}^C & (\text{Sp}(1, \mathbb{R}) \times \text{SU}(3, 3))/\mathbb{Z}_2 \times 2 & (\mathbb{R}^+ \times \text{SU}(3, 3)) \times 2 \\
E_{6(-14)}^C & (\text{Sp}(1, \mathbb{R}) \times \text{SU}(1, 5))/\mathbb{Z}_2 \times 2 & (\mathbb{R}^+ \times \text{SU}(1, 5)) \times 2 \\
E_6^C & (\text{Sp}(1, C) \times \text{SL}(6, C))/\mathbb{Z}_2 & (\text{Sp}(1, C) \times \text{SL}(5, C))/\mathbb{Z}_2 \times \mathbb{Z}_5 \\
E_{6(6)}^C & (\text{Sp}(1, \mathbb{R}) \times \text{SL}(6, \mathbb{R}))/\mathbb{Z}_2 \times 2 & (\text{Sp}(1, \mathbb{R}) \times \mathbb{R}^+ \times \text{SL}(5, \mathbb{R})) \times 2 \\
E_6^C & (\mathbb{C}^* \times \text{Spin}(10, C))/\mathbb{Z}_4 & (\mathbb{C}^* \times \text{Spin}(8, C))/\mathbb{Z}_2 \times \mathbb{Z}_4 \\
E_{6(6)} & (\mathbb{R}^+ \times \text{Spin}(5, 5)) \times 2 & (\mathbb{R}^+ \times \mathbb{R}^+ \times \text{Spin}(4, 4)) \times 2^2 \\
E_{6(2)} & (U(1) \times \text{Spin}(6, 4))/\mathbb{Z}_4 & (U(1) \times \mathbb{R}^+ \times \text{Spin}(5, 3))/\mathbb{Z}_2 \times 2 \\
E_{6(-14)} & (U(1) \times \text{Spin}(2, 8))/\mathbb{Z}_4 & (U(1) \times \mathbb{R}^+ \times \text{Spin}(1, 7))/\mathbb{Z}_2 \times 2 \\
E_{6(-26)} & \mathbb{R}^+ \times \text{Spin}(9, 1) & (\mathbb{R}^+ \times \mathbb{R}^+ \times \text{Spin}(8)) \times 2^2 \\
\end{array}
\]

0.1. Introduction

Let $G$ be a simple Lie group, $g$ the Lie algebra of $G$ and $G^C$, $g^C$ their complexifications, respectively. In a 2-graded decomposition $g = g_2 \oplus g_1 \oplus g_0$, we know that there exists $Z \in g_0$, called a characteristic element, such that each $g_k$ is the $k$-eigenspace of $\text{ad} Z : g \to g$. In $g^C$, let

$$
\tilde{\gamma} = \exp \frac{2\pi i}{2} \text{ad} Z, \quad \tilde{\gamma}_3 = \exp \frac{2\pi i}{3} \text{ad} Z.
$$

Then $\tilde{\gamma}, \tilde{\gamma}_3$ are automorphisms of order 2, 3 of $g^C$, respectively. In view of the above consideration, we explain our methods. Let $\tilde{\gamma}_3$ be an automorphism of order 3 of $g^C$ (all automorphisms of order 3 of $g^C$ are classified [3]) and $(g^C)_0$ the 1-eigenspace of $\tilde{\gamma}_3$:

$$(g^C)_0 = (g^C)_{\tilde{\gamma}_3} = \{ X \in g^C \mid \tilde{\gamma}_3 X = X \}.$$

If \((g_C)_0\) contains \(C\) as a direct summand, then a generator \(Z\) of \(C\) is a characteristic element of \(g_C\). Using \(ad\ Z\), we decompose \(g_C\) into the eigenspaces,

\[
g_C = (g_C)^{-2} \oplus (g_C)^{-1} \oplus (g_C)_0 \oplus (g_C)^1 \oplus (g_C)^2.
\]

Next, let

\[
g_k = (g_C)^k \cap g.
\]

Then obviously we have

\[
g \supset g^{-2} \oplus g^{-1} \oplus g_0 \oplus g_1 \oplus g_2; \quad [g_k, g_l] \subset g_{k+l}.
\]

However, the above "\(\supset\)" is not necessarily "\(=\)". So we need to find (if possible) a real form \((g_C)^{\tilde{t}}\) of \(g_C\) (where \(\tilde{t}\) is a complex-conjugate involutive automorphism of \(g_C\)) such that \((g_C)^{\tilde{t}} \cong g\) satisfying \(((g_C)^k)^{\tilde{t}} \subset (g_C)^k\). Then we get a 2-graded decomposition \(g \cong (g_C)^{\tilde{t}} = g^{-2} \oplus g^{-1} \oplus g_0 \oplus g_1 \oplus g_2\), where \(g_k = (g_C)^k \cap (g_C)^{\tilde{t}}\). In this paper, for the simply connected exceptional complex simple Lie group \(G_C\) of type \(G_2, F_4\) and \(E_6\), we determine the group structures of

\[
\sum_{k} (g_C)^{\tilde{t}} = (g_C)^{\tilde{t}_2} + (g_C)^{\tilde{t}_3} + \cdots + (g_C)^{\tilde{t}_r},
\]

where \(\tilde{t}, \tilde{t}_1, \tilde{t}_2, \ldots, \tilde{t}_r\) are automorphisms of \(G_C\) associated with automorphisms \(\tilde{t}, \tilde{t}_1, \tilde{t}_2, \ldots, \tilde{t}_r\) of \(g_C\).

### 0.2. Notations and preliminaries

\(R, C = \{1, e_1\} (e_1^2 = -1), H = C \oplus Ce_2 = \{1, e_1, e_2, e_3\}\) denote the fields of real, complex, quaternion numbers and \(C' = \{1, e_1'\} (e_1'^2 = 1), H' = C' \oplus C'e_2\) denote the \(R\)-algebras of split complex, split quaternion numbers, respectively. Let \(R^{+} = R - \{0\}, R_{+} = \{x \in R \mid x > 0\}\).

For an \(R\)-vector space \(V\), its complexification \(\{u + iv \mid u, v \in V\}\) is denoted by \(V^C\). The complex conjugation in \(V^C\) is denoted by \(\tau\):

\[
\tau(u + iv) = u - iv.
\]

\(R^C\) is briefly denoted by \(C\) and let \(C^* = C - \{0\}\). For a \(K\)-vector space \(V (K = R, C), \text{Iso}_K(V)\) denotes all of \(K\)-linear isomorphisms of \(V\). For a \(K\)-linear mapping \(f\) of \(V, V_f\) (or \(V^f\)) denotes \(\{v \in V \mid f(v) = v\}\).

For a topological group \(G, G^0\) denotes the connected component containing the identity of \(G\) and \(G = G^0 \times r = G^0 \times \{1, g_1, \ldots, g_{r-1}\}\) means that \(G\) has \(r\) connected components such that \(G = G^0 \cup g_1G^0 \cup \cdots \cup g_{r-1}G^0\). When \(G\) is a transformation group of a space \(X, \{g \in G \mid gx_k = x_k, k = 1, \ldots, m\}\).
Let $G$ be a group and $\sigma$ an automorphism of $G$, then $G^\sigma$ denotes \( \{ g \in G \mid \sigma(g) = g \} \). For $s \in G$, $G^s$ denotes \( \{ g \in G \mid sg^{-1} = g \} \).

\[ E = \text{diag}(1, \ldots, 1) \in M(n, \mathbb{R}). \]

\[ I_1 = \text{diag}(-1, 1, \ldots, 1), \quad I_2 = \text{diag}(-1, -1, 1, \ldots, 1), \ldots \in M(n, \mathbb{R}). \]

\[ I = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \text{moreover } I = \text{diag}(I, \ldots, I), \quad J = \text{diag}(J, \ldots, J) \in M(2n, \mathbb{R}). \]

\[ GL(n, K) = \{ A \in M(n, K) \mid \det A \neq 0 \}, \quad K = \mathbb{R}, \mathbb{C}, \]
\[ SL(n, K) = \{ A \in M(n, K) \mid \det A = 1 \}, \quad K = \mathbb{R}, \mathbb{C}, \]
\[ SO(n, K) = \{ A \in M(n, K) \mid {}^tAA = E, \quad \det A = 1 \}, \quad K = \mathbb{R}, \mathbb{C}, \quad SO(n) = SO(n, \mathbb{R}), \]
\[ O(m, n-m) = \{ A \in M(n, \mathbb{R}) \mid {}^tA I_mA = I_m \}, \]
\[ U(n, K) = \{ A \in M(n, K) \mid A^*A = E \}, \quad K = \mathbb{C}, \mathbb{C}', \mathbb{C}^C, U(n) = U(n, \mathbb{C}), \]
\[ SU(m, n-m) = \{ A \in M(n, \mathbb{C}) \mid {}^t(\tau A) I_mA = I_m, \quad \det A = 1 \}, \]
\[ SU^*(2n, \mathbb{C}^C) = \{ A \in (2n, \mathbb{C}^C) \mid JA = \bar{A}J, \quad \det A = 1 \}, \]
\[ Sp(n, \mathbb{H}^C) = \{ A \in M(n, \mathbb{H}^C) \mid A^*A = E \}, \]
\[ Sp(n, K) = \{ A \in M(n, K) \mid {}^tAJA = J \}, \quad K = \mathbb{R}, \mathbb{C}. \]

Then we have the following isomorphism ([5]).

\[ U(n, \mathbb{C}^C) \cong GL(n, \mathbb{C}), \quad U(n, \mathbb{C}') \cong GL(n, \mathbb{R}) \quad \text{(in particular, } U(1, \mathbb{C}^C) \cong \mathbb{C}^* \text{, } U(1, \mathbb{C}') \cong \mathbb{R}^* \text{).} \]

\[ SU^*(2n, \mathbb{C}^C) \cong SL(2n, \mathbb{C}), \quad Sp(n, \mathbb{H}^C) \cong Sp(n, \mathbb{C}), \quad Sp(n, \mathbb{H}') \cong Sp(n, \mathbb{R}). \]

The Lie algebra of a Lie group $G$ is denoted by the corresponding German small letter $\mathfrak{g}$.

In the following arguments, the following two Lemmas are useful.

**Lemma 0.1.** Let $G$ be a simply connected Lie group and $\sigma$ an automorphism of $G$ of finite order, then $G^\sigma$ is connected.

**Lemma 0.2.** Let $\varphi : G \to G'$ be a homomorphism of Lie groups satisfying

\begin{itemize}
  \item[(1)] $G'$ connected
  \item[(2)] $\ker \varphi$ is discrete
  \item[(3)] $\dim \mathfrak{g} = \dim \mathfrak{g}'$
\end{itemize}

Then $\varphi$ is onto. (Thus we have an isomorphism $G' \cong G/\ker \varphi$).

### Group $G_2$

#### 1.1. Lie groups of type $G_2$ and their Lie algebras

Let $\mathfrak{e} = \mathbb{H} \oplus \mathbb{H}e$ (resp. $\mathfrak{e}' = \mathbb{H} \oplus \mathbb{H}e'$) be the Cayley algebra (resp. the split Cayley algebra) over $\mathbb{R}$ with the multiplication

\[ (m_1 + n_1e)(m_2 + n_2e) = (m_1m_2 - \overline{n_2n_1}) + (n_1\overline{m_2} + n_2m_1)e, \]

(resp. $(m_1 + n_1'e')(m_2 + n_2'e') = (m_1m_2 + \overline{n_2n_1}) + (n_1\overline{m_2} + n_2m_1)e'$),
the conjugation $\overline{m + ne} = \overline{m} - ne$ (resp. $\overline{m + ne'} = \overline{m} - ne'$) and the inner product $(x, y) = \frac{1}{2}(xy + yx)$.

The connected linear Lie groups of type $G_2$ are given by

$$G_2^C = \{ \alpha \in \text{Iso}_C(\mathfrak{c}^C) \mid \alpha(xy) = (\alpha x)(\alpha y) \},$$
$$G_2 = \{ \alpha \in \text{Iso}_R(\mathfrak{c}) \mid \alpha(xy) = (\alpha x)(\alpha y) \},$$
$$G_{2(2)} = \{ \alpha \in \text{Iso}_R(\mathfrak{c}') \mid \alpha(xy) = (\alpha x)(\alpha y) \}.$$

$G_2^C$ and $G_2$ are simply connected. We define $R$-linear transformations $\gamma, \gamma_1, \gamma_2$ and $\gamma_3$ of $\mathfrak{c}$ by

$$\gamma(m + ne) = m - ne,$$
$$\gamma_1(m + ne) = \gamma_1 m + (\gamma_1 n)e,$$
$$\gamma_2(m + ne) = \gamma_2 m + (\gamma_2 n)e,$$
$$\gamma_3(m + ne) = m + (\omega_1 n)e,$$

$m + ne \in H \oplus He = \mathfrak{c}$, where $\gamma_1, \gamma_2 : H \to H$ are defined as

$$\gamma_1(x + ye_2) = x + ye_2, \quad \gamma_2(x + ye_2) = x - ye_2, \quad x + ye_2 \in C \oplus C e_2 = H,$$

respectively, and $\omega_1 = -\frac{1}{2} + \frac{\sqrt{3}}{2} e_1 \in C \subset H \subset \mathfrak{c}$. Then $\gamma, \gamma_1, \gamma_2, \gamma_3 \in G_2 \subset G_2^C$ and $\gamma^2 = \gamma_1^2 = \gamma_2^2 = 1, \gamma_3^3 = 1$. ($\gamma_1, \gamma_2$ are denoted by $\gamma_C, \gamma_H$, respectively, in [5]).

Since $\mathfrak{c} = (\mathfrak{c}^C)_{\tau}, \mathfrak{c}' = (\mathfrak{c}^C)_{\tau\gamma}$ we have

**Proposition 1.1.** $G_2 = (G_2^C)^\tau, G_{2(2)} \cong (G_2^C)^{\tau\gamma}$. ($((G_2^C)^\tau$ is defined by $(G_2^C)^\tau = \{ \alpha \in G_2^C \mid \tau \alpha \tau = \alpha \}$ etc.).

Let $\mathfrak{so}(8) = \mathfrak{so}(\mathfrak{c})$ be the Lie algebra of the group $SO(8) = SO(\mathfrak{c}) = \{ \alpha \in \text{Iso}_R(\mathfrak{c}) \mid (\alpha x, \alpha y) = (x, y), \det \alpha = 1 \}$:

$$\mathfrak{so}(8) = \{ D \in \text{Hom}_R(\mathfrak{c}, \mathfrak{c}) \mid (Dx, y) + (x, Dy) = 0 \}.$$

Let $\{ e_0 = 1, e_1, e_2, e_3, e_4 = e, e_5, e_6, e_7 \}$ be a canonical $R$-basis of $\mathfrak{c}$ (cf. the right figure) and define an $R$-linear mapping $G_{kl}$ of $\mathfrak{c}$ satisfying

$$G_{kl} e_k = -e_l, \quad G_{kl} e_l = e_k, \quad G_{kl} e_m = 0, \text{ otherwise}.$$

Then $G_{kl} \in \mathfrak{so}(8)$ and $\{ G_{kl} \mid 0 \leq k < l \leq 7 \}$ is an $R$-basis of $\mathfrak{so}(8)$.
LEMMA 1.2. The Lie bracket in \( \mathfrak{so}(8) \) is given by

\[
[G_{kl}, G_{lm}] = G_{km}, \quad [G_{kl}, G_{mn}] = 0, \quad k, l, m, n \text{ are distinct.}
\]

The Lie algebra \( \mathfrak{g}_2 \) of the group \( G_2 \):

\[
\mathfrak{g}_2 = \{ D \in \text{Hom}_R(\mathcal{C}, \mathcal{C}) \mid D(xy) = (Dx)y + x(Dy) \}
\]
is a subalgebra of \( \mathfrak{so}(8) \).

PROPOSITION 1.3 ([1]). The following 7 elements are additive generators of the Lie algebra \( \mathfrak{g}_2 \):

\[
\begin{align*}
\lambda G_{23} + \mu G_{45} + \nu G_{67}, & \quad -\lambda G_{13} - \mu G_{46} + \nu G_{57}, \\
\lambda G_{12} + \mu G_{47} + \nu G_{56}, & \quad -\lambda G_{15} + \mu G_{26} - \nu G_{37}, \quad \lambda, \mu, \nu \in \mathbb{R}, \\
\lambda G_{14} - \mu G_{27} - \nu G_{36}, & \quad -\lambda G_{17} - \mu G_{24} + \nu G_{35}, \quad \lambda + \mu + \nu = 0.
\end{align*}
\]

1.2. Subgroups of type \( C_1^C \oplus C_1^C \) and \( C \oplus C_1^C \) of \( G_2^C \)

In the complexified Lie algebra \( \mathfrak{g}_2^C \) of \( \mathfrak{g}_2 \), let \( Z = i(G_{45} - G_{67}) \). Then we can easily verify that

\[
\gamma = \exp \frac{2\pi i}{2} Z, \quad \gamma_3 = \exp \frac{2\pi i}{3} Z.
\]

Since \( \gamma, \gamma_1 \) are conjugate in \( G_2 \) ([5] Proposition 1.2.3), we have

PROPOSITION 1.4. \( G_2(2) = (G_2^C)^{\gamma_1} \cong (G_2^C)^{\gamma_1} \).

LEMMA 1.5. In the Lie algebra \( \mathfrak{so}(8, C) \), we have

\[
\left\{ \begin{array}{c}
\tau G_{kl} \tau = G_{kl} \\
\tau (iG_{kl}) \tau = -iG_{kl}
\end{array} \right. \quad \gamma_1 G_{kl} \gamma_1 = \left\{ \begin{array}{c}
G_{kl}, \quad k + l \text{ is even} \\
-G_{kl}, \quad k + l \text{ is odd}
\end{array} \right.
\]

THEOREM 1.6. The 2-graded decomposition of \( \mathfrak{g}_2(2) = (\mathfrak{g}_2^C)^{\gamma_1} \) (or \( \mathfrak{g}_2^C \)),

\[
\mathfrak{g}_2(2) = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2
\]

with respect to \( \text{ad} Z, \quad Z = i(G_{45} - G_{67}) \), is given by

\[
\mathfrak{g}_0 = \left\{ i(G_{45} - G_{67}), \quad i(-2G_{23} + G_{45} + G_{67}), \quad 2G_{13} - G_{46} + G_{57}, \quad i(-2G_{12} + G_{47} + G_{56}) \right\}
\]
PROOF. We can verify that $g_k$ is the $k$-eigenspace of $\text{ad} \, Z : g_2 \rightarrow g_2$ (Lemma 1.2). Moreover each element $X$ of $g_2 \subset C$ is invariant under $\text{Ad}(\gamma_1)$, that is, $\gamma_1 X \gamma_1 = X$ (Lemma 1.5).

Hence the 2-graded decomposition of $g_2 \subset C$ over $C$ induces also a 2-graded decomposition of $g_2(2) = (g_2)^{\gamma_2}$ over $R$. (Note. $\{X_1, \ldots, X_m\} = g_k$ is a basis of $g_k$ and the number of the right side is the dimension of $g_k$).

Now, we shall determine the group structures of

$$\begin{align*}
(G_2^C)_{\text{ev}} &= (G_2^C)^{\gamma}, \quad (G_2^C)_0 = (G_2^C)^{\gamma_3}.
\end{align*}$$

**Theorem 1.7.**

1. $(G_2^C)_{\text{ev}} \cong (\text{Sp}(1, C) \times \text{Sp}(1, C))/\mathbb{Z}_2$, $\mathbb{Z}_2 = \{(1, 1), (-1, -1)\}$.
2. $(G_2^C)_0 \cong (\mathbb{C}^* \times \text{Sp}(1, C))/\mathbb{Z}_2$, $\mathbb{Z}_2 = \{(1, 1), (-1, -1)\}$.

**Proof.**

1. We define $\varphi : \text{Sp}(1, H^C) \times \text{Sp}(1, H^C) \rightarrow (G_2^C)^{\gamma} = (G_2^C)_{\text{ev}}$ by

$$\varphi(p, q)(m + me) = qmq + (q)e, \quad m + me \in H^C \oplus H^C e = \mathcal{C}^C.$$

Then $\varphi$ is well-defined, is a homomorphism and $\ker \varphi = \mathbb{Z}_2$. Since $(G_2^C)^{\gamma}$ is connected and $\dim_C(\text{sp}(1, H^C) \oplus \text{sp}(1, H^C)) = 3 + 3 = 6 = 4 + 2 = \dim_C((G_2^C)_{\text{ev}})$ (Theorem 1.6), $\varphi$ is onto. Therefore $(G_2^C)_{\text{ev}} \cong (\text{Sp}(1, H^C) \times \text{Sp}(1, H^C))/\mathbb{Z}_2 \cong (\text{Sp}(1, C) \times \text{Sp}(1, C))/\mathbb{Z}_2$.

2. The restriction mapping $\varphi : U(1, C^C) \times \text{Sp}(1, H^C) \rightarrow (G_2^C)^{\gamma_3} = (G_2^C)_0$ of $\varphi$ above (1) is well-defined (because $\gamma_3 = \varphi(\omega_1, 1)$) and $\ker \varphi = \mathbb{Z}_2$. Since $(G_2^C)^{\gamma_3}$ is connected and $\dim_C(\text{u}(1, C^C) \oplus \text{sp}(1, H^C)) = 1 + 3 = 4 = \dim_C((G_2^C)_0)$ (Theorem 1.6), $\varphi$ is onto. Therefore $(G_2^C)_0 \cong (U(1, C^C) \times \text{Sp}(1, H^C))/\mathbb{Z}_2 \cong (\mathbb{C}^* \times \text{Sp}(1, C))/\mathbb{Z}_2$.

**1.2.1. Subgroups of type $C_{1(1)} \oplus C_{1(1)}$ and $R \oplus C_{1(1)}$ of $G_2(2)$**

We shall determine the group structures of

$$(G_{2(2)})_{\text{ev}} = (G_2^C)^{\gamma} \cap (G_2^C)^{\gamma_1}, \quad (G_{2(2)})_0 = (G_2^C)^{\gamma_3} \cap (G_2^C)^{\gamma_1}.$$
Lemma 1.8. \( \varphi : Sp(1, H^C) \times Sp(1, H^C) \rightarrow G_2^C \) of Theorem 1.7.(1) satisfies

1. \( \gamma = \varphi(-1, 1), \quad \gamma_1 = \varphi(e_2, e_2), \quad \gamma_2 = \varphi(e_1, e_1). \)
2. \( \tau \varphi(p, q) \tau = \varphi(\tau p, \tau q), \quad \gamma_1 \varphi(p, q) \gamma_1 = \varphi(\gamma_1 p, \gamma_1 q). \)

Theorem 1.9. (1) \((G_2(2))_{ev} \cong (Sp(1, R) \times Sp(1, R))/Z_2 \times \{1, \gamma_2\}, \)
\( Z_2 = \{(1, 1), (-1, -1)\}. \)

(2) \((G_2(2))_0 \cong (R^+ \times Sp(1, R)) \times \{1, \gamma_2\}. \)

Proof. (1) For \( \alpha \in (G_2(2))_{ev} \subset (G_2^C)^G \), there exist \( p, q \in Sp(1, H^C) \) such that \( \alpha = \varphi(p, q) \) (Theorem 1.7.(1)). From \( \gamma_1 \tau \alpha \tau \gamma_1 = \alpha \), we have \( \varphi(\gamma_1 \tau p, \gamma_1 \tau q) = \varphi(p, q) \) (Lemma 1.8.(2)). Hence

\[ \gamma_1 \tau p = p, \quad \gamma_1 \tau q = q \quad \text{or} \quad \gamma_1 \tau p = -p, \quad \gamma_1 \tau q = -q. \]

In the former case, \( p, q \in Sp(1, H^C) \). Hence the group in the former case is

\[ (Sp(1, H^C) \times Sp(1, H^C))/Z_2 \cong (Sp(1, R) \times Sp(1, R))/Z_2. \]

In the latter case, \( p = q = e_1 \) satisfy these conditions and \( \varphi(e_1, e_1) = \gamma_2 \) (Lemma 1.8.(1)). Therefore

\[ (G_2(2))_{ev} \cong (Sp(1, R) \times Sp(1, R))/Z_2 \times \{1, \gamma_2\}. \]

(2) For \( \alpha \in (G_2(2))_0 \subset (G_2^C)^G \), there exist \( a \in U(1, C^C) \) and \( q \in Sp(1, H^C) \) such that \( \alpha = \varphi(a, q) \) (Theorem 1.7.(2)). From \( \gamma_1 \tau a \gamma_1 = \alpha \), we have \( \varphi(\gamma_1 \tau a, \gamma_1 \tau q) = \varphi(a, q) \) (Lemma 1.8.(2)). Hence

\[ \gamma_1 \tau a = a, \quad \gamma_1 \tau q = q \quad \text{or} \quad \gamma_1 \tau a = -a, \quad \gamma_1 \tau q = -q. \]

In the former case, \( a \in U(1, C^C) \) and \( q \in Sp(1, H^C) \). Hence the group in the former case is

\[ (U(1, C^C) \times Sp(1, H^C))/Z_2 \cong (R^+ \times Sp(1, R))/Z_2 \cong R^+ \times Sp(1, R). \]

In the latter case, \( a = q = e_1 \) satisfy these conditions and \( \varphi(e_1, e_1) = \gamma_2 \) (Lemma 1.8.(1)). Therefore

\[ (G_2(2))_0 \cong (R^+ \times Sp(1, R)) \times \{1, \gamma_2\}. \]

Group \( F_4 \)

2.1. Lie groups of type \( F_4 \) and their Lie algebras

Let

\[ \mathfrak{j}(3, K) = \{ X \in M(3, K) \mid X^* = X \}, \quad K = C, C', C^C \quad (or \ H) \]
\[ \mathfrak{j}(1, 2, K) = \{ X \in M(3, K) \mid I_1 X^* I_1 = X \}, \]

be the Jordan algebras with the Jordan multiplication \( X \circ Y \), the inner product \((X, Y)\) and the Freudenthal multiplication \( X \times Y \),

\[ X \circ Y = \frac{1}{2} (XY + YX), \quad (X, Y) = \text{tr}(X \circ Y), \]
\[ X \times Y = \frac{1}{2} (2X \circ Y - \text{tr}(X)Y - \text{tr}(Y)X + (\text{tr}(X)\text{tr}(Y) - (X, Y))E), \]
respectively. From now on, we briefly denote $\mathfrak{J} = \mathfrak{J}(3, \mathbb{C})$, $\mathfrak{J'} = \mathfrak{J}(3, \mathbb{C'})$, $\mathfrak{J}^C = \mathfrak{J}(3, \mathbb{C}^C)$. In $\mathfrak{J}$, $\mathfrak{J'}$ or $\mathfrak{J}^C$, we use the following notations.

$$
E_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad E_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad E_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix},
$$

$$
F_1(x) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & x \\ 0 & x & 0 \end{pmatrix}, \quad F_2(x) = \begin{pmatrix} 0 & 0 & \bar{x} \\ 0 & 0 & 0 \\ x & 0 & 0 \end{pmatrix}, \quad F_3(x) = \begin{pmatrix} 0 & x & 0 \\ \bar{x} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.
$$

The tables of the Jordan and the Freudenthal multiplications among them are given as follows.

$$
\begin{cases}
E_k \circ E_k = E_k \\
E_k \circ F_k(x) = 0 \\
F_k(x) \circ F_k(y) = (x,y)(E_{k+1} + E_{k+2}),
\end{cases}
$$

$$
\begin{cases}
E_k \circ E_l = 0, \quad k \neq l \\
E_k \circ F_l(x) = \frac{1}{2} F_l(x), \quad k \neq l \\
F_k(x) \circ F_{k+1}(y) = \frac{1}{2} F_{k+2}(xy),
\end{cases}
$$

where the indices are considered as mod 3.

The connected linear Lie groups of type $F_4$ are given by

$$
F_4^C = \{ \alpha \in \text{Iso}_C(\mathfrak{J}(3, \mathbb{C}^C)) \mid \alpha(X \circ Y) = \alpha X \circ \alpha Y \},
$$

$$
F_4 = \{ \alpha \in \text{Iso}_R(\mathfrak{J}(3, \mathbb{C})) \mid \alpha(X \circ Y) = \alpha X \circ \alpha Y \},
$$

$$
F_{4(4)} = \{ \alpha \in \text{Iso}_R(\mathfrak{J}(3, \mathbb{C'})) \mid \alpha(X \circ Y) = \alpha X \circ \alpha Y \},
$$

$$
F_{4(-20)} = \{ \alpha \in \text{Iso}_R(\mathfrak{J}(1, 2, \mathbb{C})) \mid \alpha(X \circ Y) = \alpha X \circ \alpha Y \}.
$$

$F_4^C$, $F_4$ and $F_{4(-20)}$ are simply connected. In the definitions of the groups above, the condition $\alpha(X \circ Y) = \alpha X \circ \alpha Y$ is equivalent to

$$
\alpha(X \times Y) = \alpha X \times \alpha Y.
$$

Note that $\alpha \in F_4^C$ satisfies

$$
(\alpha X, \alpha Y) = (X, Y), \quad \text{tr}(\alpha X) = \text{tr}(X).
$$

The group $F_4^C$ contains $G_2^C$ as a subgroup by the following way. For $\alpha \in G_2^C$, we define $\bar{\alpha} : \mathfrak{J}^C \to \mathfrak{J}^C$ by

$$
\bar{\alpha} \begin{pmatrix} \xi_1 & x_3 & \bar{x}_2 \\ \bar{x}_3 & \xi_2 & x_1 \\ x_2 & \bar{x}_1 & \xi_3 \end{pmatrix} = \begin{pmatrix} \xi_1 & \alpha x_3 & \bar{\alpha} \bar{x}_2 \\ \bar{\alpha} x_3 & \xi_2 & \alpha x_1 \\ \alpha x_2 & \bar{\alpha} \bar{x}_1 & \xi_3 \end{pmatrix},
$$
then $\bar{\alpha} \in F_4^C$ and $G_2^C \cong \{ \bar{\alpha} \mid \alpha \in G_2^C \} \subset F_4^C$. Similarly we have $G_2 \subset F_4$. In particular, $\gamma, \gamma_1, \gamma_2, \gamma_3 \in G_2 \subset F_4 \subset F_4^C$.

We define $R$-linear transformations $\sigma, \sigma'$ and $\sigma_3$ of $\mathfrak{g}$ by

$$\sigma X = \sigma \begin{pmatrix} \xi_1 & x_3 & \overline{x_2} \\ \overline{x_3} & \xi_2 & x_1 \\ x_2 & \overline{x_1} & \xi_3 \end{pmatrix} = \begin{pmatrix} \xi_1 & -x_3 & -\overline{x_2} \\ -\overline{x_3} & \xi_2 & x_1 \\ -x_2 & -\overline{x_1} & \xi_3 \end{pmatrix},$$

$$\sigma'X = \begin{pmatrix} \xi_1 & x_3 & \overline{x_2} \\ \overline{x_3} & \xi_2 & x_1 \\ x_2 & \overline{x_1} & \xi_3 \end{pmatrix},$$

$$\sigma_3X = \begin{pmatrix} \xi_1 & x_3\omega_1 & \overline{\omega_1\overline{x_2}} \\ \overline{x_3\omega_1} & \xi_2 & \overline{\omega_1x_1\overline{\omega_1}} \\ \omega_1x_2 & \omega_1\overline{x_1}\omega_1 & \xi_3 \end{pmatrix}.$$  

$(\omega_1 = -\frac{1}{2} + \frac{\sqrt{3}}{2}e_1)$, respectively. Then $\sigma, \sigma', \sigma_3 \in F_4 \subset F_4^C$ and $\sigma^2 = \sigma'^2 = 1, \sigma_3^3 = 1$.

Since $\mathfrak{g}(3,\mathbb{C}) = (\mathfrak{g}(3,\mathbb{C}))^{\tau}, \mathfrak{g}(3,\mathbb{C})^\prime = (\mathfrak{g}(3,\mathbb{C}))^{\tau,\gamma}, \mathfrak{g}(1,2,\mathbb{C}) \cong (\mathfrak{g}(3,\mathbb{C}))^{\tau,\gamma}, \mathfrak{g}(4,20) = (\mathfrak{g}(4,C))^{\tau,\gamma}$.

Proposition 2.1. $F_4 = (F_4^C)^{\tau}, F_4(4) = (F_4^C)^{\tau,\gamma} \cong (F_4^C)^{\tau,\gamma_1}, F_4(20) = (F_4^C)^{\tau,\gamma}$.  

The Lie algebra $\mathfrak{f}_4$ of the group $F_4$ is given by

$$\mathfrak{f}_4 = \{ \delta \in \text{Hom}_R(\mathfrak{g}, \mathfrak{g}) \mid \delta(X \circ Y) = \delta X \circ Y + X \circ \delta Y \}. $$

The subalgebra $\mathfrak{d}_4$ of $\mathfrak{f}_4$:

$$\mathfrak{d}_4 = \{ \delta \in \mathfrak{f}_4 \mid \delta E_k = 0, k = 1, 2, 3 \}$$

is isomorphic to the Lie algebra $\text{so}(8) = \text{so}(\mathbb{C})$ by the correspondence

$$D_1 \in \text{so}(8) \rightarrow \delta = \delta(D_1, D_2, D_3) \in \mathfrak{d}_4,$$

$$\delta \begin{pmatrix} \xi_1 & x_3 & \overline{x_2} \\ \overline{x_3} & \xi_2 & x_1 \\ x_2 & \overline{x_1} & \xi_3 \end{pmatrix} = \begin{pmatrix} 0 & D_3x_3 & \overline{D_2x_2} \\ D_3x_3 & 0 & D_1x_1 \\ D_2x_2 & D_1x_1 & 0 \end{pmatrix},$$

where $D_2, D_3$ are elements of $\text{so}(8)$ determined by the principle of triality

$$(D_1x)y + x(D_2y) = \overline{D_3(xy)}, \quad x, y \in \mathbb{C}.$$  

From now on, we identify $D_1 \in \text{so}(8)$ with $\delta(D_1, D_2, D_3) \in \mathfrak{d}_4 \subset \mathfrak{f}_4$.

Let $(\mathfrak{M}^-)_0 = \{ A \in M(3, \mathbb{C}) \mid A^* = -A, \text{tr}(A) = 0 \}$. For $A \in (\mathfrak{M}^-)_0$, we define an $R$-linear mapping $\tilde{A}$ of $\mathfrak{g}$ by

$$\tilde{A}X = \frac{1}{2}[A, X] = \frac{1}{2}(AX -XA).$$
Then we can verify that \( \tilde{A} \in f_4 \). In \((\mathfrak{M}^-)_{a}\), we use the following notations.

\[
A_1(a) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & a \\ 0 & \tilde{a} & 0 \end{pmatrix}, \quad A_2(a) = \begin{pmatrix} 0 & 0 & -\tilde{a} \\ 0 & 0 & 0 \\ -\tilde{a} & 0 & 0 \end{pmatrix}, \quad A_3(a) = \begin{pmatrix} 0 & a & 0 \\ -\tilde{a} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.
\]

The action of \( \tilde{A}_k(a) \in f_4 \) on \( \mathfrak{J} \) is given as follows.

\[
\left\{
\begin{array}{ll}
\tilde{A}_k(a)E_k = 0 & \\
\tilde{A}_k(a)E_{k+1} = -\frac{1}{2}F_{k}(a) & \\
\tilde{A}_k(a)E_{k+2} = \frac{1}{2}F_{k}(a) & \\
\end{array}
\right.
\]

\[
\left\{
\begin{array}{ll}
\tilde{A}_k(a)F_k(x) = (a,x)(E_{k+1} - E_{k+2}) & \\
\tilde{A}_k(a)F_{k+1}(x) = \frac{1}{2}F_{k+2}(a\tilde{x}) & \\
\tilde{A}_k(a)F_{k+2}(x) = -\frac{1}{2}F_{k+1}(\tilde{x}\tilde{a}) & \\
\end{array}
\right.
\]

**Proposition 2.2 ([1]).** Any element \( \delta \) of the Lie algebra \( f_4 \) can be uniquely expressed as

\[
\delta = D + \tilde{A}_1(a_1) + \tilde{A}_2(a_2) + \tilde{A}_3(a_3), \quad D \in \mathfrak{so}(8), \; a_k \in \mathcal{C}.
\]

### 2.2. Subgroups of type \( C_1 \times C_3 \) and \( C \times C_3 \) of \( F_4^C \)

In the complexified Lie algebra \( f_4^C \) of \( f_4 \), let \( Z = i(G_{45} - G_{67}) \). Then, as is shown in \( g_2^C \), we have

\[
\gamma = \exp \frac{2\pi i}{2} Z, \quad \gamma_3 = \exp \frac{2\pi i}{3} Z.
\]

**Lemma 2.3.** In the Lie algebra \( f_4^C \), we have

1. \( [D_1, \tilde{A}_1(a)] = \tilde{A}_1(D_1a), \; [D_1, \tilde{A}_2(a)] = \tilde{A}_2(D_2a), \; [D_1, \tilde{A}_3(a)] = \tilde{A}_3(D_3a) \)

where \( D_1 = \delta(D_1, D_2, D_3) \in \mathfrak{so}(8) \subseteq f_4^C \).

2. \( \tau \tilde{A}_k(a)\tau = \tilde{A}_k(\tau a), \quad \gamma_1 \tilde{A}_k(a)\gamma_1 = \tilde{A}_k(\gamma_1 a) \)

**Theorem 2.4.** The 2-graded decomposition of \( f_4(4) = (f_4^C)^{\gamma_1} \) (or \( f_4^C \)),

\[
f_4(4) = g_2 \oplus g_{-1} \oplus g_0 \oplus g_1 \oplus g_2
\]

with respect to \( \text{ad} \; Z \), \( Z = i(G_{45} - G_{67}) \), is given by

\[
g_0 = \left\{ \begin{array}{ll}
G_{46} - G_{57}, \; i(G_{47} + G_{56}), \; \tilde{A}_1(1), \; \tilde{A}_2(1), \; \tilde{A}_3(1), \\
iG_{01}, \; G_{02}, \; iG_{03}, \; i\tilde{A}_1(e_1), \; i\tilde{A}_2(e_1), \; i\tilde{A}_3(e_1), \\
iG_{12}, \; G_{13}, \; iG_{23}, \; \tilde{A}_1(e_2), \; \tilde{A}_2(e_2), \; \tilde{A}_3(e_2), \\
iG_{45}, \; iG_{67}, \; i\tilde{A}_1(e_3), \; i\tilde{A}_2(e_3), \; i\tilde{A}_3(e_3) \end{array} \right\}
\]
\( \mathfrak{g}_1 = \{ (G_{46} + G_{57}) - i(G_{47} - G_{56}) \} \)

\( \mathfrak{g}_2 = \{ (G_{46} + G_{57}) - i(G_{47} - G_{56}) \} \)

\( \mathfrak{g}_1 = \tau(\mathfrak{g}_1^2), \quad \mathfrak{g}_2 = \tau(\mathfrak{g}_2^2) \).

**Proof.** Note that for \( D_1 = G_{45} - G_{67} \in \mathfrak{so}(8) \) we have \( D_2 = D_3 = G_{45} - G_{67} \). Then we can prove this theorem in a way similar to Theorem 1.6, using Lemmas 1.5 and 2.3.

Now, we shall determine the group structures of

\[
(F_4)^{C^2} = (F_4)^C, \quad (F_4)_0 = (F_4)^C^{\gamma_3}
\]

To an element \( X = \begin{pmatrix} \xi_1 & x_3 & \bar{x}_2 \\ \bar{x}_3 & \xi_2 & x_1 \\ x_2 & \bar{x}_1 & \xi_3 \end{pmatrix} \in \mathfrak{J}(3, \mathbb{C}) \), we associate an element \( M + n \in \mathfrak{J}(2, \mathbb{H}) \oplus \mathbb{H}^3 \) such that

\[
\begin{pmatrix} \xi_1 & m_3 & \bar{m}_2 \\ \bar{m}_3 & \xi_2 & m_1 \\ m_2 & \bar{m}_1 & \xi_3 \end{pmatrix} + (n_1, n_2, n_3), \quad x_k = m_k + n_k e \in \mathbb{H} \oplus \mathbb{H} e = \mathbb{C}.
\]

Then \( \mathfrak{J}(3, \mathbb{H}) \oplus \mathbb{H}^3 \) has a multiplication

\[
(M_1 + n_1) \times (M_2 + n_2) = \left( M_1 \times M_2 - \frac{1}{2} (n_1^* n_2 + n_2^* n_1) \right) - \frac{1}{2} (n_1 M_2 + n_2 M_1)
\]

corresponding to the multiplication \( X \times Y \) of \( \mathfrak{J}(3, \mathbb{C}) \).

**Theorem 2.5.**

1. \( (F_4)^{C^2} \) isomorphic to \( (Sp(1, C) \times Sp(3, C)) / \mathbb{Z}_2, \quad \mathbb{Z}_2 = \{(1, E), (-1, -E)\} \).

2. \( (F_4)^0 \cong (C^* \times Sp(3, C)) / \mathbb{Z}_2, \quad \mathbb{Z}_2 = \{(1, E), (-1, -E)\} \).

**Proof.** 1. We define \( \varphi : Sp(1, \mathbb{H}^C) \times Sp(3, \mathbb{H}^C) \rightarrow (F_4)^\gamma = (F_4)^C \) by

\[
\varphi(p, A)(M + n) = AM A^* + pnA^*, \quad M + n \in \mathfrak{J}(3, \mathbb{H}^C) \oplus (\mathbb{H}^C)^3 = \mathfrak{J}(3, \mathbb{C}).
\]

Then \( \varphi \) is well-defined, is a homomorphism and \( \text{Ker } \varphi = \mathbb{Z}_2 \). Since \( (F_4)^\gamma \) is connected and \( \dim_{\mathbb{C}}(sp(1, \mathbb{H}^C) \oplus sp(3, \mathbb{H}^C)) = 3 + 21 = 24 = 22 + 2 \times 1 = \dim_{\mathbb{C}}((f_4)^{C^2}) \) (Theorem 2.4), \( \varphi \) is onto. Therefore \( (F_4)^C \) isomorphic to \( (Sp(1, \mathbb{H}^C) \times Sp(3, \mathbb{H}^C)) / \mathbb{Z}_2 \cong (Sp(1, C) \times Sp(3, C)) / \mathbb{Z}_2 \).
2. graded decompositions of exceptional Lie algebras

(2) The restriction mapping \( \varphi : U(1, \mathbb{C}^c) \times Sp(3, \mathbb{H}^c) \to (F_4^c)^{\gamma_3} = (F_4^c)_0 \) of \( \varphi \) above (1) is well-defined (because \( \gamma_3 = \varphi(\omega_1, E) \)) and \( \text{Ker} \varphi = \mathbb{Z}_2 \). Since \((F_4^c)^{\gamma_3}\) is connected and \(\dim_{\mathbb{C}}(u(1, \mathbb{C}^c) \oplus sp(3, \mathbb{H}^c)) = 1 + 21 = 22 = \dim_{\mathbb{C}}((f_4^c)_0)\) (Theorem 2.4), \( \varphi \) is onto. Therefore \((F_4^c)_0 \cong (U(1, \mathbb{C}^c) \times Sp(3, \mathbb{H}^c))/\mathbb{Z}_2 \cong (\mathbb{C}^* \times Sp(3, \mathbb{C}))/\mathbb{Z}_2 \).

2.2.1. Subgroups of type \( C_{1(1)} \oplus C_{3(3)} \) and \( R \oplus C_{3(3)} \) of \( F_{4(4)} \)

We shall determine the group structures of

\[(F_{4(4)})_{ev} = (F_4^c)^{\gamma} \cap (F_4^c)^{\gamma_1}, \quad (F_{4(4)})_0 = (F_4^c)^{\gamma} \cap (F_4^c)^{\gamma_1}.
\]

**Lemma 2.6.** \( \varphi : Sp(1, \mathbb{H}^c) \times Sp(3, \mathbb{H}^c) \to F_4^c \) of Theorem 2.5.(1) satisfies

1. \( \gamma = \varphi(-1, E), \quad \gamma_1 = \varphi(e_2, e_2E), \quad \gamma_2 = \varphi(e_1, e_1E). \)
2. \( \tau \varphi(p, A) \tau = \varphi(\tau p, \tau A), \quad \gamma_1 \varphi(p, A) \gamma_1 = \varphi(\gamma_1 p, \gamma_1 A). \)

**Theorem 2.7.** (1) \((F_{4(4)})_{ev} \cong (Sp(1, \mathbb{R}) \times Sp(3, \mathbb{R}))/\mathbb{Z}_2 \times \{1, \gamma_2\}, \quad \mathbb{Z}_2 = \{(1, E), (-1, -E)\}.
\]
(2) \((F_{4(4)})_0 \cong (\mathbb{R}^+ \times Sp(3, \mathbb{R}))/\{1, \gamma_2\}.
\]

**Proof.** We can prove this theorem in a way similar to Theorem 1.9, using Lemma 2.6.

2.3. Subgroups of type \( B_4^c \) and \( C \oplus B_3^c \) of \( F_4^c \)

In the Lie algebra \( f_4^c \), let \( Z = -2i G_{01} \). Then we can easily verify that

\[ \sigma = \exp \frac{2\pi i}{2} Z, \quad \sigma_3 = \exp \frac{2\pi i}{3} Z. \]

**Theorem 2.8.** The 2-graded decomposition of \( f_4(4) = (f_4^c)^{\gamma_1} \) (or \( f_4^c \)),

\[ f_4(4) = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2 \]

with respect to \( \text{ad} Z, Z = -2i G_{01}, \) is given by

\[ \mathfrak{g}_0 = \begin{cases} iG_{01}, \quad iG_{23}, \quad G_{24}, \quad G_{25}, \quad G_{26}, \quad iG_{27}, \quad \tilde{A}_1(e_2), \quad \tilde{A}_1(e_3), \\ iG_{34}, \quad G_{35}, \quad iG_{36}, \quad G_{37}, \quad \tilde{A}_1(e_4), \quad \tilde{A}_1(e_5), \\ iG_{45}, \quad G_{46}, \quad iG_{47}, \quad \tilde{A}_1(e_6), \quad \tilde{A}_2(e_7), \\ iG_{56}, \quad G_{57}, \quad \\ iG_{67}, \quad \end{cases} \]

\[ \mathfrak{g}_{-1} = \begin{cases} \tilde{A}_2(1 + ie_1), \quad \tilde{A}_2(e_2 + ie_3), \quad \tilde{A}_2(e_4 + ie_5), \quad \tilde{A}_2(e_6 + ie_7), \\ \tilde{A}_3(1 + ie_1), \quad \tilde{A}_3(e_2 - ie_3), \quad \tilde{A}_3(e_4 - ie_5), \quad \tilde{A}_3(e_6 - ie_7) \end{cases} \]
\[ \mathfrak{g}_{-2} = \left\{ G_{02} - iG_{12}, iG_{03} + G_{13}, G_{04} - iG_{14}, \quad \tilde{A}_1(1 - ie_1) \right\} \]
\[ \mathfrak{g}_1 = \tau(\mathfrak{g}_{-1}), \quad \mathfrak{g}_2 = \tau(\mathfrak{g}_{-2}). \]

**PROOF.** Note that for \( D_1 = -2G_{01} \in \mathfrak{so}(8) \) we have
\[ D_2 = G_{01} + G_{23} + G_{45} + G_{67}, \quad D_3 = G_{01} - G_{23} - G_{45} - G_{67}. \]

Then we can prove this theorem in a way similar to Theorem 2.4, using Lemmas 1.5 and 2.3.

Now, we shall determine the group structures of
\[ (F_4^C)_{E_1,F_1(1),F_1(e_1)} \]

**PROPOSITION 2.9.** (1) \( (F_4^C)_{E_1,F_1(1),F_1(e_1)} \cong \text{Spin}(7, \mathbb{C}). \)
(2) \( (F_4^C)_{E_1,F_1(1)} \cong \text{Spin}(8, \mathbb{C}). \)
(3) \( (F_4^C)_{E_1} \cong \text{Spin}(9, \mathbb{C}). \)

**PROOF.** (1) The group \( (F_4^C)_{E_1,F_1(1),F_1(e_1)} \) acts on the \( \mathbb{C} \)-vector space
\[ (V^C)^7 = \{ X \in J^C \mid E_1 \circ X = 0, \text{tr}(X) = 0, (F_1(1), X) = (F_1(e_1), X) = 0 \} \]
\[ = \left\{ X = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \xi & x \\ 0 & \bar{x} & -\xi \end{pmatrix} \mid \xi \in \mathbb{C}, x \in \mathbb{C}^C, (1, x) = (e_1, x) = 0 \right\} \]
with the norm \( (X, X)/2 = \xi^2 + x\bar{x} \). The connectedness of \( (F_4^C)_{E_1,F_1(1),F_1(e_1)} \) follows from
\[ (F_4^C)_{E_1,E_2,E_3,F_1(1),F_1(e_1)} = (F_4^C)_{E_1,E_2,E_3,F_1(1),F_1(e_1)} \cong \text{Spin}(6, \mathbb{C}) \cong SU(4, \mathbb{C}^C). \]

Hence we can define a homomorphism \( \pi : (F_4^C)_{E_1,F_1(1),F_1(e_1)} \rightarrow SO(7, \mathbb{C}) = SO((V^C)^7) \) by \( \pi(\alpha) = \alpha|(V^C)^7 \). \( \ker \pi = \{ 1, \sigma \} = \mathbb{Z}_2 \). Since
\[ \dim_c((F_4^C)_{E_1,F_1(1),F_1(e_1)}) = \dim_c(\text{Spin}(6, \mathbb{C})) + \dim_c((S^C)^6) = 15 + 6 = 21 = \dim_c(SO(7, \mathbb{C})), \]
\( \pi \) is onto. Therefore \( (F_4^C)_{E_1,F_1(1),F_1(e_1)} \) is \( \text{Spin}(7, \mathbb{C}) \) as a double covering group of \( SO(7, \mathbb{C}). \)
(2) The group \((F_4^C)_{E_1,F_1(1)}\) acts on the \(C\)-vector space
\[
(V^C)^8 = \{X \in \mathfrak{g}^C \mid E_1 \circ X = 0, \operatorname{tr}(X) = 0, (F_1(1), X) = 0\}
= \{X = \xi E_2 - \xi E_3 + F_1(x) \mid \xi \in C, x \in \mathfrak{c}^C, (1, x) = 0\}
\]
with the norm \((X, X)/2 = \xi^2 + x\bar{x}\). The connectedness of \((F_4^C)_{E_1,F_1(1)}\) follows from
\[
(F_4^C)_{E_1,F_1(1)}/(F_4^C)_{E_1,F_1(1),F_1(\epsilon_1)} \simeq (S^C)^7 = \{X \in (V^C)^8 \mid (X, X) = 2\}.
\]
Therefore, in a similar way as in (1), \((F_4^C)_{E_1,F_1(1)}\) is \(\text{Spin}(9, C)\) as a covering group of \(SO(8, C) = SO((V^C)^8)\).

(3) The group \((F_4^C)_{E_1}\) acts on the \(C\)-vector space
\[
(V^C)^9 = \{X \in \mathfrak{g}^C \mid E_1 \circ X = 0, \operatorname{tr}(X) = 0\}
= \{X = \xi E_2 - \xi E_3 + F_1(x) \mid \xi \in C, x \in \mathfrak{c}^C\}
\]
with the norm \((X, X)/2 = \xi^2 + x\bar{x}\). The connectedness of \((F_4^C)_{E_1}\) follows from
\[
(F_4^C)_{E_1}/(F_4^C)_{E_1,F_1(1)} \simeq (S^C)^8 = \{X \in (V^C)^9 \mid (X, X) = 2\}.
\]
Therefore, in a similar way as in (1), \((F_4^C)_{E_1}\) is \(\text{Spin}(9, C)\) as a covering group of \(SO(9, C) = SO((V^C)^9)\).

We define a mapping \(D : U(1, C^C) \to F_4^C\) by
\[
D(a) = \begin{pmatrix}
\xi_1 & x_3 & \bar{x}_2 \\
\bar{x}_3 & \xi_2 & x_1 \\
x_2 & \bar{x}_1 & \xi_3
\end{pmatrix}
= \begin{pmatrix}
\xi_1 & x_3 a & \bar{a}x_2 \\
\bar{x}_3 a & \xi_2 & \bar{a}x_1 \bar{a} \\
a x_2 & \bar{a}x_1 a & \xi_3
\end{pmatrix}.
\]
Then \(\sigma = D(-1)\) and \(\sigma_3 = D(\omega_1)\).

**Theorem 2.10.** (1) \((F_4^C)_{ev} \cong \text{Spin}(9, C)\).
(2) \((F_4^C)_0 \cong (C^* \times \text{Spin}(7, C))/\mathbb{Z}_2\), \(\mathbb{Z}_2 = \{(1, 1), (-1, \sigma)\}\).

**Proof.** (1) Since \((f_4^C)_{ev} = (f_4^C)_{E_1} = \{\delta \in f_4^C \mid \delta E_1 = 0\}\) (Theorem 2.8) and \((F_4^C)^\sigma = (F_4^C)_{ev} = (F_4^C)_{E_1} \cong \text{Spin}(9, C)\) (Proposition 2.9(3)). (Another proof is found in [5] Theorem 2.4.3).

(2) Let \(\text{Spin}(7, C) = (F_4^C)_{E_1,F_1(1),F_1(\epsilon_1)}\) (Proposition 2.9(1)). We define \(\varphi : U(1, C^C) \times \text{Spin}(7, C) \to (F_4^C)^{\sigma_3}\) by
\[
\varphi(a, \beta) = D(a) \beta.
\]
Since \( iG_{01} \) is a center of \( \{f_4^C\}_0 \) (Theorem 2.8), \( D(a) \in U(1, C^C) \) and \( \beta \in Spin(7, C) \) commute. Hence \( \varphi \) is well-defined and is a homomorphism. \( \text{Ker} \varphi = Z_2. \) Since \( \{f_4^C\}^{\sigma_3} \) is connected and \( \dim_C(u(1, C^C) \oplus \text{spin}(7, C)) = 1 + 21 = 22 = \dim_C((f_4^C)_0) \) (Theorem 2.8), \( \varphi \) is onto. Therefore \( (f_4^C)_0 \cong (U(1, C^C) \times \text{spin}(7, C))/Z_2 \cong (C^* \times Spin(7, C))/Z_2. \)

2.3.1. Subgroups of type \( B_{4(4)} \) and \( R \oplus B_{3(3)} \) of \( F_{4(4)} \)

We shall determine the group structures of

\[
(f_4(4))_{ev} = (f_4^C)_{\sigma} \cap (f_4^C)^{\tau_{\gamma_1}}, \quad (f_4(4))_0 = (f_4^C)^{\sigma_3} \cap (f_4^C)^{\tau_{\gamma_1}}.
\]

**Lemma 2.11.** In the group \( F_4^C \), we have

\[
\tau D(a) \tau = D(\tau a), \quad \gamma_1 D(a) \gamma_1 = D(\bar{a}).
\]

**Theorem 2.12.**

1. \( (f_4(4))_{ev} \cong \text{spin}(4, 5). \)

2. \( (f_4(4))_0 \cong (R^* \times \text{spin}(3, 4)) \times \{1, \sigma^4\}. \)

**Proof.**

1. The group \( (f_4(4))_{ev} = (f_4^C)_{\sigma} \cap (f_4^C)^{\tau_{\gamma_1}} = (f_4^C)_{E_1} \cap (f_4^C)^{\tau_{\gamma_1}} \) (Theorem 2.10.(1)) acts on the \( R \)-vector space

\[
V^{4,5} = \{X \in (3^C)_{\tau_{\gamma_1}} \mid E_1 \circ X = 0, \text{tr}(X) = 0\}
\]

\[
= \{X = \xi E_2 - \xi E_3 + F_1(x) \mid \xi \in R, x \in (C^C)_{\tau_{\gamma_1}} = C^i\}
\]

with the norm \( (X, X)/2 = \xi^2 + \bar{x} \xi. \) Since \( (f_4(4))_{ev} = ((f_4^C)^{\sigma})^{\tau_{\gamma_1}} \) (because \( \sigma \) and \( \tau_{\gamma_1} \) commute) \( = (\text{spin}(9, C))^{\tau_{\gamma_1}} \) (Theorem 2.10.(1)), \( (f_4(4))_{ev} \) is connected. Hence we can define a homomorphism \( \pi : (f_4(4))_{ev} \rightarrow O(4, 5)^0 = O(V^{4,5})^0 \) by \( \pi(\alpha) = (1, \sigma) \). Since \( \dim((f_4(4))_{ev}) = 22 + 2 \times 7 \) (Theorem 2.8) \( = 36 = \dim(\text{so}(4, 5)), \pi \) is onto. Therefore \( (f_4(4))_{ev} \) is \( \text{spin}(4, 5) \) (not simply connected) as a double covering group of \( O(4, 5)^0. \)

2. For \( \alpha \in (f_4(4))_0 \subset (f_4^C)^{\sigma_3}, \) there exist \( a \in U(1, C^C) \) and \( \beta \in Spin(7, C) \) such that \( \alpha = \varphi(\alpha, \beta) = D(a) \beta \) (Theorem 2.10.(2)). From \( \gamma_1 \tau a \tau \gamma_1 = \alpha, \) we have

\[
\begin{cases}
\gamma_1 \tau D(a) \tau \gamma_1 = D(a) \\
\gamma_1 \tau \beta \tau \gamma_1 = \beta
\end{cases}
\]

\[
\begin{cases}
\gamma_1 \tau D(a) \tau \gamma_1 = D(-a) \\
\gamma_1 \tau \beta \tau \gamma_1 = \sigma \beta
\end{cases}
\]

In the former case, from \( D(\tau a) = D(a) \) (Theorem 2.11), we have \( \tau a = a, \) hence \( a \in U(1, C^C) \cong R^*. \) The group \( (\text{spin}(7, C))^{\tau_{\gamma_1}} = ((f_4^C)_{E_1, F_1(1), F_1(e_1)})^{\tau_{\gamma_1}} \) (Proposition 2.9.(1)) acts on the \( R \)-vector space

\[
V^{3,4} = \{X \in (3^C)_{\tau_{\gamma_1}} \mid E_1 \circ X = 0, \text{tr}(X) = 0, (F_1(1), X) = (F_1(e_1), X) = 0\}
\]

\[
= \{X = \xi E_2 - \xi E_3 + F_1(x) \mid \xi \in R, x \in (C^C)_{\tau_{\gamma_1}} = C^i, (1, x) = (e_1, x) = 0\}
\]
with the norm \((X, X)/2 = \xi^2 + x\bar{x}\). Since \((Spin(7, C))^\tau\) is connected, in a similar way as in (1), \((Spin(7, C))^\tau\) is \(spin(3, 4)\) (not simply connected) as a double covering group of \(O(3, 4)^0 = O(V^3, A)^0\). Hence the group of the former case is \((R^* \times spin(3, 4))/Z_2 (Z_2 = \{(1, 1), (-1, \sigma)\}) \cong R^+ \times spin(3, 4)\). In the latter case, \(a = e_1\) and \(\beta = \sigma'D(-e_1)\) satisfy these conditions and \(\varphi(e_1, \sigma'D(-e_1)) = \sigma'\). Therefore \((F_4(4))_0 \cong (R^* \times spin(3, 4)) \times \{1, \sigma'\}\).

2.3.2. Subgroups of type B4(-20) and R \(\oplus\) B3(-21) of F4(-20)

We define \(\delta \in F_4\) by \(\delta = \exp \frac{\pi}{2} \tilde{A}_1(e_1)\). This \(\delta\) is an \(R\)-linear transformation of \(J\) satisfying

\[
\begin{align*}
E_1 & \to E_1 \\
E_2 + E_3 & \to E_2 + E_3 \\
F_1(e_k) & \to F_1(e_k), \ k \neq 1,
\end{align*}
\]

And we define an \(R\)-linear transformation \(\sigma_1\) of \(J\) by \(\sigma_1 = \delta^{-1} \sigma' \delta\), that is,

\[
\sigma_1 \begin{pmatrix}
\xi_1 & x_3 & x_2 \\
x_3 & \xi_2 & x_1 \\
x_2 & x_1 & \xi_3
\end{pmatrix} = \begin{pmatrix}
\xi_1 & \bar{e}_1 x_2 & x_3 e_1 \\
\bar{e}_1 x_2 & \xi_3 & \bar{e}_1 x_1 e_1 \\
x_3 e_1 & e_1 x_1 e_1 & \xi_2
\end{pmatrix}.
\]

Obviously \(\sigma_1 \in F_4 \subset F_4^C\) and \(\sigma_1^2 = 1\). Since \(\sigma, \sigma'\) and \(\sigma_1\) are conjugate with each other in \(F_4\) ([5] Proposition 2.2.3), we have

**Proposition 2.13.** \(F_4(-20) = (F_4^C)^{\tau_\sigma} \cong (F_4^C)^{\tau \sigma'} \cong (F_4^C)^{\tau \sigma_1}\).

**Lemma 2.14.** In the Lie algebra \(f_4^C\), we have

1. \(\sigma_1 G_{kl} \sigma_1 = -G_{kl}, \ k = 1\) or \(l = 1, \ \sigma_1 G_{kl} \sigma_1 = G_{kl}\), otherwise,
2. \(\sigma_1 \tilde{A}_1(a) \sigma_1 = -\tilde{A}_1(e_1 a e_1), \ \sigma_1 \tilde{A}_2(a) \sigma_1 = -\tilde{A}_2(e_1 a), \ \sigma_1 \tilde{A}_3(a) \sigma_1 = -\tilde{A}_3(e_1 a)\).

**Theorem 2.15.** The 2-graded decomposition of \(f_4(-20) = (f_4^C)^{\tau \sigma_1}\) (or \(f_4^C\)),

\[
f_{4(-20)} = g_{-2} \oplus g_{-1} \oplus g_0 \oplus g_1 \oplus g_2
\]

with respect to \(ad Z, Z = -2iG_{01}\), is given by

\[
g_0 = \left\{ iG_{01}, \ G_{23}, \ G_{24}, \ G_{25}, \ G_{26}, \ G_{27}, \ \tilde{A}_1(e_2), \ \tilde{A}_1(e_3), \ G_{34}, \ G_{35}, \ G_{36}, \ G_{37}, \ \tilde{A}_1(e_4), \ \tilde{A}_1(e_5), \ G_{45}, \ G_{46}, \ G_{47}, \ \tilde{A}_1(e_6), \ \tilde{A}_2(e_7) \right\}
\]

\[
g_0 = \left\{ G_{56}, \ G_{57}, \ G_{67} \right\}
\]

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We can prove this theorem in a way similar to Theorem 2.8, using Lemma 2.14.

We shall determine the group structures of

\[ (F_{4(-20)})_{\text{ev}} = (F_4^C)^{\sigma_1} \cap (F_4^C)^{\tau_{\sigma_1}}, \quad (F_{4(-20)})_0 = (F_4^C)^{\sigma_3} \cap (F_4^C)^{\tau_{\sigma_1}}. \]

**Lemma 2.16.** In the group \( F_4^C \), we have

\[ \sigma_1 D(a) \sigma_1 = D(\bar{a}). \]

**Theorem 2.17.** (1) \( (F_{4(-20)})_{\text{ev}} \cong \text{Spin}(8,1) \).

(2) \( (F_{4(-20)})_0 \cong (\mathbb{R}^+ \times \text{Spin}(7)) \times \{1, \sigma'\} \).

**Proof.** (1) The group \( (F_{4(-20)})_{\text{ev}} = (F_4^C)^{\sigma} \cap (F_4^C)^{\tau_{\sigma_1}} = (F_4^C)_{E_1} \cap (F_4^C)^{\tau_{\sigma_1}} \) (Theorem 2.10.(1)) acts on the \( \mathbb{R} \)-vector space

\[ V^{8,1} = \{ X \in (3^C)_{\tau_{\sigma_1}} \mid E_1 \circ X = 0, \text{tr}(X) = 0 \} \]

with the norm \( (X, X)/2 = -\xi^2 + x\bar{x} = -\xi^2 + x_1^2 - y\bar{y} \). Since \( (F_{4(-20)})_{\text{ev}} = (F_4^C)^{\sigma} \cap (F_4^C)^{\tau_{\sigma_1}} = (F_4^C)^{\sigma} \cap (F_4^C)^{\tau_{\sigma_1}} \) (because \( \sigma \) and \( \tau_{\sigma_1} \) commute) \( (\text{Spin}(9, C))^{\tau_{\sigma_1}} \) (Theorem 2.10.(1)), \( (F_{4(-20)})_{\text{ev}} \) is connected. Hence, in a similar way as in Theorem 2.12.(1), \( (F_{4(-20)})_{\text{ev}} \) is \( \text{Spin}(8,1) \) as a double covering group of \( O(8,1)^9 = O(V^{8,1})^9 \).

(2) For \( \alpha \in (F_{4(-20)})_0 \subset (F_4^C)^{\sigma_3} \), there exist \( a \in U(1, C^C) \) and \( \beta \in \text{Spin}(7, C) \) such that \( \alpha = \varphi(a, \beta) = D(a)\beta \) (Theorem 2.10.(2)). From \( \sigma_1 \tau_\sigma \sigma_1 = \alpha \), we have

\[
\begin{align*}
\{ \sigma_1 \tau D(a) \tau_{\sigma_1} = D(a) \quad &\text{or} \quad \{ \sigma_1 \tau D(a) \tau_{\sigma_1} = D(-a) \\
\sigma_1 \beta \tau_{\sigma_1} = \beta \quad &\text{or} \quad \sigma_1 \beta \tau_{\sigma_1} = \beta \}
\end{align*}
\]

In the former case, from \( D(\tau \bar{a}) = D(a) \) (Lemma 2.16), we have \( \tau \bar{a} = a \), hence \( a \in U(1, C^C) \cong \mathbb{R}^* \). The group \( (\text{Spin}(7, C))^{\tau_{\sigma_1}} = ((F_4^C)^{\tau_{\sigma_1}})_{E_1, F_1(1), F_1(1)} \) (Proposition...
2.9. \( (1) \) acts on the \( \mathbb{R} \)-vector space

\[
V^7 = \{ X \in (\mathbb{R}^3)^{\sigma_1} \mid E_1 \circ X = 0, \, \text{tr}(X) = 0, \, (F_1(1), X) = (F_1(e_1), X) = 0 \} \\
= \{ X = i\xi E_2 - i\xi E_3 + iF_1(y) \mid \xi \in \mathbb{R}, \, y \in \mathbb{C}, (1, y) = (e_1, y) = 0 \}
\]

with the norm \(- (X, X)/2 = \xi^2 + y\bar{y} \). Hence, in a way as in Theorem 2.12.(2), \((\text{Spin}(7, C)))^{\sigma_1}\) is \(\text{Spin}(7)\) as a double covering group of \(\text{SO}(7) = \text{SO}(V^7)\). Hence the group of the former case is \((\mathbb{R}^* \times \text{Spin}(7))/\mathbb{Z}_2 (\mathbb{Z}_2 = \{(1, 1), (-1, a)\}) \cong \mathbb{R}^+ \times \text{Spin}(7)\). In the latter case, \(a = e_1\) and \(\beta = \sigma' D(-e_1)\) satisfy these conditions and \(\varphi(e_1, \sigma' D(-e_1)) = \sigma'\). Therefore \((F_{4(-20)})_0 \cong (\mathbb{R}^* \times \text{Spin}(7)) \times \{1, \sigma'\}\).

### Group \(E_6\)

#### 3.1. Lie groups of type \(E_6\) and their Lie algebras

In \(\mathfrak{J}, \mathfrak{J}'\) or \(\mathfrak{J}^C\), we define a trilinear form \((X, Y, Z)\) and the determinant \(\det X\) by

\[(X, Y, Z) = (X, Y \times Z), \quad \det X = \frac{1}{3} (X, X, X)\]

and in \(\mathfrak{J}^C\), hermitian inner products \((X, Y), (X, Y)\gamma\) and \((X, Y)\sigma\) by

\[(X, Y) = (\tau X, Y), \quad (X, Y)\gamma = (\gamma \tau X, Y), \quad (X, Y)\sigma = (\sigma \tau X, Y), \]

respectively.

The connected universal linear Lie groups of type \(E_6\) are given by

\[
E_6^C = \{ \alpha \in \text{Iso}_C(\mathfrak{J}^C) \mid \det \alpha X = \det X \}, \\
E_6 = \{ \alpha \in \text{Iso}_C(\mathfrak{J}^C) \mid \det \alpha X = \det X, \, (\alpha X, \alpha Y) = (X, Y) \}, \\
E_{6(6)} = \{ \alpha \in \text{Iso}_R(\mathfrak{J}') \mid \det \alpha X = \det X \}, \\
E_{6(2)} = \{ \alpha \in \text{Iso}_C(\mathfrak{J}^C) \mid \det \alpha X = \det X, \, (\alpha X, \alpha Y)\gamma = (X, Y)\gamma \}, \\
E_{6(-14)} = \{ \alpha \in \text{Iso}_C(\mathfrak{J}^C) \mid \det \alpha X = \det X, \, (\alpha X, \alpha Y)\sigma = (X, Y)\sigma \}, \\
E_{6(-26)} = \{ \alpha \in \text{Iso}_R(\mathfrak{J}) \mid \det \alpha X = \det X \}.
\]

\(E_6^C, E_6\) and \(E_{6(-26)}\) are simply connected. For \(\alpha \in \text{Iso}_C(\mathfrak{J}^C)\), \(\cdot \alpha\) denotes the transpose of \(\alpha : \, (\cdot \alpha X, Y) = (X, \alpha Y)\). Then, in the definitions of the groups above, the condition \(\det \alpha X = \det X\) is equivalent to

\[(\alpha X, \alpha Y, \alpha Z) = (X, Y, Z) \quad \text{or} \quad \alpha^{-1}(X \times Y) = \alpha X \times \alpha Y.
\]

Since for \(\alpha \in E_6^C\) we have \(\cdot \alpha^{-1} \in E_6^C\) ([5] Lemma 3.2.1), we define an automorphism \(\lambda\) of \(E_6^C\) by

\[\lambda(\alpha) = \cdot \alpha^{-1}, \quad \alpha \in E_6^C.
\]
The group $E_6^C$ contains $F_4^C$ as a subgroup by
\[ F_4^C = (E_6^C)^\lambda = \{ \alpha \in E_6^C \mid (\alpha X, \alpha Y) = (X, Y) \} \]
\[ = \{ \alpha \in E_6^C \mid \alpha E = E \}. \]

Similarly we have $F_4 \subset E_6$. In particular, $\gamma, \gamma_1, \gamma_2, \gamma_3, \sigma, \sigma', \sigma_1, \sigma_3 \in F_4 \subset E_6 \subset E_6^C$.

From the definitions of the groups above, we have

**PROPOSITION 3.1.** $E_6 = (E_6^C)^{\lambda \gamma}$, $E_6(6) = (E_6^C)^{\lambda \gamma_1}$, $E_6(2) = (E_6^C)^{\lambda \gamma_2}$, $E_6(14) = (E_6^C)^{\lambda \gamma_3}$, $E_6(-26) = (E_6^C)^{\gamma}$.

$((E_6^C)^{\lambda \gamma}$ is defined by $(E_6^C)^{\lambda \gamma} = \{ \alpha \in E_6^C \mid \tau^t \alpha^{-1} \tau = \alpha \}$ etc.).

The Lie algebra $\mathfrak{e}_6^C$ of the group $E_6^C$ is given by
\[ \mathfrak{e}_6^C = \{ \phi \in \text{Hom}_C(\mathfrak{j}^C, \mathfrak{j}^C) \mid (\phi X, X, X) = 0 \}. \]

Let $\mathfrak{j}_0^C = \{ T \in \mathfrak{j}^C \mid \text{tr}(T) = 0 \}$. For $T \in \mathfrak{j}_0^C$, we define a $C$-linear mapping $\tilde{T}$ of $\mathfrak{j}^C$ by
\[ \tilde{T}X = T \circ X = \frac{1}{2}(TX + XT). \]

Then we can verified that $\tilde{T} \in \mathfrak{e}_6^C$.

**PROPOSITION 3.2 ([1]).** Any element $\phi$ of the Lie algebra $\mathfrak{e}_6^C$ can be uniquely expressed as
\[ \phi = \delta + \tilde{T}, \quad \delta \in \mathfrak{j}_4^C, \; T \in \mathfrak{j}_0^C. \]

### 3.2. Subgroups of type $C_1^C \oplus A_5^C$ and $C \oplus A_5^C$ of $E_6^C$

In the Lie algebra $\mathfrak{e}_6^C$, let $Z = i(G_{45} - G_{67})$. Then, as is shown in $\mathfrak{g}_2^C$ or $\mathfrak{j}_4^C$, we have
\[ \gamma = \exp \frac{2\pi i}{2} Z, \quad \gamma_3 = \exp \frac{2\pi i}{3} Z. \]

**LEMMA 3.3.** In the Lie algebra $\mathfrak{e}_6^C$, we have
\begin{enumerate}
  \item $[D_1, \tilde{F}_1(a)] = \tilde{F}_1(D_1a)$, $[D_1, \tilde{F}_2(a)] = \tilde{F}_2(D_2a)$, $[D_1, \tilde{F}_3(a)] = \tilde{F}_3(D_3a)$,
  \[
  [D_1, (E_k - E_{k+1})^\gamma] = 0
  \]
  where $D_1 = \delta(D_1, D_2, D_3) \in \mathfrak{so}(8) \subset \mathfrak{j}_4^C \subset \mathfrak{e}_6^C$.
  \item $\tau \tilde{F}_k(a) \tau = \tilde{F}_k(\tau a), \quad \tau(E_k - E_{k+1})^\gamma \tau = (E_k - E_{k+1})^\gamma$,
  \[
  \gamma_1 \tilde{F}_k(a) \gamma_1 = \tilde{F}_k(\gamma_1 a), \quad \gamma_1(E_k - E_{k+1})^\gamma \gamma_1 = (E_k - E_{k+1})^\gamma.
  \]
\end{enumerate}
Theorem 3.4. The 2-graded decomposition of $e_{6(6)} = (e_6^C)^{\gamma_{11}}$ (or $e_6^C$),

$$e_{6(6)} = g_{-2} \oplus g_{-1} \oplus g_0 \oplus g_1 \oplus g_2$$

with respect to $ad Z$, $Z = i(G_{45} - G_{67})$, is given by

$$g_0 = \left\{ \begin{array}{l}
G_{46} - G_{57}, \ iG_{47} + G_{56}, \ iG_{01}, \ iG_{02}, \ iG_{03}, \ iG_{12}, \ iG_{13}, \\
iG_{23}, \ iG_{24}, \ iG_{25}, \ iG_{26}, \ iG_{35}, \ iG_{36}, \ iG_{37}, \\
\end{array} \right\}$$

$$g_{-1} = \left\{ \begin{array}{l}
G_{04} + iG_{05}, \ iG_{06} - iG_{07}, \ iG_{14} - G_{15}, \ iG_{16} + iG_{17}, \\
iG_{24} + iG_{25}, \ G_{26} - iG_{27}, \ G_{28} - G_{37}, \ G_{34} - G_{35}, \\
\end{array} \right\}$$

$$g_{-2} = \{(G_{46} + G_{57}) - i(G_{47} - G_{56})\}$$

PROOF. We can prove this theorem in a way similar to Theorem 2.4, using Lemma 3.3.

Now, we shall determine the group structures of

$$(E_6^C)_{ev} = (E_6^C)^{\gamma}, \ (E_6^C)_0 = (E_6^C)^{\eta}.$$
su*(6, C) = 3 + 35 = 38 = 36 + 2 = \text{dim}_C((e_6 C)_\text{ev}) (\text{Theorem 3.4}), \psi \text{ is onto. Therefore } (E_6 C)_\gamma \cong (Sp(1, H) \times SU^*(6, C))/Z_2.

**Theorem 3.5.** (1) \((E_6 C)_\text{ev} \cong (Sp(1, C) \times SL(6, C))/Z_2, Z_2 = \{(1, E), (-1, -E)\}.

(2) \((E_6 C)_0 \cong (C^* \times SL(6, C))/Z_2, Z_2 = \{(1, E), (-1, -E)\}.

**Proof.** (1) Note that \(f : SL(6, C) \to SU^*(6, C)\) given by \(f(A) = \varepsilon A - \varepsilon JAJ\), where \(\varepsilon = \frac{1}{2}(1 + i e_1)\), is an isomorphism, and we define \(\varphi : Sp(1, H) \times SL(6, C) \to (E_6 C)_\gamma = (E_6 C)_\text{ev} \) by

\[
\varphi(p, A) = \psi(p, f(A)).
\]

Then \(\varphi\) is also a homomorphism, onto and \(\text{Ker } \varphi = Z_2\). Therefore \((E_6 C)_\text{ev} \cong (Sp(1, H) \times SL(6, C))/Z_2 \cong (Sp(1, C) \times SL(6, C))/Z_2\).

(2) The restriction mapping \(\varphi : U(1, C) \times SL(6, C) \to (E_6 C)_\gamma = (E_6 C)_0\) of \(\varphi\) above (1) is well-defined and \(\text{Ker } \varphi = Z_2\). Since \((E_6 C)_\gamma\) is connected and \(\text{dim}_C(U(1, C) \oplus \text{sl}(6, C)) = 1 + 35 = 36 = \text{dim}_C((e_6 C)_0)\) (Theorem 3.4), \(\varphi\) is onto. Therefore \((E_6 C)_0 \cong (U(1, C) \times SL(6, C))/Z_2 \cong (C^* \times SL(6, C))/Z_2\).

3.2.1. **Subgroups of type \(C_1(1) \oplus A_5(5)\) and \(R \oplus A_5(5)\) of \(E_6(6)\)**

We shall determine the group structures of

\[
(E_6(6))_{\text{ev}} = (E_6 C)_\gamma \cap (E_6 C)^{\gamma_1}, \quad (E_6(6))_0 = (E_6 C)^{\gamma_3} \cap (E_6 C)^{\gamma_1}.
\]

**Lemma 3.6.** \(\varphi : Sp(1, H) \times SL(6, C) \to E_6 C\) of Theorem 3.5.(1) satisfies

(1) \(\gamma = \varphi(-1, E), \quad \gamma_1 = \varphi(e_2, J), \quad \gamma_2 = \varphi(e_1, e_1 I)\).

(2) \(\tau \varphi(p, A) = \varphi(\tau p, -J(\tau A)J), \quad \gamma_1 \varphi(p, A) = \varphi(\gamma_1 p, -JAJ)\).

**Theorem 3.7.** (1) \((E_6(6))_{\text{ev}} \cong (Sp(1, R) \times SL(6, R))/Z_2 \times \{1, \gamma_2\}, Z_2 = \{(1, E), (-1, -E)\}.

(2) \((E_6(6))_0 \cong (R^+ \times SL(6, R)) \times \{1, \gamma_2\}.

**Proof.** We can prove this theorem in a way similar to Theorem 1.9 or Theorem 2.7, using Lemma 3.6.

3.2.2. **Subgroups of type \(C_1(1) \oplus A_5(1)\) and \(R \oplus A_5(1)\) of \(E_6(2)\)**

**Lemma 3.8.** In the Lie algebra \(e_6 C\), we have

\[
\begin{align*}
\lambda(G_{kl}) = G_{kl} & \quad \lambda(\tilde{F}_k(a)) = -\tilde{F}_k(a) \\
\lambda(A_k(a)) = \tilde{A}_k(a) & \quad \lambda((E_k - E_{k+1})\sim) = -(E_k - E_{k+1})\sim.
\end{align*}
\]

**Theorem 3.9.** The 2-graded decomposition of \(e_6(2) = (e_6 C)^{\lambda+\gamma_1}\) (or \(e_6 C\)),

\[
e_6(2) = g_{-2} \oplus g_{-1} \oplus g_0 \oplus g_1 \oplus g_2.
\]
with respect to \( \text{ad } Z, Z = i(G_{45} - G_{67}) \), is given by

\[
g_0 = \begin{cases} 
  iG_{45}, iG_{67}, & \tilde{A}_1(1), \tilde{A}_1(e_1), \tilde{A}_1(e_2), \tilde{A}_1(e_3), \\
  G_{46} - G_{57}, i(G_{47} + G_{56}), & \tilde{A}_2(1), \tilde{A}_2(e_1), \tilde{A}_2(e_2), \tilde{A}_2(e_3), \\
  iG_{01}, iG_{02}, iG_{03}, & \tilde{A}_3(1), \tilde{A}_3(e_1), \tilde{A}_3(e_2), \tilde{A}_3(e_3), \\
  iG_{12}, iG_{13}, & \tilde{F}_1(1), \tilde{F}_1(e_1), \tilde{F}_1(e_2), \tilde{F}_1(e_3), \\
  iG_{23}, & \tilde{F}_2(1), \tilde{F}_2(e_1), \tilde{F}_2(e_2), \tilde{F}_2(e_3), \\
  i(E_1 - E_2)^{\sim}, i(E_2 - E_3)^{\sim}, & iF_3(1), iF_3(e_1), iF_3(e_2), iF_3(e_3) 
\end{cases}
\]

\[\begin{align*}
g_{-1} &= \begin{cases} 
  G_{04} + iG_{05}, & G_{06} - iG_{07}, \quad iG_{14} - G_{15}, \quad iG_{16} + G_{17}, \\
  G_{24} + iG_{25}, & G_{26} - iG_{27}, \quad iG_{34} - G_{35}, \quad iG_{36} + G_{37}, \\
  \tilde{A}_1(e_4 + ie_5), & \tilde{A}_2(e_4 + ie_5), \quad \tilde{A}_3(e_4 + ie_5), \\
  \tilde{A}_1(e_6 - ie_7), & \tilde{A}_2(e_6 - ie_7), \quad \tilde{A}_3(e_6 - ie_7), \\
  i\tilde{F}_1(e_4 + ie_5), & i\tilde{F}_2(e_4 + ie_5), \quad i\tilde{F}_3(e_4 + ie_5), \\
  i\tilde{F}_1(e_6 - ie_7), & i\tilde{F}_2(e_6 - ie_7), \quad i\tilde{F}_3(e_6 - ie_7) 
\end{cases}
\end{align*}\]

\[\begin{align*}
g_{-2} &= \{(G_{46} + G_{57}) - i(G_{47} - G_{56})\} \\
g_1 &= \tau(g_{-1})\tau, \quad g_2 = \tau(g_{-2})\tau.
\end{align*}\]

**Proof.** We can prove this theorem in a way similar to Theorem 3.4, using Lemma 3.8.

We shall determine the group structures of

\[(E_{6(2)})_{ev} = (E_6^C)^{\gamma} \cap (E_6^C)^{\tau_\gamma_1}, \quad (E_{6(2)}^C)_0 = (E_6^C)^{\gamma_3} \cap (E_6^C)^{\tau\gamma_1}.
\]

**Lemma 3.10.** \( \varphi : Sp(1, H^C) \times SL(6, C) \to E_6^C \) of Theorem 3.5.(1) satisfies

\[t^{\varphi(p, A)^{-1}} = \varphi(p, -J^tA^{-1}J).
\]

**Theorem 3.11.** (1) \((E_{6(2)})_{ev} \cong (Sp(1, R) \times SU(3,3))/Z_2 \times \{1, \gamma_2\}, Z_2 = \{(1, E), (-1, -E)\}.
\]

(2) \((E_{6(2)}^C)_0 \cong (R^+ \times SU(3,3))/Z_2 \times \{1, \gamma_2\}.
\]

**Proof.** (1) For \( \alpha \in (E_{6(2)})_{ev} \subset (E_6^C)^{\gamma} \), there exist \( p \in Sp(1, H^C) \) and \( A \in SL(6, C) \) such that \( \alpha = \varphi(p, A) \) (Theorem 3.5.(1)). From \( \gamma_1\tau^t\alpha^{-1}\tau\gamma_1 = \alpha \), we have \( \varphi(\gamma_1\tau p, -J^t(\tau A)^{-1}J) = \varphi(p, A) \) (Lemmas 3.6.(2), 3.10). Hence

\[\gamma_1\tau p = p, \quad -J^t(\tau A)^{-1}J = A \quad \text{or} \quad \gamma_1\tau p = -p, \quad -J^t(\tau A)^{-1}J = -A.
\]

In the former case, \( p \in Sp(1, H^t) \cong Sp(1, R) \) and the group \( \{A \in SL(6, C) \mid -J^t(\tau A)^{-1}J = A\} \) is \( \{A \in SL(6, C) \mid t^{(\tau A)JA = J} \cong \{A \in SL(6, C) \mid
\]
(1) The restriction mapping \( \varphi : U(1, C^C) \times SL(6, C) \to (E_6^C)^{\gamma_3} \) of \( \varphi \) above (1) induces, in a similar way as in Theorems 1.9.(2) or 2.7.(2), an isomorphism \( (E_6(2))_{ev} \cong (R^+ \times SU(3, 3)) \times \{1, \gamma_2\} \).

3.2.3. Subgroups of type \( C_{1(1)} \oplus A_5(-15) \) and \( R \oplus A_5(-15) \) of \( E_6(-14) \)

We define \( \delta' \in F_4 \subset E_6, \delta_1' \in E_6 \) by \( \delta' = \exp \frac{\pi}{2} A_1(e_4), \delta_1' = \exp \frac{\pi}{2} iF_1(e_6) \), respectively. \( \delta' \) is an \( R \)-linear transformation of \( J \) defined as similar to \( \delta \) in \( F_4 \) of Section 2.3.2 and \( \delta_1' \) is a \( C \)-linear transformation of \( J^C \) satisfying

\[
\sigma_2 \begin{pmatrix}
\xi_1 \\
x_3 \\
x_2 \\
\xi_2 \\
x_1 \\
\xi_3
\end{pmatrix} = \begin{pmatrix}
\xi_1 \\
-\xi_2 \\
-\xi_3
\end{pmatrix}
\begin{pmatrix}
i(x_3e_4)e_6 \\
-\xi_2 \\
-e_6(e_4x_1e_4)e_6
\end{pmatrix}
\begin{pmatrix}
i(e_6e_4x_2) \\
-e_6(e_4x_1e_4)e_6 \\
-\xi_3
\end{pmatrix}.
\]

Then \( \sigma_2 \in E_6 \) and \( \sigma_2^2 = 1 \). Since \( \sigma, \sigma' \) and \( \sigma_2 \) are conjugate with each other in \( E_6 \), we have

**Proposition 3.12.** \( E_6(-14) = (E_6^C)^{\lambda r \sigma} \cong (E_6^C)^{\lambda r \sigma_2} \).

We shall determine the group structures of

\( (E_6(-14))_{ev} = (E_6^C)^{\gamma} \cap (E_6^C)^{\lambda r \sigma_2}, \quad (E_6(-14))_0 = (E_6^C)^{\gamma_3} \cap (E_6^C)^{\lambda r \sigma_2} \).

**Lemma 3.13.** In the Lie algebra \( e_6^C \), we have

\[
(1) \begin{cases}
\sigma_2G_{k4}\sigma_2 = -G_{k4}, \quad k \neq 6 \\
\sigma_2G_{k6}\sigma_2 = -G_{k6}, \quad k \neq 4,
\end{cases}
\quad \sigma_2G_{kl}\sigma_2 = G_{kl}, \text{ otherwise},
\]

\[
(2) \begin{cases}
\tau_2A_1(a)\sigma_2 = \tilde{A}_1(e_6(e_4ae_4)e_6) \\
\tau_2\tilde{A}_2(a)\sigma_2 = i\tilde{F}_2(e_6(e_4a)) \\
\tau_2\tilde{A}_3(a)\sigma_2 = -i\tilde{F}_3((ae_4)e_6),
\end{cases}
\begin{cases}
\sigma_2\tilde{F}_1(a)\sigma_2 = \tilde{F}_1(e_6(e_4ae_4)e_6) \\
\sigma_2\tilde{F}_2(a)\sigma_2 = i\tilde{A}_2(e_6(e_4a)) \\
\sigma_2\tilde{F}_3(a)\sigma_2 = -i\tilde{A}_3((ae_4)e_6) \\
\sigma_2(E_k - E_{k+1})\sigma_2 = (E_k - E_{k+1}).
\end{cases}
\]
Theorem 3.14. The 2-graded decomposition of $\mathfrak{e}_6(-14) = (\mathfrak{e}_6^C)^{\lambda \tau \sigma_2}$ (or $\mathfrak{e}_6^C$),

$$\mathfrak{e}_6(-14) = \mathfrak{g}_2 \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2$$

with respect to $\text{ad} \, Z$, $Z = i(G_{45} - G_{67})$, is given by

\[
\begin{align*}
\mathfrak{g}_0 &= \left\{ G_{46} - G_{57}, \, i(G_{47} + G_{56}), \, iG_{45}, \, iG_{67}, \, i(E_1 - E_2)^\sim, \\
& \quad G_{01}, \, G_{02}, \, G_{03}, \, G_{12}, \, G_{13}, \, G_{23}, \, i(E_2 - E_3)^\sim, \\
& \quad \tilde{A}_1(1), \, \tilde{A}_1(e_1), \, \tilde{A}_1(e_2), \, \tilde{A}_1(e_3), \, i\tilde{F}_1(1), \, i\tilde{F}_1(e_1), \, i\tilde{F}_1(e_2), \, i\tilde{F}_1(e_3), \\
& \quad \tilde{A}_2(1) + i\tilde{F}_2(e_2), \, i\tilde{A}_2(1) + \tilde{F}_2(e_2), \, \tilde{A}_2(e_2) - i\tilde{F}_2(1), \, i\tilde{A}_2(e_2) - \tilde{F}_2(1), \\
& \quad \tilde{A}_3(e_1) + i\tilde{F}_3(e_3), \, i\tilde{A}_3(e_1) + \tilde{F}_3(e_3), \, \tilde{A}_3(e_3) - i\tilde{F}_3(e_1), \, i\tilde{A}_3(e_3) - \tilde{F}_3(e_1), \\
& \quad i\tilde{A}_2(e_3) - \tilde{F}_2(e_1), \, \tilde{A}_2(e_3) + i\tilde{F}_2(e_1), \, i\tilde{A}_3(e_1) + \tilde{F}_3(e_2), \, i\tilde{A}_3(e_1) + \tilde{F}_3(e_2), \\
& \quad \tilde{A}_3(e_2) - i\tilde{F}_3(1), \, i\tilde{A}_3(e_2) - \tilde{F}_3(1), \, \tilde{A}_3(e_1) - i\tilde{F}_3(e_3), \, i\tilde{A}_3(e_1) - \tilde{F}_3(e_3) \right\} \\
\mathfrak{g}_{-1} &= \left\{ iG_{04} - G_{05}, \, iG_{06} + G_{07}, \, iG_{14} - G_{15}, \, iG_{16} + G_{17}, \\
& \quad iG_{24} - G_{25}, \, iG_{26} + G_{27}, \, iG_{34} - G_{35}, \, iG_{36} + G_{37}, \\
& \quad \tilde{A}_1(e_4 + ie_5), \, \tilde{A}_1(e_4 - ie_5), \, \tilde{F}_1(e_4 + ie_5), \, \tilde{F}_1(e_4 - ie_5), \\
& \quad \tilde{A}_2(e_4 + ie_5) - i\tilde{F}_2(e_6 - ie_7), \, \tilde{A}_2(e_4 + ie_5) + i\tilde{F}_2(e_6 - ie_7), \\
& \quad \tilde{A}_3(e_4 + ie_5) + i\tilde{F}_3(e_6 - ie_7), \, \tilde{A}_3(e_4 + ie_5) - i\tilde{F}_3(e_6 + ie_5), \\
& \quad \tilde{A}_3(e_6 - ie_7) - i\tilde{F}_3(e_4 + ie_5), \, \tilde{A}_3(e_6 - ie_7) + i\tilde{F}_3(e_4 + ie_5) \right\} \\
\mathfrak{g}_2 &= \{ (G_{46} + G_{57}) - i(G_{47} - G_{56}) \} \\
\mathfrak{g}_1 &= \tau(\mathfrak{g}_{-1}) \tau, \quad \mathfrak{g}_2 = \tau(\mathfrak{g}_{-2}) \tau.
\end{align*}
\]

Proof. We can prove this theorem in a way similar to Theorem 3.9, using Lemmas 3.8 and 3.13.

In the identification $\mathfrak{J}^C = \mathfrak{J}(3, \mathfrak{H}^C) \oplus (\mathfrak{H}^C)^3$ in Section 2.2, we have

$$\sigma_2(M + n) = \sigma_2 X = \sigma_2 \begin{pmatrix} x_1 & \xi_1 & x_2 \\ x_3 & \xi_2 & x_1 \\ x_2 & \xi_3 \end{pmatrix}$$

$$\begin{pmatrix} \xi_1 & i(x_3 e_4) e_6 & ie_6(e_4 x_2) \\ i(x_3 e_4) e_6 & -\xi_2 & -e_6(e_4 x_1 e_4) e_6 \\ ie_6(e_4 x_2) & -e_6(e_4 x_1 e_4) e_6 & -\xi_3 \end{pmatrix}$$

$$\begin{pmatrix} \xi_1 & -ie_2 m_3 & im_2 e_2 \\ -ie_2 m_3 & -\xi_2 & -m_1 \\ im_2 e_2 & -m_1 & -\xi_3 \end{pmatrix} + (\gamma_1 n_1, i(\gamma_1 n_2) e_2, i(\gamma_1 n_3) e_2)$$
LEMMA 3.15. \( \varphi : Sp(1, H^C) \times SL(6, C) \to E_6^C \) of Theorem 3.5.(1) satisfies 
\[
\sigma_2 \varphi(p, A) \sigma_2 = \varphi(\eta_1 p, -PAP), \quad P = \text{diag}(J, i, i, i, i, i) \in M(6, C).
\]

THEOREM 3.16. (1) \((E_6(-14))_{ev} \cong (Sp(1, R) \times SU(1, 5)) / Z_2 \times \{1, \gamma_2\}, \ Z_2 = \{(1, \text{e}), (-1, -\text{e})\}.

(2) \((E_6(-14))_0 \cong (R^+ \times SU(1, 5)) \times \{1, \gamma_2\}.

PROOF. (1) For \( \alpha \in (E_6(-14))_{ev} \subset (E_6^C)_{\gamma} \), there exist \( p \in Sp(1, H^C) \) and \( A \in SL(6, C) \) such that \( \alpha = \varphi(p, A) \) (Theorem 3.5.(1)). From \( \sigma_2 \tau^\alpha \tau_2 = \alpha \), we have \( \varphi(\gamma_1 \tau p, -P^t(\tau A)^{-1} P) = \varphi(p, A) \) (Lemmas 3.6.(2), 3.10, 3.15). Hence

\[
\gamma_1 \tau p = p, \quad -P^t(\tau A)^{-1} P = A \quad \text{or} \quad \gamma_1 \tau p = -p, \quad -P^t(\tau A)^{-1} P = -A.
\]

In the former case, \( p \in Sp(1, H^C) \cong Sp(1, R) \) and the group \( \{A \in SL(6, C) \mid -P^t(\tau A)^{-1} P = A\} \) is \( \{A \in SL(6, C) \mid P^t(\tau A) P = A\} \cong \{A \in SL(6, C) \mid P^t(\tau A) I_1 A = I_1 \} \cong SU(1, 5) \). Hence the group of the former case is \((Sp(1, R) \times SU(1, 5))/Z_2\). In the latter case, \( p = e_1 \) and \( A = e_1 I \) satisfy these conditions and \( \varphi(e_1, e_1 I) = \gamma_2 \) (Lemma 3.6.(1)). Therefore \((E_6(-14))_{ev} \cong (Sp(1, R) \times SU(1, 5))/Z_2 \times \{1, \gamma_2\} \).

(2) The restriction mapping \( \varphi : U(1, C^C) \times SL(6, C) \to (E_6^C)^{\gamma_3} \) of \( \varphi \) above (1) induces, in a similar way as in Theorems 1.9.(1), 2.7.(2) or 3.11.(2), an isomorphism \((E_6(-14))_0 \cong (R^+ \times SU(1, 5)) \times \{1, \gamma_2\} \).

3.3. Subgroups of type \( C_1^C \oplus A_5^C \) and \( C_1^C \oplus C \oplus A_4^C \) of \( E_6^C \)

Let \( \omega = \exp \frac{2\pi i}{3}, \mu = \exp \frac{2\pi i}{6}, \nu = \exp \frac{2\pi i}{9} \in C \) and let \( A_\mu = \text{diag}(\mu^5, \mu^{-1}, \ldots, \mu^{-1}) \) and \( A_\nu = \text{diag}(\nu^5, \nu^{-1}, \ldots, \nu^{-1}) \in SL(6, C) \). We define \( \mu_2, \mu_3 \in E_6 \subset E_6^C \) by 

\[
\mu_2 = \varphi(1, A_\mu), \quad \mu_3 = \varphi(1, A_\nu),
\]
respectively, where \( \varphi : Sp(1, H^C) \times SL(6, C) \to E_6^C \) is the mapping defined in Theorem 3.5.1(1). Now, in the Lie algebra \( \mathfrak{e}_6^C \), let \( Z = i(G_{45} + G_{67}) + \frac{2}{3}(2E_1 - E_2 - E_3)^\sim \). Then we can verify that

\[
\omega^2 = \mu_2 = \exp 2\pi i \frac{Z}{2}, \quad \mu_3 = \exp 2\pi i \frac{Z}{3}.
\]

**Lemma 3.17.** In the Lie algebra \( \mathfrak{e}_6^C \), we have

\[
[(\xi_1 E_1 + \xi_2 E_2 + \xi_3 E_3)^\sim, \tilde{A}_k(a)] = \frac{1}{2}(\xi_{k+1} - \xi_{k+2})\tilde{F}_k(a),
\]

\[
[(\xi_1 E_1 + \xi_2 E_2 + \xi_3 E_3)^\sim, \tilde{F}_k(a)] = \frac{1}{2}(\xi_{k+1} - \xi_{k+2})\tilde{A}_k(a),
\]

\( \xi_1 + \xi_2 + \xi_3 = 0 \).

**Theorem 3.18.** The 2-graded decomposition of \( \mathfrak{e}_6(6) = (\mathfrak{e}_6^C)^{\gamma_1} \) (or \( \mathfrak{e}_6^C \)),

\[
\mathfrak{e}_6(6) = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2
\]

with respect to \( \text{ad} Z \), \( Z = i(G_{45} + G_{67}) + \frac{2}{3}(2E_1 - E_2 - E_3)^\sim \), is given by

\[
\mathfrak{g}_0 = \begin{cases}
G_{46} + G_{57}, i(G_{47} - G_{56}), iG_{45}, iG_{67}, (E_2 - E_3)^\sim,
G_{01}, G_{02}, iG_{03}, iG_{12}, G_{13}, iG_{23}, (2E_1 - E_2 - E_3)^\sim,
\tilde{A}_1(1), i\tilde{A}_1(e_1), \tilde{A}_1(e_2), i\tilde{A}_1(e_3), \tilde{F}_1(1), i\tilde{F}_1(e_1), i\tilde{F}_1(e_2), i\tilde{F}_1(e_3),
\tilde{F}_2(1 + ie_1), \tilde{A}_2(1 - ie_1) + \tilde{F}_2(1 - ie_1),
\tilde{A}_3(1 + ie_1) - \tilde{F}_3(1 + ie_1), \tilde{A}_3(1 - ie_1) + \tilde{F}_3(1 - ie_1),
\end{cases}
\]

\[
\mathfrak{g}_{-1} = \begin{cases}
G_{04} + iG_{05}, G_{06} + iG_{07}, iG_{14} - G_{15}, iG_{16} - G_{17},
G_{24} + iG_{25}, G_{26} + iG_{27}, iG_{34} - G_{35}, iG_{36} - G_{37},
\tilde{A}_1(e_4 + ie_5), \tilde{A}_1(e_6 + ie_7), \tilde{F}_1(e_4 + ie_5), \tilde{F}_1(e_6 + ie_7),
\tilde{F}_2(e_4), i\tilde{A}_2(e_5) + i\tilde{F}_2(e_5), \tilde{A}_2(e_6) + \tilde{F}_2(e_6),
i\tilde{A}_3(e_7) + i\tilde{F}_3(e_7), \tilde{A}_3(e_5) - \tilde{F}_3(e_5), i\tilde{A}_3(e_6) - i\tilde{F}_3(e_6),
\end{cases}
\]

\[
\mathfrak{g}_{-2} = \begin{cases}
(G_{46} - G_{57}) + i(G_{47} + G_{56}),
\tilde{A}_2(1 + ie_1) + \tilde{F}_2(1 + ie_1), \tilde{A}_2(e_2 - ie_3) + \tilde{F}_2(e_2 - ie_3),
\tilde{A}_3(1 - ie_1) - \tilde{F}_3(1 - ie_1), \tilde{A}_3(e_2 - ie_3) - \tilde{F}_3(e_2 - ie_3)
\end{cases}
\]

\[
\mathfrak{g}_1 = \tau(\lambda(\mathfrak{g}_{-1}))^\tau, \quad \mathfrak{g}_2 = \tau(\lambda(\mathfrak{g}_{-2}))^\tau.
\]

**Proof.** Note that for \( D_1 = G_{45} + G_{67} \in \mathfrak{so}(8) \) we have \( D_2 = G_{01} - G_{23}, D_3 = -G_{01} - G_{23} \). Then we can prove this theorem in a way similar to Theorem 3.4, using Lemma 3.17.
Now, we shall determine the group structures of

\[(E_6^C)_{ev} = (E_6^C)^{\mu_2}, \quad (E_6^C)_0 = (E_6^C)^{\mu_3}.\]

**Theorem 3.19.** (1) \((E_6^C)_{ev} \cong (Sp(1, C) \times SL(6, C))/Z_2, \quad Z_2 = \{(1, E), (-1, -E)\}.\)

(2) \((E_6^C)_0 \cong (Sp(1, C) \times C^* \times SL(5, C))/(Z_2 \times Z_5), \quad Z_5 = \{(1, 1, E), (-1, -1, -E), (\kappa, \kappa^k E), \quad k = 0, 1, \ldots, 4\} (\kappa = \exp \frac{2\pi i}{5}).\)

**Proof.** (1) Since \(\mu_2 = \omega_2^2 = w^2\gamma\) and \(\omega_2^1\) is a center of \(E_6^C\), \((E_6^C)_{ev} = (Sp(1, C) \times SL(6, C))/Z_2\) (Theorem 3.5.(1)).

(2) The restriction mapping \(\varphi : Sp(1, H^C) \times S(GL(1, C) \times GL(5, C)) \rightarrow (E_6^C)^{\mu_3}\) of \(\varphi\) of Theorem 3.5.(2) is well-defined and \(\text{Ker}\ \varphi = \{(1, E), (-1, -E)\} = Z_2\). Since \((E_6^C)^{\mu_3}\) is connected and \(\dim_{C}(sp(1, C) \oplus gl(1, C) \oplus gl(5, C)) = 3 + 1 + 24 = 28 = \dim_{C}(e_6^C)_0\) (Theorem 3.18), \(\varphi\) is onto. Hence \((E_6^C)^{\mu_3} \cong (Sp(1, H^C) \times S(GL(1, C) \times GL(5, C)))/Z_5\). Since \(h : C^* \times SL(5, C) \rightarrow S(GL(1, C) \times GL(5, C)), h(z) = \text{induces an isomorphism}\) \((C^* \times SL(5, C))/Z_5\) \((Z_5 = \{(\kappa^k E), (1, 1, E), (-1, -1, -E)\}) \rightarrow S(GL(1, C) \times GL(5, C))\), we have \((E_6^C)^{\mu_3} \cong (Sp(1, C) \times C^* \times SL(5, C))/Z_2 \times Z_5).\)

**3.3.1. Subgroups of type \(C_1(1) \oplus A_5(5)\) and \(C_1(1) \oplus R \oplus A_4(4)\) of \(E_6(6)\)**

We shall determine the group structures of

\[(E_6(6))_{ev} = (E_6(6))^{\mu_2} \cap (E_6(6))^{\gamma_1}, \quad (E_6(6))_0 = (E_6(6))^{\mu_3} \cap (E_6(6))^{\gamma_1}.\]

**Theorem 3.20.** (1) \((E_6(6))_{ev} \cong (Sp(1, R) \times SL(6, R))/Z_2 \times \{(1, \gamma_2)\}, \quad Z_2 = \{(1, E), (-1, -E)\}.\)

(2) \((E_6(6))_0 \cong (Sp(1, R) \times R^+ \times SL(5, R)) \times \{(1, \gamma_2)\}.\)

**Proof.** (1) \((E_6(6))_{ev} = (E_6(6))^{\mu_2} \cap (E_6(6))^{\gamma_1} = (E_6(6))^{\gamma_1} \cap (E_6(6))^{\gamma_1} \cong (Sp(1, R) \times SL(6, C))/Z_2 \times \{(1, \gamma_2)\} \text{ (Theorem 3.7.(1)).}\)

(2) For \(\alpha \in (E_6(6))_0 \subset (E_6(6))^{\mu_3}\), there exist \(p \in Sp(1, H^C)\) and \(A \in S(GL(1, C) \times GL(5, C))\) such that \(\alpha = \varphi(p, A)\) (Theorem 3.19.(2)). From \(\gamma_1 \tau \alpha \gamma_1 = \alpha\), we have \(\varphi(\gamma_1 \tau p, \gamma_1 \tau A) = \varphi(p, A)\) (Lemma 3.6.(2)). Hence

\[\gamma_1 \tau p = p, \quad \gamma_1 \tau A = A \quad \text{or} \quad \gamma_1 \tau p = -p, \quad \gamma_1 \tau A = -A.\]

The group of the former case is \((Sp(1, R) \times S(GL(1, R) \times GL(5, R)))/Z_2, \quad Z_2 = \{(1, E), (-1, -E)\}) \cong (SL(1, R) \times R^+ \times SL(5, R))/Z_2 \quad Z_2 = \{(1, 1, E), (-1, -1, -E)\}) \cong Sp(1, R) \times R^+ \times SL(5, R). p = e_1\) and \(A = e_1 I\) satisfy the latter case and \(\varphi(e_1, e_1 I) = \gamma_2\) (Lemma 3.6.(1)). Therefore \((E_6(6))_0 \cong (Sp(1, R) \times R^+ \times SL(5, R)) \times \{(1, \gamma_2)\}.\)
3.4. Subgroups of type $C \oplus D_5^C$ and $C \oplus C \oplus D_4^C$ of $E_6^C$

In the Lie algebra $\mathfrak{e}_6^C$, let $Z = -2iG_{01}$. Then, as is shown in $f_4^C$, we have

$$\sigma = \exp \frac{2\pi i}{Z}, \quad \sigma_3 = \exp \frac{2\pi i}{3Z}.$$ 

**Theorem 3.21.** The 2-graded decomposition of $\mathfrak{e}_6(6) = (\mathfrak{e}_6^C)^{\gamma_11}$ (or $\mathfrak{e}_6^C$),

$$\mathfrak{e}_6(6) = \mathfrak{g}_2 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2$$ 

with respect to $\text{ad} Z$, $Z = -2iG_{01}$, is given by

$$\mathfrak{g}_0 = \begin{cases} iG_{01}, & iG_{23}, G_{24}, iG_{25}, G_{26}, iG_{27}, \tilde{A}_1(e_2), \tilde{F}_1(e_2), \\ iG_{34}, & G_{35}, iG_{36}, G_{37}, \tilde{A}_1(e_3), i\tilde{F}_1(e_3), \\ iG_{45}, & G_{46}, iG_{47}, \tilde{A}_1(e_4), \tilde{F}_1(e_4), \\ iG_{56}, & G_{57}, \tilde{A}_1(e_5), i\tilde{F}_1(e_5), \\ iG_{67}, & \tilde{A}_1(e_6), \tilde{F}_1(e_6), \end{cases}$$

$$\mathfrak{g}_{-1} = \begin{cases} A_2(1 + ie_1), & \tilde{A}_2(e_2 + ie_3), A_2(e_4 + ie_5), \tilde{A}_2(e_6 + ie_7), \\ \tilde{A}_3(1 + ie_1), & \tilde{A}_3(e_2 - ie_3), \tilde{A}_3(e_4 - ie_5), \tilde{A}_3(e_6 - ie_7), \\ \tilde{F}_2(1 + ie_1), & \tilde{F}_2(e_2 + ie_3), \tilde{F}_2(e_4 + ie_5), \tilde{F}_2(e_6 + ie_7), \\ \tilde{F}_3(1 + ie_1), & \tilde{F}_3(e_2 - ie_3), \tilde{F}_3(e_4 - ie_5), \tilde{F}_3(e_6 - ie_7), \end{cases}$$

$$\mathfrak{g}_{-2} = \begin{cases} iG_{02} - iG_{12}, & iG_{03} + G_{13}, G_{04} - iG_{14}, \tilde{A}_1(1 - ie_1), \\ iG_{05} + G_{15}, & G_{06} - iG_{16}, iG_{07} + iG_{17}, \tilde{F}_1(1 - ie_1) \end{cases}$$

$$\mathfrak{g}_1 = \tau(\mathfrak{g}_{-1}), \quad \mathfrak{g}_2 = \tau(\mathfrak{g}_{-2})\tau.$$

**Proof.** We can prove this theorem in a way similar to Theorem 2.8, using Lemma 3.3.

We shall determine the group structures of

$$(E_6^C)_{\text{ev}} = (E_6^C)^\sigma, \quad (E_6^C)_0 = (E_6^C)^{\sigma_3}.$$ 

**Proposition 3.22.** (1) $(E_6^C)_{E_1,F_1(1),F_1(e_1)} \cong \text{Spin}(8,C)$.

(2) $(E_6^C)_{E_1} \cong \text{Spin}(10,C)$.

**Proof.** (1) The group $(E_6^C)_{E_1,F_1(1),F_1(e_1)}$ acts on the $C$-vector space

$$(V^C)^8 = \{ X \in \mathfrak{g}^C \mid 4E_1 \times (E_1 \times X) = X, (E_1, F_1(1), X) = (E_1, F_1(e_1), X) = 0 \}$$

$$= \{ X = \xi E_2 + \eta E_3 + F_1(x) \mid \xi, \eta \in C, x \in \mathfrak{g}^C, (1, x) = (e_1, x) = 0 \}$$
with the norm \((E_1, X, X) = \xi \eta - x \bar{x}\). The connectedness of \((E_6^C)_{E_1, F_1(1), F_1(e_1)}\) follows from

\[(E_6^C)_{E_1, F_1(1), F_1(e_1)} / (F_4^C)_{E_1, F_1(1), F_1(e_1)} \cong (S^C)^7 = \{ X \in (V^C)^8 \mid (E_1, X, X) = 1 \}\]

(Proposition 2.9.(1)). Hence we can define a homomorphism \(\pi:\)

\[(E_6^C)_{E_1, F_1(1), F_1(e_1)} \to SO(8, C) = SO((V^C)^8))\]

by \(\pi(\alpha) = \alpha |(V^C)^8\). Ker \(\pi = \{1, \sigma\} = Z_2\). Since \(\text{dim}_C((E_6^C)_{E_1, F_1(1), F_1(e_1)}) = \text{dim}_C((F_4^C)_{E_1, F_1(1), F_1(e_1)}) + \text{dim}_C((S^C)^7) = 21 + 7 = 28 = \text{dim}_C(SO(8, C))\), \(\pi\) is onto. Therefore \((E_6^C)_{E_1, F_1(1), F_1(e_1)}\) is \(Spin(8, C)\) as a double covering group of \(SO(8, C)\).

(2) The group \((E_6^C)_{E_1}\) acts on the \(C\)-vector space

\[(V^C)^{10} = \{ X \in \mathfrak{C} \mid 4E_1 \times (E_1 \times X) = X \} = \{ X = \xi E_2 + \eta E_3 + F_1(x) \mid \xi, \eta \in C, x \in \mathfrak{C} \} \]

with the norm \((E_1, X, X) = \xi \eta - x \bar{x}\). The connectedness of \((E_6^C)_{E_1}\) follows from

\[(E_6^C)_{E_1} / (F_4^C)_{E_1} \cong (S^C)^9 = \{ X \in (V^C)^{10} \mid (E_1, X, X) = 1 \}\]

(Proposition 2.9.(3)). Hence we can define a homomorphism \(\pi:\)

\[(E_6^C)_{E_1} \to SO(10, C) = SO((V^C)^{10})\) by \(\pi(\alpha) = \alpha |(V^C)^{10}\). Ker \(\pi = \{1, \sigma\} = Z_2\). Since \(\text{dim}_C((E_6^C)_{E_1}) = \text{dim}_C((F_4^C)_{E_1}) + \text{dim}_C((S^C)^7) = 36 + 9 = 45 = \text{dim}_C(SO(10, C))\), \(\pi\) is onto. Therefore \((E_6^C)_{E_1}\) is \(Spin(10, C)\) as a double covering group of \(SO(10, C)\).

We define a mapping \(\phi: C^* \to E_6^C\) by

\[
\phi(\theta) = \begin{pmatrix}
\xi_1 & x_3 & \bar{x}_2 \\
\bar{x}_3 & \xi_2 & x_1 \\
x_2 & \bar{x}_1 & \xi_3
\end{pmatrix} = \begin{pmatrix}
\theta^4 \xi_1 & \theta x_3 & \theta \bar{x}_2 \\
\theta \bar{x}_3 & \theta^{-2} \xi_2 & \theta^{-2} x_1 \\
\theta x_2 & \theta^{-2} \bar{x}_1 & \theta^{-2} \xi_3
\end{pmatrix}.
\]

**Theorem 3.23.** (1) \((E_6^C)^{ev} \cong C^* \times Spin(10, C))/Z_4, Z_4 = \{(1, 1), (-1, \sigma), (i, \phi(-i)), (-i, \phi(i))\}.

(2) \((E_6^C)^0 \cong (C^* \times C^* \times Spin(8, C))/Z_4 \times Z_2, Z_4 = \{(1, 1, 1), (-1, -1, 1), (i, e_1, \phi(-i)D(-e_i)), (-i, -e_1, \phi(i)D(e_i))\}, Z_2 = \{(1, 1, 1), (1, -1, \sigma)\}.

**Proof.** (1) Let \(Spin(10, C) = (E_6^C)_{E_1}\) (Proposition 3.22.(2)). We define \(\varphi: C^* \times Spin(10, C) \to (E_6^C)^\sigma\) by

\[
\varphi(\theta, \beta) = \phi(\theta) \beta.
\]
\(\varphi\) is well-defined. In fact, \((E_6^C)_E \subset (E_6^C)^{\sigma}\) (because \((e_6^C)_E \subset (e_6^C)^{ev}\) (Theorem 3.21)) and \(\phi(\theta) \in (E_6^C)^{\sigma}\). Since \(\phi(\theta)\) and \(\beta \in \text{Spin}(10, C)\) commute, \(\phi\) is a homomorphism. \(\text{Ker} \varphi = Z_4\). Since \((E_6^C)^{\sigma}\) is connected and \(\text{dim}_C((e_6^C)^{ev}) = 1 + 45 = 46\) (Theorem 3.21), \(\varphi\) is onto. Therefore \((E_6^C)^{ev} \cong (C^* \times \text{Spin}(10, C))/Z_4\).

(2) Let \(\text{Spin}(8, C) = (E_6^C)_{E_1, F_1(1), F_1(e_1)}\) (Proposition 3.22.(1)). We define \(\varphi : C^* \times U(1, C^C) \times \text{Spin}(8, C) \to (E_6^C)^{\sigma_2}\) by

\[
\varphi(\theta, a, \beta) = \phi(\theta)D(a)\beta.
\]

\(\varphi\) is well-defined (because \(\sigma_2 = \varphi(1, (1, l, 1))\)). Since \(D(a)\) commute with \(\phi(\theta)\) and \(\beta\) (because \(iG_0\) is a center of \((e_6^C)^0\) (Theorem 3.21)), \(\varphi\) is a homomorphism. \(\text{Ker} \varphi = \{(1, 1, 1), (-1, -1, 1), (1, -1, \sigma), (-1, 1, \sigma), (i, e_1, \varphi(-i)D(-e_1)), (i, -e_1, \varphi(\sigma)D(e_1)), (-i, -e_1, \varphi(i)D(e_1)), (-i, e_1, \varphi(i)D(-e_1))\} = Z_4 \times Z_2\). Since \((E_6^C)^{\sigma_2}\) is connected and \(\text{dim}_C((e_6^C)^0) = 1 + 1 + 28 = 30 = \text{dim}_C((e_6^C)^0)\) (Theorem 3.21), \(\varphi\) is onto. Therefore \((E_6^C)^0 \cong (C^* \times U(1, C^C) \times \text{Spin}(8, C))/Z_4 \times Z_2\).

3.4.1. Subgroups of type \(R \oplus D_5(5)\) and \(R \oplus R \oplus D_4(4)\) of \(E_6(6)\)

We shall determine the group structures of

\[
(E_6(6))_{ev} = (E_6^C)^{\sigma} \cap (E_6^C)^{\tau_{\gamma_1}}, \quad (E_6(6))_0 = (E_6^C)^{\sigma_2} \cap (E_6^C)^{\tau_{\gamma_1}}.
\]

We define \(\rho \in E_6 \subset E_6^C\) by \(\rho = \varphi(1, I_2')\) where \(I_2' = \text{diag}(1, -1, 1, -1, 1, 1)\) and \(\varphi : \text{Sp}(1, H^C) \times \text{SL}(6, C) \to E_6^C\) is the mapping defined in Theorem 3.5.(1). The explicit form of \(\rho\) is

\[
\rho \begin{pmatrix}
\xi_1 \\
3 \\
x_3 \\
\xi_2 \\
3 \\
x_1 \\
\xi_3 \\
3 \\
x_2
\end{pmatrix} = \begin{pmatrix}
-\xi_1 & e_1x_3e_1 & -i\xi_2e_1 \\
e_1x_3e_1 & -\xi_2 & ie_1x_1 \\
-ie_2e_1 & ie_1x_1 & \xi_3 \\
3 & 3 & 3
\end{pmatrix}.
\]

Lemma 3.24. In the group \(E_6^C\), we have

1. \(\tau\phi(\theta)\tau = \phi(\tau\theta)\), \(\gamma_1\phi(\theta)\gamma_1 = \phi(\theta)\).
2. \(\tau\rho\tau = \sigma'\rho\), \(\gamma_1\rho\gamma_1 = \sigma'\rho\).

Theorem 3.25. (1) \((E_6(6))_{ev} \cong (R^+ \times \text{spin}(5, 5)) \times \{1, \rho\}\).
(2) \((E_6(6))_0 \cong (R^+ \times R^+ \times \text{spin}(4, 4)) \times \{1, \sigma'\} \times \{1, \rho\}\).

Proof. (1) For \(\alpha \in (E_6(6))_{ev} \subset (E_6^C)^{\tau}\), there exist \(\theta \in C^*\) and \(\beta \in \text{Spin}(10, C)\) such that \(\alpha = \varphi(\theta, \beta) = \phi(\theta)\beta\) (Theorem 3.23.(1)). From \(\gamma_1\tau\alpha\tau\gamma_1 = \alpha\), we have

\[
\begin{align*}
\gamma_1\tau\phi(\theta)\tau\gamma_1 & = \phi(\theta) \quad (\text{i}) \\
\gamma_1\tau\beta\tau\gamma_1 & = \beta \quad (\text{ii}) \\
\phi(-\theta) & = \sigma\beta \quad (\text{iii}) \\
\phi(i\theta) & = \phi(-i\theta) \quad (\text{iv}) \quad \phi(i\beta).
\end{align*}
\]
(i) From $\phi(\tau \theta) = \phi(\theta)$ (Lemma 3.24.(1)), we have $\tau \theta = \theta$, hence $\theta \in R^*$. The group $(\text{Spin}(10, C))^{\tau_1} = ((E_6^C)_{E_1})^{\tau_1}$ (Proposition 3.22.(2)) $= ((E_6^C)^{\tau_1})_{E_1}$ acts on the $R$-vector space

$$V^{5,5} = \{ X \in (3^C)^{\tau_1} \mid 4E_1 \times (E_1 \times X) = X \}$$

$$= \{ X = \xi E_2 + \eta E_3 + F_1(x) \mid \xi, \eta \in R, x \in (C^C)^{\tau_1} = C' \}$$

with the norm $(E_1, X, X) = \xi \eta - x\bar{x}$. Hence $(\text{Spin}(10, C))^{\tau_1}$ is $\text{spin}(5, 5)$, in a similar way as in Theorem 2.12.(2), as a double covering group of $O(5, 5)^0 = O(5.5)^0$. Therefore the group of the case (i) is $(R^* \times \text{spin}(5, 5))/\mathbb{Z}_2$ ($\mathbb{Z}_2 = \{(1,1), (-1, a)\}) \cong R^+ \times \text{spin}(5, 5)$.

(ii) $\tau \theta = -\theta$, $\gamma_1 \tau \beta \gamma_1 = \sigma \beta$. Then $\theta = i$ and $\beta = \phi(-i)$ satisfy these conditions and $\phi(i, \phi(-i)) = 1$.

(iii) $\varphi\left(\frac{1-i}{\sqrt{2}}, \rho \phi\left(\frac{1+i}{\sqrt{2}}\right)\right) = \rho$.

Therefore $(E_{6(6)})_{ev} \cong (R^* \times \text{spin}(5, 5)) \times \{1, \rho\}$.

(2) For $a \in (E_{6(6)})_{ev} \subset (E_6^C)^{\sigma_5}$, there exist $\theta \in C^*$, $a \in U(1, C')$ and $\beta \in \text{Spin}(8, C)$ such that $\alpha = \varphi(\theta, a, \beta) = \phi(\theta)D(a)\beta$ (Theorem 3.23.(2)). From $\gamma_1 \tau \alpha \gamma_1 = \alpha$, we have

$$\begin{aligned}
&\left\{ 
\begin{array}{l}
\gamma_1 \tau \phi(\theta) \gamma_1 = \phi(\theta) \\
\gamma_1 \tau D(a) \gamma_1 = D(a) \\
\gamma_1 \tau \beta \gamma_1 = \beta
\end{array}
\right.
&\left\{ 
\begin{array}{l}
\phi(-\theta) \\
D(-a) \\
\beta
\end{array}
\right.
&\left\{ 
\begin{array}{l}
\phi(\theta) \\
D(-a) \\
\sigma \beta
\end{array}
\right.

\end{aligned}$$

$$\begin{aligned}
&\left\{ 
\begin{array}{l}
\phi(-\theta) \\
D(a) \\
\sigma \beta
\end{array}
\right.
&\left\{ 
\begin{array}{l}
\phi(-i\theta) \\
D(-e_1 a) \\
\phi(i)D(e_1)\beta
\end{array}
\right.
&\left\{ 
\begin{array}{l}
\phi(-i\theta) \\
D(-e_1 a) \\
\phi(-i)D(e_1)\beta
\end{array}
\right.

\end{aligned}$$

(i) From $\phi(\tau \theta) = \phi(\theta)$, $D(\tau \bar{a}) = D(a)$ (Lemma 3.24.(1)), we have $\tau \theta = \theta$, $\tau \bar{a} = a$, hence $\theta \in R^*$, $a \in U(1, C') \cong R^*$, respectively. The group $(\text{Spin}(8, C))^{\tau_1} = ((E_6^C)^{\tau_1})_{E_1, F_1(1), F_1(e_1)}$ (Proposition 3.22.(1)) acts on the $R$-vector space

$$V^{4,4} = \{ X \in (3^C)^{\tau_1} \mid 4E_1 \times (E_1 \times X) = X, (E_1, F_1(1), X) = (E_1, F_1(e_1), X) = 0 \}$$

$$= \{ X = \xi E_2 + \eta E_3 + F_1(x) \mid \xi, \eta \in R, x \in (C^C)^{\tau_1} = C', (x, e_1, x) = 0 \}$$

with the norm $(E_1, X, X) = \xi \eta - x\bar{x}$. Hence $(\text{Spin}(8, C))^{\tau_1}$ is $\text{spin}(4, 4)$, in a similar way as in (1). Therefore the group of the case (i) is $(R^* \times R^* \times \text{spin}(4, 4))/(Z_2 \times Z_2)$ $((Z_2 \times Z_2) = \{(1,1,1), (-1, -1, 1) \} \times \{(1,1,1), (1, -1, \sigma)\}) \cong R^+ \times R^+ \times \text{spin}(4, 4)$.
(ii) $\theta = i$, $a = e_1$ and $\beta = \phi(-i)D(-e_1)$ satisfy these conditions and $\varphi(i, e_1, \phi(-i)D(-e_1)) = 1$.

Similarly, using Lemma 3.24, we have

(iii) $\varphi(1, e_1, \sigma' D(-e_1)) = \sigma'$.

(iv) $\varphi(i, 1, \sigma' \phi(-i)) = \sigma'$.

(v) $\varphi \left( \frac{1 - i}{\sqrt{2}}, \frac{1 + e_1}{\sqrt{2}}, \rho \phi \left( \frac{1 - i}{\sqrt{2}} \right) D \left( \frac{1 - e_1}{\sqrt{2}} \right) \right) = \rho$.

(vi) $\varphi \left( \frac{1 - i}{\sqrt{2}}, \frac{1 + e_1}{\sqrt{2}}, \rho \phi \left( \frac{1 - i}{\sqrt{2}} \right) D \left( \frac{1 + e_1}{\sqrt{2}} \right) \right) = \rho$.

(vii) $\varphi \left( \frac{1 - i}{\sqrt{2}}, \frac{1 - e_1}{\sqrt{2}}, \sigma' \rho \phi \left( \frac{1 + i}{\sqrt{2}} \right) D \left( \frac{1 + e_1}{\sqrt{2}} \right) \right) = \sigma' \rho$.

(viii) $\varphi \left( \frac{1 + i}{\sqrt{2}}, \frac{1 + e_1}{\sqrt{2}}, \sigma' \rho \phi \left( \frac{1 - i}{\sqrt{2}} \right) D \left( \frac{1 - e_1}{\sqrt{2}} \right) \right) = \sigma' \rho$.

Therefore $(E_6(6))_0 \cong (R^+ \times R^+ \times \text{spin}(4, 4)) \times \{1, \sigma', \rho, \sigma' \rho\}$.

### 3.4.2. Subgroups of type $iR \oplus D_{5(3)}$ and $iR \oplus R \oplus D_{4(2)}$ of $E_6(2)$

**Theorem 3.26.** The 2-graded decomposition of $e_{6(2)} = (e_6^C)^{\lambda_{17}}$ (or $e_6^C$),

$$e_{6(2)} = g_{-2} \oplus g_{-1} \oplus g_0 \oplus g_1 \oplus g_2$$

with respect to $ad = -2iG_{01}$, is given by

$$g_0 = \begin{cases} 
    iG_{01}, iG_{23}, G_{24}, iG_{25}, G_{26}, iG_{27}, \tilde{A}_1(e_2), \tilde{F}_1(e_2), \\
iG_{34}, G_{35}, iG_{36}, G_{37}, \tilde{A}_1(e_3), \tilde{F}_1(e_3), \\
iG_{45}, G_{46}, iG_{47}, \tilde{A}_1(e_4), \tilde{F}_1(e_4), \\
iG_{56}, G_{57}, \tilde{A}_1(e_5), \tilde{F}_1(e_5), \\
iG_{67}, \tilde{A}_1(e_6), \tilde{F}_1(e_6), \\
i(2E_1 - E_2 - E_3)^\sim, i(E_2 - E_3)^\sim, \tilde{A}_1(e_7), \tilde{F}_1(e_7) \end{cases}$$

$$g_{-1} = \begin{cases} 
    \tilde{F}_2(1 + i e_1), \tilde{F}_2(e_2 + i e_3), \tilde{A}_2(e_4 + i e_5), \tilde{A}_2(e_6 + i e_7), \\
\tilde{A}_3(1 + i e_1), \tilde{A}_3(e_2 - i e_3), \tilde{A}_3(e_4 - i e_5), \tilde{A}_3(e_6 - i e_7), \\
i\tilde{F}_3(1 + i e_1), i\tilde{F}_3(e_2 + i e_3), i\tilde{F}_3(e_4 + i e_5), i\tilde{F}_3(e_6 + i e_7), \\
i\tilde{F}_3(1 + i e_1), i\tilde{F}_3(e_2 - i e_3), i\tilde{F}_3(e_4 - i e_5), i\tilde{F}_3(e_6 - i e_7) \end{cases}$$

$$g_{-2} = \begin{cases} 
    G_{02} - iG_{12}, iG_{03} + G_{13}, G_{04} - iG_{14} \tilde{A}_1(1 - i e_1), \\
iG_{05} + G_{15}, G_{06} - iG_{16}, G_{07} + iG_{17} i\tilde{F}_1(1 - i e_1) \end{cases}$$

$g_1 = \tau(g_{-1})\tau, \ g_2 = \tau(g_{-2})\tau$.

**Proof.** We can prove this theorem in a way similar to Theorem 3.21, using Lemma 3.8.
We shall determine the group structures of

\[(E_6(2))_{ev} = (E_6^C)_{ev} \cap (E_6^C)^{\lambda \tau \gamma_1}, \quad (E_6(2))_0 = (E_6^C)^{\sigma_3} \cap (E_6^C)^{\lambda \tau \gamma_1}.
\]

**Lemma 3.27.** In the group \(E_6^C\), we have

\[t^t \phi(\theta)^{-1} = \phi(\theta^{-1}).\]

**Theorem 3.28.** (1) \((E_6(2))_{ev} \cong (U(1) \times \text{spin}(6,4))/\mathbb{Z}_4\), \(\mathbb{Z}_4 = \{(1,1), (-1, \sigma), (i, \phi(-i)), (-i, \phi(i))\}\).

(2) \((E_6(2))_0 \cong (U(1) \times \mathbb{R}^+ \times \text{spin}(5,3))/\mathbb{Z}_2 \times \{1, a'\}, \mathbb{Z}_2 = \{(1,1,1), (-1,1,1,1)\}\).

**Proof.** (1) For \(\alpha \in (E_6(2))_{ev} \subset (E_6^C)_{ev}\), there exist \(\theta \in C^*\) and \(\beta \in \text{Spin}(10, C)\) such that \(\alpha = \phi(\theta)\beta\) (Theorem 3.23.(1)). From \(\gamma_1 \tau^t \alpha^{-1} \tau \gamma_1 = \alpha\), we have

\[\gamma_1 \tau^t \phi(\theta)^{-1} \tau \gamma_1 = \phi(\theta), \quad \phi(-\theta), \quad \phi(i\theta) \quad \text{or} \quad \phi(-i\theta).
\]

Hence \(\tau \theta = 1 = (1,1)\), \(i \theta = (i, (i))\), \((i, (i))\), \(\tau \theta = 1 = (1,1)\), \(i \theta = (i, (i))\), \(i \theta = (i, (i))\).

Therefore \((E_6(2))_{ev} \cong (U(1) \times \text{spin}(6,4))/\mathbb{Z}_4\).

(2) For \(\alpha \in (E_6(2))_0 \subset (E_6^C)_{ev}\), there exist \(\theta \in C^*\), \(\beta \in U(1, C')\) and \(\beta \in \text{Spin}(8, C)\) such that \(\alpha = \phi(\theta)\beta\) (Theorem 3.23.(2)). From \(\gamma_1 \tau^t \alpha^{-1} \tau \gamma_1 = \alpha\), in a similar way as in (1), we have \(\theta \in U(1)\) and

\[\gamma_1 \tau^t D(a)^{-1} \tau \gamma_1 = D(a)\quad \text{or} \quad \gamma_1 \tau^t D(a)^{-1} \tau \gamma_1 = D(-a).
\]

In the former case, from \(D(\tau a) = D(a)\) (Lemma 2.11), we have \(\tau a = \bar{a}\), hence \(a \in U(1, C') \cong \mathbb{R}^*\). The group \((\text{Spin}(8, C))^{\lambda \tau \gamma_1} = ((E_6^C)^{\lambda \tau \gamma_1})_{F_1, F_1(1), F_1(e_1)}\) (Proposition 3.22.(2)) acts on the \(R^1\)-vector space

\[V^{6,4} = \{X \in \mathbb{C}^C \mid 2E_1 \times X = -\tau \gamma_1 X\}
\]

with the norm \((E_1, X, X) = -\xi_2 E_3 + F_1(x) \mid \xi \in C, x \in (C')_{\tau \gamma_1} = \mathbb{C}'\) in a similar way as in Theorem 2.12.(1). Therefore \((E_6(2))_0 \cong (U(1) \times \text{spin}(5,3))/\mathbb{Z}_4\).

(2) For \(\alpha \in (E_6(2))_0 \subset (E_6^C)_{ev}\), there exist \(\theta \in C^*\), \(\beta \in U(1, C')\) and \(\beta \in \text{Spin}(8, C)\) such that \(\alpha = \phi(\theta)\beta\) (Theorem 3.23.(2)). From \(\gamma_1 \tau^t \alpha^{-1} \tau \gamma_1 = \alpha\), in a similar way as in (1), we have \(\theta \in U(1)\) and

\[\gamma_1 \tau^t \beta - \tau^{t} \gamma_1 = \beta, \quad \gamma_1 \tau^t \beta^{-1} \tau \gamma_1 = \beta.
\]

In the former case, from \(D(\tau a) = D(a)\) (Lemma 2.11), we have \(\tau a = \bar{a}\), hence \(a \in U(1, C') \cong \mathbb{R}^*\). The group \((\text{Spin}(8, C))^{\lambda \tau \gamma_1} = ((E_6^C)^{\lambda \tau \gamma_1})_{F_1, F_1(1), F_1(e_1)}\) (Proposition 3.22.(1)) acts on the \(R^1\)-vector space

\[V^{5,3} = \{X \in \mathbb{C}^C \mid 2E_1 \times X = -\gamma_1 \tau X, (E_1, F_1(1), X) = (E_1, F_1(e_1), X) = 0\}
\]

with the norm \((E_1, X, X) = -\xi_2 E_3 + F_1(x) \mid \xi \in C, x \in (C')_{\tau \gamma_1} = \mathbb{C}'\). (1, x) = (e_1, x) = 0\}

in a similar way as in Theorem 2.12.(1). Therefore the group of the former case is...
(U(1) × R⁺ × spin(5,3))/Z₂ × Z₂ = \{(1,1,1),(-1,1,1)\} × \{(1,1,1),(-1,1,\sigma)\} \cong (U(1) × R⁺ × spin(5,3))/Z₂. In the latter case, \(\theta = 1, a = e_1\) and \(\beta = \sigma'D(-e_1)\) satisfy these conditions and \(\varphi(1,e_1,\sigma'D(-e_1)) = \sigma'.\) Therefore \((E_{6(2)})_0 \cong (U(1) × R⁺ × spin(5,3))/Z₂ × \{1,\sigma'\}.

3.4.3. Subgroups of type \(iR ⊕ D_{5(-13)}\) and \(iR ⊕ R ⊕ D_{4(-14)}\) of \(E_{6(-14)}\)

**Lemma 3.29.** In the Lie algebra \(\mathfrak{e}_6^C\), we have

\[
\begin{align*}
\sigma_1 \tilde{F}_1(a)\sigma_1 &= \tilde{F}_1(a^e_1), \\
\sigma_1 \tilde{F}_2(a)\sigma_1 &= \tilde{F}_3(a^e_1), \\
\sigma_1 \tilde{F}_3(a)\sigma_1 &= \tilde{F}_2(a^e_1), \\
\sigma_1(\xi_1 E_1 + \xi_2 E_2 + \xi_3 E_3)\sim \sigma_1 &= (\xi_1 E_1 + \xi_2 E_2 + \xi_3 E_3)\sim, \\
\xi_1 + \xi_2 + \xi_3 &= 0.
\end{align*}
\]

**Theorem 3.30.** The 2-graded decomposition of \(\mathfrak{e}_6(-14) = (\mathfrak{e}_6^C)^{\lambda'\sigma_1}\) (or \(\mathfrak{e}_6^C\)),

\[
\mathfrak{e}_6(-14) = \mathfrak{g}_- ⊕ \mathfrak{g}_- ⊕ \mathfrak{g}_0 ⊕ \mathfrak{g}_1 ⊕ \mathfrak{g}_2
\]

with respect to \(\text{ad} Z, Z = -2iG_{01}\), is given by

\[
\mathfrak{g}_0 = \begin{cases}
  iG_{01}, G_{23}, G_{24}, G_{25}, G_{26}, G_{27}, \tilde{A}_1(e_2), \tilde{F}_1(e_2), \\
  G_{34}, G_{35}, G_{36}, G_{37}, \tilde{A}_1(e_3), \tilde{F}_1(e_3), \\
  G_{45}, G_{46}, G_{47}, \tilde{A}_1(e_4), \tilde{F}_1(e_4), \\
  G_{56}, G_{57}, \tilde{A}_1(e_5), \tilde{F}_1(e_5), \\
  G_{67}, \tilde{A}_1(e_6), \tilde{F}_1(e_6), \\
  i(2E_1 - E_2 - E_3)\sim, (E_2 - E_3)\sim, \tilde{A}_1(e_7), \tilde{F}_1(e_7)
\end{cases}
\]

\[
\mathfrak{g}_- = \begin{cases}
  \tilde{A}_2(1+i e_1) - i\tilde{A}_3(1+i e_2), & i\tilde{A}_2(1+i e_1) - \tilde{A}_3(1+i e_2), \\
  \tilde{A}_2(e_2 + i e_3) + i\tilde{A}_3(e_2 - i e_3), & i\tilde{A}_2(e_2 + i e_3) + \tilde{A}_3(e_2 - i e_3), \\
  \tilde{A}_2(e_4 + i e_5) + i\tilde{A}_3(e_4 - i e_5), & i\tilde{A}_2(e_4 + i e_5) + \tilde{A}_3(e_4 - i e_5), \\
  \tilde{A}_2(e_6 + i e_7) + i\tilde{A}_3(e_6 - i e_7), & i\tilde{A}_2(e_6 + i e_7) + \tilde{A}_3(e_6 - i e_7), \\
  \tilde{F}_2(1+i e_1) - i\tilde{F}_3(1+i e_2), & i\tilde{F}_2(1+i e_1) - \tilde{F}_3(1+i e_2), \\
  \tilde{F}_2(e_2 + i e_3) + i\tilde{F}_3(e_2 - i e_3), & i\tilde{F}_2(e_2 + i e_3) + \tilde{F}_3(e_2 - i e_3), \\
  \tilde{F}_2(e_4 + i e_5) + i\tilde{F}_3(e_4 - i e_5), & i\tilde{F}_2(e_4 + i e_5) + \tilde{F}_3(e_4 - i e_5), \\
  \tilde{F}_2(e_6 + i e_7) + i\tilde{F}_3(e_6 - i e_7), & i\tilde{F}_2(e_6 + i e_7) + \tilde{F}_3(e_6 - i e_7)
\end{cases}
\]

\[
\mathfrak{g}_- = \begin{cases}
  G_{02} - iG_{12}, G_{03} - iG_{13}, G_{04} - iG_{14}, \tilde{A}_1(1-i e_1), \\
  G_{05} - G_{15}, G_{06} - iG_{16}, G_{07} - iG_{17}, \tilde{F}_1(1-i e_1)
\end{cases}
\]

\[
\mathfrak{g}_1 = \tau(\mathfrak{g}_-)\tau, \quad \mathfrak{g}_2 = \tau(\mathfrak{g}_-)\tau.
\]

**Proof.** We can prove this theorem in a way similar to Theorem 2.15, using Lemmas 2.14 and 3.29.
We shall determine the group structures of

\[ (E_6(-14))_{ev} = (E_6^C)_{\sigma} \cap (E_6^C)^{\lambda \tau \sigma_1}, \quad (E_6(-14))_{0} = (E_6^C)_{\sigma_3} \cap (E_6^C)^{\lambda \tau \sigma_1}. \]

**Lemma 3.31.** In the group \( E_6^C \), we have

\[ \sigma_1 \phi(\theta) \sigma_1 = \phi(\theta). \]

**Theorem 3.32.**

1. \( (E_6(-14))_{ev} \cong (U(1) \times \text{spin}(2,8))/\mathbb{Z}_4 \), \( \mathbb{Z}_4 = \{(1, 1), (-1, \sigma), (i, \phi(-i)), (-i, \phi(i))\} \).
2. \( (E_6(-14))_{0} \cong (U(1) \times \mathbb{R}^+ \times \text{Spin}(1,7))/\mathbb{Z}_2 \times \{1, \sigma'\}, \mathbb{Z}_2 = \{(1, 1, 1), (-1, 1, \sigma)\} \).

**Proof.**

1. For \( \alpha \in (E_6(-14))_{ev} \subset (E_6^C)_{\sigma} \), there exist \( \theta \in C^* \) and \( \beta \in \text{Spin}(10, C) \) such that \( \alpha = \phi(\theta) \beta \) (Theorem 3.23.(1)). From \( \sigma_1 \tau^t \alpha^{-1} \tau \sigma_1 = \alpha \), we have

\[ \sigma_1 \tau^t \phi(\theta)^{-1} \tau \sigma_1 = \phi(\theta), \quad \phi(-\theta), \quad \phi(i\theta) \text{ or } \phi(-i\theta). \]

Hence \( \tau \theta^{-1} = \theta, -\theta, i\theta \) or \(-i\theta\) (Lemmas 3.24.(1), 3.27, 3.31), but the last three cases are impossible. Hence \( \theta \in U(1) = \{\theta \in C \mid \theta(\tau \theta) = 1\} \). The group \( (\text{Spin}(10, C))^{\lambda \tau \sigma_1} = ((E_6^C)^{\lambda \tau \sigma_1})_{E_1} \) acts on the \( \mathbb{R} \)-vector space

\[ V^{2,8} = \{X \in \mathbb{C} \mid 2E_2 \times X = -\tau \sigma_1 X\} \]

with the norm \( \langle E_1, X, X \rangle = -\xi \eta - x\bar{x} = -\xi \eta - x_1^2 + y\bar{y} \). Hence \( (\text{Spin}(10, C))^{\lambda \tau \sigma_1} \) is \( \text{spin}(2,8) \), in a similar way as in Theorem 2.12.(1). Therefore \( (E_6(-14))_{ev} \cong (U(1) \times \text{spin}(2,8))/\mathbb{Z}_4 \).

2. For \( \alpha \in (E_6(-14))_{0} \subset (E_6^C)_{\sigma_3} \), there exist \( \theta \in C^*, a \in U(1, C^C) \) and \( \beta \in \text{Spin}(8, C) \) such that \( \alpha = \varphi(\theta,a,\beta) = \phi(\theta)D(a)\beta \) (Theorem 3.23.(2)). From \( \sigma_1 \tau^t \alpha^{-1} \tau \sigma_1 = \alpha \), in a similar way as in (1), we have \( \theta \in U(1) \) and

\[ \sigma_1 \tau^t \beta = \beta \quad \text{or} \quad \sigma_1 \tau^t \beta = \beta. \]

In the former case, from \( D(\tau a) = D(a) \), (Lemmas 2.11, 2.16), we have \( \tau a = a \), hence \( a \in U(1, C^C) \cong \mathbb{R}^* \). The group \( (\text{Spin}(8, C))^{\lambda \tau \sigma_1} = ((E_6^C)^{\lambda \tau \sigma_1})_{E_1,F_1(1),F_1(1)} \) (Proposition 3.22.(1)) acts on the \( \mathbb{R} \)-vector space

\[ V^{1,7} = \{X \in \mathbb{C} \mid 2E_2 \times X = -\tau \sigma_1 X, (E_1,F_1(1),X) = (E_1,F_1(1),X) = 0\} \]

\[ = \{X = i\xi E_2 + i\eta E_3 + F_1(iy) \mid \xi, \eta \in \mathbb{R}, y \in \mathbb{C}, (1,y) = (e_1,y) = 0\} \]
with the norm \((E_1, X, X) = -\xi \eta + yy\). Hence \((Spin(8, C))^{A_{27}}\) is \(Spin(1, 7)\), in a similar way as in Theorem 2.12. Therefore the group of the former case is \((U(1) \times R^+ \times Spin(1, 7))/(Z_2 \times Z_2) (Z_2 \times Z_2 = \{(1,1,1), (1,1,1), (1,1,1)\}) \cong (U(1) \times R^+ \times Spin(1, 7))/Z_2\). In the latter case, \(\theta = 1\), \(a = e_1\) and \(\beta = \sigma' D(-e_1)\) satisfy these conditions and \(\varphi(1, e_1, \sigma' D(-e_1)) = \sigma'\). Therefore \((E_6(-14))_0 \cong (U(1) \times R^+ \times Spin(1, 7))/Z_2 \times \{1, \sigma'\}.

### 3.4.4. Subgroups of type \(R \oplus D_{5(-27)}\) and \(R \oplus R \oplus D_{4(-28)}\) of \(E_6(-26)\)

Define \(\delta_1 \in E_6 \subset E_6^C\) by \(\delta_1 = \exp \frac{\pi}{2} i F_1(1)\) (which is similar to the one defined in Section 3.2.3). We define a complex-conjugate linear transformation \(\tau_1\) of \(3^C\) by

\[
\tau_1 X = \delta_1^{-1} \tau \delta_1 \begin{pmatrix}
\xi_1 & x_3 & \overline{x_2} \\
x_3 & \xi_2 & x_1 \\
\overline{x_2} & \overline{x_1} & \xi_3
\end{pmatrix} = \begin{pmatrix}
\tau \xi_1 & -i\tau \overline{x_2} & -i\tau x_3 \\
-i\tau x_2 & -\tau \xi_3 & -\tau \overline{x_1} \\
-i\tau \overline{x_3} & -\tau x_1 & -\tau \xi_2
\end{pmatrix}.
\]

Since \(\tau\) and \(\tau_1\) are related with \(\tau_1 = \delta_1^{-1} \tau \delta_1\), we have

**Proposition 3.33.** \(E_6(-26) = (E_6^C)^\tau \cong (E_6^C)^{\tau_1}\).

**Lemma 3.34.** In the Lie algebra \(e_6^C\), we have

1. \(\tau_1 G_{0l} \tau_1 = -G_{0l}\), \(\tau_1 G_{k0} \tau_1 = G_k,\) \(kl \neq 0,\)
2. \(\tau_1 \tilde{A}_1(a) \tau_1 = -\tilde{A}_1(\tau a),\) \(\tau_1 \tilde{A}_2(a) \tau_1 = -i \tilde{F}_3(\tau a),\) \(\tau_1 \tilde{A}_3(a) \tau_1 = i \tilde{F}_2(\tau a),\)
3. \(\tau_1 (\xi_1 E_1 + \xi_2 E_2 + \xi_3 E_3)^\tau = ((\tau \xi_1) E_1 + (\tau \xi_2) E_2 + (\tau \xi_3) E_3)^\tau,\) \(\xi_1 + \xi_2 + \xi_3 = 0.\)

**Theorem 3.35.** The 2-graded decomposition of \(e_6(-26) = (e_6^C)^{\tau_1}\) (or \(e_6^C\)),

\[
e_6(-26) = e_6^C = g_{-2} \oplus g_{-1} \oplus g_0 \oplus g_1 \oplus g_2
\]

with respect to \(ad Z, Z = -2iG_{01}\), is given by

\[
g_0 = \left\{ iG_{01}, G_{23}, G_{24}, G_{25}, G_{26}, G_{27}, \tilde{A}_1(e_2), i\tilde{F}_1(e_2), G_{34}, G_{35}, G_{36}, G_{37}, \tilde{A}_1(e_3), i\tilde{F}_1(e_3), G_{45}, G_{46}, G_{47}, \tilde{A}_1(e_4), i\tilde{F}_1(e_4), G_{56}, G_{57}, \tilde{A}_1(e_5), i\tilde{F}_1(e_5), G_{67}, \tilde{A}_1(e_6), i\tilde{F}_1(e_6), (2E_1 - E_2 - E_3)^\tau, i(E_2 - E_3)^\tau, \tilde{A}_1(e_7), i\tilde{F}_1(e_7) \right\}
\]
We can prove this theorem in a way similar to Theorem 3.21, using Lemma 3.34.  

We shall determine the group structures of  

\[ g_{-1} = \begin{cases} 
\mathbb{A}_2(1 + i e_1) - i \mathcal{F}_3(1 + i e_2), & i \mathbb{A}_2(1 + i e_1) - \mathcal{F}_3(1 + i e_2), \\
\mathbb{A}_2(e_2 + i e_3) + i \mathcal{F}_3(e_2 - i e_3), & i \mathbb{A}_2(e_2 + i e_3) + \mathcal{F}_3(e_2 - i e_3), \\
\mathbb{A}_2(e_2 + i e_5) + i \mathcal{F}_3(e_4 - i e_5), & i \mathbb{A}_2(e_2 + i e_5) + \mathcal{F}_3(e_4 - i e_5), \\
\mathbb{A}_2(e_6 + i e_7) + i \mathcal{F}_3(e_6 - i e_7), & i \mathbb{A}_2(e_6 + i e_7) + \mathcal{F}_3(e_6 - i e_7), \\
\mathbb{A}_2(1 + i e_1) + i \mathcal{F}_2(1 + i e_2), & i \mathbb{A}_2(1 + i e_1) + \mathcal{F}_2(1 + i e_2), \\
\mathbb{A}_2(e_2 + i e_3) - i \mathcal{A}_3(e_2 - i e_3), & i \mathbb{A}_2(e_2 + i e_3) - \mathcal{A}_3(e_2 - i e_3), \\
\mathbb{A}_2(e_4 + i e_5) - i \mathcal{A}_3(e_4 - i e_5), & i \mathbb{A}_2(e_4 + i e_5) - \mathcal{A}_3(e_4 - i e_5), \\
\mathbb{A}_2(e_6 + i e_7) - i \mathcal{A}_3(e_6 - i e_7), & i \mathbb{A}_2(e_6 + i e_7) - \mathcal{A}_3(e_6 - i e_7), \\
\end{cases} \]

Theorem 3.37.  

\[ g_{-2} = \begin{cases} 
\mathbb{A}_2(1 + i e_1) + i G_{12}, & i G_{13}, G_{03} + G_{14}, i \mathcal{A}_1(1 - i e_1), \\
\mathbb{A}_2(1 + i e_1) + i G_{04} + G_{14}, i \mathcal{A}_1(1 - i e_1), \\
\mathbb{A}_2(1 + i e_1) + i G_{05} + G_{15}, i G_{06} + G_{16}, i G_{07} + i G_{17}, i \mathcal{A}_1(1 - i e_1), \\
\end{cases} \]

\[ g_1 = \tau(\lambda(g_{-1})) \tau, \quad g_2 = \tau(\lambda(g_{-2})) \tau. \]

Proof. We can prove this theorem in a way similar to Theorem 3.21, using Lemma 3.34.

In the group \( E_6 \), we have

\[ \tau_1 \phi(\theta) \tau_1 = \phi(\tau \theta), \quad \tau_1 D(a) \tau_1 = D(\tau a), \]

\[ \tau_1 \sigma' \tau_1 = \sigma \sigma', \quad \tau_1 \rho \tau_1 = \rho \phi(i) D(e_1). \]

Theorem 3.37.  

(1) \( (E_6(-26)) \cong \mathbb{R}^+ \times \text{Spin}(9,1). \)

(2) \( (E_6(26)) \cong (\mathbb{R}^+ \times \mathbb{R}^+ \times \text{Spin}(8)) \times \{(1, \sigma') \times \{1, \rho\}\}. \)

Proof. (1) For \( \alpha \in (E_6(-26)) \subseteq (E_6) \), there exist \( \theta \in C^+ \) and \( \beta \in \text{Spin}(10, C) \) such that \( \alpha = \phi(\theta) \beta \) (Theorem 3.23.(1)). \( (E_6(-26)) \cong (E_6) \cong (E_6^\sigma \cap (E_6)^{\tau_1} = (E_6)^{\sigma} \cap (E_6)^{\tau_1}) \) (because \( \sigma \) and \( \tau_1 \) commute) \( \cong (E_6)^{\sigma} \) is connected. Hence, from \( \tau_1 \alpha \tau_1 = \alpha \), we have only one case

\[ \tau_1 \phi(\theta) \tau_1 = \phi(\theta), \quad \tau_1 \beta \tau_1 = \beta. \]

From \( \phi(\tau \theta) = \phi(\theta) \) (Lemma 3.36), we have \( \tau \theta = \theta \), hence \( \theta \in \mathbb{R}^+ \). The group \( (\text{Spin}(10, C))^{\tau_1} = ((E_6)^{\tau_1})_{E_1} \) (Proposition 3.22.(2)) acts on the \( R \)-vector space

\[ V_{\theta} = \left\{ X \in \left( \mathbb{R} \right)^{\tau_1} \mid 4 E_1 \times (E_1 \times X) = X \right\} 
= \left\{ X = \xi E_2 - \tau \xi E_3 + F_1(x) \mid \xi \in C, x = ix_1 + y, x_1 \in R, y \in C, (1, y) = 0 \right\} \]

where \( \mathbb{R} \) and \( \mathbb{C} \) are the fields of real numbers and complex numbers, respectively.
with the norm \((E_1, X, X) = -\xi(\tau \xi) - x^2 = -\xi(\tau \xi) + x_1^2 - y\bar{y}\). Hence \((Spin(10, C))_{\tau_1}\) is \(Spin(9, 1)\), in a similar way as in Theorem 2.12. Therefore \((E_{6(26)})_{ev} \cong (R^* \times Spin(9, 1))/Z_2 (Z_2 = \{(1, 1), (-1, a)\}) \cong R^* \times Spin(9, 1)\).

(2) For \(\alpha \in (E_{6(26)})_{ev} \subset (E_6^C)_{\sigma_3}\), there exist \(\theta \in C^*, a \in U(1, C^C)\) and \(\beta \in Spin(8, C)\) such that \(\alpha = \varphi(\theta, a, \beta) = \phi(\theta)D(a)\beta\) (Theorem 3.23.(2)). From \(\tau_1 \alpha \tau_1 = \alpha\), we have

\[
\begin{align*}
&\phi(-\theta) \\
&\phi(i\theta) \\
&\phi(-i\theta) \\
&\phi(i)D(e_1)a \\
&\phi(i)D(e_1)\beta \\
&\phi(-i)D(e_1)\beta
\end{align*}
\]

(i) From \(\phi(\tau \theta) = \phi(\theta), D(\tau \bar{a}) = D(a)\) (Lemma 3.36), we have \(\tau \theta = \theta, \tau \bar{a} = a\), hence \(\theta \in R^*, a \in U(1, C^C) \cong R^*,\) respectively. The group \((Spin(8, C))_{\tau_1}(E_6^C)_{\tau_1}\) acts (Proposition 3.22.(1)) on the \(R\)-vector space

\[
V^8 = \{X \in (S^C)_{\tau_1} \mid 4E_1 \times (E_1 \times X) = X, (E_1, F_1(1), X) = (E_1, F_1(e_1), X) = 0\}
\]

\[
= \{X = \xi E_2 - \tau \xi E_3 + F_1(y) \mid \xi \in C, y \in C, (1, y) = (e_1, y) = 0\}
\]

with the norm \(-(E_1, X, X) = \xi(\tau \xi) + y\bar{y}\). Hence \((Spin(8, C))_{\tau_1}\) is \(Spin(8)\), in a similar way as in Theorem 2.12. Therefore the group of the case (i) is \((R^* \times R^* \times Spin(8))/\langle Z_2 \times Z_2 \rangle (Z_2 \times Z_2 = \{(1, 1, 1), (-1, -1, 1)\} \times \{(1, 1, 1), (1, -1, 1)\}) \cong R^* \times R^* \times Spin(8)\).

(ii) \(\varphi(i, e_1, \phi(-i))D(-e_1) = 1\).

(iii) \(\varphi(1, e_1, \sigma' D(-e_1)) = \sigma'\) (Lemma 3.36).

(iv) \(\varphi(i, 1, \sigma' \phi(-i)) = \sigma'\).

(v) \(\varphi\left(\frac{1 - i}{\sqrt{2}}, \frac{1 - e_1}{\sqrt{2}}, \rho \phi\left(\frac{1 + i}{\sqrt{2}}\right)D\left(\frac{1 + e_1}{\sqrt{2}}\right)\right) = \rho\) (Lemma 3.36).

(vi) \(\varphi\left(\frac{1 + i}{\sqrt{2}}, \frac{1 + e_1}{\sqrt{2}}, \rho \phi\left(\frac{1 - i}{\sqrt{2}}\right)D\left(\frac{1 - e_1}{\sqrt{2}}\right)\right) = \rho\).

(vii) \(\varphi\left(\frac{1 - i}{\sqrt{2}}, \frac{1 + e_1}{\sqrt{2}}, \sigma' \rho \phi\left(\frac{1 + i}{\sqrt{2}}\right)D\left(\frac{1 - e_1}{\sqrt{2}}\right)\right) = \sigma' \rho\) (Lemma 3.36).

(viii) \(\varphi\left(\frac{1 + i}{\sqrt{2}}, \frac{1 - e_1}{\sqrt{2}}, \sigma' \rho \phi\left(\frac{1 - i}{\sqrt{2}}\right)D\left(\frac{1 + e_1}{\sqrt{2}}\right)\right) = \sigma' \rho\).

Therefore \((E_{6(26)})_0 \cong (R^* \times R^* \times Spin(8)) \times \{1, \sigma', \rho, \sigma' \rho\}\).
Reference