Graph link invariants of isolated singularities of holomorphic vector fields in $\mathbb{C}^2$ III

By Nobuatsu Oka

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§ Introduction

Camacho, Sad and Lins Neto showed that every holomorphic vector field defined in $\mathbb{C}^2$ with an isolated singularity (the origin $o$ of $\mathbb{C}^2$) has a complex analytic subvariety passing through its singularity invariant by the vector field. The subvariety is called a separatrix of the vector field (see [1]). Since the separatrices are analytic subvarieties of $\mathbb{C}^2$ (i.e., plane curves), the intersection between a small $S^3$ around the origin and the separatrices is a graph link. So this link is represented by a graph called a splice diagram. These facts motivate us to define certain topological invariants around an isolated singularity of a holomorphic vector field in $\mathbb{C}^2$. In fact, we obtained in [10] and [11] certain topological invariants from the above intersection. So our invariants are induced from the graph link appearing in the intersection and we used the splice diagram to get the invariants. We saw that a splice diagram representing the above graph link is one of the topological invariants of the holomorphic vector field around the singularity (see Theorem A in [10]).

In this paper, we first pay attention to the multiple graph link and give an answer to the following question:

“What kind of a graph link or a multiple graph link appears as the intersection between the separatrices of a holomorphic vector field in $\mathbb{C}^2$ and a small $S^3$ around its singularity?”

Since any irreducible graph link in $S^3$ is a solvable link (see [3]), the link is represented by a splice diagram. So we can replace a graph link and a multiple graph link by a splice diagram and a splice diagram with certain multiplicities respectively when we discuss the above questions. Here the multiplicities are defined by those of the graph link defined in [11] for the vertex weight of the splice diagram. We will recall this multiplicity in §1 and will give an answer to the above question in §2, i.e., we will discuss a necessary and sufficient condition for the existence of a
vector field such that the intersection between a small $S^3$ around the singularity and its separatrices is a given graph link (or a splice diagram) with multiplicities. We give a necessary and sufficient condition for a graph link with multiplicities to be realized by a certain holomorphic vector field in the cases of the multi-Hopf link and torus links (cf. Theorem 2.4, Theorem 2.7). In Section 3, we show, using these facts, the Thurston norm of the vector field is a finer topological invariant of vector fields than the Milnor number for those cases (cf. Proposition 3.6).

§1. Preparations

In this section we recall the multiplicity of a graph link induced from a vector field. We extend the definition of a minimal plumbing diagram defined in [10] and introduce an index theorem of holomorphic vector fields by Camacho and Sad [2]. These notions will be necessary to describe theorems in §2.

First, we recall the notion of the multiplicity of graph links induced by a holomorphic vector field.

**Definition 1.1.** Let $Z$ denote a holomorphic vector field and suppose $\hat{Z}$ is a desingularized vector field of $Z$. Suppose that the link $L_Z$ is a graph link appearing in the intersections between a boundary of a plumbed 4-manifold (i.e. a small three sphere) and the separatrices of the vector field $Z$. The diagram $\Gamma_Z$ has several arrowhead vertices and each arrowhead vertex corresponds to an intersection between one of separatrices of $\hat{Z}$ and the boundary of the plumbed 4-manifold. Let $(z_1, z_2)$ be a local chart around a simple singularity of $\hat{Z}$. We assume the singularity corresponds to $(0, 0)$ in this chart. $\hat{Z}$ is written in the following normal form:

$$\hat{Z} = \{\lambda_1 z_1 + \phi_1(z_1, z_2)\} \frac{\partial}{\partial z_1} + \{\lambda_2 z_2 + \phi_2(z_1, z_2)\} \frac{\partial}{\partial z_2}, \quad \lambda_2 \not\in \mathbb{Q}_+.$$  

Here $\phi_1(z_1, z_2)$ and $\phi_2(z_1, z_2)$ mean the higher terms of the Taylor development of $\hat{Z}$ around $(0, 0)$.

If an arrowhead vertex of $\Gamma_Z$ defines the intersection between the boundary of the plumbed 4-manifold and the separatrix of the simple singularity such that $m_1\lambda_1 + m_2\lambda_2 = 0$, $m_1, m_2 \in \mathbb{N}$, its multiplicity is defined by $m_1$ or $m_2$. We shall explain this in Lemma 1.1.

The following lemma is effective for realizing the vector field of which separatrices derive an arrowhead with a given multiplicity.

**Lemma 1.1.** Suppose that the foliation $F$ around the origin of $\mathbb{C}^2$ is defined by the following differential equation:

$$\frac{dx}{dt} = \lambda_1 x + \phi_1(x, y),$$
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\[
\frac{dy}{dt} = \lambda_2 y + \Phi_2(x, y),
\]

which have a resonance condition \(\lambda_1 m_1 + \lambda_2 m_2 = 0\), then the multiplicity of the arrowhead (1) in Figure 1 of the splice diagram induced by the foliation \(F\) is the integer \(m_1\) and the multiplicity of the arrowhead (2) is an integer \(m_2\). Here the arrowhead (1) represents a component of the link induced by the separatrix \(x = 0\), and the arrowhead (2) represents a component induced by the separatrix \(y = 0\).

![Figure 1.1](image)

When the edge weights are omitted as in Figure 1.1, these weights mean 1. This lemma is clear from the definition of the multiplicity and the linear holonomy in the plane defined by \(y = 0\) (or \(x = 0\)) (cf. [11]).

When we reduce a given plumbing diagram to the simplest plumbing diagram (called a minimal plumbing diagram) or reconstruct the given plumbing diagram from the reduced graph, we call it a (-1)-blow-down-operation or a blow-up operation respectively (cf. [10]). Here this procedure of graphs corresponds to an operation which removes the complex projective space \(\mathbb{C}P^2\) from the plumbed manifold \(P\) or makes a connected sum \(\mathbb{C}P^2 \# P\) respectively. If we perform several (-1)-blow-down operations to a plumbing diagram induced by a splice diagram of a given multi-graph link, it turns into plumbing diagrams for which we can not continue a (-1)-blow-down operation. There are cases that the vertex weights become zero after performing (-1)-blow-down operations since the vertex weight decreases by 1 for one blow-down operation, and we can not continue this operation any more if the vertex of which weight is zero. Every multi-graph link, which appears as the intersection between separatrices and a small \(S^3\), must have the plumbing diagram which can be reduced to a trivial one by only (-1)-blow-down operations (cf. [8]). However, if we reduce the splice diagram representing this multi-graph link as above to a plumbing diagram, and perform (-1)-blow down procedures to this diagram with all arrowhead lines deleted, we sometimes meet with the phenomenon as above. Thus the (-1)-blow-down operations for this diagram can not be continued before reaching the one which can be reduced to a trivial diagram by only (-1)-blow-down operations. There exist certain gaps between two plumbing
diagrams which represent the same multi-link. To fill up these gaps we need some other operations. These operations are called a 0-chain absorption, a (+1)-blow-up and a (+1)-blow-down operation introduced by W.D. Neumann [8]. For these reasons we must change slightly the definition of a minimal plumbing diagram from that in [10].

**Definition 1.2 ([3] and [8]) (a 0-chain absorption).** We call the operations below (1) and (2), which change left hand diagrams to the right hand ones 0-chain absorptions.

\[
\begin{align*}
(1) & \quad 
\begin{array}{c}
\Delta_1: \\
-\varepsilon_1 & 0 & -\varepsilon_2 \\
\end{array} \\
\rightarrow & \quad 
\begin{array}{c}
\Delta_1: \\
-\varepsilon_1 & \varepsilon_2 \\
(\varepsilon_1 + \varepsilon_2) \\
\end{array}
\end{align*}
\]

\[
\begin{align*}
(2) & \quad 
\begin{array}{c}
\Delta_1: \\
-\varepsilon_1 & 0 \\
\end{array} \\
\rightarrow & \quad 
\begin{array}{c}
\Delta_1: \\
\varepsilon_1 \\
-1 \\
\end{array}
\end{align*}
\]

**Figure 1.2**

**Definition 1.3 ([3] and [8]) (a (+1)-blow-up or a blow-down operation).** The right hand diagrams are altered to the left hand ones by the following operations. We call these operations (+1)-blow-down operations, and the reverse operations as (+1)-blow-up operations (1), (2) and (3).

\[
\begin{align*}
(1) & \quad 
\begin{array}{c}
\vdots & \vdots \\
-\varepsilon_1 & \pm 1 & -\varepsilon_2 \\
\Delta_1 & \\
\end{array} \\
\leftrightarrow & \quad 
\begin{array}{c}
\vdots & \vdots \\
-\varepsilon_2 & \pm 1 \\
\Delta_1 & \\
\end{array}
\end{align*}
\]

\[
\begin{align*}
(2) & \quad 
\begin{array}{c}
\vdots \\
-\varepsilon_1 & \pm 1 \\
\Delta_1 & \\
\end{array} \\
\leftrightarrow & \quad 
\begin{array}{c}
\vdots \\
-\varepsilon_1 & \pm 1 \\
\Delta_1 & \\
\end{array}
\end{align*}
\]
DEFINITION 1.4 ([3]). We call the following operations as a splitting operation.

Here we give another definition of a minimal plumbing diagram.

DEFINITION 1.5 ([3]). If a plumbing diagram admits no blow-down, 0-chain absorption, and splitting, we call it a minimal plumbing diagram.

All irreducible links have a unique representation by the minimal plumbing diagram (cf. [3]). For reducible links, each component of the plumbing diagram which is split by doing operations as in Definition 1.4 (cf. [3]) has a unique representation.

There is a theorem obtained in [3] which is needed to prove our theorems. To describe the theorem we prepare some notations. We denote a continued fraction:

\[
\cfrac{1}{c_1 - \cfrac{1}{c_2 - \cfrac{1}{c_3 - \cdots - \cfrac{1}{c_n \cdots}}}}
\]

by \([c_1, \ldots, c_n]\) (\(c_i \in \mathbb{N}, \ i = 1, \ldots, n\)).
DEFINITION 1.6. We call the vertex of a plumbing diagram or a splice diagram of which degree is greater than three as a node vertex.

Now we extend the value of vertex weight of plumbing diagrams to the rational number. Let \( \Delta_j \) be a subgraph of \( \Delta \) which is the right side of the vertex \( v_j \) (including \( v_j \)) as in Figure 1.5 (a), (b). Pick the vertex \( v_j \) in a given graph, then the rational number \( \text{cf}(\Delta_j, v_j) \) is defined as follows.

\[
e_{j}' = e_j - \left( \frac{1}{e_1} + \frac{1}{e_2} + \cdots + \frac{1}{e_k} \right).
\]

Figure 1.5

DEFINITION 1.7 ([3]). Suppose that the weight of the vertex \( v_j \) is the integer \( -e_j \) (\( e_j > 0 \)). By performing the procedures (a) or (b) as in Figure 1.5, which delete the edges coming into \( v_j \), the positive integer \( e_j \) changes to the positive rational number \( e_{j}' = e_j - [e_1, e_2, \ldots, e_k] \) and the weight \( \text{cf}(\Delta_j, v_j) \) of the vertex \( v_j \) is defined by the rational number \( -e_{j}' \).

THEOREM 1.2 ([3]). The torus link with the splice diagram

\[
\begin{array}{c}
\text{has the following plumbing diagram. (Figure 1.6)} \\
\text{If } v_i \text{ is the outermost vertex of the } i\text{-th branch of } \Delta' \text{ which is a plumbing}
\end{array}
\]
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Diagram $\Delta$ with an arrowhead deleted, then $\text{cf}(\Delta_i, v_i) = 1/b_i$ ($i = 1, 2$) and $b_i$ (resp. $b_2$) is equal to $\left(\frac{q_1}{p_1}\right)$ (resp. $\left(\frac{p_1}{q_1}\right)$). Here the outermost vertex means a vertex which exists at the end of the right side or the left side of the diagram $\Delta'$ in Figure 1.7.

$$\frac{q_1}{p_1} = \left(\frac{q_1}{p_1}\right) - [b_{1s_1}, \ldots, b_{11}], \quad b_{1k} \geq 2 \ (k = 1, \ldots, s_1)$$

$$\frac{p_1}{q_1} = \left(\frac{p_1}{q_1}\right) - [b_{2s_2}, \ldots, b_{21}], \quad b_{2l} \geq 2 \ (l = 1, \ldots, s_2)$$

$$b = \frac{1}{p_1q_1} + [b_{11}, \ldots, b_{1s_1}] + [b_{21}, \ldots, b_{2s_2}],$$

where $\left(\frac{q_1}{p_1}\right)$ (or $\left(\frac{p_1}{q_1}\right)$) means the least integer greater than or equal to $\frac{q_1}{p_1}$ (or $\frac{p_1}{q_1}$).

Figure 1.6

Figure 1.7

In the remaining part of this section, we state an index theorem developed by C. Camacho, A. Lins Neto and P. Sad. We will use the index theorem in the next section.

Let $S$ be a compact Riemann surface embedded in a 2-dimensional complex manifold $M$. Suppose $F$ is a codimension one complex foliation on $M$ with singu-
larities \( \text{Sing}(F) \) defined in a neighborhood of \( S \) satisfying

(i) \( \text{Sing}(F) = \{ q_1, \ldots, q_r \} \subset S \), and

(ii) \( S - \text{Sing}(F) \) is a leaf of \( F \). Suppose \( q \in \text{Sing}(F) \) and \( F \) be defined near \( q \) by a Pfaffian form \( \omega \). Let \( \phi : (\mathbb{C}^2, 0) \to (M, q), \phi(0) = q \) be a local chart such that \( \{ \phi(z, 0) \mid z \in \mathbb{C} \} \subset S \). Then

\[
(\phi^* \omega)(z, w) = A(z, w)dz + B(z, w)dw
\]

where

\[
A(0, 0) = B(0, 0) = 0 \quad \text{and} \quad A(z, 0) = 0.
\]

**Definition 1.8 ([2]).** The index of \( F \) and \( S \) at \( q \in S \) is

\[
i_q(F, S) = -\text{Res} \frac{\partial}{\partial w}(A/B)(z, 0).
\]

The foliation \( F \) induced by the Pfaffian form in the above definition is equal to the one induced by the vector field such that

\[
Z = B(z, w) \frac{\partial}{\partial z} - A(z, w) \frac{\partial}{\partial w}
\]

near the point \( q \).

Let \( E \) be the normal bundle to \( S \) and \( c(E) \) the Chern class of \( E \). The index theorem is described as follows.

**Theorem 1.3 ([2]).**

\[
\sum_{q \in \text{Sing}F} i_q(F, S) = c(E).
\]

By the tubular neighborhood theorem, \( F \) induces a foliation of \( E \) near the zero section of \( E \). Let \( S_i \) \((i = 1, 2)\) be a copy of a projective line. Let \( E_i \) \((i = 1, 2)\) be a complex line bundle over \( S_i \). We paste a tubular neighborhood of the zero section of \( E_1 \) over \( S_1 \) to a zero section of \( E_2 \) over \( S_2 \). We can construct the plumbed 4-manifold of \( E_1 \) and \( E_2 \) which is a complex 2-manifold. Also the foliations in \( E_i \) \((i = 1, 2)\) induce the foliation on this plumbed 4-manifold (cf. [4] and [7]). We will apply the index theorem to the holomorphic foliations on the plumbed 4-manifold.
§2. Construction of a holomorphic vector field on $\mathbb{C}^2$ carrying a multi-link in a small $S^3$ around its singularity

In this section, we discuss the main results in this paper. We stated that a holomorphic vector field defined on a neighborhood of the origin of $\mathbb{C}^2$ has invariant curves passing through the origin (see [2]). The invariant curves are called separatrices and they are plane curves. After several blow-up operations every holomorphic vector field defined as above is desingularized (see [13]). Thus we can define a plumbing diagram which represents a final stage of the desingularization of a vector field as in [10]. For every vector field the minimal plumbing diagram is equal to one of its separatrices and it implies that the splice diagram induced by the vector field is the same graph as the one which is defined from its separatrices (see [10] and [11]). By using this splice diagram we can realize a graph ink appearing as the intersection between the separatrix and a small $S^3$ around the singularity.

Since every plane curve is desingularized after a finite numbers of blow-up operations, its plumbing diagram is equal to the one induced by the desingularization of a singularity of an algebraic curve. Thus the splice diagram of separatrices represents an algebraic link, and the splice diagram of a vector field also represents a certain algebraic link. Hence we see that the configurations of all splice diagrams of vector fields can be defined as the following diagrams (1), (2), (3) by using the results of D. Eisenbud and W.D. Neumann [3] (also see Mumford [6]).

(1) A multiple Hopf link;

\[ \text{Figure 2.1 (1)} \]

(2) A torus link;
The edge weight of two node vertices satisfy the condition

\[ a_1 b_1 - \prod_{i=1}^{n} c_i \prod_{j=1}^{m} d_j > 0. \]

And at most two edges have integer weights which are greater than two around each vertex. In general, the configurations are obtained by splicing the diagrams of type (1) and (2) satisfying the above condition for each adjacent vertices. Thus we have

**Proposition 2.1.** Suppose a holomorphic vector field \( Z \) is defined on a neighborhood of the origin \( o \in \mathbb{C}^2 \), and the origin is an isolated singularity of \( Z \). Then the intersection between the separatrices of \( Z \) and a small \( S^3 \) around the origin is a certain graph link. Its form is specified in the above three types (1), (2), and (3).

Now we consider the converse problem, i.e., “Given a multi-graph link of which configuration is one of (1), (2), and (3), does it appear as the intersection between separatrices of a certain holomorphic vector field and a small \( S^3 \) around the origin?”

To answer the above question, we show the existence of a vector field such that the intersection of its separatrices and a small \( S^3 \) around the origin \( o \) is a multi-graph link satisfying the one of the above conditions (1), (2), and (3).

**Lemma 2.2.** The links which are induced by separatrices are irreducible graph links.
Proof. The final resolution picture of a holomorphic vector fields is connected graph. So its plumbing diagram is connected. By Theorem 24.1 in [3] the weights of all vertices for this plumbing diagram are negative integers, since the plumbing diagram belongs to the category of the resolution graph of algebraic curves. If the link which is induced by separatrices is reducible, its plumbing diagram is isomorphic to a graph which has the left configuration in Definition 1.4. So the plumbing diagram must have the zero vertex. It contradicts the resolution graph of algebraic curve. Thus we perform the splitting operation to erase this zero vertex. After performing this operation, the diagram becomes a disconnected graph. So this graph does not represent any graph induced by separatrices. Thus we conclude that our link is irreducible.

Remark. By this lemma, we can prove our main results in [10] without Lemma 5 in [10].

Proposition 2.3. If the link appearing as the intersection is a multi-Hopf link with n components, then its multiplicity must be

$$(r_1, \ldots, r_k, \ldots, r_n) \quad \text{with} \quad r_1 + \cdots + r_n = -1, r_k \in \mathbb{C}.$$ 

Here the integer $r_k$ $(k = 1, \ldots, n)$ means 1 if the number $r_k$ is not a negative rational number, and the numerator of $|r_k|$ in the simple fraction, if the number $r_k$ is a negative rational number.

We see that the number $r_k$ can not be a positive rational number by calculating the index around simple singularities. Proposition 2.3 is proved as follows. In fact a multi-Hopf link is represented by the following splice diagram:

![Figure 2.2](image)

Applying Theorem 1.2 to this splice diagrams we obtain the plumbing diagram:

![Figure 2.3](image)
So the final resolution picture is obtained by this plumbing diagram as follows (cf. [10]):

![Figure 2.4](image)

We get our holomorphic vector field by performing a blow-down operation on the divisor of the above final resolution picture. Hence there exists a holomorphic vector field (or a holomorphic foliation) on a plumbed 4-manifold representing the first vector field on $\mathbb{C}^2$ by just one blow-up operation. Next, we apply the index theorem by Camacho and Sad to the vector field on the plumbed 4-manifold and we have the relation

$$r_1 + r_2 + \cdots + r_n = -1,$$

where the complex number $r_i \in \mathbb{C}$ ($i = 1, \ldots, n$) is the index around the $i$-th simple singularity of the vector field on the plumbed 4-manifold. The definition of the multiplicity on a component of the link and Lemma 1.1 implies that the multiplicity of the $i$-th vertex of the arrowhead corresponding to the $i$-th component of a multi-Hopf link is the numerator of $|r_i|$ if $r_i$ is a negative rational number or an 1, otherwise.

**Theorem 2.4 (A case of multi-Hopf link.).** Suppose a multi-Hopf link has $n$ components. Let $p_k$ and $q_k$ ($k = 1, \ldots, n$) be relatively prime positive integers satisfying the condition

$$\frac{q_1}{p_1} + \frac{q_k}{p_k} + \cdots + \frac{q_n}{p_n} = 1.$$

Let $q_k$ be the multiplicity on the $k$-th component. Then there exists a holomorphic vector field such that the multi-Hopf link with $n$ components appearing as the intersection has the multiplicity $(q_1, \ldots, q_n)$.

**Proof.** For a given multi-Hopf link of $n$ components with multiplicities $(q_1, \ldots, q_n)$, we represent it by a splice diagram:
As in the proof of Proposition 2.1, the following plumbing diagram (a) is induced by the above splice diagram by using Theorem 1.2:

So we obtain the final resolution picture (b):

Now we will construct a vector field on a plumbed manifold represented by the above plumbing diagram by using the above final resolution picture. The way of constructing the vector field is the same as that in the proof of Theorem 1 in [7]. Take $n$ arbitrary points $z_0^1, z_0^2, \ldots, z_0^n$ on a divisor represented by the final resolution picture. For $i \in \{1, \ldots, n\}$ let $D_i$ be an open disk of radius $r$ and the center $z_i^0$, where $r$ is a small positive real number satisfying $|z_i^0 - z_j^0| > 2r$ for any $i \neq j$, $1 \leq i, j \leq n$. For $j \in \{2, \ldots, n\}$ we choose a point $z'_j \in D_j - \{z_j^0\}$ and a point $z''_j \in D_1 - \{z_1^0\}$ as follows;

Let $I = [0, 1]$ and $\alpha_2, \ldots, \alpha_n : I \to \mathbb{C}$, be simple curves in the divisor satisfying the following conditions:

1. $\alpha_j(0) = z''_j$, $\alpha_j(1) = z'_j$.
2. $\alpha_i(I) \cap D_i = \emptyset$ if $i \neq j$, and both $i$ and $j$ are not equal to 1.
3. $\alpha_i(I) \cap \alpha_j(I) = \emptyset$ if $i \neq j$. 
(4) For any \( j \in \{2, \ldots, n\} \), \( \alpha_j(I) \cap D_1 \) and \( \alpha_j(I) \cap D_j \) are line segments contained in the diameters of \( D_1 \) and \( D_j \) respectively.

Let \( A_2, \ldots, A_n \) be small strips around \( \alpha_2, \ldots, \alpha_n \) respectively which satisfy the following properties:

1. \( A_i \cap D_i = \emptyset \), if \( i \neq j \) and both \( i \) and \( j \) are not equal to 1.
2. \( A_i \cap A_j = \emptyset \) if \( i \neq j \).
3. \( A_j \cap D_1 \) and \( A_j \cap D_j \) are contained in the sectors of \( D_1 \) and \( D_j \), \( 2 \leq j \leq n \).

We also set \( U = (\bigcup_{k=2}^n A_k) \cup (\bigcup_{i=1}^n D_i) \) and \( \gamma = \partial U \). By construction \( \gamma \) is obviously a simple closed curve in the divisors. Let \( T \) be a tubular neighborhood of \( \gamma \) and set \( V = (\mathbb{CP}^1 - U) \cup T \), where the divisor is a holomorphically diffeomorphic to \( \mathbb{CP}^1 \). Then \( \{A_2, \ldots, A_n, D_1, \ldots, D_n, V\} \) is a covering of \( \mathbb{CP}^1 \) by open sets. For each \( j = 2, \ldots, n \) let us consider coordinates \((z, v_j), z \in A_j, v_j \in \mathbb{C}\) for \( i = 1, \ldots, n \) and coordinates \((z, u_i), \) in \( D_i \times \mathbb{C}, z \in D_i, u_i \in \mathbb{C}\). In \( V \times \mathbb{C} \) we take coordinates.
(w, y) where w = 1/z ∈ V and y ∈ C. Now we construct a local foliation in every component V × C, D_i × C, and A_j × C (1 ≤ i ≤ n and 2 ≤ j ≤ n) and glue them together as in Theorem 1 in [7].

In A_j × C we take the horizontal foliation whose leaves are of the form v_j = constant, j = 2, . . . , n. And in V × C we take also the horizontal foliation whose leaves are y = constant. For D_k × C (k = 1, . . . , n), the foliation with an isolated singularity is induced by the following differential equation:

\[
\frac{dz}{dt} = z - z_k, \quad \frac{du_k}{dt} = \frac{\lambda_{2k}}{\lambda_{1k}} u_k,
\]

where λ_{1k}q_k + λ_{2k}p_k = 0, (k = 1, . . . , n) and λ_{1k}, λ_{2k} ∈ Z. The rational numbers λ_{2k}/λ_{1k} (k = 1, . . . , n) satisfy the equation λ_{2k}/λ_{1k} + · · · + λ_{2k}/λ_{1k} = -1. By the construction of the foliation and Lemma 1.1 we see that the multiplicity of the i-th arrowhead of the splice diagram corresponding to a separatrix defined by z = 0 in D_i × C is the integer q_i. In order to glue together the sets A_j × C and D_j × C (j = 2, . . . , n) we need the diffeomorphisms of identification for the foliated local charts similarly to Theorem 1 in [7]. Notice that A_j ∩ D_j is simply connected and z_j ∈ A_j ∩ D_j. Take the coordinate system (z, u_j) in (A_j ∩ D_j) × C where

\[
u_j = u_j \exp(q_j/p_j \log(z - z_j^0/r/2)). \quad (*)
\]

Here log is the branch of the logarithm in C− (the non-positive part of real axis) such that log(1) = 0 as in Theorem 1 [7].

From (*) we have that the leaves of foliation restricted to (A_j ∩ D_j) × C ⊂ D_j × C are level surfaces defined by u_j = constant. Let us identify the point (z, v_j) ∈ (A_j ∩ D_j) × C ⊂ A_j × C with (z, u_j) ∈ (A_j ∩ D_j) × C ⊂ D_j × C where

\[
u_j = v_j \exp(-q_j/p_j \log(z - z_j^0/r/2)). \quad (**)
\]

Thus we can glue the plaques of foliations in A_j × C to the plaques of foliations in D_j × C (see the proof of Theorem 1 [7]). To glue the plaques of the foliation in D_j ∪ A_j to the plaques of the foliation in D_1 we use the following identification map (see Theorem 1 [7]).

Let us identify the points (z, v_j) ∈ (A_j ∩ D_1) × B_j with (z, u_0) ∈ (A_j ∩ D_1) × C by u_0 = h_j(v_j) exp(-q_1/p_1 \log(z/z_j^0)), where h_j : B_j(⊂ C) → C_j(⊂ C) is a certain holomorphic diffeomorphism satisfying h(0) = 0 ∈ B_j ∩ C_j.

There exists a foliation F_U on U = A_2 ∪ · · · ∪ A_n ∪ D_1 ∪ · · · ∪ D_n. Let A = T ∩ U, where T is a tubular neighborhood of γ = ∂U. Then A is homeomorphic to an annulus. In fact, let δ be a closed curve in A which generates the fundamental group π_1(A). Then we may assume that the holonomy of δ with respect to the foliation F_U is trivial (see the proof of Theorem 1 in [7]). If we restrict the foliation
$F_U$ to the annulus $A$, it is diffeomorphic to a product foliation. So there exists a diffeomorphism $\Phi : W \to A \times D$ of some neighborhood of $A$ in a component of the divisor (homeomorphic to $\mathbb{CP}^1$) of the plumbed manifold, where $D \subset \mathbb{C}$ is a disk such that $\Phi$ sends the leaves of the foliation on $W$ onto leaves of the trivial foliations on $A \times \{a\}$, $a \in D$. The map $\Phi$ enables us to glue the foliation on $U \cup W$ to the foliation on $V$. Thus we can construct a foliation on the plumbed 4-manifold. The divisor in the plumbed manifold and the separatrices of simple singularities of the foliation constructed as above induce the final resolution picture. It is the desired one. So we can proceed the blow-down operation to the plumbed 4-manifold, and we get a foliation in a neighborhood of the origin of $\mathbb{C}^2$. The link appeared as the intersection between its separatrix and a small $S^3$ around the origin is represented by a given splice diagram with multiplicity $(q_1, \ldots, q_n)$.

**Corollary 2.5.** For a given multi-Hopf link with $n$ components with multiplicity $(\langle \alpha_1 \rangle, \ldots, \langle \alpha_n \rangle)$ satisfying the equation $\alpha_1 + \cdots + \alpha_n = 1$, $\alpha_i \in \mathbb{C}$ ($i = 1, \ldots, n$), there exists a vector field which has a multi-Hopf link having the above multiplicity.

**Proof.** In the construction of a local singular foliation in the proof of Theorem 2.4 we change local foliations of Theorem 2.4 to foliations induced by the following equation;

$$\frac{dx_i}{dt} = x_i - x_i^0, \quad \frac{dy_i}{dt} = \alpha_i(y_i - y_i^0) \quad (i = 1, \ldots, n).$$

By using the same methods as in Theorem 2.4, we can paste these foliated local charts. And we obtain the desired foliated plumbed manifold, even if we change the local singular foliations as above.

A case of torus link:

As in the case of multi-Hopf link we will first consider the multiplicity of the link appearing in the intersection, and we obtain the following proposition.

**Proposition 2.6.** If the link appearing as the intersection between the separatrices of a holomorphic vector field in $\mathbb{C}^2$ and the small three sphere $S^3$ around the isolated singularity $(0, 0) \in \mathbb{C}^2$ is a $n$-parallel $(p, q)$-torus link, then the splice diagrams defined by Figure 2.7.

![Figure 2.7](image-url)
and the minimal plumbing diagram induced by this splice diagram has only one node vertex as in Figure 2.8.

![Diagram](image)

Figure 2.8

Also, the multiplicity of this splice diagram is defined by \((r_1, \ldots, r_n)\). Here the numbers \(r_k \in \mathbb{N} \) \((k = 1, \ldots, n)\) are defined as above. Moreover, the numbers \(r_k \) \((k = 1, \ldots, n)\) satisfy the following conditions.

\[
r_1 + \cdots + r_n = -\frac{1}{pq}.
\]

**Proof.** An \(n\)-parallel \((p, q)\)-torus link represented by the splice diagram in Figure 2.9.

![Diagram](image)

Figure 2.9

Applying Theorem 1.2 to this splice diagram, we obtain the following plumbing diagram.

\[
\begin{align*}
q/p &= \left(\frac{q}{p}\right) - [c_s, c_{s-1}, \ldots, c_1], \\
p/q &= \left(\frac{p}{q}\right) - [d_r, d_{r-1}, \ldots, d_1], \\
b &= \frac{1}{pq} + [c_1, \ldots, c_s] + [d_1, \ldots, d_r],
\end{align*}
\]
\[
\left(\left\lfloor \frac{q}{p} \right\rfloor \right) \quad \text{(resp. \(\left\lfloor \frac{p}{q} \right\rfloor\)) means a least integer greater than or equal to \(\frac{q}{p}\) (resp. \(\frac{p}{q}\)).
\]

Figure 2.10

There exists an algebraic curve defined by \(x^p - y^p = 0\) which induces a splice diagram representing a \((p, q)\)-torus knot by performing the resolution of its isolated singularity \((0, 0)\). So every plumbing diagram induced by the splice diagram of torus knot can be reduced to a minimal plumbing diagram, which can be changed to a trivial plumbing diagram by performing only several \((-1)\)-blow-down operations by the uniqueness of the minimal plumbing diagram (cf. [3]).

For any pair of natural numbers \((p, q)\), the plumbing diagram induced by the splice diagram of \((p, q)\)-torus link with \(n\) components agrees with the one induced by the splice diagram of \((p, q)\)-torus knot if we remove all arrowhead lines from the both plumbing diagrams. So the above diagram with arrowhead lines deleted can be reduced to a trivial one.

Next we will check the multiplicity of components of the torus link. By using the index defined by Camacho we will calculate the index of the singularity of the foliation \(F\) induced by the next differential equation.

\[
\frac{dx}{dt} = \lambda_1 x + \phi(x, y), \\
\frac{dy}{dt} = \lambda_2 y + \psi(x, y).
\]

The index of \(F\) relative to the leaf defined by \(y = 0\) is \(\lambda_2 / \lambda_1\), and the index of \(F\) relative to the leaf defined by \(x = 0\) is \(\lambda_1 / \lambda_2\).

From this fact, it follows that if the index of the foliation around the corner \((1)\) relative to the component of the divisor \((\ast)\) in Figure 2.11 is \(\alpha\), then the index around the same corner relative to the component of the divisor \((\ast\ast)\) is \(1/\alpha\). Thus the index around the corner \((2)\) relative to the component \((\ast\ast\ast)\) is defined by the continued fraction \(-[c_1, \ldots, c_a]\), and the index around the corner \((3)\) relative to the same component is \(-[d_1, \ldots, d_r]\). Apply again the index theorem to the component of the divisor \((\ast\ast\ast)\). Then the sum of indices around the simple singularities of which separatrices are defined by arrowhead lines in the above plumbing diagram \((\ast\ast)\) is \(-b + [c_1, \ldots, c_a] + [d_1, \ldots, d_r]\) since the Chern class of the component \((\ast\ast\ast)\) of the divisor in Figure 2.11 is \(-b\). Here the plumbing diagram can not be reduced to its vertices any more if we do not remove all arrowhead lines from the diagram.

So the weight of vertices are greater than two expect for the vertex which has arrowhead lines, and the weight of the vertex which has arrowhead line is one. If the weight of the vertex having arrowhead lines is not one, the plumbing diagram with all arrowhead lines deleted can not be reduced to a trivial diagram by only
isolated singularities of holomorphic vector fields

blow-down operations. Thus we see \( b = 1 \). Also we see that the multiplicity \((\langle r_1 \rangle, \langle r_2 \rangle, \ldots, \langle r_n \rangle)\) is a set of natural numbers satisfying

\[
 r_1 + \cdots + r_n = -1 + [c_1, \ldots, c_s] + [d_1, \ldots, d_r],
\]

\((r_i \in \mathbb{C}) \ (i = 1, \ldots, n)\).

From the third equality in Figure 2.10, we see

\[
 -1 + [c_1, \ldots, c_n] + [d_1, \ldots, d_r] = -\frac{1}{pq}.
\]

So we get our result by Lemma 1.1.

\[
-1 + [c_1, \ldots, c_n] + [d_1, \ldots, d_r] = -\frac{1}{pq}.
\]

Now we discuss the following theorems which correspond to Theorem 2.4 in the case of a multi-Hopf link.

**THEOREM 2.7.** Let \( L \) be an \( n \)-parallel \((p, q)\)-torus link and its multiplicities \((m_1, \ldots, m_n)\) are defined by a set of numerators of simple fractions \( q_i/p_i \) \((i = 1, \ldots, n)\) such that

\[
\frac{q_1}{p_1} + \cdots + \frac{q_n}{p_n} = \frac{1}{pq}.
\]

Then there exists a holomorphic vector field such that the intersection between its separatrices and a small \( S^3 \) around its singularity is the link \( L \).

**PROOF.** A given torus link is represented by the following splice diagram:

(1)

Here we get the plumbing diagram defined by the graph as in Figure 1.5 if we apply Theorem 1.2 to the diagram (1). We obtain the minimal plumbing diagram (2):
Here the integers \(-b, -c_i (i=1,\ldots,s)\) and \(-d_j (j=1,\ldots,r)\) are vertex weights of the minimal plumbing diagram defined by (2).

By removing all arrowhead lines from the diagram (2), it is reduced to a trivial diagram \(\bullet\) by the same arguments in the proof of Proposition 3. Begin the procedure of blowing up at the origin of \(\mathbb{C}^2\) and continue this operation at a certain point of a component of the divisor. Notice that the divisor is created by the blowing-up operation and stop the blowing-up operations when we obtain the divisor represented by the diagram (2) with deleted arrowhead lines. Since the above blowing-up operations are chosen to be a reverse operation of the reduction from (2) to \(\bullet\), we can obtain such divisor and the plumbed 4-manifold in the neighborhood of this divisor by Grauert’s theorem [4]. Now we explain briefly the construction of the vector field on this plumbed manifold as in the proof of Theorem 2.4 (also see §2.4 in [7]).

First, we construct the separatrices of the simple singularities of a foliation (or a vector field) on the above plumbed 4-manifold corresponding to the plumbing diagram defined by (2). Using the local coordinates \((x_i, y_i)\) about the certain point \(o_i\) of the divisor \((i=1,\ldots,n)\) we construct the following differential equation around \(o_i\):

\[
\frac{dx_i}{dt} = x_i - x_i^0,
\]

\[
\frac{dy_i}{dt} = \frac{\lambda_{i2}}{\lambda_{i1}} (y_i - y_i^0).
\]

Here, \(o_i = (x_i^0, y_i^0)\) and \(\lambda_{i1}q_i + \lambda_{i2}p_i = 0 (i=1,\ldots,n)\). In completing the construction of the foliation, the separatrices representing the arrowheads in a splice diagram of a torus link are defined by the equations \(x_i = 0 (i=1,\ldots,n)\) by using local coordinates \((x_i, y_i)\). We can construct a foliation on the plumbing manifold by gluing such local singular foliations and local trivial foliations defined in the proof of Theorem 2.4. To construct the foliation we must pay attention to the construction of a local foliation around the corner as in §2.4 in [7]. Suppose that \(\{p\} = P_i \cap P_j \neq \emptyset\) (here, \(P_i\) and \(P_j\) are components of the divisor), and that the foliation \(F_i\) is written in a neighborhood \(U_p\) (in the plumbing manifold) of \(p\) as

\[
\frac{dx}{dt} = x, \quad \frac{dy}{dt} = \alpha y,
\]
where, \((x, y)\) is a coordinate system such that \(p = (0, 0)\) and \(P_i \cap U_p = \{y = 0\}\).

Similarly suppose \(F_j\) be written in a neighborhood \(U'_p\) of \(p\) in the plumbing manifold as follows:

\[
\frac{du}{dt} = u, \quad \frac{dv}{dt} = \beta v,
\]

where \(U'_p \cap P_j = \{v = 0\}\).

Moreover, suppose that \(\alpha \times \beta = 1\). Then the foliations defined by the above two equations can be glued by the diffeomorphism \(\phi(x, y) = (y, x) = (u, v)\). Therefore, we can glue the foliation near the component \(P_i\) to the foliation near the component \(P_j\). Continuing the same procedures around all corners of the divisor, we obtain the foliated plumbed 4-manifold. If we proceed the blow-down operations at the singularities of the foliated 4-manifold, then we get a foliation defined in a small neighborhood of which boundary is homeomorphic to \(S^3\). Notice that the intersection between this \(S^3\) and the separatrices of this foliation are defined by the multi-link \(L\).

**Corollary 2.8.** Given an \(n\)-parallel \((p, q)\)-torus link which has a multiplicity satisfying the conditions in Proposition 2.6, there exists a vector field defined in a neighborhood of the origin of \(\mathbb{C}^2\) so that the above multi-link appears as the intersection between its separatrices and a small \(S^3\) around the origin.

We can prove this corollary by going in the same way as in the proof of Corollary 2.5.

Another types of torus link are defined by the splice diagram:

\[
\begin{array}{ccc}
\text{ (1) } & \cdots & \text{ (2) } \\
\downarrow & & \uparrow \\
\text{ (3) } & \cdots & \text{ (3) }
\end{array}
\]

\[
p, q \in \mathbb{N}.
\]

\[
\text{G.C.D.}(p, q) = 1
\]

Figure 2.12

For these cases we have the following Proposition 2.9 (resp. Theorem 2.10) corresponding to the Proposition 2.3 (resp. Theorem 2.4). We discuss only the case of (3) in Figure 2.12. Other cases can be treated similarly.

**Proposition 2.9.** If the link appearing as the intersection between the separatrices of holomorphic vector field and the small \(S^3\) around the origin is a torus
link with \(n\)-components defined by the splice diagram (3) in Figure 2.12 then the minimal plumbing diagram is induced by Figure 2.13.

Also the multiplicities of the torus link is defined by \((\langle r_1 \rangle, \ldots, \langle r_n \rangle)\) where the definitions of \(\langle r_k \rangle\) (\(k = 1, \ldots, n\)) are the same as in Proposition 2.3, and the numbers \(r_k \in \mathbb{C}\) (\(k = 1, \ldots, n\)) satisfy the following equality

\[
r_2 + \cdots + r_{n-1} = -1 + [c_1, \ldots, c_s + r_1] + [d_1, \ldots, d_r + r_n].
\]

The proof for Proposition 2.3 ensures the truth of Proposition 2.9.

**Theorem 2.10.** Let \(L\) be a \((p, q)\)-torus link with \(n\) components defined by the following splice diagram.

Its multiplicities \((\langle r_1 \rangle, \ldots, \langle r_n \rangle)\) are defined by the following numbers \(r_i \in \mathbb{C}\) (\(i = 1, \ldots, n\)) such that

\[
r_2 + \cdots + r_{n-1} = [c_1, \ldots, c_s + r_1] + [d_1, \ldots, d_r + r_n] - 1.
\]

Here \(\langle r_1 \rangle\) (resp. \(\langle r_n \rangle\)) is the multiplicity of the arrowhead line with edge weight of \(p\) (resp. \(q\)). Then there exists a holomorphic vector field such that the \((p, q)\)-torus link with \(n\) components appearing as the intersection has the multiplicity \((\langle r_1 \rangle, \ldots, \langle r_n \rangle)\).

Theorem 2.10 is proved by going on the same way as in the proof of Theorem 2.4.
Now we consider general irreducible graph links. The splice diagram is obtained by joining two splice diagrams representing the torus link inductively under the certain conditions concerning the edge weights.

First, we will deal the next configuration of splice diagram, i.e., the diagram obtained by joining exactly two multiple torus links. (One of them may be replaced by a multi-Hopf links.)

\[ \begin{align*}
&\quad \quad \quad \quad p_1, q_1, p_2, \text{ and } q_2 \in \mathbb{N}, \\
&\quad q_1 q_2 - p_1 p_2 > 0.
\end{align*} \]

Figure 2.15

One of the two splice diagrams may have no arrowhead lines, for example the diagram represented by Figure 2.16. It defines the same link represented by the diagram in Figure 2.17.

\[ \begin{align*}
&\quad \quad \quad \quad p_1, q_1, p_2, \text{ and } q_2 \in \mathbb{N}, \\
&\quad q_1 q_2 - p_1 p_2 > 0.
\end{align*} \]

Figure 2.15

Figure 2.16

So we can exclude these cases.

We need the next theorem by D. Eisenbud and W.D. Neumann to get a plumbing diagram for a spliced link.

**Theorem 2.11** ([3]). Let \( L_1 = (S^3, K_1) \) and \( L_2 = (S^3, K_2) \) be graph links given by plumbing according to
Let $1/a_1 = cf(\Delta_1', v)$, $1/a_2 = cf(\Delta_2', w)$. If we splice along the link components corresponding to the arrowheads, then the following plumbing diagram $\Delta$ is constructed.

$$
\Delta_1 = \begin{array}{c}
v \\
- e_v
\end{array} \quad \text{and} \quad \Delta_2 = \begin{array}{c}
w \\
- e_w
\end{array}
$$

We see that the assumptions $1/a_1 = cf(\Delta_1', v)$ and $1/a_2 = cf(\Delta_2', w)$ in the above theorem are not singular conditions by Theorem 1.2 (also see [3]).

**Lemma 2.12.** Let $L_1$ be a torus link defined by the diagram:

```
\begin{array}{c}
\bullet \\
p_1 & q_1
\end{array}
```

Figure 2.19

```
\begin{array}{c}
\bullet \\
q_2 & p_2
\end{array}
```

Figure 2.20

Suppose that the link $L_1$ is represented by the next minimal plumbing diagram:

```
\begin{array}{c}
- c_1 & - c_2 & \ldots & - c_k & \ldots & - c_{m-1} & - c_m
\end{array}
```

$c_k = 1$

Figure 2.21
and the link $L_2$ is the following minimal plumbing diagram.

\[
\begin{array}{c}
\bullet \quad \bullet \quad \bullet \\
-b_1 & -b_2 & -b_t & -b_{u-1} & -b_u \\
\end{array}
\]

\[b_t = 1\]

Figure 2.22

Also we assume that the spliced link $L_1 - L_2$ is represented as follows:

\[
\begin{array}{c}
\bullet \\
p_1 & q_1 & q_2 & p_2 \\
\end{array}
\]

Figure 2.23

Then the spliced link $L_1 - L_2$ has one of the next minimal plumbing diagrams (1), (2), (3), (4) and (5).

1. $k \leq l < m$

\[
\begin{array}{c}
\bullet \\
-c_1 & -c_2 & -c_k & -c_l - b_1 + 1 & -b_2 & -b_t & -b_u \\
\end{array}
\]

2. $1 < s \leq t$

\[
\begin{array}{c}
\bullet \\
-c_1 & -c_2 & -c_k & -c_l - b_s + 1 & -b_{s+1} & -b_t & -b_u \\
\end{array}
\]

3. $k \leq l < m$, $1 < p < t$ (or $k < l < m$, $1 < p \leq t$)

\[
\begin{array}{c}
\bullet \\
-c_1 & -c_2 & -c_k & -c_l - b_p + 1 & -b_{p+1} & -b_t & -b_u \\
\end{array}
\]
(4) $k = l$, $p = t - 1$

\[
\begin{array}{c}
\cdots \\
-c_1 & c_2 & c_l & c_p - 1 & b_t & -b_u \\
\end{array}
\]

$p = t$, $k = l - 1$

\[
\begin{array}{c}
\cdots \\
-c_1 & c_2 & c_k & c_l & b_t - 1 & -b_u \\
\end{array}
\]

(5)

\[
\begin{array}{c}
\cdots \\
-c_m & b_1 + 1 \\
-c_1 & c_2 & c_k & c_{m-1} & b_2 & b_t & -b_u \\
\end{array}
\]

Figure 2.24

If the spliced link $L_1 - L_2$ is represented by the diagram in Figure 2.25,

\[
\begin{array}{c}
\cdots \\
-p_1 & q_1 & 1 & p_2 \\
\end{array}
\]

Figure 2.25

its minimal plumbing diagram is defined by one of next diagrams (1)', (2)', ..., and (5)'. In this case, if the diagram which represents the link $L_2$ has no arrowhead lines, its minimal plumbing diagrams is one of the diagrams (1)', (2)', ..., and (5)' of which arrowhead lines are deleted from the right hand vertex.
(1) \( k \leq l < m \)

(2) \( 1 < s \leq t \)

(3) \( k \leq l < m, 1 < p < t \) (or \( k < l < m, 1 < p \leq t \))

(4) \( k = l, p = t - 1 \)
PROOF. In the case of link defined by Figure 2.23, we see that the link $L_1 - L_2$ has the following plumbing diagram (*) by using Theorem 2.11.

Here put $d = 1/cf(\Delta_1, v_1) = ((p_1/q_1))$, and $e = 1/cf(\Delta_2, v_2) = ((p_2/q_2))$. Also the diagram $\Delta_1$ (resp. $\Delta_2$) is derived from the diagram in Figure 2.21 (resp. the diagram in Figure 2.22) and is defined by the diagram (A) (resp. (B)) in Figure 2.27.
Since the link $L_1 - L_2$ is an algebraic link, we see that $\left(\frac{a_1}{b_1}\right) \times \left(\frac{b_2}{a_2}\right) < 1$. Then at least one of $d$ and $e$ is defined by an integer 1 (see Figure 2.24). So the plumbing diagram (*) is not a minimal plumbing diagram. Thus we must perform the $(-1)$-blow-down operation at the vertex which is weighted by $d$ or $e$ to get the minimal plumbing diagram. Now we consider the following several cases.

Suppose that $d = 1$. We perform the blow-down operation at this vertex. Next we calculate the continued fractions

$$c_m - 1 - [c_{m-1}, \ldots, c_{k+1}] \quad \text{and} \quad e - [b_1, \ldots, b_{t-1}] .$$

Since $d - [c_m, \ldots, c_{k+1}] = a_1/b_1$, $e - [b_1, \ldots, b_{t-1}] = b_2/a_2$ and $d = 1$, then

$$c_m - 1 - [c_{m-1}, \ldots, c_{k+1}] = \frac{a_1}{b_1 - a_1},$$

$$e - 1 - [b_1, \ldots, b_{t-1}] = \frac{b_2 - a_2}{a_2} .$$

Also the condition (***) is satisfied, because $a_1b_2/b_1a_2 < 1$.

$$\frac{a_1}{b_1 - a_1} \times \frac{b_2 - a_2}{a_2} < 1 .$$

So we can continue the blow-down operations, unless the vertex weight $e$ becomes 0. Here suppose that $-e + 1 \leq -2$, the blow-down operations progress toward the vertex weighted by $-c_k$. When the weight of extra vertex weighted by $-e + 1$ becomes 0, the blow-down operations are stopped. And performing the 0-chain absorption, we obtain the diagram defined by (1).

For the case $e = 1$ and $-d + 1 \leq -2$, we treat similarly. We get the diagram defined by (2).

If the above blow-down operations stop at a certain vertex weighted by $c_l$ ($k < l < m$) before the vertex weight $-e + 1$ becomes 0, we see that $c_l \geq 3$ and the weight $-e + 1$ must become $-1$. Otherwise, it contradicts the condition (**). Then the blow-down operations start from the vertex weighted by $-c_l$ and the direction of the progress is the opposite one. After the two different blow-down
operations are repeated alternately several times, eventually they stop, since the plumbing diagram is a finite graph. Also we see that a zero vertex appears between the vertices weighted by $-ck$ and $-bt$. Otherwise, the plumbing diagram becomes the one of the next configurations.

(1) \[
\begin{array}{cccc}
\cdots & \cdots & & \\
& 0 & -a & \\
\cdots & & 0 & -1 & \cdots \\
\end{array}
\]

$a \geq 2$

(2) \[
\begin{array}{cccc}
\cdots & \cdots & & \\
& -1 & -b & \\
\cdots & & 0 & \\
\end{array}
\]

$b \geq 2$

(3) \[
\begin{array}{cccc}
\cdots & \cdots & & \\
& 0 & & 0 & \\
\end{array}
\]

This diagram does not represent any algebraic links. So the zero vertex appears between the vertex weighted by $-cs+1$ (resp. $-cs$) ($k < s < l$) and $-bp$ (resp. $-bp+1$) ($1 < p < t$). The form of the diagram is defined by the following diagrams.

\[
\begin{array}{cccc}
\cdots & \cdots & & \\
& -ck & -cs+1 & 0 & -bp & -bt & \\
\end{array}
\]

Figure 2.29

Also we perform a zero-chain absorption, and get the diagram defined by (3). In this situation, adding the condition $s = k$, the diagram is defined by the next form.
If $p = t$ (and $k < s < m$), the diagram is defined as below.

![Figure 2.30](image)

They belong to the diagram (3). Moreover, if $p + 1 = t$ or $k = s - 1$, the two vertices which are initial vertices of arrowhead lines are adjacent. The configuration of the diagram is defined by (4).

If $d = -1$ and $-e + 1 = -1$ (or $-d + 1 = -1$ and $e = -1$), we see that the diagram is defined by (1), (2), (3), and (4).

If $d = e = -1$, the diagram is defined by (5).

In the case of the splice diagram represented by Figure 2.32, the proof is almost the same to the first type of the splice diagram. So we will skip it.

![Figure 2.32](image)

We have the proposition corresponding to the Propositions 2.3 and 2.6.

**Proposition 2.13.** *If the spliced link*
appears as the intersection between separatrices of a given holomorphic vector field and a small $S^3$ around its singularity, then it satisfies the next properties (1) and (2), and its multiplicities $(\langle m_1, \ldots, m_k \rangle)$ and $(\langle n_1, \ldots, n_s \rangle)$ satisfy the property (3), where $\langle \cdot \rangle$ means the same as in Propositions 2.3 and 2.6.

(1) $a_2b_1 - a_1b_2 > 0$, where $a_1, a_2, b_1$ and $b_2$ are edge weights and are positive integers.

(2) The spliced link has the following minimal plumbing diagram.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{plumbing_diagram.png}
\caption{Figure 2.33}
\end{figure}

(3) $c_s + (\alpha_1 + \cdots + \alpha_k) = [c_{s+1}, c_{s+2}, \ldots, c_t + (\beta_1 + \cdots + \beta_r), c_{t+1}, \ldots, c_m] + [c_{s-1}, \ldots, c_1]$. $c_s = 1$ or $c_t = 1$.

Here, $\alpha_j$ ($j = 1, \ldots, k$) is the index around the singularity of which separatrix is represented by the $j$-th arrowhead line starting from the vertex $s$ of the plumbing diagram in (2) and $\beta_j$ ($j = 1, \ldots, r$) is the index around the singularity of which separatrix is represented by a $j$-th arrowhead lines starting from the vertex labeled the number $t$. Also $-c_i$ ($i = 1, \ldots, m$) is a vertex weight of the $i$-th vertex of the plumbing diagram defined in the property (2), and $\gamma_i$ ($\gamma_i \in \mathbb{C}, i = 1, \ldots, k$) denotes the following complex number:

\[
\frac{1}{\gamma_1 - \frac{1}{\gamma_2 - \frac{1}{\gamma_3 - \ldots - \frac{1}{\gamma_k}}}}
\]

**Proof.** Since the given link is the intersection between the separatrices and a small $S^3$, it is an algebraic link, and its splice diagram satisfies the property (1) by Theorem 9.4 in [3] (also see [6]). This spliced link consists of two torus links.
Isolated singularities of holomorphic vector fields

(one of them may be replaced by a multi-Hopf link). Hence the splice diagram of this link is constructed by joining two splice diagrams (a) and (b) in Figure 2.34.

![Figure 2.34](image)

Thus we get the plumbing diagram of the spliced link from the following plumbing diagrams of the two torus links in Figure 2.35,

![Figure 2.35](image)

\[
\Delta_1 = \begin{cases}
-a_{1s_1} - a_{1s_1-1} & -a \\
-a_{2s_2} - a_{2s_2} & -a_{2s_2-1}
\end{cases}
\]

\[
\Delta_2 = \begin{cases}
-b_{1r_1} - b_{1r_1-1} & -b \\
-b_{2r_2} - b_{2r_2} & -b_{2r_2-1}
\end{cases}
\]

\[
a_2/a_1 = ((a_2/a_1)) - [a_{2s_2}, a_{2s_2-1}, \ldots, a_{21}],
\]

\[
a_1/a_2 = ((a_1/a_2)) - [a_{1s_1}, a_{1s_1-1}, \ldots, a_{11}],
\]

\[
a = 1/a_1a_2 + [a_{11}, \ldots, a_{1s_1}] + [a_{21}, \ldots, a_{2s_2}],
\]

\[
b_2/b_1 = ((b_2/b_1)) - [b_{1r_1}, \ldots, b_{11}],
\]

\[
b_1/b_2 = ((b_1/b_2)) - [b_{2r_2}, \ldots, b_{21}],
\]

\[
b = 1/b_1b_2 + [b_{11}, \ldots, b_{1r_1}] + [b_{21}, \ldots, b_{2r_2}].
\]
By using Theorem 2.4 the plumbing diagram of the spliced link is represented by the following diagram in Figure 2.36.

![Figure 2.36](image)

Since we can change the diagram in Figure 2.36 into the minimal plumbing diagram by some operations concerning the reduction of a graph, then the type of the minimal plumbing diagram is one of (1), (2), ..., and (5) in Lemma 2.12. And we rewrite its all vertex weights by the integers $-c_1$, $-c_2$, ..., and $-c_m$. We get that the minimal plumbing diagram satisfies the property (2).

Take the index around the corner of a component $P_i$ of the divisor and $P_j$ in the final resolution picture corresponding to the plumbing diagram in Figure 2.33. If the index of the corner relative to $P_j$ is $\alpha$, the index of the same corner relative to $P_i$ is $1/\alpha$ as in Proposition 2.6. Using the index theorem, the sum of the index of all singularities of a component of the divisor $P$ is equal to the vertex weight of the plumbing diagram. Here this vertex corresponds to the component $P$ of the final resolution picture. The index theorem ensures the property (3).

The following theorem corresponds to Theorems 2.4 and 2.7 in the case of multi-Hopf link and torus link.

**Theorem 2.14.** Let $L$ be a spliced link with $n$-components having the following three properties.

1. The link $L$ is represented by the following splice diagram:

   ![Figure 2.37](image)

2. The edge weights $a_1$, $a_2$, $b_1$ and $b_2$ of the diagram in (1) satisfy the inequality $b_1b_2 - a_1a_2 > 0$ and are positive integers.

3. There are at most two edges such that the value of each edge weight is greater than two.

Then we obtain the following minimal plumbing diagram.
Also there exists a vector field $Z$ such that $S_Z \cap S^3 = L$, where $S_Z$ is a separatrix of $Z$ and $S^3$ is a small 3-sphere around the isolated singularity of $Z$. Multiplicities of the link $L$ are defined by $\langle \alpha_1 \rangle, \ldots, \langle \alpha_s \rangle, \langle \alpha_{s+1} \rangle, \ldots, \langle \alpha_{s+r} \rangle$, where $s + r = n$ and the number $\alpha_j \in \mathbb{C}$ ($j = 1, \ldots, n$) satisfy the condition

$$c_j + (\alpha_1 + \cdots + \alpha_s) = [c_{j-1}, \ldots, c_1 + (\alpha_{s+1} + \cdots + \alpha_{s+r}), c_{l-1}, \ldots, c_0] + [c_{j+1}, \ldots, c_m],$$

$$c_j = 1 \quad \text{or} \quad c_l = 1.$$

Here, the integers $-c_0, \ldots,$ and $-c_m$ are vertex weights of the above plumbing diagram.

REMARK. Performing several $(-1)$-blow-down operations and a 0-chain absorption, we see that the integers $c_0, c_1, \ldots, c_m$ are uniquely determined by the integers $a_1, b_1, a_2$ and $b_2$.

REMARK. The condition (3) of this theorem is a sufficient condition for the existence of a given graph link in $S^3$.

PROOF. Since the link $L$ has the condition (1) and (2), $L$ is an algebraic link by Theorem 9.4 in [3]. So there exists an algebraic curve which intersects with the small $S^3$ around its singularity at the link $L$. Also a minimal plumbing diagram of $L$ is obtained by applying some of operations i.e., $(-1)$-blow-down operation and 0-chain absorption. Thus if we remove all arrowhead lines from the diagram, then it becomes a trivial diagram by applying only some blow-down operations. Similarly to the proofs of Theorems 2.4 and 2.7 we can construct a vector field on the neighborhood of the divisor represented by a final resolution picture which is a dual graph of the above minimal plumbing diagram. Since the minimal plumbing diagram can be reduced to a trivial one by removing arrowhead lines, the neighborhood of the divisor can be changed to a neighborhood of the origin of $\mathbb{C}^2$. So we obtain a vector field on a neighborhood of the origin, and the link $L$ appears in a small $S^3$ around the origin of $\mathbb{C}^2$. 

Next we discuss the following configurations of the splice diagram:
REMARK. We can exclude the next configuration, because it does not satisfy the condition of algebraic link such that $1 - p_1 q_1 p_2 q_2 > 0$.

**Theorem 2.15.** Suppose that the link $L$ is represented by the diagram defined in (1) of Figure 2.39. Then the link $L$ has the minimal plumbing diagram defined by $(1)'$, $(2)'$, ..., and $(5)'$ in Lemma 2.12. Also there exists a vector field $Z$ such that $S_Z \cap S^3 = L$, where $S_Z$ is a separatrix of $Z$ and $S^3$ is a small 3-sphere around the isolated singularity of $Z$. 
If the minimal plumbing diagram is defined by \((1)\)', we rewrite the weights of all vertices of this diagram as follows:

\[ \langle \alpha_1 \rangle \langle \alpha_k \rangle \]
\[ \langle \alpha_{k+1} \rangle \langle \alpha_n \rangle \]

\[ -c_0 \quad -c_1 \quad -c_p \quad -c_{p+1} \]
\[ -c_l \quad -c_{l_1} \quad -c_{l_2} \quad -c_{l_{1b}} \]

Figure 2.41

then the multiplicities of this link \(L\) are defined by \(\langle \alpha_1 \rangle, \ldots, \langle \alpha_k \rangle, \langle \alpha_{k+1} \rangle, \ldots, \langle \alpha_{k+r} \rangle\) where \(r + k = n\), and the number \(\alpha_j \in \mathbb{C}\) \((j = 1, 2, \ldots, n)\) satisfies the following condition.

\[
c_l + (\alpha_{k+1} + \cdots + \alpha_{k+r}) = [c_{l_1}, c_{l_2}, \ldots, c_{l_{1a}}] + [c_{l_2}, \ldots, c_{l_{2b}}] + \cdots + [c_{l_{-1}}, c_{l_{-2}}, \ldots, c_p + (\alpha_1 + \alpha_2 + \cdots + \alpha_k), c_{p-1}, \ldots, c_0],
\]

\[ c_p = 1 \quad \text{or} \quad c_l = 1. \]

Remark. If the minimal plumbing diagram is defined by \((2)', (3)', (4)' and (5)'\) in Lemma 2.12, its multiplicities satisfy the similar conditions.

The proof is almost the same as the proof of Theorem 2.14. So we will omit it.

We discuss the existence of a vector field corresponding to a given general spliced multi-links with \(n\)-components. However, it is difficult to write down the necessary conditions for getting the multiplicities of arrowhead vertices in a general form. To get out of the difficulty we need several propositions.

First we give configurations of the splice diagrams which have three nodes as follows:

\[
(1) \quad \alpha_1 \quad \beta_1 \quad \alpha_2 \quad \beta_2 \quad \alpha_3 \quad \beta_3
\]
(2) \[ \alpha_1 \beta_1 \alpha_2 \beta_2 1 \beta_3 \]

(3) \[ \alpha_1 \beta_1 \alpha_2 1 \beta_2 \alpha_3 \beta_3 \]

(4) \[ \alpha_1 \beta_1 1 \alpha_2 1 \beta_2 \alpha_3 \]

(5) \[ \alpha_1 \beta_1 1 \beta_2 1 \alpha_3 \beta_3 \]

(6) \[ \alpha_1 1 \alpha_2 \beta_2 \alpha_3 \beta_3 \]
The symbol \[\uparrow\uparrow\] means that arrowhead line in \([\ ]\) can be removed from the node vertex. We will deal only with the first splice diagram and the second one. Remaining cases can be treated in almost the same way. So we will omit them. Now we assume that the minimal plumbing diagram derived from the splice diagram in Figure 2.43 (a) is defined by the diagram in Figure 2.43 (b) and the minimal plumbing diagram derived from the splice diagram defined by Figure 2.43 (c) is defined by the diagram in Figure 2.43 (d).
And we get the plumbing diagrams corresponding to the splice diagrams (1) and (2) by connecting the above two minimal plumbing diagrams (see Figure 2.43 (b) and (d)).

The splice diagram (1)

The splice diagram (2)
The vertices weighted by the integers $-e$ and $-d$ are extra vertices as in Lemma 2.12 and they are defined by the symbol *. Using the same methods as in Lemma 2.12, the minimal plumbing diagrams corresponding to the splice diagrams (1) and (2) can be defined as follows.

The case of the splice diagram defined by (1)

(1) $l \leq j < n$.

(2) $l < r \leq s$.

(3) $l \leq j < n$, $1 < p < s$ (or $l < j < n$, $1 < p \leq s$).

(4) $j = l$, $p = s - 1$.

$p = s$, $l = j - 1$. 
Figure 2.45

The case of the splice diagram defined by (2)

(1) \( l \leq j < n \).

(2) \( l < r \leq s \).
(3) \( l \leq j < n, 1 < r < s \) (or \( l < j < n, 1 < r \leq s \)).

(4) \( j = l, r = s - 1 \).

(5) \( r = s, l = j - 1 \).

Figure 2.46
PROPOSITION 2.16. Let $L$ be a multi-link as in Theorem 2.14 and $L_t$ be a torus link with $t$ components.

Assume that the link $L - L_t$ has the following from (#) by virtue of the splice diagram.

$$\begin{array}{cccc}
\alpha_1 & \beta_1 & \alpha_2 & \beta_2 \\
& & \alpha_3 & \beta_3 \\
\end{array}$$

Here $\beta_1 \alpha_2 - \alpha_1 \beta_2 > 0$ and $\beta_2 \alpha_3 - \alpha_2 \beta_3 > 0$.

Then the minimal plumbing diagram is one of (1), (2), (3), (4), and (5) in Figure 2.45.

Suppose that the minimal plumbing diagram is defined by (1) of Figure 2.45. Rewrite the vertex weights of all vertices $-c_0, -c_1, \ldots, -c_w$ as follows:

$$\begin{array}{cccc}
m_1 & m_r & m_{r+1} & m_{r+s} \\
\vdots & \ddots & \ddots & \ddots \\
-k & \ddots & \ddots & \ddots \\
-c_0 & -c_1 & \ldots & -c_w \\
\end{array}$$

Then there exists a vector field on a neighborhood of the origin which has the link $L - L_t$ with multiplicities $(m_1, \ldots, m_r)$, $(m_{r+1}, \ldots, m_{r+s})$, and $(m_{r+s+1}, \ldots, m_{r+s+t})$. Also its multiplicities are defined by the positive integers $\langle q_i/p_i \rangle$ $(i = 1, 2, \ldots, r + s + t)$ such that the numerators of simple fractions $q_1/p_1, \ldots, q_r/p_r, q_{r+1}/p_{r+1}, \ldots, q_{r+s}/p_{r+s}, q_{r+s+1}/p_{r+s+1}, \ldots$, and $q_{r+s+t}/p_{r+s+t}$ satisfy the following condition:

$$c_0 = \left[ c_1, c_2, \ldots, c_k - \left( \frac{q_1}{p_1} + \cdots + \frac{q_r}{p_r} \right), \ldots, \right. \\
\left. \ldots, c_l - \left( \frac{q_{r+1}}{p_{r+1}} + \cdots + \frac{q_{r+s}}{p_{r+s}} \right), \ldots, \right. \\
\left. \ldots, c_u - \left( \frac{q_{r+s+1}}{p_{r+s+1}} + \cdots + \frac{q_{r+s+t}}{p_{r+s+t}} \right), c_{u+1}, \ldots, c_w \right].$$

REMARK. If the minimal plumbing diagram is defined by other types, we can obtain similar conclusions.
PROOF. By the condition (##), the splice link $L - L_t$ is also an algebraic link.

(##) $\beta_1 \alpha_2 - \alpha_1 \beta_2 > 0, \quad \beta_2 \alpha_3 - \alpha_2 \beta_3 > 0.$

Here $\alpha_i (i = 1, 2, 3)$ and $\beta_j (j = 1, 2, 3)$ are edge weights of the diagram. Therefore, there is a minimal plumbing diagram which can be reduced to a trivial one by removing all arrowhead lines from the diagram and by performing several blow-down operations as in Theorem 2.7. Thus we construct a vector field on a neighborhood of the Theorem 2.4; this neighborhood can be changed to a certain neighborhood of the origin of $\mathbb{C}^2$. We can find a foliation with a singularity of which a leaf passes through its singularity to realize an arrowhead line corresponding to a component of link $L - L_t$. In fact, by using $(x_i, y_i) \in \mathbb{C}^2 (i = 1, 2, \ldots, r + s + t)$, the local foliations are induced by the following equations,

$$\frac{dx_i}{dt} = x_i - x_i^0, \quad \frac{dy_i}{dt} = \gamma_i (y_i - y_i^0), \quad (i = 1, \ldots, r + s + t),$$

where, $(x_i^0, y_i^0) = (0, 0)$ and the numbers $\gamma_i$ satisfy

$$c_0 = [c_1, c_2, \ldots, c_k + (\gamma_1 + \cdots + \gamma_r), \ldots, c_l + (\gamma_{r+1} + \cdots + \gamma_{r+s}), \ldots, c_u + (\gamma_{r+s+1} + \cdots + \gamma_{r+s+t}), \ldots, c_w].$$

Once the construction of the foliations are accomplished, we see that multiplicities can be defined by the integers $((\gamma_1), \ldots, (\gamma_r)), ((\gamma_{r+1}), \ldots, (\gamma_{r+s}))$ and $((\gamma_{r+s+1}), \ldots, (\gamma_{r+s+t}))$ by Lemma 1.1. Pasting the above local foliations in the same way as in Theorem 2.4, we can construct a vector field around the origin of $\mathbb{C}^2$ which satisfies the desired properties by several (-1)-blow-down operations.

Next we will discuss the case of the second type of splice diagram.
Proposition 2.17. Let $L - L_t$ be a spliced link obtained by joining the link $L$ which is the same configuration as the one in Proposition 2.16 and a torus link. Assume that the link $L - L_t$ has a next representation with the use of a spliced link (see Figure 2.49) which satisfies the condition of an algebraic link as in Proposition 2.16.

Then the minimal plumbing diagram is one of (1), (2), ..., and (5) which are defined in Figure 2.46.

Let the minimal plumbing diagram be defined by (1), and rewrite the vertex weights of all vertices by $-c_0, ..., -c_m, -a_1, ..., -a_p, -b, -b_1, ..., -b_q$ as follows:

Then there exists a vector field on a neighborhood of the origin of $\mathbb{C}^2$, which has the link with multiplicities $(m_1, ..., m_r)$, $(m_{r+1}, ..., m_{r+s})$, and $(n_{r+s+1}, ..., n_{r+s+t})$. Also its multiplicities are defined by the set of integers such that the numerators of the rational numbers $q_1/p_1, ..., q_r/s/r$ satisfy the following conditions

$$[a_1, a_2, ..., a_p] + [b_1, ..., b_q] + \left[ c_m, c_{m-1}, ..., c_l - \left( \frac{q_{r+1}}{p_{r+1}} + \cdots + \frac{q_{r+s}}{p_{r+s}} \right), ..., c_k - \left( \frac{q_1}{p_1} + \cdots + \frac{q_r}{p_r} \right), c_{k-1}, ..., c_0 \right]$$

$$= b - \left( \frac{q_{r+s+1}}{p_{r+s+1}} + \cdots + \frac{q_{r+s+t}}{p_{r+s+t}} \right).$$
We can prove this proposition proceeding in the same way as in the proof of Proposition 2.16. Thus we omit it.

Using the Propositions 2.16 and 2.17, and similar propositions for the remaining cases of splice diagrams inductively, we see that a vector field (or a foliation) which has any links of general types with a certain multiplicity can be constructed.

§3. An application to the Thurston norm of the holomorphic vector fields

First, we discuss the difference between the norm \( \| \|_1 \) defined by the ordinary multiplicity (see §4 in [10]) and the norm \( \| \|_2 \) defined in Definition 1.1 (or Definition 3 in [11]).

**Proposition 3.1.** Suppose that the separatrices of the vector field \( Z \) are defined by \( f = 0 \) and we assume that the plumbing diagram \( P \) is defined by the resolution of its singularity \( (0,0) \subset \mathbb{C}^2 \) as follows:

![Figure 3.1](image)

Here, the integers \( m_i (i = 1, \ldots, n) \) designate ordinary multiplicities along the component of the divisor and the integer defined by \( k \) means an ordinary multiplicity along the irreducible component of desingularized curves.

Suppose that G.C.D.(\( m_i, k \)) = \( d_i (i = 1, \ldots, n) \) where the integer \( n \) is the number of irreducible components of the curve \( f = 0 \). Then the relation of two Thurston norms is as follows:

If the norm defined by the ordinary multiplicity is represented by \( \| Z \|_1 = \sum (\delta_j - 2)|m_1l_{1j} + \cdots + m_nl_{nj}| \), then the norm defined by Definition 1.1 is \( \| Z \|_2 = \sum (\delta_j - 2)|(m_1/d_1)l_{1j} + \cdots + (m_n/d_n)l_{nj}| \) where \( \delta_j \) denotes the degree of the \( j \)-th vertex in the splice diagram \( \Gamma \) induced by the plumbing diagram \( P \), and \( l_{ij} \) is the linking number of \( i \)-th vertex and \( j \)-th vertex.

**Proof.** The Milnor fibration defined by \( f = 0 \) is induced by the following differential equation (1) around each intersection between the components of the curve and a component of the divisor by using local coordinates \( (x_i, y_i) \). Here, we assume that \( x_i = 0 \) defines a component of the divisor, and \( y_i = 0 \) means a component of the desingularized curve in the coordinate \( (x_i, y_i) \). Because the leaves
of the Milnor fibration are defined by $x_i^{m_i}y_i^{k_i} = c, \ c \in \mathbb{C}$ around the singularity, the foliation is given by the following equation (1).

$$\frac{dx_i}{dt} = kx_i, \quad \frac{dy_i}{dt} = -m_iy_i \quad (i = 1, \ldots, n).$$

Since the norm defined by the ordinary one is equal to the norm of the vector field $Z_f$ which induces the Milnor fibration defined by $f = 0$, it is defined by $\|Z\|_1 = \|Z_f\|_1 = \sum (\delta_j - 2)|m_1l_{1j} + \ldots + m_nl_{nj}|$ (see [3] and [10]). The multiplicity $m_i (i = 1, \ldots, n)$ is defined by the sum of the winding number of each connected component of intersections between the Seifert surface of the multi-link $L$ around the $i$-th component of $L$. Here, the link $L$ is represented by the above plumbing diagram of the resolution of the singularity of the curve defined by $f = 0$. Also the number of the connected component of the above intersections in the boundary of the tubular neighborhood of the $i$-th component of $L$ is defined by $d_i (i = 1, \ldots, n)$. So we obtain the multiplicity of Definition 1.1 to be $(m_1/d_1, \ldots, m_n/d_n)$. Thus we get our results.

**Corollary 3.2.** Suppose that the minimal plumbing diagram of a vector field is as follows:

$$\begin{align*}
&\text{(m_1) \quad (m_2) \quad (m_3) \quad (m_i) \quad (m_n)} \\
&\text{k_1 \quad k_2 \quad k_3 \quad k_i \quad k_n}
\end{align*}$$

Figure 3.2

Here $m_i$ $(i = 1, \ldots, n)$ means a multiplicity along the divisor of the irreducible component of the desingularized curve as in Proposition 3.1, and $k_i$ $(i = 1, \ldots, n)$ means a multiplicity along the component of the divisor corresponding to the vertex which is an initial vertex of the arrowhead line of which arrowhead is weighted by the integer $m_i$ $(i = 1, \ldots, n)$. We assume that $\text{G.C.D.}(m_i, k_i) = 1$ $(i = 1, \ldots, n)$. Then the above two Thurston norms coincide.

**Proof.** When we compute the Thurston norm of a given graph link by using its splice diagram, we calculate the norm of each splice component of the diagram and sum up the norms of all splice components by using Theorem in [3]. So it is enough to check the coincidence of two norms of the link represented by the one splice component (Seifert component). The Seifert component in a three sphere is represented by the diagram in Proposition 3.1. Thus from the assumptions such that $\text{G.C.D}(m_i, k_i) = 1$ $(i = 1, \ldots, l)$, we get our result.
If the separatrix of $Z$ is defined by an irreducible curve $f = 0$ near the origin of $\mathbb{C}^2$, the splice diagram of the curve $f = 0$ represents a graph knot.

**Proposition 3.3.** Suppose that a separatrix of $Z$ is an irreducible curve, and the splice diagram does not represent a trivial knot. Then the vector field $Z$ is a generalized curve, and its splice diagram has only one arrowhead vertex. And the splice diagram of the irreducible curve also has one arrowhead vertex. In this case, the two norms defined in Proposition 3.1 coincide: i.e., $\|Z\|_1 = \|Z\|_2$, if and only if the ordinary multiplicity of the splice diagram of the curve is defined by an integer 1.

Moreover, $\mu(Z) = \|Z\|_2 + 1$, where $\mu(Z)$ means the Milnor number of $Z$.

**Proof.** Let $(z_1, z_2)$ be a local chart around the singularity $(0,0) \in \mathbb{C}^2$ of $Z$. Since the splice diagram of $Z$ does not represent a trivial knot, $Z$ is neither topologically equivalent to $\lambda_1 z_1 \partial/\partial z_1 + g(z_1, z_2) \partial/\partial z_2$ nor to $(\lambda_1 z_1 + a z_2^2) \partial/\partial z_1 + (\lambda_2 z_2) \partial/\partial z_2$ ($\lambda_1 = n \lambda_2$ and $a \neq 0$), where the function $g(z_1, z_2)$ satisfies

$$g(0,0) = \frac{\partial g}{\partial z_1}(0,0) = \frac{\partial g}{\partial z_2}(0,0) = 0.$$ 

So we can exclude all vector fields of which separatrices are irreducible, but not generalized curves.

Now we discuss the final resolution picture by desingularizing a vector field $Z$. The separatrix of $Z$ is an irreducible curve and so it has only one arrowhead. The index of the singularity corresponding to the intersection of the separatrix and a component of the divisor of the final resolution picture is a rational number. In fact, we see that the index is non-zero by using the same arguments as in the proof of theorem in [2]. Hence the singularity is a non-zero simple singularity. It ensures that the vector field $Z$ is a generalized curve.

Moreover, we have that the index is a rational number, since it is defined by a finite continued fraction. Also this index agrees with the index of the Hamiltonian vector field $Z_f$. Also the splice diagram of $Z$ coincides with one of $Z_f$. Here $Z_f$ is defined by

$$\frac{dx_1}{dt} = \frac{\partial f}{dx_2}, \quad \frac{dx_2}{dt} = -\frac{\partial f}{dx_1},$$

and $f(x_1, x_2) = 0$ defines an irreducible curve which is the separatrix of $Z$. The splice diagram of the curve $f = 0$ agrees with one of $Z_f$ (see [10]). So the splice diagram of the curve $f = 0$ has only one arrowhead vertex. Also we see that the ordinary multiplicity of this arrowhead agrees with the ordinary multiplicity of the diagram of $Z_f$ (see [10]).
Consider a neighborhood $N$ of the knot defined by the above arrowhead of the splice diagram of $Z_f$. Suppose that the ordinary multiplicity of the diagram of $Z_f$ is defined by an integer $m$. The intersection between the Seifert surface of the multi-knot derived from the separatrix of $Z_f$ and the neighborhood $N$ is null homologous by this Seifert surface. So it intersects $N$ by $m$ circles which are parallel to this knot. Thus the multiplicity of the arrowhead of the splice diagram of $Z$ is an integer 1 by using the multiplicity in Definition 1.1.

The ordinary multiplicity is equal to the multiplicity of Definition 1.1, if and only if $m = 1$. So $\|Z_f\|_1 = \|Z_f\|_2$. Since the index of $Z$ is equal to the index of $Z_f$, $\|Z\|_2 = \|Z_f\|_2 = \|Z\|_1 = \|Z\|_1$, if and only if $m = 1$. From the facts $\|Z_f\|_1 + 1 = \mu(Z_f)$ and $\mu(Z_f) = \mu(Z)$ (see [1]) we obtain $\|Z\|_2 + 1 = \mu(Z)$.

**Corollary 3.4.** Suppose that the multiplicity for the splice diagram of $Z$ in Proposition 3.3 is defined by Definition 1.1. Then it is equal to the integer 1.

It is clear from the proof of Proposition 3.3.

For example, the vector field $Z_1(x, y)$ is defined by the next differential equation:

$$\frac{dx}{dt} = -3y^2,$$
$$\frac{dy}{dt} = 2x.$$ 

This vector field $Z_1(x, y)$ is a Hamiltonian vector field and its splice diagram is given by the following.

This splice diagram represents a $(2, 3)$-torus knot. The ordinary multiplicity of the arrowhead is equal to the integer 1. On the other hand, suppose that the splice diagram of a vector field represents a $(a_1, a_2)$-torus knot. When the vector field is desingularized, the only one simple singularity appears in the plumbed manifold. And its index is equal to the rational number $-1/a_1a_2$. So the multiplicity in Definition 1.1 is defined by an integer 1. So $\|Z_1\|_1 = \|Z_1\|_2$. 

![Figure 3.3](image-url)
COROLLARY 3.5. Suppose that $Z$ is a Hamiltonian vector field and its separatrices are defined by the curve $f = 0$ (not necessary irreducible). Suppose that the splice diagram of $Z$ represents a link $(S^3, S_1 \cup S_2 \cup \cdots \cup S_n)$ and that the $i$-th component $S_i$ of this link has the ordinary multiplicity $m_i$ ($i = 1, \ldots, n$). If the following conditions (*) are satisfied, then $\|Z\|_2 = \mu(Z) - 1$.

\[ (*) \quad \text{G.C.D.}(m_i, H_i) = 1 \text{ for all } i \in \{1, \ldots, n\}. \]

Here $H_i$ denotes the integer $m_1 \text{link}(S_1, S_i) + m_2 \text{link}(S_2, S_i) + \cdots + m_{i-1} \text{link}(S_i, S_{i-1}) + m_{i+1} \text{link}(S_{i+1}, S_i) + \cdots + m_n \text{link}(S_n, S_i)$.

**Proof.** Let $N^i$ be a tubular neighborhood of the $i$-th component $S_i$ belonging to $(S^3, S_1 \cup \cdots \cup S_n)$. The Seifert surface of this multi-link intersects $N^i$ by the $d$-parallel $(m_i/d_i, -H_i/d_i)$-torus link, because the intersections in $N^i$ are homologous to $-\sum m_j S_j$ by the Seifert surface of this multi-link. By the condition (*), we see $d_i = 1$. Thus the intersection is a cable knot of $S_i$. Let $\tilde{Z}$ be a desingularized vector field of $Z$. From the above fact, the numerator of the index of the simple singularity of $\tilde{Z}$ is equal to the integer $m_i$, if the separatrix intersects the boundary of the plumbed manifold (a 3-sphere) by $S_i$. We remark that all indices for $\tilde{Z}$ are rational numbers, since $Z$ is a Hamiltonian vector field. Thus we see that the multiplicity $m_i$ in Definition 1.1 is equal to the ordinary multiplicity for all $i \in \{1, 2, \ldots, n\}$. So we obtain our result.

**Proposition 3.6.** The Thurston norm of the vector field $Z$ is a finer topological invariant than Milnor number of $Z$.

**Proof.** Take a vector field which has a multi-Hopf link with $n$ components as an intersection between its separatrix and a small $S^3$ around the origin (cf. Theorem 2.4). Its final resolution picture is defined by Figure (a).

```
(a)
```

Then its minimal plumbing diagram is given by Figure (b).
And its splice diagram is Figure (c).

Here $\langle \alpha_i \rangle$ ($i = 1, \ldots, n$) is the multiplicity of the $i$-th vertex, and its multiplicity is defined by a set of the integers $\langle \alpha_1, \ldots, \alpha_n \rangle$, where $\alpha_i \in \mathbb{C}$ ($i = 1, \ldots, n$) satisfy the condition

$$\alpha_1 + \cdots + \alpha_n = -1.$$ 

Put $\alpha_i = -(q_i/p_i)$, $p_i, q_i \in \mathbb{N}$, G.C.D.($p_i, q_i$) = 1 ($i = 1, \ldots, n$) satisfying the condition $q_1/p_1 + \cdots + q_n/p_n = 1$. Then we obtain the multiplicity $(q_1, \ldots, q_n)$. Thus we can find vector fields induced by the multi-Hopf link with $n$ components with multiplicity $(q_1, \ldots, q_n)$ by using Theorem 2.4 again. Here, we represent this multi-Hopf link by a splice diagram:

We can calculate the Thurston norm of these vector fields by using the formula in Proposition 3.1. The Thurston norms are given as follows.

$$(n - 2)|q_1 + \cdots + q_n| = (q_1 + \cdots + q_n)(n - 2).$$

Let all the integers $p_i$ ($i = 1, \ldots, n$) be the same prime number. It is clear that there exist countably many vector fields which have distinct norms. On the other
hand, we can calculate the Milnor number of these vector fields by using the first blow-up formula (see p.164 in [1] for the definition of it). It is necessary to know the algebraic multiplicity of $Z$ defined in [1] by using the next formula concerning the algebraic multiplicity.

Let $\nu_Z$ be the algebraic multiplicity of $Z$. Let $o_i$ ($i = 1, \ldots, n$) be the singularities of $Z$. Then we have

$$\nu_Z + 1 = \sum_{i=1}^{n} |\text{Ch}(P)|\phi(o_i, P),$$

where $\text{Ch}(P)$ means the Chern class of a component $P$ of a divisor (see [2] for the definition of the Chern class of a divisor), and $\phi(o_i, P)$ denotes a certain multiplicity defined by Camacho, A. Lins Neto and P. Sad (see p.159 in [1]). In this case we see $\text{Ch}(P) = -1$, and $\phi(o_i, P)$ are all 1, since $o_i$ ($i = 1, \ldots, n$) are simple singularities of $Z$. So we get $\nu_Z = n - 1$. By the first blow-up formula, we obtain that the Milnor numbers of these vector field which have the multi-Hopf links with $n$ components with multiplicity $(q_1, \ldots, q_n)$ are given by $(n - 1)^2$.

Fix a separatrices of vector field which induce a multi-Hopf link with $n$ components. Suppose it is defined by $f = 0$. From the above Proposition 3.6, there exist many multi-Hopf links with $n$ components with the multiplicity $(q_1, \ldots, q_n)$ which are realized by vector fields of which separatrices are defined by $f = 0$. The multi-Hopf link with multiplicity defined by the curve $f = 0$ is unique. Therefore, the above multi-Hopf links contain the ones which can not be realized by the curve $f = 0$. Here this multiplicity means the one in Definition 1.1.

There are some examples of a graph link realized by the separatrices of certain vector fields as in Figures 3.5 and 3.6.

The following link (Figure 3.7) is not realized by any holomorphic vector fields, since it does not satisfy the condition (2) of Proposition 2.13 and it is already not realized by any analytic curves.

Next, we show the existence of another criterion which can distinguish whether the given multi-graph links can be realized by a certain vector field or not.

**Lemma 3.7.** All minimal plumbing diagrams of any graph links in $S^3$ can be reduced to the empty diagram by several $(\pm 1)$-blow-down operations and 0-chain absorptions if all arrowhead lines are removed from the minimal plumbing diagram.

This lemma is included in theorem 6-1 in [8], so we omit the proof.

A criterion is described as follows.
\begin{equation*}
\begin{aligned}
\dot{x} &= x^4y \\
\dot{y} &= x^9 + y^4 + 2x^3y^2 
\end{aligned}
\end{equation*}

Figure 3.5
\[ \begin{align*}
\dot{x} &= -x^{11} - 6x^8y^2 + 8x^6y^3 - 5x^5y^4 - 8y^7 \\
\dot{y} &= 12x^{11} + 16x^7y^3 + 11x^{10}y - 12x^5y^4
\end{align*} \]

Figure 3.6
Figure 3.7
Proposition 3.8. Given an irreducible graph link $L$ we consider the reductive process such that a minimal plumbing diagram $P$ of $L$ changes to the empty diagram by removing all arrowhead lines from $P$. If all reductive operations include a $(+1)$-blow-down operation or a 0-chain absorption, the link $L$ cannot be realized by any vector fields.

Proof. Any graph link in $S^3$ can be reduced to the empty diagram, when all arrowhead lines are removed from the minimal plumbing diagram of a given graph link by Lemma 3.7. Every graph link induced by vector field has a minimal plumbing diagram which can be reduced by only several $(-1)$-blow-down operations when we remove all arrowhead lines from it. Therefore, we get our result.

If the minimal plumbing diagram does not reduce to the empty diagram, when all arrowhead lines are deleted, then the link $L$ is not realized in $S^3$ by Lemma 3.7.

We see this criterion is effective when a graph link is represented by a plumbing diagram at the start. For example, we see by this criterion that the plumbing diagram in Figure 3.8 (also see Figure 3.9) cannot be realized by any vector fields.

\[\begin{array}{c}
\begin{array}{c}
\uparrow \\
-1 \\
\downarrow
\end{array} & \begin{array}{c}
\uparrow \\
-1 \\
\downarrow
\end{array} & -2 \\
\end{array}\]

Figure 3.8
Figure 3.9
References