Logarithmic del Pezzo surfaces of rank one with unique singular points

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0. Introduction

Let k be an algebraically closed field of characteristic zero. Let $\bar{X}$ be a normal algebraic surface with a unique quotient singular point P . Let $\pi : X \rightarrow \bar{X}$ be a minimal resolution of $\bar{X}$ and let $D = \sum_{i=1}^n D_i$ be the reduced exceptional divisor with respect to π , where the D_i are irreducible components. Then there exists uniquely an effective $\mathbf{Q}$ -divisor $D^\# = \sum_{i=1}^n \alpha_i D_i$ such that $D^\# + K_X$ is numerically equivalent to $\pi^*(K_{\bar{X}})$. Since P is a log terminal singular point (for the definition, see [3]), it follows that $0 \leq \alpha_i < 1$ for any i . Put $\text{Bk}(D) = D - D^\#$.

DEFINITION. The above pair (X, D) is *almost minimal* if, for every irreducible curve C , either $(D^\# + K_X \cdot C) \geq 0$ or the intersection matrix of $C + \text{Bk}(D)$ is not negative definite. The singular point P is *almost minimal* in $\bar{X}$ if the pair (X, D) is almost minimal.

Starting with arbitrary quotient singular points, we can construct almost minimal quotient singular points, though the singularities might be changed from the original ones.

In the present article, we assume that a pair (X, D) is almost minimal and the logarithmic Kodaira dimension $\bar{\kappa}(X - D) = -\infty$. Since $K_{\bar{X}}$ is then not numerically effective, there exists an extremal rational curve $\bar{\ell}$ on $\bar{X}$. Let ℓ be the proper transform of $\bar{\ell}$ on X . By Miyanishi-Tsunoda [7, Lemma 2.7], one of the following two cases then takes place:

(A) The intersection matrix of $\ell + \text{Bk}(D)$ is negative semi-definite, but not negative definite. Furthermore, $(\bar{\ell}^2) = 0$.

(B) The Picard number $\rho(\bar{X})$ equals 1, and $-K_{\bar{X}}$ is ample. Namely, (X, D) is a logarithmic del Pezzo surface of rank one with contractible boundary (henceforth

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S_a ($a \geq 0$) in Figure 2 stands for one of the linear chains plus the (-1) -curve E given as in Figure 4, where $m_1 \geq 3$ and $m_i \geq 1$ for $2 \leq i \leq a$.

(Type D)

Figure 2

(Type E_6)

Figure 3

In the case (B), if P is a rational double or triple singular point, then such pairs have been classified completely (see [9] and [12]). But for arbitrary quotient singularities, such pairs are not yet classified.

In the present article, we attempt to classify all the log del Pezzo surfaces of rank one with unique quotient singular points. Our main result is the following:

MAIN THEOREM. *Let $\bar{X}$ be a log del Pezzo surface of rank one with a unique singular point, say P , and let (X, D) be the minimal resolution of $\bar{X}$. Then the weighted dual graph or the configuration of D are listed in Appendices A, B and C. Furthermore, all the listed cases are realizable.*

We prove Main Theorem by using Zhang's results on log del Pezzo surfaces of rank one (cf. [11] and [12]) and analyzing singular fibers of $\mathbf{P}^1$ -fibrations. First of all, we prove in Section 1 certain results on log del Pezzo surfaces and $\mathbf{P}^1$ -fibrations. In Section 2, we classify the pairs (X, D) when P is a quotient singular point of type E_6 , E_7 or E_8 (Theorem 2.1). In Section 3, we classify the pairs (X, D) when P is a quotient singular point of type D (Theorem 3.1). Finally, in Section 4, the case where P is a cyclic quotient singular point is treated (Theorem 4.1).

Figure 4

TERMINOLOGY. A $(-n)$ -curve is a nonsingular rational curve with self intersection number $-n$. For the definitions of rods, twigs and forks, we refer to [7]. A reduced effective divisor D is called an NC (resp. SNC) divisor if D has only normal (resp. simple normal) crossings. We employ the following notation:

K_X (or simply, K): the canonical divisor on X .

$\bar{\kappa}(X - D)$: the logarithmic Kodaira dimension of an open surface $X - D$.

$\rho(X)$: the Picard number of X .

$\mathbf{F}_n$ ($n \geq 0$): a Hirzebruch surface of degree n .

M_n ($n \geq 0$): the minimal section of $\mathbf{F}_n$.

$D^\# := D - \text{Bk}(D)$.

$\#D$: the number of all irreducible components in $\text{Supp}(D)$.

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1. Preliminary results

In this section, we shall state some results in [11] and [12] and prove some results which will be used in the next sections.

Let (X, D) be the minimal resolution of a log del Pezzo surface $\overline{X}$ of rank one with a unique singular point P . Then $\rho(X) = \#D + 1$. We have the following:

LEMMA 1.1 (cf. [12, Lemma 1.1]). *Let (X, D) be as above. Then we have:*

- (1) $f^*(K_{\overline{X}}) \equiv D^\# + K_X$ and $-(D^\# + K_X)$ is nef and big. Moreover, for any irreducible curve F , $-(D^\# + K_X \cdot F) = 0$ if and only if F is a component of D .
- (2) Any $(-n)$ -curve with $n \geq 2$ is a component of D .
- (3) X is a rational surface.

LEMMA 1.2 (cf. [11, Lemma 1.4]). *There is no (-1) -curve E such that, after contracting E and consecutively (smoothly) contractible curves in $E + D$, the divisor $E + D$ becomes an admissible rational rod or fork, where the adjective “admissible” means that each irreducible component of the image of $E + D$ has self intersection number ≤ -2 .*

By Lemma 1.1 (1), one can find an irreducible curve C such that $-(C \cdot D^\# + K_X)$ attains the smallest positive value.

LEMMA 1.3. *Suppose that $|C + D + K_X| \neq \emptyset$ and P is not a rational double point. Then D is an admissible rod, i.e., P is a cyclic quotient singular point.*

PROOF. By [11, Lemma 2.1], there exists a unique decomposition of D as a sum of effective integral divisors $D = D' + D''$ such that:

- (i) $(C \cdot D_i) = (D'' \cdot D_i) = (K_X \cdot D_i) = 0$ for any component D_i of D' .
- (ii) $C + D'' + K_X \sim 0$.

If $D'' = 0$ then $D = D'$, where D' consists of (-2) -curves. This contradicts the hypothesis. Hence, $D'' \neq 0$ and $D = D''$ because $\text{Supp}(D)$ is connected. By [12, Lemma 1.7(2)] D is an admissible rod. Q.E.D.

Suppose that $|C + D + K_X| \neq \emptyset$ and $D'' = 0$ in the proof of Lemma 1.3. Such a pair (X, D) is then completely determined. Namely, we have the following result.

LEMMA 1.4 (cf. [9, Lemma 3] and [12, Lemma 5.1]). *With the notation as above, suppose that $|C + D + K_X| \neq \emptyset$ and $D'' = 0$. Then P is a rational double*

point and the dual graph of D is one of the following Dynkin graphs:

$$A_1, A_4, D_5, E_6, A_7, E_7, D_8, E_8, A_8.$$

Moreover, the above (X, D) is obtained from the Hirzebruch surface $\mathbf{F}_2$ of degree 2 by a sequence of blowing-ups. The configuration or the weighted dual graph of D is given as one of the cases in Appendices A, B and C.

Suppose that $|C + D + K_X| = \emptyset$. The divisor $C + D$ is then an SNC-divisor, consisting of nonsingular rational curves and the dual graph of $C + D$ is a tree (cf. [6, Lemma 2.1.3]). Since $\text{Supp}(D)$ is connected, $C \cap D = \emptyset$ or C meets only one irreducible component of D transversally. More precisely, we have the following:

LEMMA 1.5 (cf. [11, Lemma 2.2] and [2, Proposition 3.6]). *Suppose that $|C + D + K_X| = \emptyset$. Then (X, D) is $(\mathbf{F}_n, M_n)$, where $n = -(D^2) \geq 2$, or we may assume that C is a (-1) -curve.*

PROPOSITION 1.6. *Suppose that $|C + D + K_X| = \emptyset$ and D is an admissible rational fork. Let D_0 be the unique irreducible component of D such that $(D_0 \cdot D - D_0) = 3$. Then C does not meet D_0 .*

PROOF. Since D is a fork, C is a (-1) -curve by Lemma 1.5. Suppose that $(C \cdot D_0) = 1$. Then D_0 is a (-2) -curve by Lemma 1.2. Let $\mu : X \rightarrow \tilde{X}$ be the contraction of C and let $\tilde{D} = \mu(D)$. By an argument similar to [11, Lemma 6.4], we know that $(\tilde{X}, \tilde{D})$ is a log del Pezzo surface of rank one with non-contractible boundary (for the definition, see Miyanishi-Tsunoda [8]). By Lemma 2.6 and Theorems 4 and 6 in [8], $\tilde{D}$ consists of a fork and an admissible rational fork which are disjoint from each other. This is however a contradiction because $\text{Supp}(D)$ (and hence $\text{Supp}(\tilde{D})$) is connected. Q.E.D.

In the rest of this section we prove some lemmas concerning $\mathbf{P}^1$ -fibrations which we need later.

LEMMA 1.7 (cf. [12, Lemma 1.5(1)]). *Let $\Phi : X \rightarrow \mathbf{P}^1$ be a $\mathbf{P}^1$ -fibration on a nonsingular projective rational surface X . Suppose that there are two cross-sections H_1 and H_2 of Φ such that $(H_i^2) \leq -2$ for $i = 1, 2$. Let $\mu : X \rightarrow \mathbf{F}_m$ be the contraction of all (-1) -curves and consecutively contractible curves in singular fibers so that $(\mu_*(H_1)^2) = -m = (H_1^2)$. Then*

$$(\mu_*(H_2)^2) = m + 2(H_1 \cdot H_2).$$

PROOF. By the assumption on μ , it follows that H_1 meets none of the curves contracted by μ . Hence we have

$$(\mu_*(H_1) \cdot \mu_*(H_2)) = (H_1 \cdot H_2)$$

and

$$\mu_*(H_2) \sim \mu_*(H_1) + (m + (H_1 \cdot H_2))\ell,$$

where ℓ is a general fiber of the $\mathbf{P}^1$ -fibration $\Phi \circ \mu^{-1} : \mathbf{F}_m \rightarrow \mathbf{P}^1$. Hence

$$(\mu_*(H_2))^2 = (\mu_*(H_1))^2 + 2m + 2(H_1 \cdot H_2) = m + 2(H_1 \cdot H_2).$$

Q.E.D.

LEMMA 1.8. *Let $\mu : \mathbf{F}_n \rightarrow \mathbf{P}^1$ ($n \geq 0$) be the ruling on $\mathbf{F}_n$ and let M be an m -section ($m \geq 1$) of μ different from M_n . Then $(M^2) \geq nm^2$.*

PROOF. We may assume that $n \geq 1$. Since M is an m -section there exists an integer α such that $M \sim mM_n + \alpha\ell$, where ℓ is a general fiber of the ruling. Note that $\alpha \geq nm$ because M is irreducible and $M \neq M_n$. Hence

$$(M^2) = -nm^2 + 2\alpha m \geq nm^2.$$

Q.E.D.

2. The case of type E_n

Let (X, D) be the same pair as in Main Theorem. In the present section, we consider the case where P is a quotient singular point of type E_6 , E_7 or E_8 . The main result is stated as follows:

THEOREM 2.1. *Let $\bar{X}$ be a log del Pezzo surface of rank one with a unique singular point P and let (X, D) be the minimal resolution of $\bar{X}$. Suppose that P is a quotient singular point of type E_6 , E_7 or E_8 . Then the following assertions hold:*

(1) *If P is a rational double point, then there exists a (-1) -curve E such that the configuration of $E + D$ is one of (2), (4) and (9) in Appendix A.*

(2) *If P is not a rational double point, then the configuration of $C + D$ with the curve C as in the previous section is one of those except for (2), (4) and (9) in Appendix A.*

(3) *There exist a $\mathbf{P}^1$ -fibration $\Phi : X \rightarrow \mathbf{P}^1$ and a component H of D such that H is a cross-section of Φ and the other components of $E + D$ (if P is a rational double point) or $C + D$ (if P is not a rational double point) are contained in a unique singular fiber of Φ .*

(4) *All the cases listed in Appendix A are realizable.*

PROOF. The case where P is a rational double point follows from [9, Lemma 3]. Hence we treat the case where P is not a rational double point.

There are 15 types of Dynkin graphs for the quotient singularities of types E_6 , E_7 and E_8 (cf. [1], [5]). Our proof is to verify the assertions of the above theorem by checking these cases separately. Since the reasoning is more or less the same for all cases, we exhibit the idea of the proof in the case where the configuration of D is given as in Figure 5, where D_5 is a (-3) -curve, D_0 is a $(-a_0)$ -curve with $a_0 \geq 2$ and the other curves are (-2) -curves. The type of the quotient singular point P is E_8 .

Figure 5

As in the section one, C is an irreducible curve such that $-(C \cdot D^\# + K)$ attains the smallest positive value. Since P is not a rational double point, we may assume, by Lemmas 1.3 and 1.5, that $|C + D + K| = \emptyset$ and C is a (-1) -curve. It follows that $(C \cdot D) = 1$. Lemma 1.2 and Proposition 1.6 imply that C meets only one of the D_i ($1 \leq i \leq 4$). We consider the following four cases separately.

Case 1: $(C \cdot D_1) = 1$. By Lemma 1.2 and since $a_0 \geq 2$, we conclude $a_0 = 2$. Hence P is a rational triple singular point. By [12, Main Theorem] this is, however, not the case. For the sake of the reader's convenience, we shall reproduce the proof.

Let $S_0 = 2(C + D_0 + D_1) + D_2 + D_4$ and let $\Phi : X \rightarrow \mathbf{P}^1$ be the $\mathbf{P}^1$ -fibration defined by the linear pencil $|S_0|$. Then D_3 and D_5 are cross-sections of Φ . Since $\rho(X) = \#D + 1 = 7$ and $\#S_0 - 1 = 4$, there exists another singular fiber $S_1 = F_1 + F_2$, where F_1 and F_2 are (-1) -curves. By the count of $\rho(X)$, we know that Φ has no singular fibers other than S_0 and S_1 . Since (X, D) is almost minimal, the (-1) -curves F_1 and F_2 have to meet the cross-sections D_3 and D_5 . We may assume that $(F_1 \cdot D_3) = (F_2 \cdot D_5) = 1$. Then $(F_2 \cdot D) = (F_2 \cdot D_5) = 1$. Since $(D_5^2) = -3$, $F_2 + D$ can be contracted to a quotient singular point. This contradicts Lemma 1.2. Thus the above case cannot occur.

Case 2: $(C \cdot D_2) = 1$. By Lemma 1.2, $a_0 = 2$ or 3 . If $a_0 = 2$ then P is a rational triple singular point. By [12, Main Theorem] this is not the case. Hence $a_0 = 3$. Let $S_0 = 4(C + D_2) + 2(D_3 + D_0) + D_1 + D_4$ and let $\Phi : X \rightarrow \mathbf{P}^1$ be the $\mathbf{P}^1$ -fibration defined by the linear pencil $|S_0|$. Then D_5 is a cross-section of Φ and the fibration Φ has no singular fibers other than S_0 , since $\#S_0 - 1 = 5$ and

$\rho(X) = 7$. The configuration of $C + D$ is then given as (7) in Appendix A.

Case 3: $(C \cdot D_3) = 1$. By Lemma 1.2, $a_0 = 2$. Let $S_0 = 2(C + D_0 + D_2 + D_3) + D_1 + D_4$ and let $\Phi : X \rightarrow \mathbf{P}^1$ be the $\mathbf{P}^1$ -fibration defined by the linear pencil $|S_0|$. Then D_5 is a cross-section of Φ and S_0 is a unique singular fiber of Φ . The configuration of $C + D$ is then given as (8) in Appendix A (see also Appendix in [12]).

Case 4: $(C \cdot D_4) = 1$. By Lemma 1.2, $a_0 = 2$. Then P is a rational triple singular point, and this is not the case by [12, Main theorem].

The above argument shows that if D is as in Figure 5 then $C + D$ has the configuration (7) or (8) in Appendix A. We can construct the configurations (7) and (8) starting from the Hirzebruch surface $\mathbf{F}_3$ of degree 3.

3. The case of type D

Let (X, D) be an almost minimal pair as in Main Theorem. In this section we assume that the quotient singular point P has type D . The weighted dual graph of D is given as in Figure 6, where $a_i \geq 2$ for $i = 0, 3, \dots, r$.

Figure 6

Note that $\rho(X) = r + 2$ and $\#D = r + 1$. Our main result of this section is the following:

THEOREM 3.1. *Let $\overline{X}$ be a log del Pezzo surface of rank one with a unique singular point P and let (X, D) be the minimal resolution of $\overline{X}$. Suppose that P is a quotient singular point of type D and that the weighted dual graph of D is given as in Figure 6. Then we have the following:*

- (1) *There exists a (-1) -curve E such that $(E \cdot D) = (E \cdot D_t) = 1$, where $t = 1$ or 2 . The weighted dual graph of $E + D$ is given as in Appendix B.*
- (2) *There exists a $\mathbf{P}^1$ -fibration $\Phi : X \rightarrow \mathbf{P}^1$ such that D_4 is a cross-section of Φ and the other components of D are contained in at most two singular fibers of Φ . In particular, the D_i ($0 \leq i \leq 3$) are (-2) -curves.*
- (3) *All of the types of singularities listed in Appendix B are realizable.*

In what follows, we shall prove the above theorem. The case where P is a rational double point follows from [9, Lemma 3]. Hence we treat the case where P is not a rational double point. Let C be an irreducible curve on X such that $-(C \cdot D^\# + K)$ attains the smallest positive value. By Lemmas 1.3 and 1.5, we may assume that C is a (-1) -curve and that $|C + D + K_X| = \emptyset$. Then C meets D only in one component, say D_s ($0 \leq s \leq r$). By Proposition 1.6, $s \neq 0$, i.e., $s \geq 1$.

First, we consider the case $s = 1$ (similarly, $s = 2$). We then have the following:

LEMMA 3.2. *Suppose that $s = 1$. Then $r \geq 4$, $a_0 = a_3 = 2$ and $S_0 := 2(C + D_0 + D_1) + D_2 + D_3$ defines a $\mathbf{P}^1$ -fibration $\Phi : X \rightarrow \mathbf{P}^1$ such that Φ has at most two singular fibers. The weighted dual graph of $C + D$ is given as in Appendix B, where E is to be read as C .*

PROOF. By Lemma 1.2, $a_0 = 2$. Similarly, $a_3 = 2$. Hence $S_0 := 2(C + D_0 + D_1) + D_2 + D_3$ gives rise to a $\mathbf{P}^1$ -fibration $\Phi : X \rightarrow \mathbf{P}^1$. Since $\rho(X) = r + 2$ we know that $r \geq 4$ and D_4 is a cross-section of Φ .

Suppose that $r \geq 5$. Let S_1 be the singular fiber of Φ containing D_i ($5 \leq i \leq r$). Then,

$$\rho(X) = r + 2 \geq 2 + (\#S_0 - 1) + (\#S_1 - 1) = \#S_1 + 5.$$

Further, $\#S_1 \geq r - 3$ because S_1 contains $D_5, \dots, D_r$ and some (-1) -curves. Hence there exists a unique (-1) -curve E_1 in $\text{Supp}(S_1)$ and Φ has no singular fibers other than S_0 and S_1 . Put $E := C$. The weighted dual graph of $E + D$ is then given as in Appendix B. Q.E.D.

By Lemma 3.2, we may assume, in the subsequent arguments, that $s \geq 3$. Since $(C \cdot D^\# + K_X) < 0$ and (X, D) is almost minimal, the intersection matrix of $C + D$ is not negative definite. Furthermore, since $\rho(X) = \#D + 1 = r + 2$, the intersection matrix of $C + D$ is not negative semi-definite, either. Hence there exist an integer $e > 0$ and an effective divisor Δ_0 with $\text{Supp}(\Delta_0) \subset \text{Supp}(D)$ such that $|eC + \Delta_0|$ defines a $\mathbf{P}^1$ -fibration $\Phi : X \rightarrow \mathbf{P}^1$. Put $S_0 = eC + \Delta_0$ and put $\Delta = (\Delta_0)_{\text{red}}$. Hereafter in the present section, we retain this situation.

LEMMA 3.3. *Suppose that Δ is an admissible rational fork, i.e., $\Delta = D_0 + D_1 + D_2 + D_3 + \dots + D_i$ for some $3 \leq i \leq r$. Then we have:*

- (1) $i = r - 1$ and D_r is a 2-section of Φ . Furthermore, Φ has no singular fibers other than S_0 .
- (2) There exists a (-1) -curve E such that $(E \cdot D) = (E \cdot D_t) = 1$, where $t = 1$ or 2. Hence the weighted dual graph of $E + D$ is given as in Appendix B.

PROOF. (1) Since $\rho(X) = r + 2$, it follows that $i < r$. The weighted dual graph of $C + \Delta$ looks like that of Figure 2, where the subgraph denoted by the

encircled S_a ($a \geq 0$) is given as in Figure 4. As seen from Figure 2 the multiplicity of D_i in S_0 is equal to two. Hence D_{i+1} is a 2-section of Φ . Suppose that $i \leq r-2$. Let S_1 be a singular fiber of Φ containing $D_{i+2}, \dots, D_r$. By the count of $\rho(X)$, we know that S_1 contains a unique (-1) -curve E_1 and Φ has no singular fibers other than S_0 and S_1 . The weighted dual graph of $\text{Supp}(S_1)$ is then the one in Figure 1. Since D_{i+2} and D_r are terminal components of $\text{Supp}(S_1)$, the multiplicity of D_{i+2} in S_1 is equal to one. Then $(E_1 \cdot D_{i+1}) = 1$. Since the multiplicity of E_1 in S_1 is then equal to one, there exists another (-1) -curve in S_1 . This is a contradiction. Hence $i = r-1$. By the count of $\rho(X)$, Φ then has no singular fibers other than S_0 .

(2) Let $f : X \rightarrow \mathbf{F}_m$ be a sequence of contractions of (-1) -curves in a unique singular fiber S_0 of Φ . Then $m = 0$ or 1 . Indeed, if $m \geq 2$ then the proper transform $f'(M_m)$ of M_m is a nonsingular rational curve with $(f'(M_m))^2 \leq -2$ and $f'(M_m)$ is not contained in $\text{Supp}(D)$. This contradicts Lemma 1.1 (2).

In order to prove the assertion, we consider two cases $m = 1$ and $m = 0$ separately.

Case: $m = 1$. Put $\ell = f_*(S_0)$. Then there exists a unique fundamental point Q of f on ℓ . Let E' be the proper transform of M_1 on X . Since $f_*(D_r) \sim 2M_1 + \alpha\ell$ with $\alpha \geq 2$ we have $(f_*(D_r))^2 = -4 + 4\alpha \geq 4$. Furthermore, by the above contractions, $f_*(D_r)$ remains as a nonsingular rational curve. Since

$$-2 = (f_*(D_r))^2 + (f_*(D_r) \cdot K_{\mathbf{F}_m}) = 2(\alpha - 3),$$

we have $\alpha = 2$ and $(f_*(D_r) \cdot M_1) = 0$. In particular, $Q \notin M_1$. Hence we know that E' is a (-1) -curve with $(E' \cdot D) = (E' \cdot D_t) = 1$, where $t = 1$ or 2 .

Case: $m = 0$. Let ℓ and Q be the same as in the case $m = 1$. Put $\overline{D}_r := f_*(D_r)$. Since $\overline{D}_r \sim 2M_0 + \alpha\ell$ with $\alpha > 0$, we have $(\overline{D}_r \cdot M_0) = \alpha > 0$. Since $\overline{D}_r$ is a nonsingular rational curve and since

$$-2 = (\overline{D}_r \cdot \overline{D}_r + K_{\mathbf{F}_0}) = 2(\alpha - 2),$$

we have $\alpha = 1$. Let ℓ' be a fiber of the second $\mathbf{P}^1$ -fibration on $\mathbf{F}_0 = \mathbf{P}^1 \times \mathbf{P}^1$ through Q . The proper transform $f'(\ell')$ of ℓ' is then a (-1) -curve satisfying

$$(f'(\ell') \cdot D) = (f'(\ell') \cdot D_t) = 1,$$

where $t = 1$ or 2 .

This completes the proof of Lemma 3.3.

In view of Lemma 3.3, we next consider the case where Δ is a linear chain. The weighted dual graph of $C + \Delta$ is given as in Figure 1, where E is to be read as C .

PROPOSITION 3.4. *With the notation and assumptions as above, let D_{i_0} and D_{i_1} be the terminal components of the linear chain Δ , where $i_0 < i_1$. Then we have:*

- (1) $i_0 = 5$ and $i_1 = r$.
- (2) *There exists a (-1) -curve E ($\neq C$) such that $(E \cdot D) = (E \cdot D_t) = 1$, where $t = 1$ or 2 . The weighted dual graph of $E + D$ is given as in Appendix B.*

The proof of Proposition 3.4 consists of Lemmas 3.5 $\sim$ 3.9.

LEMMA 3.5. *With the notation as in Proposition 3.4, we have $i_0 \geq 3$. Namely, Δ does not contain D_0 .*

PROOF. Suppose to the contrary that Δ contains D_0 . We may assume that Δ does not contain D_2 . Indeed, suppose that Δ contains D_1 and D_2 . Since Δ is a linear chain, then $\Delta = D_0 + D_1 + D_2$ and $(C \cdot D) = (C \cdot \Delta) = (C \cdot D_0) = 1$. This contradicts Proposition 1.6. Our proof differs depending on whether Δ contains D_1 or not.

(I) Suppose that Δ contains D_1 . The dual graph of Δ looks like that of Figure 1 and the multiplicity m of D_0 in S_0 is larger than 1. Then D_2 is an m -section ($m \geq 2$) of Φ . Furthermore, $m = 2$. Indeed, in the course of contracting C and successively contractible curves in the fiber $S_0 = eC + \Delta_0$, when the image of D_0 becomes a (-1) -curve, the fiber consists of one (-1) -curve and two (-2) -curves, one of which is the image of D_1 . Hence the multiplicity of the (-1) -curve, which is equal to m , must be 2. We consider the following three cases separately.

Case (I-1): $i_1 = r$. Since $\rho(X) = r + 2$ and $\#S_0 = r + 1$, the $\mathbf{P}^1$ -fibration Φ has no singular fibers other than S_0 . Let $f : X \rightarrow \mathbf{F}_n$ be a sequence of contractions of (-1) -curves and subsequently contractible curves in $\text{Supp}(S_0)$. We call such a sequence of contractions a *successive contraction* of (-1) -curves. We may and shall assume that $f_*(S_0) = f_*(D_1)$, since the multiplicity of D_1 in S_0 is equal to one. By Lemma 1.1 (2), $n \leq 1$. Put $\overline{D}_2 = f_*(D_2)$. By the construction of f , $\overline{D}_2$ is a nonsingular rational curve with $(\overline{D}_2^2) = 0$. Hence $n = 0$ by Lemma 1.8.

Write $\overline{D}_2 \sim 2M_0 + \alpha l$ with $\alpha \geq 0$. Since $(\overline{D}_2^2) = 0$, it follows that $\alpha = 0$. Hence $\overline{D}_2 \sim 2M_0$. This contradicts the irreducibility of the divisor $\overline{D}_2$.

Case (I-2): $i_1 = r - 1$. In this case, D_r is a cross-section of Φ . Since $\rho(X) = r + 2$ and $\#S_0 = r$, Φ has another singular fiber S_1 which consists of two (-1) -curves, say E_0 and E_1 , and has no singular fibers other than S_0 and S_1 . We may assume that $(E_0 \cdot D_r) = 1$.

Claim 1. $(E_0 \cdot D_2) = (E_1 \cdot D_2) = 1$.

PROOF. We have $(E_0 \cdot D_2) = 1$ or 0 . Indeed, if $(E_0 \cdot D_2) = 2$ then $(E_1 \cdot D) = 0$. This contradicts Lemma 1.2. Suppose that $(E_0 \cdot D_2) = 0$. Then $(E_1 \cdot D_2) = 2$ because

$S_1 = E_0 + E_1$ and D_2 is a 2-section of Φ . Let $g : X \rightarrow \mathbf{F}_n$ be a successive contractions of (-1) -curves in S_0 and S_1 such that $g_*(S_0) = g_*(D_{r-1})$ and $g_*(S_1) = g_*(E_1)$. Then,

$$n = -(D_r^2) - 1 = a_r - 1 \geq 1.$$

Put $\overline{D}_2 := g_*(D_2)$. Then $\overline{D}_2$ is a nonsingular rational curve with $(\overline{D}_2^2) = 0$. This contradicts Lemma 1.8. Hence $(E_0 \cdot D_2) = (E_1 \cdot D_2) = 1$.

Let $f : X \rightarrow \mathbf{F}_n$ be a successive contractions of (-1) -curves in S_0 and S_1 such that $f_*(S_0) = f_*(D_{r-1})$ and $f_*(S_1) = f_*(E_0)$. Then $(f_*(D_2)^2) = 1$. This also contradicts Lemma 1.8 because $n = -(D_r^2) = -a_r \geq 2$.

Case (I-3): $i_1 \leq r - 2$. D_{i_1+1} is a cross-section of Φ . Let S_1 be a singular fiber of Φ which contains $D_{i_1+2}, \dots, D_r$. Now we prove the following assertion.

Claim 2. S_1 contains two (-1) -curves, say E_0 and E_1 .

PROOF. S_1 contains at most two (-1) -curves because $\rho(X) = r + 2$, $\#S_0 = i_1 + 1$ and $\#S_1 \geq r - i_1$. Suppose that S_1 contains only one (-1) -curve, say E . The weighted dual graph of $\text{Supp}(S_1)$ then looks like that of Figure 1. By the count of $\rho(X)$, there exists the third singular fiber $S_2 = E_0 + E_1$, where E_0 and E_1 are (-1) -curves. Note that S_0 , S_1 and S_2 exhaust all singular fibers of Φ . Since $(D_2 \cdot S_0) = 2$, the multiplicity of E in S_1 is accordingly equal to 2. For, if the multiplicity is 1 there should exist another (-1) -curve in S_1 . This implies that $i_1 = r - 4$, $S_1 = 2(E + D_{r-1}) + D_{r-2} + D_r$ and $a_{r-2} = a_{r-1} = a_r = 2$ (cf. Figure 1).

Now we may assume that $(E_0 \cdot D_{r-3}) = 1$. Then $(E_0 \cdot D_2) \leq 1$. Indeed, if $(E_0 \cdot D_2) = 2$ then $(E_1 \cdot D) = 0$, a contradiction. Let $f : X \rightarrow \mathbf{F}_n$ be a successive contraction of (-1) -curves in S_0 , S_1 and S_2 such that $f_*(S_0) = f_*(D_{r-4})$, $f_*(S_1) = f_*(D_{r-2})$ and $f_*(S_2) = f_*(E_1)$. Then $n = -(D_{r-3}^2) - 1 \geq 1$ and $M_n = f_*(D_{r-3})$. $\overline{D}_2 = f_*(D_2)$ is a nonsingular rational curve and

$$(\overline{D}_2^2) = (D_2^2) + 2 + 3 + \{(E_0 \cdot D_2)\}^2 \leq 4.$$

Hence, by Lemma 1.8, we know that $n = 1$, $(E_0 \cdot D_2) = 1$ and $(E_1 \cdot D_2) = 1$. The weighted dual graph of all singular fibers of Φ and D_2 and D_{r-3} looks like that of Figure 7.

Let $T_0 = E + D_2 + E_0$. The complete linear system $|T_0|$ defines a $\mathbf{P}^1$ -fibration $\Psi : X \rightarrow \mathbf{P}^1$ for which D_0 , D_{r-3} and D_{r-1} are cross-sections and E_1 is a 2-section. Let T_1 be a singular fiber containing $D_3, \dots, D_{r-4}$ and C and let T_2 be a fiber containing D_1 . Then T_1 (resp. T_2) contains D_r (resp. D_{r-2}). Indeed, suppose that T_1 does not contain D_r . By Lemma 1.1 (2), there exists a (-1) -curve in $\text{Supp}(T_1)$ which meets D_{r-1} . Hence $\#T_1 \geq r - 4$. Furthermore, D_r and D_{r-2} cannot belong

Figure 7

to the same fiber of Ψ . This implies that there is the fourth singular fiber T_3 which contains at least 3 components. Then,

$$\begin{aligned} r &\geq (\#T_0 - 1) + (\#T_1 - 1) + (\#T_2 - 1) + (\#T_3 - 1) \\ &\geq 2 + r - 5 + 2 + 2 \\ &\geq r + 1. \end{aligned}$$

This is a contradiction. Thus T_1 contains D_r . Next suppose that T_2 does not contain D_{r-2} . Then Ψ has the fourth singular fiber T_3 containing D_{r-2} . Since $\#T_2 \geq 3$ and $\#T_3 \geq 3$, we have

$$\begin{aligned} r &\geq (\#T_0 - 1) + (\#T_1 - 1) + (\#T_2 - 1) + (\#T_3 - 1) \\ &\geq r + 2. \end{aligned}$$

This is a contradiction. By the count of $\rho(X)$, all singular fibers of Ψ are exhausted by T_0 , T_1 and T_2 . Furthermore, we know that T_1 has two (-1) -curves and T_2 a unique (-1) -curve. Let $\bar{C}$ be a (-1) -curve in $\text{Supp}(T_1)$ other than C and let E_2 be a (-1) -curve in $\text{Supp}(T_2)$. Then $T_2 = D_1 + 2E_2 + D_{r-2}$.

We have $(\bar{C} \cdot D_r) = (\bar{C} \cdot D_3 + D_{r-4}) = 1$. This is because $C + D_3 + \cdots + D_{r-4}$ (the middle part of the older fiber S_0) is smoothly contractible to a point and the (-1) -curve $\bar{C}$ must link two disjoint parts $C + D_3 + \cdots + D_{r-4}$ and D_r of the fiber T_1 . Suppose that $(\bar{C} \cdot D_{r-4}) = 1$. Then there exists a successive contractions of (-1) -curves in T_0 , T_1 and T_2 , say $g : X \rightarrow \mathbf{F}_n$, such that $g_*(T_0) = g_*(E_0)$, $g_*(T_1) = g_*(D_{r-4})$ and $g_*(T_2) = g_*(D_{r-2})$. Then $n = -(g_*(D_{r-3})^2) = -a_{r-3} \geq 2$, $(g_*(D_{r-1})^2) = -2 + 3 = 1$. This contradicts Lemma 1.8. If $(\bar{C} \cdot D_3) = 1$ then we have a contradiction similarly.

This completes the proof of Claim 2.

Let E_0 and E_1 be new (-1) -curves contained in S_1 . Then Φ has exactly two singular fibers S_0 and S_1 . We may assume that $(E_0 \cdot D_2) > 0$. Since D_2 is a 2-section of Φ , the multiplicity m_0 of E_0 in S_1 is equal to one or two and $(E_0 \cdot D_2) = 1$ or 2. Now we consider the following three subcases separately:

Subcase (I-3-1): $m_0 = 2$. Then $(E_0 \cdot D_2) = 1$. Let $\mu : X \rightarrow X'$ be a successive contraction of (-1) -curves in S_1 such that $\mu_*(E_0)$ is a (-1) -curve and $\mu_*(E_0)$ is a unique (-1) -curve of $\mu_*(S_1)$. Since $\text{Supp}((S_1)_{\text{red}} - E_0 - E_1)$ is a linear chain, the weighted dual graph of $\mu_*(S_1)$ looks as in Figure 1. So, since $m_0 = 2$ one can construct a successive contraction of (-1) -curves in S_0 and S_1 , say $f : X \rightarrow \mathbf{F}_n$, such that $f_*(D_2)$ is a nonsingular rational curve with $(f_*(D_2))^2 = 3$. Hence $n = 0$ by Lemma 1.8. Then $\overline{D}_2 := f_*(D_2) \sim 2M_0 + \alpha\ell$ with $\alpha > 0$. On the other hand, we have $3 = (\overline{D}_2^2) = 4\alpha$. This is a contradiction.

Subcase (I-3-2): $m_0 = 1$ and $(E_0 \cdot D_2) = 2$. Let $f : X \rightarrow \mathbf{F}_n$ be a successive contraction of (-1) -curves in S_0 and S_1 such that $f_*(S_0) = f_*(D_1)$ and $f_*(S_1) = f_*(E_0)$. Then $\overline{D}_2 := f_*(D_2)$ is a nonsingular rational curve with $(\overline{D}_2^2) = 0$. Hence $n = 0$ by Lemma 1.8. Then $\overline{D}_2 \sim 2M_0 + \alpha\ell$ with $\alpha \geq 0$. Since $(\overline{D}_2^2) = 0$, it follows that $\alpha = 0$. This is a contradiction.

Subcase (I-3-3): $m_0 = 1$ and $(E_0 \cdot D_2) = 1$. It then follows that $(E_1 \cdot D_2) = 1$ and the multiplicity of E_1 in S_1 is equal to one. Since $\text{Supp}(S_1 - E_0 - E_1) = D_{i_1+2} + \cdots + D_r$ is a linear chain, the weighted dual graph of S_1 is given as in Figure 8, where we assume that $(E_0 \cdot D_{i_1+2}) = (E_1 \cdot D_r) = 1$.

Figure 8

Then $S_1 = E_0 + D_{i_1+2} + \cdots + D_r + E_1$. Let $f : X \rightarrow \mathbf{F}_n$ be a successive contraction of (-1) -curves in S_0 and S_1 such that $f_*(S_0) = f_*(D_{i_1})$ and $f_*(S_1) = f_*(E_0)$. Then $n = -(D_{i_1+1}^2) - 1 \geq 1$ and $f_*(D_{i_1+1}) = M_n$. Furthermore, $\overline{D}_2 := f_*(D_2)$ is a nonsingular rational curve with $(\overline{D}_2^2) = r - i_1 \geq 1$. $\overline{D}_2 \sim 2M_n + \alpha\ell$ with $\alpha \geq 2n$, where ℓ is a fiber of the ruling on $\mathbf{F}_n$. Then, by the construction of f , $(\overline{D}_2 \cdot M_n) = 1$ and hence $\alpha = 2n + 1$. This implies that $p_a(\overline{D}_2) = n \geq 1$, where $p_a(\overline{D}_2)$ is the arithmetic genus of $\overline{D}_2$. This is a contradiction.

(II) We continue the proof of Lemma 3.5. From the arguments in (I), we know that Δ does not contain D_1 . Since the multiplicity of D_0 in S_0 is then equal to one (with the notation before Lemma 3.3), D_1 and D_2 are cross-sections of Φ . Put $\Delta = D_0 + D_3 + \cdots + D_{i_1}$ ($i_1 \geq 4$). As in the case (I), we consider the following

three cases separately.

Case (II-1): $i_1 = r$. Since $\rho(X) = r + 2$ and $\#S_0 = r$, there exists a singular fiber $S_1 = E_0 + E_1$ of Φ , where E_0 and E_1 are (-1) -curves. Then Φ has no singular fibers other than S_0 and S_1 . Assume that $(E_0 \cdot D_1) \geq 1$. Then, by Lemma 1.2, $(E_0 \cdot D_1) = (E_1 \cdot D_2) = 1$ and $(E_0 \cdot D_2) = (E_1 \cdot D_1) = 0$. Let $f : X \rightarrow \mathbf{F}_n$ be a successive contraction of (-1) -curves in S_0 and S_1 such that $f_*(S_0) = f_*(D_0)$ and $f_*(S_1) = f_*(E_0)$. Then $f_*(D_1)$ and $f_*(D_2)$ have negative self-intersection numbers. This is a contradiction.

Case (II-2): $i_1 = r - 1$. Then D_r becomes a cross-section of Φ . By Lemma 1.1 (2) and by the count of $\rho(X)$, there exist two other singular fibers $S_1 = E_0 + E_1$ and $S_2 = F_0 + F_1$ where E_0, E_1, F_0 and F_1 are (-1) -curves. Assume that $(E_0 \cdot D_1) = (F_0 \cdot D_1) = 1$. Let $f : X \rightarrow \mathbf{F}_n$ be a successive contraction of (-1) -curves in S_0, S_1 and S_2 such that $f_*(S_0) = f_*(D_0)$, $f_*(S_1) = f_*(E_0)$ and $f_*(S_2) = f_*(F_0)$. Then $n = 2$ and $M_2 = f_*(D_1)$. Meanwhile, $\overline{D}_2 = f_*(D_2)$ is a cross-section of the ruling $\Phi \circ f^{-1} : \mathbf{F}_2 \rightarrow \mathbf{P}^1$ with $(\overline{D}_2^2) \leq 0$. This contradicts Lemma 1.8.

Case (II-3): $i_1 \leq r - 2$. Then D_{i_1+1} is a cross-section of Φ . Let S_1 be a singular fiber of Φ containing $D_{i_1+2}, \dots, D_r$. By the count of $\rho(X)$, S_1 contains at most three (-1) -curves. Since D_1 is a cross-section of Φ , there exists a (-1) -curve, say E_1 , in S_1 such that E_1 meets D_1 . Since the multiplicity of E_1 in S_1 is then equal to one, S_1 has two or three (-1) -curves. We consider the following two subcases separately:

Subcase (II-3-1). Assume that S_1 contains three (-1) -curves E_1, E_2 and E_3 in S_1 . Then Φ has no other singular fibers. We claim that $(E_1 \cdot D_2) = 0$. Indeed, suppose to the contrary that $(E_1 \cdot D_2) = 1$. Let $f : X \rightarrow \mathbf{F}_n$ be a successive contraction of (-1) -curves in S_0 and S_1 such that $f_*(S_0) = f_*(D_0)$ and $f_*(S_1) = f_*(E_1)$. Then $f_*(D_1)$ and $f_*(D_2)$ are (-2) -curves on a Hirzebruch surface. This is a contradiction. We may assume that $(E_2 \cdot D_2) = 1$.

Let $g : X \rightarrow X'$ be a successive contraction of (-1) -curves in S_1 such that $S'_1 := g_*(S_1)$ contains only two (-1) -curves $g_*(E_i)$ ($i = 1, 2$). Since $\text{Supp}((S_1)_{\text{red}} - E_1 - E_2 - E_3)$ is a linear chain, the weighted dual graph of S'_1 looks like that of Figure 8. Then $g_*(D_{i_1+1})$ meets a terminal component of $\text{Supp}((S'_1)_{\text{red}} - g_*(E_1 + E_2))$, say G . We may assume that $(g_*(E_1) \cdot G) = 1$. Let $h : X' \rightarrow \mathbf{F}_n$ be a successive contraction of (-1) -curves in $S'_0 := g_*(S_0)$ and S'_1 such that $h_*(S'_0) = (h \circ g)_*(D_0)$ and $h_*(S'_1) = (h \circ g)_*(E_1)$. Put $\overline{D}_j := (h \circ g)_*(D_j)$ ($j = 1, 2$ and $i_1 + 1$). Then $(\overline{D}_1^2) = -2$ and $\overline{D}_1 = M_2$. Furthermore, $(\overline{D}_1 \cdot \overline{D}_2) = (\overline{D}_1 \cdot \overline{D}_{i_1+1}) = 0$ and $(\overline{D}_2 \cdot \overline{D}_{i_1+1}) = 1$ by the constructions of g and h . Then $\overline{D}_2 \sim \overline{D}_{i_1+1}$ and $(\overline{D}_2^2) = 2$. This is a contradiction.

Subcase (II-3-2). Assume that S_1 contains only two (-1) -curves E_1 and E_2 and that $(E_1 \cdot D_1) = 1$. Then there exists another singular fiber $S_2 = F_1 + F_2$,

where F_1 and F_2 are (-1) -curves. By an argument similar to Subcase (II-3-1), it follows that $(E_1 \cdot D_2) = 0$, $(E_2 \cdot D_2) = 1$ and the weighted dual graph of S_1 looks like that of Figure 8. Hence we may assume that the configuration of S_1 is given as in Figure 9, where $D_{i_1+2}, D_{i_1+3}, \dots, D_r$ are (-2) -curves.

Figure 9

Now assume that $(F_1 \cdot D_{i_1+1}) = 1$. Let $f : X \rightarrow \mathbf{F}_n$ be a successive contraction of (-1) -curves in S_0 , S_1 and S_2 such that $f_*(S_0) = f_*(D_0)$, $f_*(S_1) = f_*(E_1)$ and $f_*(S_2) = f_*(F_1)$. Put $\bar{D}_j = f_*(D_j)$ ($j = 1, 2$ and $i_1 + 1$). Since $(\bar{D}_1^2) \leq -1$, we have $n = -(\bar{D}_1^2) \geq 1$. On the other hand, by the construction of f , $(\bar{D}_{i_1+1}^2) = -a_{i_1+1} + 2 \leq 0$. This contradicts Lemma 1.8.

This completes the proof of Lemma 3.5.

LEMMA 3.6. Δ does not contain D_3 . Namely, $i_0 \geq 4$.

PROOF. Suppose that Δ contains D_3 . Put $\Delta = D_3 + D_4 + \dots + D_{i_1}$. Then D_0 and D_{i_1+1} (if they exist) are cross-sections of Φ . Let S_j be a singular fiber of Φ containing D_j ($j = 1, 2$). Then $S_1 \neq S_2$ because D_0 is a cross-section of Φ . Suppose that $i_1 \geq r - 1$. By Lemma 1.1 (2), S_i ($i = 1, 2$) consists of one (-2) -curve D_i and two (-1) -curves. Then

$$\rho(X) = r + 2 \geq 2 + (\#S_0 - 1) + (\#S_1 - 1) + (\#S_2 - 1) \geq r + 3.$$

This is a contradiction. Hence $i_1 \leq r - 2$.

Let S_3 be a singular fiber of Φ containing $D_{i_1+2}, \dots, D_r$. By the count of $\rho(X)$, S_3 coincides with either S_1 or S_2 . Assume that $S_1 = S_3$. By Lemma 1.1 (2), $S_2 = F_1 + D_2 + F_2$, where F_1 and F_2 are (-1) -curves. By the count of $\rho(X)$, all singular fibers of Φ are exhausted by S_0 , S_1 and S_2 . Furthermore, S_1 has a unique (-1) -curve, say E_1 , and $(E_1 \cdot D_1) = 1$. We claim the following assertion.

Claim. $(E_1 \cdot D_r) = 1$. Furthermore, $i_1 = r - 2$ and $a_r = -2$.

PROOF. This is clear because the (-2) -curve D_1 has multiplicity 1 in S_1 .

Let $f : X \rightarrow \mathbf{F}_n$ be a successive contraction of (-1) -curves in S_0, S_1 and S_2 such that $f_*(S_0) = f_*(D_3)$, $f_*(S_1) = f_*(D_1)$ and $f_*(S_2) = f_*(D_2)$. Then $n = -a_0 \geq 2$ and $M_n = f_*(D_0)$. On the other hand,

$$(f_*(D_{r-1})^2) = -a_{r-1} + 3 \leq 1$$

because of the construction of f . This contradicts Lemma 1.8.

Q.E.D.

We prove the following lemma which will be necessary later.

LEMMA 3.7. *Let F be a singular fiber of a $\mathbf{P}^1$ -fibration $\Phi : X \rightarrow B$. Let $E_1, \dots, E_s$ be all (-1) -curves in $\text{Supp}(F)$. Suppose that the following conditions are satisfied:*

- (i) $s \leq 2$.
- (ii) $\text{Supp}((F)_{\text{red}} - \sum_{i=1}^s E_i)$ consists of two connected components A and B .
- (iii) B has a component B_0 whose multiplicity in F is equal to one.
- (iv) The dual graph of A is given as in Figure 10, where $(A_1^2) = (A_2^2) = -2$ and $r \geq 3$. Moreover, the multiplicity of A_r in F is equal to one.

Figure 10

Then we have:

- (1) $s = 2$.
- (2) Suppose that $(E_1 \cdot A) = (E_1 \cdot B) = 1$. Then $(E_2 \cdot A) = (E_2 \cdot A_\ell)$, where $\ell = 1$ or 2 .

PROOF. Let E_1 be a (-1) -curve in $\text{Supp}(F)$ such that $(E_1 \cdot A) = (E_1 \cdot B) = 1$. Put $T := A_3 + \dots + A_r$.

(1) Suppose that $s = 1$. Since $(E_1 \cdot A) = 1$, we may assume that $(E_1 \cdot A_2) = 0$. Let $f : X \rightarrow Y$ be a successive contraction of (-1) -curves in F such that $f_*(A_3)$ is a (-1) -curve. Then $f_*(A_2)$ remains as a (-2) -curve.

Assume that $r = 3$. By the condition (iv), $f_*(F)$ has another (-1) -curve. In particular, $f_*(A_1 + B) \neq 0$. Hence $f_*(A_3)$ is not a terminal component of $\text{Supp } f_*(F)$. This is a contradiction. Assume that $r \geq 4$. By the condition of f , we

know that $f_*(A_1) \neq 0$ or $f_*(A_4 + \cdots + A_r) \neq 0$. Since there are no three components meeting a (-1) -curve, $\text{Supp } f_*(F)$ is a linear chain and $f_*(B) = 0$. On the other hand, by the condition (iii), the multiplicity of $f_*(A_3)$ in $f_*(F)$ is equal to one. This is a contradiction because $f_*(A_3)$ is not a terminal component of $\text{Supp } f_*(F)$. Hence we have $s = 2$.

(2) Let E_2 be another (-1) -curve in $\text{Supp}(F)$. Then $(E_2 \cdot A) = 0$ or $(E_2 \cdot B) = 0$, for otherwise F contains a loop. We prove the following:

Claim. $(E_2 \cdot A) = 1$.

PROOF. Suppose that $(E_2 \cdot B) = 1$. Let $f' : X \rightarrow Y'$ be a successive contraction of (-1) -curves in F such that $f'_*(E_1)$ is a (-1) -curve and that $f'_*(E_2 + B)$ contains no (-1) -curves. By the condition (ii), such a birational morphism f' exists. Then $\text{Supp } f'_*(F)$ contains a unique (-1) -curve $f'_*(E_1)$ and satisfies the conditions (i) $\sim$ (iv). This contradicts the assertion (1).

In order to prove (2), it suffices to show that $(E_2 \cdot T) = 0$. Suppose that $(E_2 \cdot T) = 1$. Let $\bar{f} : X \rightarrow \bar{Y}$ be a successive contraction of (-1) -curves in F such that $\bar{f}_*(E_1)$ is a (-1) -curve and that $\text{Supp } \bar{f}_*(T)$ contains no (-1) -curves. If $\bar{f}_*(A_3) = 0$, then we have a (partial) successive contraction of (-1) -curves in F , say $f_1 : X \rightarrow Y_1$, such that $f_{1*}(E_1)$ and $f_{1*}(A_3)$ are (-1) -curves. Since $f_{1*}(A_1)$ and $f_{1*}(A_2)$ are (-2) -curves, it follows that $f_{1*}(F) = f_{1*}(A_1 + A_2 + 2A_3)$. This is a contradiction. Hence $\bar{f}_*(A_3) \neq 0$. Then $\bar{f}_*(A_1)$ and $\bar{f}_*(A_2)$ are (-2) -curves. Thus $\bar{f}_*(F)$ contains a unique (-1) -curve $\bar{f}_*(E_1)$ and satisfies the conditions (i) $\sim$ (iv). This contradicts the assertion (1). Hence we may assume that $(E_2 \cdot A_2) = 0$.
Q.E.D.

LEMMA 3.8. Δ does not contain D_4 .

PROOF. Suppose that Δ contains D_4 . Then D_3 is a cross-section of Φ . Put $\Delta = D_4 + D_5 + \cdots + D_{i_1}$. Let S_1 be a singular fiber of Φ containing D_0 , D_1 and D_2 . We consider the following three cases separately.

Case 1: $i_1 = r$. By the count of $\rho(X)$, we know that all singular fibers of Φ are exhausted by S_0 and S_1 and that S_1 has a unique (-1) -curve, say E_1 . Since $E_1 + D_0 + D_1 + D_2$ can be contracted to a nonsingular rational curve with self-intersection number zero, it follows that $(E_1 \cdot D) = (E_1 \cdot D_0) = 1$, $a_0 = 2$, and $S_1 = 2(E_1 + D_0) + D_1 + D_2$. This is a contradiction because D_0 meets D_3 which is a cross-section of Φ .

Case 2: $i_1 = r - 1$. Since D_r is a cross-section of Φ , there exists a (-1) -curve, say E_1 , in S_1 such that $(E_1 \cdot D_r) = 1$. Then, by the count of $\rho(X)$, it follows that S_1 has two (-1) -curves and Φ has no singular fibers other than S_0 and S_1 . Let E_2 be another (-1) -curve in $\text{Supp}(S_1)$. Note that $(E_2 \cdot D) = (E_2 \cdot D_0 + D_1 + D_2) = 1$.

Suppose that E_2 meets D_0 . It then follows from Lemma 1.2 that $(D_0^2) = -2$. Then $S_1 = 2(E_2 + D_0) + D_1 + D_2$, and this contradicts the existence of E_1 . This implies that E_2 meets either D_1 or D_2 . We may assume $(E_2 \cdot D_2) = 1$. Then $(D_0^2) = -2$ by Lemma 1.2. This implies that $(E_1 \cdot D_1) = 1$.

Let $\mu : X \rightarrow \mathbf{F}_n$ be a successive contraction of (-1) -curves in S_0 and S_1 such that $\mu_*(S_0) = \mu_*(D_4)$ and $\mu_*(S_1) = \mu_*(D_0)$. Then $n = -a_3 \geq 2$ and $(\mu_*(D_r)^2) = -a_r + 3 \leq 1$, which contradict Lemma 1.8.

Case 3: $i_1 \leq r - 2$. Let S_2 be a singular fiber of Φ containing $D_{i_1+2}, \dots, D_r$. Since D_3 and D_{i_1+1} are cross-sections and by the count of $\rho(X)$, it follows that $S_1 = S_2$ and S_1 has at most two (-1) -curves. Hence S_1 satisfies the conditions (i) $\sim$ (iv) in Lemma 3.7 with $A := D_0 + D_1 + D_2$ and $B := D_{i_1+2} + D_{i_1+3} + \dots + D_r$. By Lemma 3.7 (1), there exist two (-1) -curves E_1 and E_2 in $\text{Supp}(S_0)$. Assume that $(E_1 \cdot A) = (E_1 \cdot B) = 1$. By Lemma 3.7 (2) we may assume that $(E_2 \cdot D_2) = 1$. Since $(E_2 \cdot D) = (E_2 \cdot A) = 1$ and by Lemma 1.2, it follows that $a_0 = 2$. This is a contradiction because the intersection matrix of $E_1 + E_2 + A$ is then not negative definite.

This completes the proof of Lemma 3.8.

Finally, we consider the case where $i_0 \geq 5$.

LEMMA 3.9. *Suppose that $i_0 \geq 5$. Then $i_0 = 5$, $i_1 = r$ and there exists a (-1) -curve E such that $(E \cdot D) = (E \cdot D_t) = 1$, where $t = 1$ or 2 . The weighted dual graph of $E + C + D$ is given in Appendix B, where E_1 is to be read as C .*

PROOF. The component D_{i_0-1} is a cross-section of Φ . Let S_1 be a singular fiber of Φ containing $D_0, \dots, D_{i_0-2}$. As in the proof of Lemma 3.8, we consider the following three cases separately.

Case 1: $i_1 = r$. By the count of $\rho(X)$, all singular fibers of Φ are then exhausted by S_0 and S_1 . Furthermore, S_1 has a unique (-1) -curve, say E . Suppose that $(E \cdot D_0 + D_3 + \dots + D_{i_0-2}) = 1$. Since $E + D_0 + D_1 + D_2 + D_3 + \dots + D_{i_0-2}$ can be contracted to a nonsingular rational curve with self-intersection number zero, the weighted dual graph of $\text{Supp}(S_1)$ is the one in Figure 2. This is a contradiction because D_{i_0-2} meets D_{i_0-1} , which is a cross-section of Φ . Hence $(E \cdot D_1) = 1$ or $(E \cdot D_2) = 1$. Assume that $(E \cdot D_1) = 1$. Since $(E \cdot D) = (E \cdot D_1) = 1$, it follows from Lemma 1.2 and the almost minimality of (X, D) that $a_0 = a_3 = 2$. Then $S_1 = 2(E + D_0 + D_1) + D_2 + D_3$ and hence $i_0 = 5$. The weighted dual graph of $E + C + D$ is then given in Appendix B, where E_1 is to be read as C .

Case 2: $i_1 = r - 1$. Since D_r is a cross-section of Φ , S_1 has a (-1) -curve, say E_1 , such that $(E_1 \cdot D_r) = 1$. Then S_1 has another (-1) -curve E_2 . By the count of $\rho(X)$, S_1 has exactly two (-1) -curves. Then $(E_2 \cdot D) = (E_2 \cdot D_0 + D_1 + \dots + D_{i_0-2}) = 1$. Suppose that $(E_2 \cdot D_1) = 1$. By the argument as in

Case 1, we know that $a_0 = a_3 = 2$ and $S_1 = 2(E_2 + D_0 + D_1) + D_2 + D_3$. This contradicts the existence of E_1 . Similarly, $(E_2 \cdot D_2) = 1$ is impossible. Hence $(E_2 \cdot D_0 + D_3 + \cdots + D_{i_0-2}) = 1$. Since the multiplicity of E_1 in S_1 is equal to one, we can find a birational morphism $f : X \rightarrow X'$ which is a successive contraction of (-1) -curves in S_1 starting with the contraction of E_2 such that $f_*(D_0)$ and $f_*(E_1)$ are (-1) -curves. By the condition on f , it follows that $f_*(D_1)$ and $f_*(D_2)$ are (-2) -curves. Hence $f_*(S_1) = f_*(2D_0 + D_1 + D_2)$, which is a contradiction.

Case 3: $i_1 \leq r - 2$. By the count of $\rho(X)$, S_1 contains at most two (-1) -curves and $\{D_{i_1+2}, D_{i_1+3}, \dots, D_r\}$. Put $A := D_0 + D_1 + D_2 + \cdots + D_{i_0-2}$ and $B := D_{i_1+2} + D_{i_1+3} + \cdots + D_r$. Then S_1 satisfies the conditions (i) $\sim$ (iv) in Lemma 3.7. By Lemma 3.7 (1), $\text{Supp}(S_1)$ has two (-1) -curves E_1 and E_2 . Assume that $(E_1 \cdot A) = (E_1 \cdot B) = 1$. By Lemma 3.7 (2), we may assume that $(E_2 \cdot D_1) = 1$. By arguing as in Case 2, we have a contradiction.

This completes the proof of Lemma 3.9.

Proposition 3.4 is thus proved. The assertion (3) of Theorem 3.1 is then clear. Theorem 3.1 is thus verified.

4. Cyclic quotient singular points

Let (X, D) be an almost minimal pair as in Main Theorem. In this section, we treat the case where P has a cyclic quotient singularity. The weighted dual graph of D is given in Figure 11, where $a_i \geq 2$ for $i = 1, 2, \dots, r$.

Figure 11

Our main result of this section is the following:

THEOREM 4.1. *Let $\bar{X}$ be a log del Pezzo surface of rank one with a unique singular point P and let (X, D) be the minimal resolution of $\bar{X}$. Suppose that P is a cyclic quotient singular point. Then we have the following:*

(1) *If $(X, D) \neq (\mathbf{F}_n, M_n)$ then the weighted dual graph of D is given by one of (a), (b) and (c) in Appendix C.*

(2) *If the weighted dual graph of D is given by one of (a) and (b) in Appendix C, then there exist a $\mathbf{P}^1$ -fibration $\Phi : X \rightarrow \mathbf{P}^1$ and an irreducible component H of D such that H is a cross-section of Φ and the other irreducible components of D are contained in singular fibers of Φ . Furthermore, Φ has at most two singular*

fibers.

(3) If the weighted dual graph of D is given by (c) in Appendix C, then there exists a $\mathbf{P}^1$ -fibration $\Psi : X \rightarrow \mathbf{P}^1$ and two irreducible components H_1 and H_2 such that H_1 and H_2 are cross-sections of Ψ and the other irreducible components of D are contained in singular fibers of Ψ . Furthermore, Ψ has exactly two singular fibers.

(4) All of the types of singularities listed in Appendix C are realizable.

In what follows, we prove the above theorem.

LEMMA 4.2. With the notation and assumptions as above, either Theorem 4.1 is true or there exists a (-1) -curve E such that $(E \cdot D) = 1$.

PROOF. Let C be an irreducible curve such that $-(C \cdot D^\# + K)$ attains the smallest positive value (cf. §1). If $|C + D + K| = \emptyset$ then, by Lemma 1.5, it follows that either C is a (-1) -curve with $(C \cdot D) = 1$ or $(X, D) = (\mathbf{F}_n, M_n)$, where $n = -(D^2) \geq 2$.

Suppose that $|C + D + K| \neq \emptyset$. By Lemma 1.3 and its proof, we know that either $C + D + K \sim 0$, or $C + K \sim 0$ and D is a rod of (-2) -curves. If the latter case takes place, the assertion follows from [9, Lemma 3]. Suppose that $C + D + K \sim 0$. Since $|C + K| = |-D| = \emptyset$, C is a nonsingular rational curve by [6, Lemma 2.1.3]. Furthermore, $(C \cdot D) = -(C \cdot C + K) = 2$ and $(D_i \cdot C + D - D_i) = -(D_i \cdot K + D_i) = 2$ for any irreducible component D_i of D .

If $(C \cdot D_i) = 2$ for some $1 \leq i \leq r$, then D is irreducible and the configuration of $C + D$ is one of the next two in Figure 12.

Figure 12

Since D is irreducible, $\rho(X) = 2$. Hence $(X, D) = (\mathbf{F}_n, M_n)$ ($n = -(D^2) \geq 2$). Let ℓ be a fiber of the ruling on $\mathbf{F}_n$. Then $C \sim M_n + (n+2)\ell$ because $C + D \sim -K_{\mathbf{F}_n}$. We then have

$$-(C \cdot D^\# + K_{\mathbf{F}_n}) = \left(M_n + (n+2)\ell \cdot \left(2 - \frac{n-2}{n} \right) M_n + (n+2)\ell \right)$$

$$\begin{aligned}
 &= n + 2 + \frac{2(n+2)}{n} \\
 &> 4
 \end{aligned}$$

because $n \geq 2$. On the other hand,

$$-(\ell \cdot D^\# + K_X) = 2 - \frac{n-2}{n} \leq 2.$$

This contradicts the hypothesis that $-(C \cdot D^\# + K)$ attains the smallest positive value. Hence D is reducible and $(C \cdot D_1) = (C \cdot D_r) = 1$.

Suppose that $C + D$ is not a divisor of normal crossings. Then $r = 2$ and the configuration of $C + D$ is given as in Figure 13.

Figure 13

Since C is a nonsingular rational curve, we have $(C^2) \geq -1$ by Lemma 1.2. If $(C^2) \geq 0$ then the Riemann-Roch theorem yields that

$$\dim |C| \geq (C^2) + 1 \geq 1.$$

This will also imply that $|C|$ is base point free. Hence there exists a member C' of $|C|$ such that C' is irreducible and $C' + D$ is an SNC-divisor. Suppose that C is a (-1) -curve. Since $K \sim -(C + D_1 + D_2)$, we have

$$(K^2) = (C + D_1 + D_2)^2 = (C^2) + (D_1^2) + (D_2^2) + 6 \leq 1.$$

Hence $\rho(X) = 10 - (K^2) \geq 9$. On the other hand, $\rho(X) = 3$ because $r = 2$ and $\rho(X) = r + 1$. This is a contradiction.

Hence we have the following:

LEMMA 4.3. *Suppose that $C + D + K \sim 0$ and P is not a rational double point. Then by taking a suitable member of $|C|$, we may assume that $C + D$ is an SNC-divisor.*

PROOF OF LEMMA 4.2 CONTINUED. Since D is reducible X is not relatively minimal. So the set Σ of all (-1) -curves on X is not empty. If E is in $\Sigma - \{C\}$

then

$$0 = (E \cdot C + D + K) = (E \cdot C) + (E \cdot D) - 1$$

and hence $(E \cdot D) = 1$ and $(E \cdot C) = 0$ by Lemma 1.2. Lemma 4.2 is true. So we may assume that $\Sigma = \{C\}$. Let Φ be any $\mathbf{P}^1$ -fibration on X . Then by Lemma 1.2 and the count of $\rho(X)$, Φ has exactly one singular fiber S_0 , $\text{Supp } S_0 = \text{Supp}(C + D - D_i)$ where $i = 1$ or r , and D_i is a cross-section. Clearly, Theorem 4.1 is true.

This completes the proof of Lemma 4.2.

If X is relatively minimal then $(X, D) = (\mathbf{F}_n, M_n)$ for some $n \geq 2$. So, in the subsequent arguments, we assume that X is not relatively minimal. Then we may assume that there exists a (-1) -curve E such that $(E \cdot D) = 1$ by Lemma 4.2. By the Hodge index theorem and the fact that $\rho(X) = \#(D) + 1$, the intersection matrix of $E + D$ is neither negative definite nor negative semi-definite. Hence there exist an effective divisor Δ_0 with $\text{Supp}(\Delta_0) \subset \text{Supp}(D)$ and an integer $e > 0$ such that $S_0 := eE + \Delta_0$ defines a $\mathbf{P}^1$ -fibration $\Phi : X \rightarrow \mathbf{P}^1$. Put $\Delta := (\Delta_0)_{\text{red}} = D_i + D_{i+1} + \cdots + D_j$ ($i < j$). The weighted dual graph of $C + D$ looks like that of Figure 1. Hereafter in the present section, we retain this situation.

Suppose that $i = 1$. Then $j < r$ and D_{j+1} is a cross-section of Φ . Assume that $i = 1$ and $j = r - 1$. Then $\#S_0 = r$. Since $\rho(X) = r + 1 = (\#S_0 - 1) + 2$, it follows that Φ has no singular fibers other than S_0 . The weighted dual graph of $E + D$ is then given by (a) in Appendix C. Assume that $i = 1$ and $j < r - 1$. Let S_1 be a singular fiber of Φ containing $D_{j+2}, \dots, D_r$. By the count of $\rho(X)$, S_1 has a unique (-1) -curve, say E_1 . Furthermore, Φ has no singular fibers other than S_0 and S_1 . The weighted dual graph of $E + D + E_1$ is then given by (b) in Appendix C.

Next suppose that $i > 1$. If $j = r$ then relabel the D_i in Δ in the reversed order. Then the argument in the case $i = 1$ is used to determine the weighted dual graph of $E + D$ (or $E + D + E_1$). Thus, in the case $i = 1$ or $j = 1$, the assertion (2) of Theorem 4.1 holds. So, we assume that $i > 1$ and $j < r$ in the subsequent arguments. Then the following assertion holds.

LEMMA 4.4. *With the notation and assumptions as above, we have $i > 2$ or $j < r - 1$.*

PROOF. Suppose that $i = 2$ and $j = r - 1$. Then D_1 and D_r are cross-sections of Φ . By the count of $\rho(X)$ and Lemma 1.1 (2), there exists another singular fiber $S_1 = F_0 + F_1$, where F_0 and F_1 are (-1) -curves. We may assume that $(F_1 \cdot D_1 + D_r) \leq 1$. Then we reach a contradiction to Lemma 1.2 or the fact that $\rho(X) = \#(D) + 1$. Q.E.D.

By virtue of Lemma 4.4, we may assume that $i > 2$ in the subsequent arguments. Let S_1 be a singular fiber of Φ containing $D_1, \dots, D_{i-2}$ and let S_2 be a

singular fiber of Φ containing $D_{j+2}, D_{j+3}, \dots, D_r$ if $j < r - 1$.

LEMMA 4.5. *With the notation and the assumptions as above, the following assertions hold:*

- (1) *If $j < r - 1$ then $S_1 = S_2$.*
- (2) *The $\mathbf{P}^1$ -fibration Φ has no singular fibers other than S_0 and S_1 .*
- (3) *S_1 has exactly two (-1) -curves.*

PROOF. (1) Suppose to the contrary that $S_1 \neq S_2$. Then S_1 (resp. S_2) has a (-1) -curve E_1 (resp. E_2) such that $(E_1 \cdot D_{j+1}) = 1$ (resp. $(E_2 \cdot D_{i-1}) = 1$). Since the multiplicity of E_i in S_i ($i = 1, 2$) is then equal to one, S_1 and S_2 have at least two (-1) -curves each. We have

$$(\#S_0 - 1) + (\#S_1 - 1) + (\#S_2 - 1) \geq r.$$

This contradicts $\rho(X) = r + 1$.

(2) Note that $(\#S_0 - 1) + (\#S_1 - 1) \geq r - 2$. The $\mathbf{P}^1$ -fibration Φ therefore has at most one singular fiber other than S_0 and S_1 . Suppose that Φ has another singular fiber, say S_2 . Lemma 1.1 (2) implies that the singular fiber S_2 consists of two (-1) -curves, say F_0 and F_1 . By Lemma 1.2, we may assume that $(F_0 \cdot D_{i-1}) = (F_1 \cdot D_{j+1}) = 1$ and $(F_1 \cdot D_{i-1}) = (F_0 \cdot D_{j+1}) = 0$. By the count of $\rho(X)$, S_1 has a unique (-1) -curve, say E_1 . Then there exists a birational morphism $f : X \rightarrow \mathbf{F}_n$ which is a successive contraction of (-1) -curves in S_0, S_1 and S_2 such that $f_*(S_0) = f_*(D_j)$, $f_*(S_1) = f_*(D_{j+2})$ and $f_*(S_2) = f_*(F_1)$. We know that $n = -a_{j+1} \geq 2$ and $f_*(D_{j+1}) = M_n$. On the other hand, since S_0 and S_1 have only one (-1) -curve each, $(f_*(D_{i-1}))^2 = (D_{i-1}^2) + 3 \leq 1$. This contradicts Lemma 1.8.

(3) By the assertion (2) and the count of $\rho(X)$, the assertion (3) is clear.

Q.E.D.

Put $A = D_1 + \dots + D_{i-2}$ ($\neq 0$) and $B = D_{j+2} + \dots + D_r$ if $j \leq r - 2$. Let E_1 be a (-1) -curve in $\text{Supp}(S_1)$ such that $(A \cdot E_1) = (E_1 \cdot D_r) = 1$ if $j = r - 1$ and $(A \cdot E_1) = (B \cdot E_1) = 1$ if $j \leq r - 2$. We prove the following:

LEMMA 4.6. *With the notation and assumptions as above, we have*

$$(E_1 \cdot D_1) = (E_1 \cdot D_r) = 1.$$

PROOF. First, we consider the case $j = r - 1$. The multiplicity of E_1 in S_1 is then equal to one because D_r is a cross-section of Φ . By Lemma 4.5 (3), we can find a birational morphism $f : X \rightarrow Y$ which is a successive contraction of (-1) -curves in S_1 such that $f_*(D_{i-2})$ and $f_*(E_1)$ are all (-1) -curves in $\text{Supp } f_*(S_1)$. Since the multiplicities of $f_*(D_{i-2})$ and $f_*(E_1)$ in $f_*(S_1)$ are equal to one and $\text{Supp } f_*((S_1)_{\text{red}} - D_{i-2} - E_1)$ is a linear chain, the weighted dual graph of $f_*(S_1)$

looks like that of Figure 8. By Lemma 4.5 (3), the number of the fundamental points of f^{-1} is equal to one. Let it be P' . Furthermore, we know that $P' \in f_*(D_{i-2})$ and $P' \notin f_*(E_1)$. So, by the condition on f , $(E_1 \cdot D_1) = 1$. In this case, the assertion is thus verified.

Next consider the case $j \leq r - 2$. Let E_2 be a (-1) -curve in S_1 other than E_1 . We consider the case $(E_2 \cdot A) = 1$. The case $(E_2 \cdot B) = 1$ can be treated in a similar way. Since $(E_1 \cdot A) = (E_1 \cdot B) = 1$ and $A, B \neq 0$, the multiplicity of E_1 in $S_1 > 1$. Let $h : X \rightarrow \bar{X}_1$ be a successive contraction of (-1) -curves in $E_2 + A$, which starts with the contraction of E_2 , such that $h_*(E_1)$ is a (-1) -curve and $h_*(E_2 + A)$ has no (-1) -curves. Since S_1 has two (-1) -curves and $(S_1)_{\text{red}} - E_1 - E_2$ has two connected components A and B , such a morphism h exists. Then $h_*(E_2 + A) \neq 0$. We prove the following:

Claim. $\text{Supp } h_*(S_1)$ is a linear chain.

Let A_k (resp. B_ℓ) be an irreducible component of $h_*(A)$ (resp. $h_*(B)$) which $h_*(E_1)$ meets. Suppose to the contrary that $\text{Supp } h_*(S_1)$ is not a linear chain. A_k is not a terminal component of $\text{Supp } h_*(A)$ or B_ℓ is not a terminal component of $\text{Supp } h_*(B)$. We may assume that the first case takes place. The latter case can be treated similarly.

Let $h_2 : \bar{X}_1 \rightarrow \bar{X}_2$ be a successive contraction of (-1) -curves in $h_*(E_1 + B)$ such that $h_{2*}(A_k)$ is a (-1) -curve. Then $\text{Supp}(h_2 \circ h)_*(S_1)$ is a linear chain, i.e., $(h_2 \circ h)_*(E_1 + B) = 0$. Furthermore, the multiplicity of $h_{2*}(A_k)$ in $(h_2 \circ h)_*(S_1)$ is then equal to one. This is a contradiction because $h_{2*}(A_k)$ is not a terminal component of $\text{Supp}(h_2 \circ h)_*(S_1)$ by the condition on h_2 .

Hence $\text{Supp } h_*(S_1)$ is a linear chain.

Since $\text{Supp } h_*(S_1)$ is a linear chain by Claim and contains a unique (-1) -curve $h_*(E_1)$, $h_*(S_1)$ contains exactly two multiplicity 1 components: they are end components of $\text{Supp } h_*(S_1)$. Hence we know that $(E_1 \cdot B) = (E_1 \cdot D_r) = 1$. In order to prove Lemma 4.6, it suffices to show that $(E_1 \cdot A) = (E_1 \cdot D_1) = 1$. Suppose that $(E_1 \cdot A) = (E_1 \cdot D_s) = 1$ for $s \geq 2$. Let t be an integer such that $(E_2 \cdot A) = (E_2 \cdot D_t) = 1$. If $s \leq t$ then the multiplicity of $h_*(D_s)$ in $h_*(S_1)$ is equal to one. This is a contradiction because $h_*(D_s)$ is not a terminal component of $\text{Supp } h_*(S_1)$. Hence $s > t$. The configuration of $S_1 + D_{i-1} + D_{j+1}$ is then given as in Figure 14. Put $\bar{D}_m := h_*(D_m)$ for $m = s, s+1, \dots, i-2, j+1, j+2, \dots, r$ and put $\bar{S}_i := h_*(S_i)$ for $i = 0, 1$. By Claim (2) we have $h_*(E_2 + A) = \sum_{m=k}^{i-2} \bar{D}_m$. Let $\tilde{h} : \bar{X}_1 \rightarrow \mathbf{F}_n$ be a successive contraction of (-1) -curves in $\bar{S}_0$ and $\bar{S}_1$ such that $\tilde{h}_*(\bar{S}_0) = (\tilde{h} \circ h)_*(D_j)$ and $\tilde{h}_*(\bar{S}_1) = \tilde{h}_*(\bar{D}_{j+2})$. It then follows that $n = -(D_{j+1}^2) \geq 2$, $M_n = (\tilde{h} \circ h)_*(D_{j+1})$ and $\tilde{h}_*(\bar{D}_{i-1})^2 = (D_{i-1}^2) + 2 \leq 0$. This contradicts Lemma 1.8.

Figure 14

This completes the proof of Lemma 4.6.

In what follows, we shall precisely describe the weighted dual graph of $\text{Supp}(S_1)$, when $i > 2$ and $j < r$.

Suppose that $B = \emptyset$, i.e., $j = r - 1$. Let $\mu : X \rightarrow X'$ be a successive contraction of (-1) -curves in $E_2 + A$ such that $\mu_*(D_{i-2})$ becomes a (-1) -curve. The configuration of $\text{Supp } \mu_*(S_1)$ is then given as in Figure 15, where m is the number of irreducible components of $\text{Supp } \mu_*(S_1)$. Note that, by the condition on μ , $(\mu_*(D_{i-1}))^2 = (D_{i-1}^2)$.

Figure 15

Let $\mu' : X' \rightarrow \mathbf{F}_n$ be a successive contraction of (-1) -curves in $\mu_*(S_0)$ and $\mu_*(S_1)$ such that $(\mu' \circ \mu)_*(S_0) = (\mu' \circ \mu)_*(D_j)$ and $(\mu' \circ \mu)_*(S_1) = (\mu' \circ \mu)_*(E_1)$. Then $n = a_{j+1} \geq 2$, $M_n = (\mu' \circ \mu)_*(D_{j+1})$ and $(\mu' \circ \mu)_*(D_{i-1})^2 = m - a_{i-1}$. Lemma 1.7 yields that

$$m - a_{i-1} = (\mu' \circ \mu)_*(D_{i-1})^2 = n = a_{j+1}.$$

The weighted dual graph of $\text{Supp}(S_1)$ is then given as in Figure 16, where for the definition of the subgraph denoted by the encircled $\tilde{T}_\gamma$ ($\gamma \geq 0$), see Appendix C.

Figure 16

Hence the weighted dual graph of $E + E_1 + E_2 + D$ is given by (c) in Appendix C.

Next suppose that $B \neq \emptyset$. Then there exists a successive contraction of (-1) -curves in S_1 , say $\mu : X \rightarrow \tilde{X}$, which starts with the contraction of E_1 , such that $\mu_*(E_2)$ and one of $\mu_*(D_{i-2})$ and $\mu_*(D_{j+2})$ are all (-1) -curves in $\text{Supp } \mu_*(S_1)$. Indeed, if such a morphism μ does not exist, then one can find a successive contraction of (-1) -curves in S_1 , say $f : X \rightarrow Y$, such that $f_*(D_{j+2}) \neq 0$, $f_*(D_{i-2}) \neq 0$ and $f_*(S_1)$ has a unique (-1) -curve $f_*(E_2)$. By Lemma 4.6, $\text{Supp}(f_*(S_1)_{\text{red}} - f_*(E_2))$ is a linear chain. There exists a birational morphism $g : Y \rightarrow \mathbf{F}_\ell$ which is a successive contraction of (-1) -curves in $f_*(S_0)$ and $f_*(S_1)$ such that $(g \circ f)_*(S_0) = (g \circ f)_*(D_j)$ and $(g \circ f)_*(S_1) = (g \circ f)_*(D_{j+2})$. It then follows that $\ell = a_{j+1} \geq 2$, $M_\ell = (g \circ f)_*(D_{j+1})$ and $(g \circ f)_*(D_{i-1})^2 = -a_{i-1} + 2 \leq 0$. This contradicts Lemma 1.8. So, by arguing as in the case $B = \emptyset$, the weighted dual graph of $\text{Supp}(S_1)$ is given as in Figure 17, where for the definition of the subgraphs denoted by the encircled T_β ($\beta \geq 1$) and $\tilde{T}_\gamma$ ($\gamma \geq 0$), see Appendix C.

Figure 17

Hence the weighted dual graph of $E + E_1 + E_2 + D$ is given by (c) in Appendix C.

Thus, the assertion (3) of Theorem 4.1 is verified. The assertion (4) of Theorem 4.1 is then clear. Theorem 4.1 is thus verified.

REMARK 4.7. Assume that the weighted dual graph of D is given by (c) in Appendix C. We can construct such a pair (X, D) starting from the Hirzebruch surface $\mathbf{F}_2$ of degree 2.

Appendix

A. The case of type E_n

In the following list of configurations, a solid line stands for a component of $\text{Supp}(D)$; the self-intersection number -2 of a (-2) -curve of D is omitted; a line with $*$ on it is not contained in any fiber of the vertical $\mathbf{P}^1$ -fibration $\Phi : X \rightarrow \mathbf{P}^1$.

(Type E_6)

(Type E_7)

(Type E_8)

Figure 18

B. The case of type D

Figure 19

In the above list of the weighted dual graphs, $m \geq 2$ and the subgraph denoted by the encircled A_a ($a \geq 1$) is given as in Figure 1, where $m_1 \geq 2$, $m_i \geq 1$ for $2 \leq i \leq a$ and E is to be read as E_1 .

C. Cyclic quotient singularities

Figure 20

Figure 21

We omit the case where $(X, D) = (\mathbf{F}_n, M_n)$. In the above list of the weighted dual graphs, the subgraph denoted by the encircled A_α ($\alpha \geq 1$) is given as in Figure 1, where $m_1 \geq 3$ and $m_i \geq 1$ for $2 \leq i \leq \alpha$, the subgraph denoted by the encircled T_β ($\beta \geq 0$) is given as in the above Figure 21, where $m'_1 \geq 3$ and $m'_i \geq 1$ for $2 \leq i \leq \beta$, and the subgraph denoted by the encircled $\tilde{T}_\gamma$ is given S_γ as in Figure 4, where m_i (resp. E) is to be read as $\tilde{m}_i$ (resp. E_2), $\tilde{m}_1 \geq 3$ and $\tilde{m}_i \geq 1$ for $2 \leq i \leq \gamma$.

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