Necessary conditions for the hyperbolicity of 2 × 2 systems

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1. Introduction

In this note we study the Cauchy problem for first order differential operators $L$ acting on $C^\infty(\Omega, \mathbb{C}^2)$, that is $2 \times 2$ systems, where $\Omega$ is an open set in $\mathbb{R}^{n+1}$. We are interested in necessary conditions in order that the Cauchy problem for $L$ be well posed, that is, necessary conditions for $L$ to be a hyperbolic system. The fact that the symmetrizable systems are always hyperbolic and that there exists a class of hyperbolic systems which are not symmetrizable causes several troubles when we try to understand the hyperbolicity in a unified way, including the symmetrizable systems.

Let $L$ be an $m \times m$ first order hyperbolic system acting on $C^\infty(\Omega, \mathbb{C}^m)$ and let the determinant $h$ of the principal part $L_1$ of $L$ vanish of order at least two at $z=(x, \xi)$. When the rank of $L_1(z)$ is $m-1$ some necessary conditions for $L$ to be hyperbolic are available (see [10], [2], [5] and the references given there). Although [Theorem 1.1, 10] and [Theorem 2.1, 2] give fairly precise necessary conditions for hyperbolicity if the rank of $L_1(z)$ is $m-1$, nothing is implied if the rank of $L_1(z)$ is less than $m-1$, in particular $L_1(z)=0$ for instance. So to make more detailed studies in this direction we measure the order of the zeros of $h$ and $L_1$ in an “anisotropic” way. In this note we prove that $L$, an invariantly defined matrix symbol associated to $L$, introduced in [7] and [17], must vanish in the same anisotropic order as $h$ vanishes modulo terms of the form $L_1Q$ with some meromorphic $Q$ whose anisotropic vanishing order is less than half of $h$’s (see Theorem 2.1 below). We also give a necessary condition on $L_1$ required for the Cauchy problem for $L + B$ to be well posed for every $B \in C^\infty(\Omega, M(2, \mathbb{C}))$, that is the condition for $L$ to be strongly hyperbolic. We prove that every entry of $L_1$ must vanish in the same anisotropic order as $h$ vanishes. This implies that, for any entry $l$, the second order scalar operator $h + l$ verifies the so called Ivrii-Petkov condition [Theorem 4.1, 5] (see Theorem 2.2 below in which more is involved).

To prove these results, we construct an asymptotic solution $U_\lambda$ to the Cauchy
problem for $L(\lambda)$, which results from $L$ by a dilation of local coordinates such as $x \mapsto (\lambda^{-p_0}x_0, \ldots, \lambda^{-p_n}x_n)$ with $p_i \in \mathbb{Q}^{+}$ which contradicts an a priori estimate derived from the well-posedness assumption for the Cauchy problem. Let $z = (0, e_n)$. We look for $U_\lambda$ in the form $(\omega L + C)(\lambda)V_\lambda$ where $V_\lambda$ to be constructed. The main feature here is that we admit, in general, meromorphic $C$ to make the coefficient of $D_n$ in $P = L(\omega L + C)$ to be diagonal. This simplifies very much our construction of an asymptotic solution although the singularities of $C$ at the origin yields positive powers to $\lambda$ in the resulting operator $P(\lambda)$. The main point is that, taking a suitable basis for $C^2$, we can choose such a $C$ so that we can keep the resulting positive powers of $\lambda$ under control.

2. Notations and statement of the result

Let $\Omega$ denote an open subset of $\mathbb{R}^{n+1}$, $0 \in \Omega$, and $x = (x_0, x') = (x_0, \ldots, x_n)$ the generic point of $\Omega$. Let $L(x, D)$ be a differential operator acting on $C^\infty(\Omega, \mathbb{C}^2)$. We consider the Cauchy problem for $L(x, D)$ where the initial data are assigned on surfaces $\{x_0 = \text{const.}\}$, which we shall assume to be non-characteristic for $L$:

$$\begin{cases}
L(x, D)u = f, & x \in \Omega \cap \{x_0 < \text{const.}\}, \\
u|_{x_0=\text{const.}} = g(x'), & x' \in \Omega \cap \{x_0 = \text{const.}\}
\end{cases}$$  

(2.1)

here $f \in \mathcal{D}'(\Omega)$, $g \in \mathcal{D}'(\Omega \cap \{x_0 = \text{const.}\})$. We recall that the Cauchy problem (2.1) is said to be well-posed in $\Omega \cap \{x_0 < t\} =: \Omega^t$ if

(a) for every $f \in C^\infty(\Omega, \mathbb{C}^2)$, there is a $u \in \mathcal{C}'(\Omega, \mathbb{C}^2)$ such that $L(x, D)u = f$ in $\Omega^t$

(b) for every $u \in \mathcal{C}'(\Omega, \mathbb{C}^2)$ such that $L(x, D)u = 0$ in $\Omega \cap \{x_0 < t\}$, we have $u = 0$ in $\Omega^t$.

Our purpose is to give necessary conditions for the hyperbolicity or the strong hyperbolicity of $L(x, D)$. We recall that the operator $L$ is said to be strongly hyperbolic near the origin if for all $B \in C^\infty(\Omega, \text{hom}(\mathbb{C}^2, \mathbb{C}^2))$ the Cauchy problem (2.1) is well-posed for $L(x, D) + B(x)$ in $\Omega^t$ for every small $t$.

Let $L_1(x, \xi)$ be the principal symbol of $L$, i.e. the part homogeneous of degree 1 with respect to $\xi \in \mathbb{R}^{n+1} = \mathbb{R}_{\xi_0} \times \mathbb{R}_\xi'$ and denote by $h(x, \xi) = \det L_1(x, \xi)$, then $h$ is a polynomial in $\xi$ of degree 2. It is well known that for the Cauchy problem (2.1) to be well posed $h(x, \xi) = 0$ has only real zeros in $\xi_0$ for every $(x, \xi') \in \Omega \times \mathbb{R}^n$, which will be assumed throughout the paper.

Since $L$ is a differential operator, we may write

$$L(x, D) = L_1(x, D) + L_0(x),$$
where $L_0 \in C^\infty(\Omega, \text{hom}(\mathbb{C}^2, \mathbb{C}^2))$ and

$$L_1(x, D) = \sum_{j=0}^{n} A^j(x)D_j.$$  

Here $A^0$ in (2.2) is non singular since the surfaces $\{x_0 = \text{const.}\}$ are non characteristic for $L$. In what follows we suppose that $A^0(x) = I_2$ so we can write

$$L_1(x, D) = D_0 I_2 - \sum_{j=1}^{n} A^j(x)D_j$$
$$= D_0 I_2 - A(x, D').$$

In order to state our result we need the following lemma and definitions.

**Lemma 2.1.** Let $\rho$ be a double point of $h$, i.e. $h(\rho) = 0$ and $dh(\rho) = 0$. Then, via a (real analytic) change of $x$ variables preserving the plane $\{x_0 = 0\}$, we can assume that

(i) $\rho = (0, e_n)$,
(ii) $\text{Tr} A(x, \xi') = 0$.

**Proof.** The statement given in (i) can be proved with standard arguments. We prove (ii). Consider the change of $x$ variables defined by

$$\begin{align*}
y_0 &= x_0, \\
y_j &= \varphi_j(x), & 1 \leq j \leq n,
\end{align*}$$

and the corresponding contragradient transform in the $\xi$ variables

$$\begin{align*}
\xi_0 &= \eta_0 + \sum_{\mu=1}^{n} \frac{\partial \varphi_\mu}{\partial x_0} \eta_\mu, \\
\xi_j &= \sum_{\mu=1}^{n} \frac{\partial \varphi_\mu}{\partial x_j} \eta_\mu, & 1 \leq j \leq n.
\end{align*}$$

So the transformed symbol becomes

$$L(y, \eta) = -\left(\eta_0 + \sum_{\mu=1}^{n} \frac{\partial \varphi_\mu}{\partial x_0} \eta_\mu\right) + \sum_{j=1}^{n} \sum_{\mu=1}^{n} A^j(y) \frac{\partial \varphi_\mu}{\partial x_j} \eta_\mu$$
$$= -\eta_0 + \sum_{\mu=1}^{n} \left\{ -\frac{\partial \varphi_\mu}{\partial x_0} + \sum_{j=1}^{n} A^j(x) \frac{\partial \varphi_\mu}{\partial x_j} \right\} \eta_\mu.$$
Thus it is sufficient to solve the following set of Cauchy problems

\[
\begin{cases}
\text{Tr} \left\{ -\frac{\partial \varphi_{\mu}}{\partial x_0} + \sum_{j=1}^{n} A^j (y) \frac{\partial \varphi_{\mu}}{\partial x_j} \right\} = 0 \\
\varphi_{\mu}(0, x') = x_{\mu}.
\end{cases}
\]

Note that, if \( A^j \) are real analytic functions, then \( \varphi_{\mu} \) is real analytic.

**Definition 2.1.** Let \( K(x, \xi) \) be a \((2 \times 2)\) matrix of meromorphic functions. Let \( p, q \in \mathbb{Q}^+ \), positive rational numbers, such that \( q < p + 1 \). Denote \( \delta = (1 + p - q)^{-1}, \mu = 1 + \delta q \) and \( \sigma = (\sigma_0, \ldots, \sigma_n) \) with \( \sigma_j = \delta p \) for \( 0 \leq j \leq n - 1 \), \( \sigma_n = \delta q \). We define

\[
K_\lambda(y, \eta) = K(\lambda^{-\sigma} y, \lambda^\sigma \eta + \lambda^\mu e_n)
\]

where \( \lambda^{-\sigma} y = (\lambda^{-\sigma_0} y_0, \ldots, \lambda^{-\sigma_n} y_n) \).

**Definition 2.2.** Let \( K(x, \xi) \) be as in Definition 2.1. We can write

\[
K_\lambda(y, \eta) = \sum_{j=a}^{\infty} K_j(y, \eta) \lambda^{-\varepsilon j}, \quad \varepsilon \in \mathbb{Q}^+,
\]

where \( K_s(y, \eta) \) is not identically zero and define

\[
\text{Ord} [K_\lambda] = -s \varepsilon.
\]

Finally we introduce the symbols

\[
\mathcal{L}(x, \xi) = \left( L_0(x) + i \sum_{j=0}^{n} \frac{\partial^2 L_1}{\partial x_j \partial \xi_j} (x, \xi) \right)^{co} L_1(x, \xi) - i \sum_{j=0}^{n} \{ L_1, \{ L_1 \} (x, \xi) \}
\]

and

\[
\mathcal{L}^t(x, \xi) = i \sum_{j=0}^{n} \frac{\partial^2 L_1}{\partial x_j \partial \xi_j} (x, \xi)^{co} L_1(x, \xi) - \sum_{j=0}^{n} \{ L_1, \{ L_1 \} (x, \xi) \}.
\]

It can be proved (see [17]) that the symbol \( \mathcal{L}(x, \xi) \) is invariant under a change of the \( x \) variables and for a change of bases in \( \mathbb{C}^2 \) modulo terms of the form \( L_1(x, \xi) Q(x, \xi) \), where \( Q(x, \xi) \) is a suitable \( 2 \times 2 \) smooth matrix.

**Theorem 2.1.** Let \( L \) be as above with the real analytic matrix coefficients \( A^j \). Suppose that there exist \( p, q \in \mathbb{Q}^+, q < p + 1 \) such that

\[
\text{Ord} [h_\lambda] = 2\delta p.
\]
Then if the Cauchy problem (2.1) is well posed in $\Omega^t$ for every small $t$ then there is a meromorphic $C$ such that $\text{Ord } C_{\lambda} \leq \delta p$

(L) \hspace{1cm} \text{Ord } (L + L_1 C)_{\lambda} \leq 2\delta p.

**Theorem 2.2.** Let $L_1$ be as above with the real analytic matrix coefficients $A_j$. Suppose that there exist $p, q \in \mathbb{Q}^+$, $q < p + 1$ such that

$$\text{Ord } [h_{\lambda}] = 2\delta p.$$  

Then if $L_1$ is strongly hyperbolic we have

(C1) \hspace{1cm} \text{Ord } L_{1\lambda} \leq 2\delta p,$$

and there exists a meromorphic $C$ such that $\text{Ord } C_{\lambda} \leq \delta p$ and

(C2) \hspace{1cm} \text{Ord } (L^2 + L_1 C)_{\lambda} \leq 2\delta p.$$

Some comments to Theorem 2.2 are in order:

**Remark 2.1.** (i) In order to point out the meaning of our conditions it is useful to exploit conditions (2.3) and (C1). From the definition we have

$$h_{\lambda}(y, \eta) = h(\lambda^{-\delta p} \tilde{y}, \lambda^{-\delta q} y_n, \lambda^{\delta p} \tilde{\eta}, (\lambda^{\delta q} \eta_n + \lambda^\mu))$$

where $y = (\tilde{y}, y_n)$ and $\eta = (\tilde{\eta}, \eta_n)$. Expanding the right hand side of the previous equation we obtain

$$\sum_{w(\tilde{\alpha}, \beta) > -N} \frac{1}{\tilde{\alpha}!|\beta|!} h^{(\tilde{\alpha})}_{(\beta)}(0, e_n(1 + \eta_n \lambda^{-1})) \tilde{y}^{\tilde{\alpha}} y_n^{\eta_n} \tilde{\eta}^{\tilde{\eta}} \lambda^{w(\tilde{\alpha}, \beta)} + O(\lambda^{-N})$$

where

$$w(\tilde{\alpha}, \beta) = -\delta p|\tilde{\beta}| - \delta q \beta_n + \delta p|\tilde{\alpha}| + (2 - |\tilde{\alpha}|)(1 + \delta q).$$

Putting $2 = 2\delta(1 + p - q)$ and $|\tilde{\alpha}| = |\tilde{\alpha}|\delta(1 + p - q)$ we obtain

$$w(\tilde{\alpha}, \beta) = -\delta(p|\tilde{\beta}| + q \beta_n + |\tilde{\alpha}| - 2) + 2\delta p$$

thus, recalling that $h^{(\tilde{\alpha})}_{(\beta)}(0, e_n(1 + \eta_n \lambda^{-1})) = h^{(\tilde{\alpha})}_{(\beta)}(0, e_n) + O(\lambda^{-1})$ we have that condition (2.3) is equivalent to

$$h^{(\tilde{\alpha})}_{(\beta)}(0, e_n) = 0, \text{ for } p|\tilde{\beta}| + q \beta_n + |\tilde{\alpha}| < 2.$$
We now let \( L_1(x, \xi) = (l_{ij}(x, \xi)) \). The same arguments as above prove that \( \text{Ord} L_{1\lambda} \leq 2\delta p \) implies
\[
I_0^{(a)}(0, e_n) = 0 \quad \text{for} \quad p(|\tilde{\beta}| + 1) + q\beta_n + |\tilde{\alpha}| < 1.
\]
This shows that \( h(x, D) + l_{ij}(x, D) \) verifies the Ivrii-Petkov condition ([Theorem 4.1, 5]).

(ii) Assume the hypothesis of Theorem 2.2. If we have
\[
\text{Ord}[L_1 L^\dagger]_\lambda \leq 3\delta p,
\]
then condition (C2) is satisfied. In fact, if we define
\[
C(x, \xi) = \frac{-\text{co}L_1(x, \xi)L^\dagger(x, \xi)}{h(x, \xi)}
\]
(i.e. the formal solution of the equation \( L^\dagger + L_1 C = 0 \)), \( C(x, \xi) \) is meromorphic (as a quotient of analytic functions), moreover it trivially satisfies (C2) and \( \text{Ord} C_\lambda \leq \delta p \).

(iii) Condition (C) is satisfied when \( L_1 \) is a symmetric system. Consider the operator defined by
\[
L(x, D) = -D_0 I_2 + \begin{bmatrix}
a_{11}(x, D') & a_{12}(x, D') \\
a_{21}(x, D') & -a_{11}(x, D')
\end{bmatrix}, \quad a_{12} = a_{21}.
\]
Then we have
\[
h(x, \xi) = \xi_0^2 - (a_{11}(x, \xi')^2 + a_{12}(x, \xi')^2).
\]
Assume that \( \text{Ord} h_\lambda \leq 2\delta p \), then we have that
\[
\text{Ord}[(a_{11}(x, \xi')^2 + a_{12}(x, \xi')^2)_\lambda] \leq 2\delta p,
\]
because in (2.4) no cancellation can occur between the leading parts of \((\xi_0^2)_\lambda\) and \((a_{11}(x, \xi')^2 + a_{12}(x, \xi')^2)_\lambda\). Moreover (2.5) implies that
\[
a_{11\lambda} = O(\lambda^{\delta p}), \quad a_{12\lambda} = O(\lambda^{\delta p}).
\]
Then
\[
A_\lambda = O(\lambda^{\delta p}),
\]
where \( A(x, \xi') \) denotes the principal symbol of \( A(x, D') \), and hence
\[
\text{Ord}(L_{1\lambda}) = \text{Ord}(\text{co}L_{1\lambda}) \leq \delta p.
\]
Thus condition (C1) is satisfied. Moreover noting that
\[
(\partial_{x_j} \partial_{\xi_j} L_1)_{\lambda} = \partial_{y_j} \partial_{n_j} (L_{1\lambda}),
\]
\[
(\partial_{x_j} \partial_{\xi_j} L_1)_{\lambda} = \partial_{y_j} \partial_{n_j} (L_{1\lambda}) \partial_{n_j} L_{1\lambda}
\]
we see easily
\[
L_\lambda^4 = O(\lambda^{2dp})
\]
by (2.6). Thus the condition (C2) is satisfied with \( C = 0 \).

(iv) Assume that \( A(x, e_n)^2 = O \) and we can choose \( p \) so that \( p < 1 \) in Theorem 2.1. Let us set \( L = \sum_{j=0}^{n} \Phi^j(x)\xi_j \) then the condition (L) implies that
\[
(\Phi^n(x)\xi_n - A^n(x)C(x, \xi)\xi_n)_{\lambda} = O(\lambda^{2dp}).
\]
Multiply \( A^n(x) \) from the left to \( H = \Phi^n(x) - A^n(x)C(x, \xi) \) we have
\[
A^n(x)\Phi^n(x) = A^n(x)H, \quad (H\xi_n)_{\lambda} = O(\lambda^{2dp})
\]
because \( A^n(x)^2 = O \). Since \( A^n(x), \Phi^n(x) \) are smooth near the origin we get
\[
(A^n(x)\Phi^n(x)\xi_n)_{\lambda} = A^n(0)\Phi^n(0)\lambda^{1+\delta_{p}} + o(\lambda^{1+\delta_{p}})
\]
and \( (A^n(x)H\xi_n)_{\lambda} = O(\lambda^{2dp}) \). Then it follows that \( A^n(0)\Phi^n(0) = O(\lambda^{2dp-1+\delta}) = O(\lambda^{-\delta(1-p)}) \). This gives \( A^n(0)\Phi^n(0) = O \) because \( p < 1 \). Noting that \( (\partial L_1 L)(0, e_n) = A^n(0)\Phi^n(0) \) we get
\[
(\partial L_1 L)(0, e_n) = O.
\]
In particular, the case of constant multiplicity we may assume that \( h(x, \xi) = \xi_0^2 \) and hence we can take arbitrary \( p \in \mathbb{Q}^+ \) with \( p < 1 \). Then applying the above remark to every \( (x, 0, \xi') \) we conclude that \( (\partial L_1 L)(x, 0, \xi') = O \) so that \( (\partial L_1 L) \) is divisible by \( \xi_0 \). This is exactly the Levi condition (see [16]).

(v) Section 5 below is devoted to the discussion of the invariance of our condition.

We conclude this section with an example.

**Example.** (a) Let

\[
L(x, D) = D_0I_2 - \left[ \begin{array}{cc} x_n & x_n + x^m_n \\ -x_n + x^m_n & -x_n \end{array} \right] D_n
\]

where \( m \geq 2 \). The principal symbol is
\[
L_1(x, \xi) = \left[ \begin{array}{cc} \xi_0 - x_n \xi_n & -(x_n + x^m_n)\xi_n \\ -(-x_n + x^m_n)\xi_n & \xi_0 + x_n \xi_n \end{array} \right]
\]
and \( h(x, \xi) = \det L_1(x, \xi) = \xi_0^2 - x_n^{2n} \xi_n^2 \). Thus \((0, e_n)\) is a double characteristic of \( h \). Choosing \( p \in \mathbb{Q}^+ \), \( q = 1/(m - \epsilon) \) with small enough \( \epsilon \in \mathbb{Q}^+ \), we have

\[
\text{Ord}[h] = 2\delta_p
\]

(see Remark 2.1 (i)). Now the condition \((C_1)\) in Theorem 2.2 is violated. In fact we have

\[
\text{Ord}[(L_1)_{11\lambda}] = \text{Ord}[\lambda(-y_n(1 + \eta_n\lambda^{-1}) + \eta_0\lambda^{-p-1})] = 1
\]

since \( \delta_p - 1 = \delta(q - 1) < 0 \). On the other hand

\[
2\delta_p = 2\frac{p}{1 + p - q} = 2\frac{(m - \epsilon)p}{(m - \epsilon)(1 + p) - 1} < 1
\]

if we choose \( p \) small enough. So the operator defined in (2.7) is not strongly hyperbolic.

(b) Consider now the operator

\[
L(x, D) + \begin{bmatrix} b_{11}(x) & 0 \\ b_{21}(x) & 0 \end{bmatrix}
\]

where \( L(x, D) \) is defined in (2.7). We choose \( q = 1/(m - \epsilon) \), \( \epsilon \in \mathbb{Q}^+ \), \( p < (1 - \epsilon)q \).

It is easy to see that (see Remark 2.1 (i))

\[
\mathcal{L}_\lambda \equiv \begin{bmatrix} x_n b_{11}(x) \xi_n & x_n b_{11}(x) \xi_n \\ x_n b_{21}(x) \xi_n & x_n b_{21}(x) \xi_n \end{bmatrix}
\]

\[
(L_1 C)_\lambda \equiv \begin{bmatrix} -(c_{11} + c_{21})x_n \xi_n & -(c_{12} + c_{22})x_n \xi_n \\ (c_{11} + c_{21})x_n \xi_n & (c_{12} + c_{22})x_n \xi_n \end{bmatrix}
\]

modulo \( O(\lambda^{2\delta p}) \). Thus the condition \((L)\) can be written as

\[
\begin{bmatrix} (b_{11} - (c_{11} + c_{21}))x_n \xi_n & (b_{11} - (c_{12} + c_{22}))x_n \xi_n \\ (b_{21} + (c_{11} + c_{21}))x_n \xi_n & (b_{21} + (c_{12} + c_{22}))x_n \xi_n \end{bmatrix} \equiv 0
\]

modulo \( O(\lambda^{2\delta p}) \) for \( C = (c_{ij})_{1 \leq i, j \leq 2} \) such that \( \text{Ord} C \leq \delta p \). Again, from Remark 2.1 (i), we have that (2.9) is equivalent to

\[
\begin{align*}
& b_{11} - (c_{11} + c_{21}) = x_n^{m-1} t_{11}(x), & b_{21} + (c_{11} + c_{21}) = x_n^{m-1} t_{21}(x), \\
& b_{11} - (c_{12} + c_{22}) = x_n^{m-1} t_{12}(x), & b_{21} + (c_{12} + c_{22}) = x_n^{m-1} t_{22}(x).
\end{align*}
\]

In particular, (2.10) implies that

\[
(b_{11} + b_{21}) = x_n^{m-1} \hat{t}(x).
\]

We can show that this condition is also sufficient for the Cauchy problem for (2.8) to be well posed.
3. Preliminaries

In this section we will simplify the notation and we will write $L(x, \xi)$ instead of $L_1(x, \xi)$. Consider the non singular matrix (change of basis in $\mathbb{C}^2$)

$$T = \begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix}$$

and define the operator

$$L^\#(x, D) = T^{-1}L(x, D)T = D_0 - A^\#(x, D'),$$

where $A^\#(x, D') = T^{-1}A(x, D')T = [a^\#_{ij}]_{1 \leq i, j \leq 2}$. It can be easily seen that in this new coordinate setting we have $a^\#_{11} = -a_{11}^\#$ and $a^\#_{12} = a_{12}^\#$, so $h(x, \xi) = \xi_1^2 + |a_{11}^\#|^2 - |a_{12}^\#|^2$. Since $h(x, \xi)$ has only real zeros we see that

$$|a_{11}^\#| \leq |a_{12}^\#|.$$ 

In what follows we will write $L$ and $A$ instead of $L^\#$ and $A^\#$.

In this section we will rewrite our condition (L). In what follows, the following definition will be useful.

**Definition 3.1.** Let $F(x, \xi)$ and $G(x, \xi)$ be as in Definition 2.1. We say that $F(x, \xi)$ is equal to $G(x, \xi)$ modulo $O(\varepsilon^2)$ and write

$$F(x, \xi) \equiv_{2\delta_p} G(x, \xi)$$

if and only if

$$\text{Ord } [F(x, \xi) - G(x, \xi)] \leq 2\delta_p.$$

**Proposition 3.1.** Assuming (2.3), the condition (L) is equivalent to the following condition: there is a meromorphic $C$ with $\text{Ord } C \leq \delta_p$ such that

$$\text{Ord } \left( \frac{1}{i} \sum_{j=0}^{n} \partial_{\xi_j} L_1 L_1^{\co} L_1 + L_1 C + \text{Tr}(L_1^{\co} B) I_2 \right) \leq 2\delta_p.$$ 

**Proof.** We first rewrite $L$. From the definition of cofactor matrix we have

$$\partial_{x_j \xi_j}^2 h I_2 = \partial_{x_j \xi_j}^2 L_1^{\co} L_1$$

$$= \partial_{x_j \xi_j}^2 L_1^{\co} L_1 + \partial_{x_j} L_1^{\co} \partial_{x_j} L_1^{\co} L_1 + \partial_{x_j} L_1^{\co} \partial_{\xi_j} L_1 + L_1 \partial_{x_j \xi_j}^2 L_1$$

$$= \partial_{x_j \xi_j}^2 L_1^{\co} L_1 + 2 \partial_{x_j} L_1^{\co} \partial_{x_j} L_1$$

$$- \partial_{\xi_j} L_1 \partial_{x_j} L_1^{\co} L_1 + \partial_{x_j} L_1 \partial_{\xi_j} L_1^{\co} L_1 + L_1 \partial_{x_j \xi_j}^2 L_1.$$
If we resolve the previous identity in $\partial_{\xi_j} L_1 \partial_{\xi_j}^{co} L_1$, multiply by $\frac{1}{i}$ and then take the sum over $j$, we get:

$$\frac{1}{i} \sum_{j=0}^{n} \partial_{\xi_j} L_1 \partial_{\xi_j}^{co} L_1 = \frac{i}{2} \langle \partial_x, \partial_{\xi_j} \rangle L_1^{co} L_1 - \frac{i}{2} \{ L_1, ^{co} L_1 \} + \frac{i}{2} L_1 \langle \partial_x, \partial_{\xi_j} \rangle^{co} L_1 - \frac{i}{2} \langle \partial_x, \partial_{\xi_j} \rangle h I_2$$

$$= \mathcal{L}^f + \frac{i}{2} L_1 \langle \partial_x, \partial_{\xi_j} \rangle^{co} L_1 - \frac{i}{2} \langle \partial_x, \partial_{\xi_j} \rangle h I_2.$$

Since $B^{co} L_1 + L_1^{co} B = \text{Tr}(L_1^{co} B) I_2$ we obtain

$$\frac{1}{i} \sum_{j=0}^{n} \partial_{\xi_j} L_1 \partial_{\xi_j}^{co} L_1 + \text{Tr}(L_1^{co} B) I_2$$

$$= \mathcal{L} + L_1 \left( ^{co} B + \frac{i}{2} \langle \partial_x, \partial_{\xi_j} \rangle^{co} L_1 \right) - \frac{i}{2} \langle \partial_x, \partial_{\xi_j} \rangle h I_2.$$

With

$$\tilde{C} = ^{co} B + \frac{i}{2} \langle \partial_x, \partial_{\xi_j} \rangle^{co} L_1$$

we have $\text{Ord} \tilde{C} \leq 0$ because $\langle \partial_x, \partial_{\xi_j} \rangle^{co} L_1$ depends analytically only on $x$. On the other hand we have

$$\text{Ord}[\langle \partial_x, \partial_{\xi_j} \rangle h]_\lambda \leq 2\delta p,$$

because $\text{Ord}[h_\lambda] = 2\delta p$. Summing up we have proved

$$\frac{1}{i} \sum_{j=0}^{n} \partial_{\xi_j} L_1 \partial_{\xi_j}^{co} L_1 + \text{Tr}(L_1^{co} B) I_2 \equiv_{2\delta p} \mathcal{L} + L_1 \tilde{C}$$

where $\text{Ord} \tilde{C} \leq 0$ and hence the result. □

**Lemma 3.2.** If $C_\lambda = O(\lambda^{\delta p})$ then

$$\left( \frac{1}{i} \sum_{j=0}^{n} \partial_{\xi_j} L_1 \partial_{\xi_j}^{co} L_1 + L_1 C + \text{Tr}(L_1^{co} B) I_2 \right)_\lambda$$

$$\equiv_{2\delta p} (L(A^n)\xi_n - A^n(x)C\xi_n - \text{Tr}(A^n[^{co} B]) I_2\xi_n)_\lambda.$$
PROOF. In fact we have
\[ \sum_{j=0}^{n} \partial_{\xi_j} L_1 \partial_{x_j} \co L_1 = \partial_{x_0} A^n \xi_n - \sum_{j=1}^{n} A^j(x) \partial_{x_j} A^n \xi_n \]
\[ + \sum_{\mu=1}^{n-1} \partial_{x_0} A^\mu \xi_\mu - \sum_{j=1}^{n-1} A^n \partial_{x_n} A^j \xi_j \]
\[ - \sum_{j=1}^{n-1} \sum_{\mu=1}^{n-1} A^j \partial_{x_j} A^\mu \xi_\mu. \]

It is clear that
\[ \left( \sum_{\mu=1}^{n-1} \partial_{x_0} A^\mu \xi_\mu - \sum_{j=1}^{n-1} A^n \partial_{x_n} A^j \xi_j - \sum_{j=1}^{n-1} \sum_{\mu=1}^{n-1} A^j \partial_{x_j} A^\mu \xi_\mu \right) \equiv_{2\delta_p} 0, \]

hence
\[ \left( \frac{1}{i} \sum_{j=0}^{n} \partial_{\xi_j} L_1 \partial_{x_j} \co L_1 \right) \equiv_{2\delta_p} \left( \frac{1}{i} \partial_{x_0} A^{(n)} \xi_n - \frac{1}{i} \sum_{j=1}^{n} A^j(x) \partial_{x_j} A^n \xi_n \right) \equiv_{2\delta_p} (L(A^n \xi_n)). \]

On the other hand
\[ (L_1 C)_\lambda \equiv_{2\delta_p} -(AC)_\lambda, \quad (\Tr(L_1 \co B))_\lambda \equiv_{2\delta_p} -(\Tr(A^\co B))_\lambda \]
because \((\xi_0)_\lambda = O(\lambda^{\delta_p}),\ C_\lambda = O(\lambda^{\delta_p}).\) Moreover
\[ (AC)_\lambda \equiv_{2\delta_p} (A^n C \xi_n)_\lambda, \quad (\Tr(A^\co B))_\lambda \equiv_{2\delta_p} (\Tr(A^n \co B))_\lambda \]
because
\[ \left( \sum_{j=1}^{n-1} A^j \xi_j C \right)_\lambda = O(\lambda^{2\delta_p}), \quad \left( \sum_{j=1}^{n-1} A^j \xi_j \co B \right)_\lambda = O(\lambda^{\delta_p}). \]

Set \(A^n(x) = (a^n_{ij}(x))_{1 \leq i,j \leq 2}\) and
\[ F(x) := L(A^n)(x) = D_0 A^n(x) - \sum_{j=1}^{n} A^j(x) D_j A^n(x) = (f_{ij})_{1 \leq i,j \leq 2}. \]
LEMMA 3.3. Let $\text{Ord } h_\lambda = 2\delta p$. Assume that there is a meromorphic $C$ with $\text{Ord } C_\lambda \leq \delta p$ such that

\begin{equation}
\text{Ord } [F(x)\xi_n - A^n(x)\xi_n C(x, \xi) - \text{Tr}(A^n \xi_n^{\text{co}} B) I_2]_\lambda \leq 2\delta p
\end{equation}

then we have

\begin{equation}
\text{Ord } \left[ \left( f_{11} - \frac{a_{11}^n f_{21}}{a_{21}^n} - \text{Tr}(A^n [^\text{co} B]) \right) \xi_n \right]_\lambda \leq 2\delta p,
\end{equation}

\begin{equation}
\text{Ord } \left[ \left( f_{22} - \frac{a_{22}^n f_{12}}{a_{12}^n} - \text{Tr}(A^n [^\text{co} B]) \right) \xi_n \right]_\lambda \leq 2\delta p.
\end{equation}

Conversely if (3.4) holds, then there is a meromorphic $C(x)$, depending only on $x$, such that (3.3) is satisfied and $\text{Ord } C_\lambda \leq 2\delta p$.

PROOF. Set

$$F\xi_n - A^n \xi_n C(x, \xi) - \text{Tr}(A^n \xi_n^{\text{co}} B) I_2 = \begin{bmatrix} r_{11}(x, \xi) & r_{12}(x, \xi) \\ r_{21}(x, \xi) & r_{22}(x, \xi) \end{bmatrix} \xi_n,$$

$$\text{Ord } (r_{ij}(x, \xi) \xi_n)_\lambda \leq 2\delta p.$$

It follows from the equations verified by the off diagonal entries that

\begin{align*}
a_{11}^n c_{12} + a_{12}^n c_{22} &= f_{12} - r_{12} \\
a_{21}^n c_{11} + a_{22}^n c_{21} &= f_{21} - r_{21}
\end{align*}

and we get

\begin{align*}
c_{11} &= \frac{f_{21} - r_{21} - a_{22}^n c_{21}}{a_{21}^n}, \\
c_{22} &= \frac{f_{12} - r_{12} - a_{11}^n c_{12}}{a_{12}^n},
\end{align*}

and then

$$A^n C = \begin{bmatrix}
a_{11}^n f_{21} - a_{11}^n r_{21} - \det A^n c_{21} & f_{12} - r_{12} \\
& f_{21} - r_{21} & a_{22}^n f_{12} - a_{22}^n r_{12} - \det A^n c_{12} / a_{12}^n
\end{bmatrix}.$$

Summing up we obtain
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$$F - A^n C - \text{Tr}(A^n[coB])I_2 = \begin{bmatrix} f_{11} - \frac{a_{11}^n f_{21}}{a_{21}^n} - \text{Tr}(A^n[coB]) & 0 \\ 0 & f_{22} - \frac{a_{22}^n f_{12}}{a_{12}^n} - \text{Tr}(A^n[coB]) \end{bmatrix}$$

where $g = -\det A^n$. Since $|a_{11}^n| \leq |a_{21}^n|, |a_{22}^n| \leq |a_{12}^n|$, Ord $r_{ij} \xi_n \xi$ $\lambda \leq 2\delta p$ ($1 \leq i, j \leq 2$) we have that

$$\left( \frac{a_{11}^n r_{21}}{a_{21}^n} \xi_n \lambda \right) = \left( \frac{a_{11}^n}{a_{21}^n} \right) \lambda \left( r_{21} \xi_n \lambda \right) \equiv_{2\delta p} 0$$

and similarly

$$\left( \frac{a_{22}^n r_{12}}{a_{12}^n} \xi_n \lambda \right) \equiv_{2\delta p} 0.$$

Next we show that

$$\left( g \frac{c_{12}^n}{a_{12}^n} \xi_n \lambda \right) \equiv_{2\delta p} 0 \quad \text{and} \quad \left( g \frac{c_{21}^n}{a_{21}^n} \xi_n \lambda \right) \equiv_{2\delta p} 0.$$

Recalling that $a_{12}^n = a_{21}^n$ we write

$$(a_{12}^n \xi_n) \lambda = \lambda^\mu (f(x) + o(1)), \quad (a_{21}^n \xi_n) \lambda = \lambda^\nu (\bar{f}(x) + o(1)).$$

When $\mu \leq \delta p$ we have

$$\left( g \frac{c_{21}^n}{a_{21}^n} \xi_n \lambda \right) = \left( g \frac{\xi_n^2}{a_{21}^n} \xi_n \right) \lambda \equiv_{2\delta p} 0,$$

because

$$\left| g \frac{\xi_n^2}{a_{21}^n} \xi_n \right| \leq |a_{21}^n \xi_n|.$$

Using the same arguments as before we prove that

$$\left( g \frac{c_{12}^n}{a_{12}^n} \xi_n \lambda \right) \equiv_{2\delta p} 0.$$
When $\mu > \delta p$ we have
\[
\left( \frac{c_{21}}{a_{21}^n} \xi_n \right)_\lambda = O(1), \quad \left( \frac{c_{12}}{a_{12}^n} \xi_n \right)_\lambda = O(1),
\]
because $\text{Ord} C\lambda \leq \delta p$. On the other hand, from $\text{Ord} h = 2\delta p$ it follows that $\text{Ord}(g\xi_n^2)\lambda \leq 2\delta p$. So we have
\[
\left( g \frac{c_{21}}{a_{21}^n} \xi_n \right)_\lambda = (g\xi_n^2)_\lambda \left( \frac{c_{21}}{a_{21}^n} \xi_n \right)_\lambda \equiv_{2\delta p} 0,
\]
and similarly we get
\[
\left( g \frac{c_{12}}{a_{12}^n} \xi_n \right)_\lambda \equiv_{2\delta p} 0.
\]
Thus
\[
(F\xi_n - A^n\xi_n C(x, \xi) - \text{Tr}(A^n[\omega B])\xi_n I_2)_\lambda \equiv_{2\delta p} 0
\]
\[
= \begin{bmatrix}
  \left( f_{11} - \frac{a_{11}^n f_{21}}{a_{21}^n} - \text{Tr}(A^n[\omega B]) \right) \xi_n & 0 \\
  0 & \left( f_{22} - \frac{a_{22}^n f_{12}}{a_{12}^n} - \text{Tr}(A^n[\omega B]) \right) \xi_n
\end{bmatrix}_\lambda
\]
and this proves that (3.3) implies (3.4).

To prove the converse, we study $F$ more precisely. Assume (3.4) and define $C$ by
\[
C(x) = \begin{bmatrix} f_{21} & 0 \\ a_{21}^n & f_{12} \end{bmatrix}.
\]
Then it is clear that
\[
[F - A^n C - \text{Tr}(A^n[\omega B]) I_2] \xi_n
\]
\[
= \begin{bmatrix}
  \left( f_{11} - \frac{a_{11}^n f_{21}}{a_{21}^n} - \text{Tr}(A^n[\omega B]) \right) \xi_n & 0 \\
  0 & \left( f_{22} - \frac{a_{22}^n f_{12}}{a_{12}^n} - \text{Tr}(A^n[\omega B]) \right) \xi_n
\end{bmatrix}
\]
which is $\equiv_{2\delta p} 0$ by the assumption (3.4). Thus, to prove the converse, it is enough to show that $\text{Ord} C\lambda \leq \delta p$. Recall that
\[
f_{st} = D_0 a_{st}^n - \sum_{j=1}^n \left( a_{s1}^j D_j a_{1t}^n + a_{s2}^j D_j a_{2t}^n \right),
\]
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with $1 \leq s, t \leq 2$. We start studying

$$f_{21} = \frac{D_0 a_{21}^n}{a_{21}^n} - \sum_{j=1}^{n} \left( \frac{a_{21}^j}{a_{21}^n} D_j a_{11}^n + \frac{a_{22}^j}{a_{21}^n} D_j a_{21}^n \right).$$

We first have

$$\text{Ord} \left( \frac{D_j a_{11}^n}{a_{11}^n} \right)_{\lambda} \leq \delta p, \quad 0 \leq j \leq n - 1. \quad (3.6)$$

In fact, let $(a_{1k}^n)_{\lambda} = \lambda^\mu (f(x) + o(1)), f(x) \neq 0$, then

$$D_j (a_{1k}^n)_{\lambda} = \lambda^\mu (D_j f(x) + o(1)).$$

On the other hand

$$D_j (a_{1k}^n)_{\lambda} = D_j a_{ik}^n (\lambda^{-\sigma} y)$$

$$= \frac{1}{\lambda} \lambda^{-\delta p} \frac{\partial}{\partial (\lambda^{-\delta p} y_j)} a_{ik}^n (\lambda^{-\sigma} y)$$

$$= \lambda^{-\delta p} (D_j a_{ik}^n)_{\lambda}, \quad 0 \leq j \leq n - 1.$$

Thus it follows that $(D_j a_{ik}^n)_{\lambda} = \lambda^{\delta p+\mu} (D_j f(y) + o(1)) = O(\lambda^{\delta p+\mu}).$ This proves $(3.6)$. It is clear that $(D_j a_{1k}^n)_{\lambda} = O(1), 0 \leq k \leq n$ because $a_{ik}^n$ is smooth. We now show that

$$\text{Ord} \left( \frac{f_{21}}{a_{21}^n} \right)_{\lambda} \leq \delta p. \quad (3.7)$$

Indeed it follows from $(3.6)$ that

$$\text{Ord} \left( \frac{D_0 a_{21}^n}{a_{21}^n} \right)_{\lambda} \leq \delta p.$$

On the other hand we have

$$\left( \frac{a_{21}^j}{a_{21}^n} D_j a_{11}^n \right)_{\lambda} = (a_{21}^j)_{\lambda} \left( \frac{a_{11}^n}{a_{21}^n} \right)_{\lambda} \left( \frac{D_j a_{11}^n}{a_{11}^n} \right)_{\lambda} = O(\lambda^{\delta p})$$

for $0 \leq j \leq n - 1$ because

$$\text{Ord} (a_{21}^n)_{\lambda} \leq 0, \quad \text{Ord} \left( \frac{a_{11}^n}{a_{21}^n} \right)_{\lambda} \leq 0 \quad \text{from} \ (3.1)$$
and for $j = n$

$$\left( a_{21}^n \frac{D_n a_{11}^n}{a_{21}^n} \right)_\lambda = (D_n a_{11}^n)_\lambda = O(1).$$

In the same way we obtain

$$\text{Ord} \left( a_{22}^j \frac{D_j a_{21}^n}{a_{21}^n} \right)_\lambda \leq \delta p \quad \text{for} \quad 0 \leq j \leq n - 1, \quad \text{Ord} \left( a_{22}^n \frac{D_j a_{21}^n}{a_{21}^n} \right)_\lambda \leq 0.$$

Thus we have proved (3.7). Similarly we obtain

$$\text{Ord} \left( \frac{f_{12}}{a_{12}^n} \right)_\lambda \leq \delta p.$$

4. Proof of the theorems

In order to prove our results, we assume that (L) does not hold, and we will construct an asymptotic solution for $L$ violating the a priori estimate in Proposition 4.2 below.

To do this, consider the operator (we still write $L$ instead of $L^\#$)

$$K(x, D) = (L(x, D) + B(x))(M(x, D) + c^L B(x) + C(x))$$

where

$$L(x, D) = D_0 - A(x, D'), \quad M(x, D) = c^L L(x, D) = D_0 + A(x, D'),$$

$B$ is a $C^\infty$ $2 \times 2$ matrix, and $C$ is given by (3.5).

In what follows we will construct an asymptotic solution for $K$ and from this solution we will extract the desired solution for $L + B$.

First consider the principal symbol of $K$

$$\sigma(K)(x, \xi) = h(x, \xi) I_2$$

$$+ \left[ \frac{i}{4} \sum_{j=0}^n \partial_{\xi_j} L \partial_{x_j} M + \text{Tr}(L^c B)(x, \xi) I_2 + L(x, \xi) C(x) \right]$$

$$+ L^c B(x) + L(C)(x) + B(x) c^L B(x) + B(x) C(x).$$
Assume that our condition does not hold. Recall that \( \text{Ord} C_\lambda \leq \delta p \). As before it is easy to see that

\[
\left( \frac{1}{i} \sum_{j=0}^{n} \partial_{\xi_j} L \partial_{\xi_j} M + LC + \text{Tr}(L^c B)I_2 \right)_{\lambda}
\]

\( \equiv 2\delta p \left( L(A^n)\xi_n - A^n C \xi_n - \text{Tr}(A^n \{^c B\})\xi_n I_2 \right)_{\lambda} \)

\( \equiv 2\delta p \left( (F - \text{Tr}(A^n \{^c B\}) - A^n \{^c C\})\xi_n \right)_{\lambda} \)

\[
\begin{pmatrix}
 f_{11} - \frac{a_{11}^n f_{21}}{a_{21}^n} - \text{Tr}(A^n \{^c B\}) & 0 \\
 0 & f_{22} - \frac{a_{22}^n f_{12}}{a_{12}^n} - \text{Tr}(A^n \{^c B\})
\end{pmatrix} \xi_n
\]

Then we have either

(4.1) \( \text{Ord} \left[ \left( f_{11} - \frac{a_{11}^n f_{21}}{a_{21}^n} - \text{Tr}(A^n \{^c B\}) \right) \xi_n \right]_{\lambda} > 2\delta p \)

or

(4.2) \( \text{Ord} \left[ \left( f_{22} - \frac{a_{22}^n f_{12}}{a_{12}^n} - \text{Tr}(A^n \{^c B\}) \right) \xi_n \right]_{\lambda} > 2\delta p. \)

Summing up we have the following

**Lemma 4.4.** Suppose that \((L)\) does not hold. Then we have

(4.3) \( K(x, D) = h(x, D)I_2 + \begin{bmatrix} \alpha(x) & 0 \\ 0 & \beta(x) \end{bmatrix} D_n + R(x, D) + S(x), \)

where \( R(x, D) \) is a first order differential operator such that

\( \text{Ord}[R(x, \xi)]_{\lambda} \leq 2\delta p, \)

\( S(x) = L(C) + BC + \{\text{smooth functions}\} \)

and

(4.4) \( \text{Ord}(\alpha(x)\xi_n)_{\lambda} > 2\delta p \)

or

(4.5) \( \text{Ord}(\beta(x)\xi_n)_{\lambda} > 2\delta p. \)

**Lemma 4.5.** Suppose that \((C_1)\) or \((C_2)\) does not hold. Then, choosing \( B \) suitably, the same conclusion as in Lemma 4.4 holds.
PROOF. If $\text{Ord} L > 2\delta p$ we can take $B(x)$ such that $\text{Ord}[\text{Tr}(L^{\sigma}B)] > 2\delta p$. Then eventually changing $B$ (actually replacing $B$ by $tB$ and getting $t$ large) we have either (4.1) or (4.2). If there is no $C$ with $\text{Ord} C \leq \delta p$ such that $(C_2)$ is verified, then taking $C$ as above and $B = 0$, we have a fortiori either (4.1) or (4.2).

Let us write

$$a_{12}^n(\lambda^{-\sigma}y) = \lambda^\mu \{a_{12}^n(y) + o(1)\}.$$ 

Since $\bar{a}_{12}^n(y)$ does not vanish identically we can choose a bounded open set $U$ in $\mathbb{R}^{n+1}_y$ so that

$$|\bar{a}_{12}^n(y)| \geq c > 0$$

in $U$ with some $c > 0$. Then $K_{x}(y, D)$ is well defined in $U$ for large $\lambda$. Assume that (4.4) holds. Since $(\alpha(x)\xi_n)_x = \alpha(\lambda^{-\sigma}y)(\lambda^{1+\delta q} + \lambda^{\delta q}n)$ we have

$$(\alpha(x)\xi_n)_x(y, D_y) = \lambda^{1-\delta q}\alpha(\lambda^{-\sigma}y) + \lambda^{\delta q}\alpha(\lambda^{-\sigma}y)D_y.$$ 

Set

$$(4.6) \quad \alpha(\lambda^{-\sigma}y) = \lambda^{-\delta \mu}(\alpha(y) + o(1)),$$

where $\mu = \mu(p, q)$ is linear in $p$ and $q$. Then by our assumption

$$1 + \delta q - \delta \mu > 2\delta p$$

which is equivalent to

$$(4.7) \quad 1 > p + \mu.$$ 

**Lemma 4.6.** Given $\mu(p, q)$ as in (4.6) then we have

$$\mu(p, q) \geq 0.$$ 

**Proof.** Recall that

$$\alpha(x) = f_{11}(x) - \frac{a_{11}^n(x)f_{21}(x)}{a_{21}^n(x)} - \text{Tr}(A^n[A^{\sigma}B])_x(x)$$ 

and

$$a_{11}^n(x)f_{21}(x) = \frac{a_{11}^n}{a_{21}^n} D_0 a_{21}^n - \sum_{j=1}^{n} \left( a_{21}^n a_{11}^n D_j a_{11}^n + a_{22}^n a_{21}^n D_j a_{21}^n \right).$$
Since
\[ \left| \frac{a_{11}}{a_{21}} \right| \leq 1 \]
and \( \text{Tr}(A^n[\omega B]) \) is smooth, it is clear that \( \mu(p, q) \geq 0 \). \( \square \)

**Lemma 4.7.** Assume that
\[ \mu(p, q) + p < 1, \quad 1 + p > q. \]
Then there is \( \hat{p} \in \mathbb{Q}^+ \) with \( p \leq \hat{p} < 1 \) such that \( \mu(\hat{p}, q) + \hat{p} < 1 \) and
\[ 2q - 3\hat{p} - 1 - \mu(\hat{p}, q) < 0. \]

**Proof.** Set
\[ f(p) = 1 - p - \mu(p, q), \quad g(p) = 2q - 3p - 1 - \mu(p, q). \]
Note that \( \mu(p, q) \) is continuous in \( p \), so even \( f(p) \) and \( g(p) \) are continuous functions of \( p \).

Suppose that
\[ f(p) > 0, \quad g(p) < 0. \]
We have
\[ f(\bar{p}) - g(\bar{p}) = 2(1 + \bar{p} - q) > 0, \quad \text{if} \quad \bar{p} \geq p \]
and
\[ f(1) \leq 0, \quad g(1) < 0 \]
because \( \mu(1, q) \geq 0 \) and \( q < 2 \) (this follows immediately from \( \mu(p, q) + p < 1 \) and \( 1 + p > q \)).

Let \( p^* < 1 \) be such that \( g(p^*) = 0, g(\bar{p}) < 0, p^* < \bar{p} \leq 1 \). Since \( p \leq p^* \)
\[ f(p^*) - g(p^*) = 2(1 + p^* - q) \geq 2(1 + p - q) > 0 \]
taking \( \hat{p} > p^* \) close to \( p^* \) we have
\[ f(\hat{p}) > 0, \quad g(\hat{p}) < 0. \]
By the previous lemma (eventually replacing $p$ by $\hat{p}$) we may suppose that
\[
\mu(p, q) + p < 1, \quad 2q - 3p - 1 - \mu(p, q) < 0.
\]
Now we define $\kappa > 0$ by
\[
1 + \delta q - \delta \mu = 2\delta p + 2\kappa,
\]
then
\[
\delta q - \delta \mu = 2\delta p + 2\kappa - 1.
\]
On the other hand
\[
2\kappa - 2 = \delta(2q - 3p - 1 - \mu) < 0
\]
and hence $0 < \kappa < 1$.

Now we will examine the behavior of every single term in (4.3) under symplectic dilations and conjugation with the exponential function defined by
\[
E(y, \lambda) = \exp \left( i\gamma \lambda y_n + i \sum_{\nu} \mu^j(y) \lambda^{\nu(\nu+1)} \right)
\]
where $\gamma \in \mathbb{R}$ and the functions $\mu^j$ are analytic. We also define
\[
E_1(y, \lambda) = \exp (i\gamma \lambda y_n),
\]
\[
E_2(y, \lambda) = \exp \left( i \sum_{\nu} \mu^j(y) \lambda^{\nu(\nu+1)} \right)
\]
so we have $E(y, \lambda) = E_1(y, \lambda)E_2(y, \lambda)$.

**Definition 4.1.** Let $P(x, D)$ be a differential operator (scalar or matrix). Denote, with the same notations as in Definition 2.1,
\[
P(\gamma, e)(y, \eta) = P(\gamma \sigma y, \lambda^\sigma \eta).
\]

**Remark 4.1.** We have the following relation between the dilation of Definition 2.1 and the dilation of Definition 4.1 plus the conjugation with the exponential $E_1(y, \lambda)$:
\[
E_1(y, \lambda)^{-1} P(\lambda)(y, D_y) E_1(y, \lambda)
\]
\[
= P(\lambda)(y, \tilde{D}, \gamma \lambda + D_n)
\]
\[
= P(\lambda^{-\delta p} y, \lambda^{-\delta q} y_n, \lambda^{\delta p} \tilde{D}, \lambda^{\delta q} D_n + \gamma \lambda^{1+\delta q})
\]
\[
=: P(\lambda, \gamma)(y, D_y).
\]
In what follows we write $P_\lambda(y, D_y) \equiv P_\lambda$, $\gamma$ where $\gamma \in \mathbb{R}$ will be determined later.

Finally we state the a priori inequality:

**Proposition 4.2.** Assume that $\Omega \ni 0$ and the Cauchy problem for $L$ is well posed in $\Omega_t$ for every small $t$. Then for any compact $W \subset \mathbb{R}^{n+1}$ and $T > 0$ there are positive constants $C$, $\bar{\lambda}$ and $r \in \mathbb{N}$ such that

$$|u|_{C^0(W')} \leq C\lambda^{\bar{\sigma}r}|L(\lambda)u|_{C^r(W')}$$

with $\bar{\sigma} = \max_j \sigma_j$ for every $u \in C_0^\infty(W', C^2)$, $\lambda > \bar{\lambda}$, $|t| < T$.

For the proof we refer to [5], [11].

Let $\tau$ be the common denominator of $\delta$, $p$, $q$ and set $\nu = \tau\kappa \in \mathbb{N}$ ($\theta\nu = \kappa$, $\theta = \frac{1}{\tau}$). We start examining

\[ E(y, \lambda)^{-1}(\alpha(x)\xi_n)(y, D_y)E(y, \lambda) = E_2(y, \lambda)^{-1}(\alpha(x)\xi_n)(y, D_y)E_2(y, \lambda). \]

Recall that

\[ (\alpha(x)\xi_n)(y, D_y) = \gamma\lambda^{1+\delta q}\alpha(\lambda^{-\sigma}y) + \lambda^{\delta q}\alpha(\lambda^{-\sigma}y)D_n. \]

For the first term in the right hand side of (4.8) we have

\[ \gamma\lambda^{1+\delta q}\alpha(\lambda^{-\sigma}y) = \lambda^{2\delta p+2\kappa}\sum_{j=0} \lambda^{-j}\alpha_j(y) \]

where $\alpha_0(y) = \gamma\alpha_0(y)$. Let $U$ be the open set defined after Lemma 4.4. Shrinking $U$ if necessary we may suppose $\bar{\alpha}_0(y) \neq 0$ in $U$.

For the second term in the right hand side of (4.8) we have

\[ E_2(y, \lambda)^{-1}\lambda^{\delta q}\alpha(\lambda^{-\sigma}y)D_nE_2(y, \lambda) = \lambda^{2\delta p+2\kappa}\sum_{k=\tau - \nu} \lambda^{-k}\Theta_k(\alpha_i, l^j)|i + j \leq k) \]

\[ + \lambda^{\delta q-\delta \mu}\sum_{j=0} \lambda^{-j}\alpha_j(y)D_n \]

because

\[ \delta q - \delta \mu = 2\delta p + 2\kappa - 1 < 2\delta p + \kappa, \]

\[ 2\delta p + 2\kappa - 1 + \theta\nu = 2\delta p + 2\kappa - \theta(\tau - \nu). \]
Summing up, we have

\[ E_2(y, \lambda)^{-1} (\alpha(x) \xi_n) \lambda(y, D_y) E_2(y, \lambda) \]

\[ = \lambda^{2\delta p + 2\kappa} \sum_{j=0}^{c} (\alpha_j(y) + H_j(\alpha_i, l^k | i + k \leq j, k \geq 1)) \lambda^{-j\theta} \]

\[ + \lambda^{2\delta p + \kappa} \sum_{j=1}^{c} \lambda^{-j\theta} h_j(y, D_y). \]

For the principal part we have

\[ h_\lambda(y, D_y) = \lambda^{2\delta p} \sum_{j=0}^{c} \lambda^{-j\theta} h_j(y, D_y) \]

where \( h_j(y, D_y) \) are second order operators and \( h_0(y, D_y) = D_0^2 - \sum_{1 \leq i, j \leq n-1} c_{ij}(y) D_i D_j \), then

\[ E_2(y, \lambda)^{-1} h_\lambda(y, D_y) E_2(y, \lambda) \]

\[ = \lambda^{2\delta p + 2\kappa} \sum_{j=1}^{\nu} L_j(l^1, \ldots, l^j) \lambda^{-(j-1)\theta} \]

\[ + \lambda^{2\delta p + \kappa} \left\{ \sum_{j=0}^{n} \partial_{l^j} h_0(y, l^1_y) D_j + O(\lambda^{-\theta}) \right\} \]

where

\[ L_j(l^1, \ldots, l^j) = \sum_{\mu=0}^{n} \partial_{l^j} h_0(y, l^1_y) \partial_{x_\mu} l^j + K_j(l^1, \ldots, l^{j-1}), \]

\[ L_1(l^1) = h_0(y, l^1_y). \]

Let \( R(x, D) \) be a \( 2 \times 2 \) matrix of first order differential operators with \( \text{Ord} R_\lambda \leq 2\delta p \), i.e.

\[ R_\lambda(y, D_y) = \lambda^{2\delta p} \sum_{j=0}^{c} \lambda^{-j\theta} R_j(y, D_y), \]

where \( R_j(y, D_y) \) is a \( 2 \times 2 \) matrix of first order differential operators.

Again, we have

\[ E_2(y, \lambda)^{-1} R_\lambda(y, D_y) E_2(y, \lambda) = \lambda^{2\delta p + \kappa} \sum_{j=0}^{c} \Psi_j(l^1, \ldots, l^\nu) \lambda^{-j\theta} \]

\[ + \lambda^{2\delta p} \sum_{j=0}^{c} R_j(y, D_y) \lambda^{-j\theta}. \]
At last we have to study \( S(x) = L(C) + BC + \{\text{smooth}\} \). It is clear that \((BC + \{\text{smooth}\})_\lambda = O(\lambda^{\delta p})\). We show that \((L(C))_\lambda = O(\lambda^{2\delta p})\).

The same argument as before proves that

\[
\left( \frac{D_j D_i a^m_{\mu \nu}}{a^m_{\mu \nu}} \right)_{\lambda} = O(\lambda^{2\delta p}), \quad 1 \leq i, j \leq n - 1, \ 1 \leq \mu, \nu \leq 2.
\]

Write

\[
f_{21} = \frac{D_0 a_{21}^n}{a_{21}^n} - \sum_{j=1}^{n-1} a_{21}^j \frac{D_j a_{11}^n}{a_{21}^n} + a_{22}^j \frac{D_j a_{21}^n}{a_{21}^n} - a_{21}^n \frac{D_n a_{11}^n}{a_{21}^n} - a_{22}^n \frac{D_n a_{21}^n}{a_{21}^n}
\]

then it is easy to check that

\[
\left( D_0 \left[ \frac{D_0 a_{21}^n}{a_{21}^n} - \sum_{j=1}^{n-1} a_{21}^j \frac{D_j a_{11}^n}{a_{21}^n} + a_{22}^j \frac{D_j a_{21}^n}{a_{21}^n} \right] \right)_{\lambda} = O(\lambda^{2\delta p}).
\]

We turn to the term

\[
D_0(D_n a_{21}^n) + D_0 \left( a_{22}^n \frac{D_n a_{21}^n}{a_{21}^n} \right),
\]

and write

\[
D_0 \left( a_{22}^n \frac{D_n a_{21}^n}{a_{21}^n} \right) = \frac{D_0 a_{22}^n}{a_{21}^n} D_n a_{21}^n + \frac{a_{22}^n}{a_{21}^n} D_0 D_n a_{21}^n - \frac{a_{22}^n}{a_{21}^n} D_n a_{21}^n.
\]

Thus we get

\[
\left[ D_0 \left( a_{22}^n \frac{D_n a_{21}^n}{a_{21}^n} \right) \right]_{\lambda} = O(\lambda^{2\delta p}),
\]

and finally

\[
\left( D_0 \frac{f_{21}}{a_{21}^n} \right)_{\lambda} = O(\lambda^{2\delta p}).
\]

In a similar way we obtain

\[
\left( D_j \frac{f_{21}}{a_{21}^n} \right)_{\lambda} = O(\lambda^{2\delta p}), \quad 0 \leq j \leq n - 1,
\]

\[
\left( D_j \frac{f_{12}}{a_{12}^n} \right)_{\lambda} = O(\lambda^{2\delta p}), \quad 0 \leq j \leq n - 1.
\]
This proves that

\[
(D_0C - \sum_{j=1}^{n-1} A^j(x)D_jC) = O(\lambda^{2\delta p}).
\]

It remains to examine the term

\[ A^n(x)D_nC. \]

Recall that

\[
A^n(x)D_nC = \begin{bmatrix}
    a_{11}^n D_n a_{21}^{2n} f_{21}^n & a_{12}^n D_n a_{12}^{12} f_{12}^n \\
    a_{21}^n D_n a_{21}^{2n} f_{21}^n & a_{22}^n D_n a_{12}^{12} f_{12}^n
\end{bmatrix}
\]

For example we study the entry \((1,1):\)

\[
a_{11}^n D_n f_{21}^n = a_{11}^n D_n \left( \frac{D_0 a_{21}^n}{a_{21}^{2n}} - \sum_{j=1}^{n} \left\{ a_{21}^j \frac{D_j a_{11}^n}{a_{21}^{2n}} + a_{22}^j \frac{D_j a_{21}^n}{a_{21}^{2n}} \right\} \right).
\]

First we have

\[
a_{11}^n D_n \left( \frac{D_0 a_{21}^n}{a_{21}^{2n}} \right) = a_{11}^n D_n D_0 a_{21}^n - a_{11}^n D_n a_{21}^n D_0 a_{21}^n
\]

and hence

\[
\left[ a_{11}^n D_n \left( \frac{D_0 a_{21}^n}{a_{21}^{2n}} \right) \right] = O(\lambda^{\delta p}).
\]

On the other hand

\[
a_{11}^n D_n \left[ a_{21}^j \frac{D_j a_{11}^n}{a_{21}^{2n}} \right] = a_{11}^n (D_n a_{21}^j) \frac{D_j a_{11}^n}{a_{21}^{2n}} - a_{11}^n a_{11}^j \frac{D_j a_{11}^n}{a_{21}^{2n}} D_n a_{21}^n
\]

hence

\[
\left[ a_{11}^n D_n \left( \frac{a_{21}^j D_j a_{11}^n}{a_{21}^{2n}} \right) \right] = O(\lambda^{\delta p}), \quad 1 \leq j \leq n - 1.
\]

Similarly we get

\[
\left[ a_{11}^n D_n \left( \frac{a_{22}^j D_j a_{21}^n}{a_{21}^{2n}} \right) \right] = O(\lambda^{\delta p}), \quad 1 \leq j \leq n - 1.
\]
Noting that

\[ a_{11}^n D_n \left( \frac{a_{11}^n D_n a_{11}^n}{a_{21}^n} \right) = a_{11}^n D_n^2 a_{11}^n, \]

and

\[ a_{11}^n D_n \left( \frac{a_{11}^n a_{21}^n}{a_{21}^n} \right) = \frac{a_{11}^n}{a_{21}^n} (D_n a_{21}^n)(D_n a_{21}^n) + \frac{a_{11}^n a_{21}^n}{a_{21}^n} (D_n a_{21}^n)^2 - \frac{a_{11}^n}{a_{21}^n} (D_n a_{21}^n)^2 \]

we obtain

\[ \left[ a_{11}^n D_n \left( \frac{a_{11}^n a_{21}^n}{a_{21}^n} \right) \right]_\lambda = O(1), \]

and then

\[ \left( a_{11}^n D_n \left\{ \frac{a_{11}^n a_{21}^n}{a_{21}^n} + \frac{a_{11}^n a_{21}^n}{a_{21}^n} \right\} \right) = O(1). \]

So we conclude that

\[ \left( a_{11}^n D_n \frac{f_{21}}{a_{21}^n} \right)_\lambda = O(\lambda^{5p}). \]

With similar arguments we can prove that \((A^n D_n C)_\lambda = O(\lambda^{5p})\). Thus we have proved that \((L(C))_\lambda = O(\lambda^{2p})\) and hence

\[ S_\lambda(y) = \lambda^{2p} \sum_{j=0}^{\lambda^{-j^{(1)}}} S_j(y). \]

We summarize

\[ (4.9) \]

\[ E(y, \lambda)^{-1} \left[ (L + B) \left( \alpha L + \alpha B + C \right) \right]_\lambda E(y, \lambda) = E_2(y, \lambda)^{-1} \left\{ (L + B)(\alpha L + \alpha B + C) \right\}_\lambda E_2(y, \lambda) = \lambda^{2p+2\kappa} \]

\[ \times \sum_{j=1}^\nu \left[ \begin{array}{c} \left[ \mathcal{L}_j(l^1, \ldots, l^j) + \alpha_{j-1}(y) \\
+ H_{j-1}(\alpha_i, t^k | i + k \leq j - 1) \end{array} \right]_0 \right] \lambda^{-(j-1)\theta} \]

\[ + \lambda^{2p+\kappa} \left( \mathcal{L}(y, D_y) + O(\lambda^{-\theta}) \right). \]
Let us denote
\[
\Phi_j(l^1, \ldots, l^j, \alpha_1, \ldots, \alpha_{j-1}) := \mathcal{L}_j(l^1, \ldots, l^j) + \alpha_{j-1}(y) + H_{j-1}(\alpha_i, l^k | i + k \leq j - 1).
\]

Then we can rewrite the right hand side of (4.9) as
\[
\lambda^{2\delta p + 2\kappa} \sum_{j=1}^{\nu} \begin{bmatrix}
\Phi_j(l^1, \ldots, l^j, \alpha_1, \ldots, \alpha_{j-1}) & 0 \\
0 & \Phi_j(l^1, \ldots, l^j, \beta_1, \ldots, \beta_{j-1})
\end{bmatrix} \lambda^{-(j-1)\theta} + \lambda^{2\delta p + \kappa} (\mathcal{L}(y, D_y) + O(\lambda^{-\theta}))
\]
where \(\mathcal{L}(y, D_y)\) has the form
\[
\mathcal{L}(y, D_y) = \begin{bmatrix}
\sum_{j=0}^{n} \partial_{\xi_j} h_0(y, l^1_2) D_j & 0 \\
0 & \sum_{j=0}^{n} \partial_{\xi_j} h_0(y, l^1_2) D_j
\end{bmatrix} + \tilde{C}(y)
\]
\[
= 2l^1_{y_0} D_0 I_2 + \sum_{j=1}^{n} C_j(y) D_j + \tilde{C}(y).
\]

Recall
\[
\gamma \lambda^{1+\delta \rho} \alpha(\lambda^{-\sigma} y) = \lambda^{2\delta p + 2\kappa} \sum_{j=0}^{\nu} \lambda^{-j\theta} \alpha_j(y)
\]
\[
\gamma \lambda^{1+\delta \rho} \beta(\lambda^{-\sigma} y) = \lambda^{2\delta p + 2\kappa} \sum_{j=0}^{\nu} \lambda^{-j\theta} \beta_j(y)
\]
and \(\alpha_0(y) = \gamma \bar{\alpha}_0(y), \bar{\alpha}_0(y) \neq 0\) in \(U\). We have to consider two cases

1. \(\alpha_j(y) = \beta_j(y)\) in \(U\), \(0 \leq j \leq \nu - 1\),
2. \(\exists k \leq \nu - 1, \exists \bar{y} \in U\) such that
   \[
   \alpha_j(y) = \beta_j(y) \quad \text{in} \ U, \quad 0 \leq j \leq k - 1,
   \]
   \[
   \alpha_k(\bar{y}) \neq \beta_k(\bar{y}).
   \]

In case (2) we can take \(W_1 \subset U\), a neighborhood of \(\bar{y}\), so that
\[
|\alpha_k(y) - \beta_k(y)| \geq \delta c > 0 \quad \text{in} \ W_1.
\]

We can choose \(\gamma \in \mathbb{R}\) and \(W_2 \subset W_1\), a neighborhood of \(\bar{y}\), in such a way that the equation
\[
\Phi_1(y, \zeta) = h_0(y, \zeta) + \gamma \bar{\alpha}_0(y) = 0
\]
has a root $\zeta_0 = F(y, \zeta_1, \ldots, \zeta_n)$ with $\Re F < -\delta_0 < 0$ for $y \in W_2$ and $(\zeta_1, \ldots, \zeta_n)$ near 0 because $h_0(0, \zeta_0, 0, \ldots, 0) = \zeta_0^2$.

We solve the Cauchy problem

$$
\begin{aligned}
\left\{ 
I_{y_0}^1 = F(y, l^1_{y_1}, \ldots, l^1_{y_n}), \\
|l^1|_{y_0=\bar{y}_0} = i|y' - \bar{y}'|^2,
\end{aligned}
$$

$$
\tag{4.10}
\begin{aligned}
l^1 |_{y_0=\bar{y}_0} = 0, \quad y' = (y_1, \ldots, y_n).
\end{aligned}
$$

We find $l^j$ as the solution of ($2 \leq j \leq \nu$)

$$
\begin{aligned}
\Phi_j(l^1, \ldots, l^j, \alpha_1, \ldots, \alpha_{j-1}) = \sum_{\mu=0}^{n} \partial_{\xi_\mu} h_0(y, l^1_{y_1}) \partial_{\xi_\mu} l^i + \alpha_{j-1}(y) \\
+ H_{j-1}(l^1, \ldots, l^{j-1}, \alpha_1, \ldots, \alpha_{j-2}) = 0
\end{aligned}
$$

$$
\tag{4.11}
\begin{aligned}
l^j |_{y_0=\bar{y}_0} = 0.
\end{aligned}
$$

Note that the solutions of (4.10) and (4.11) are analytic.

In case (1) we have

$$
\Phi_j(l^1, \ldots, l^j, \beta_1, \ldots, \beta_{j-1}) = 0, \quad 0 \leq j \leq \nu.
$$

In case (2) we have

$$
\Phi_j(l^1, \ldots, l^j, \beta_1, \ldots, \beta_{j-1}) = 0 \quad \text{for} \quad j = 0, \ldots, k,
$$

$$
|\Phi_{k+1}(l^1, \ldots, l^{k+1}, \beta_1, \ldots, \beta_k)| \geq c > 0 \quad \text{in} \quad W_2
$$

because, if $\alpha_i = \beta_i$, $0 \leq i \leq j - 2$, then

$$
\Phi_j(l^1, \ldots, l^j, \alpha_1, \ldots, \alpha_{j-1}) - \Phi_j(l^1, \ldots, l^j, \beta_1, \ldots, \beta_{j-1}) = \alpha_{j-1} - \beta_{j-1}.
$$

Finally we have

$$
\tag{4.12}
E(y, \lambda)^{-1}(L + B)(\lambda)^{\langle c_0L + c_0B + C \rangle}(\lambda) E(y, \lambda)
$$

$$
= \begin{bmatrix}
0 & 0 \\
0 & \sum_{j=k+1}^{\nu} \Phi_j(l^1, \ldots, l^j, \beta_1, \ldots, \beta_{j-1}) \lambda^{-(j-1)\theta}
\end{bmatrix} \lambda^{2p+2\kappa}
$$

$$
+ \lambda^{2p+\kappa} \left( L(y, D_y) + O(\lambda^{-\theta}) \right).
$$

We apply this operator to

$$
\tag{4.13}
\sum_{j=0}^{\nu} v_j \lambda^{-j\theta}, \quad v_j = \begin{bmatrix} v_j^I \\ v_j^J \end{bmatrix}.
$$
In case (1) we see that
\[ E(y, \lambda)^{-1}(L + B)(\lambda)^{(\infty) L + \infty B + C)(\lambda)}E(y, \lambda) \sum_{j=0} v_j \lambda^{-j \theta} = 0 \]
leads us to
\[ \mathcal{L}(y, D_y) v_q + R_q(v_1, \ldots, v_{q-1}) = 0. \]
In case (2) take \( v^{(II)}_0 = \cdots = v^{(II)}_{\nu+k-1} = 0 \) and note that
\[
\sum_{j=k+1}^{\nu} \Phi_j(l^1, \ldots, l^j, \beta_1, \ldots, \beta_{j-1}) \lambda^{-(j-1) \theta} \sum_{j=0} v^{(II)}_j \lambda^{-j \theta} \\
= \lambda^{-\kappa} \sum_{q=0}^{q+(\nu-k)} \left( \sum_{i=q+1}^{q+\nu+1-i} \Phi_{q+\nu+1-i} v^{(II)}_i \right) \lambda^{-q \theta}.
\]
Thus the equation obtained by applying (4.12) to (4.13) is reduced to
\[
\begin{cases}
\mathcal{L}_{11}(x, D)v^I_q + H^1_q(v^I_0, \ldots, v^I_{q-1}) + G^1_q(v^{(II)}_{\nu-k}, \ldots, v^{(II)}_{q-1}) = 0 \\
\Phi_{k+1} v^{(II)}_{q+(\nu-k)} + H^2_q(v^I_0, \ldots, v^I_q) + G^2_q(v^{(II)}_{\nu-k}, \ldots, v^{(II)}_{q+(\nu-k)-1}) = 0
\end{cases}
\]
for \( q = 0, 1, 2, \ldots \). The second equation is solved by
\[ v^{(II)}_{q+(\nu-k)} = -\Phi^{-1}_{k+1} \left[ H^2_q(v^I_0, \ldots, v^I_q) + G^2_q(v^{(II)}_{\nu-k}, \ldots, v^{(II)}_{q+(\nu-k)-1}) \right]. \]
We finally get
\[ E(y, \lambda)^{-1}(L + B)(\lambda)^{(\infty) L + \infty B + C)(\lambda)}E(y, \lambda) \sum_{j=0} v_j \lambda^{-j \theta} \sim 0. \]
In order to obtain an asymptotic solution to \((L + B)(\lambda)\) let us write
\[ E(y, \lambda)^{-1}(L + B)(\lambda)E(y, \lambda) \left\{ E(y, \lambda)^{-1}(\infty L + \infty B + C)(\lambda)E(y, \lambda) \sum_{j=0} v_j \lambda^{-j \theta} \right\} \sim 0 \]
and define (for a suitable \( \bar{k} \in \mathbb{Q}^+ \))
\[ \lambda^{\bar{k}} \sum_{j=0} u_j \lambda^{-j \theta} := E(y, \lambda)^{-1}(\infty L + \infty B + C)(\lambda)E(y, \lambda) \sum_{j=0} v_j \lambda^{-j \theta}. \]
We must examine that we can take initial data \(v_0\) so that \(u_0(\tilde{y}) \neq 0\). Note that

\[
E(y, \lambda)^{-1} [\sigma L(\lambda) E(y, \lambda)] = \lambda^{\delta p + \kappa} l_{y_0}^1 I_2 - \sum_{j=1}^{n-1} A_j (\lambda^{-\sigma} y) \lambda^{\delta p + \kappa} l_{y_j}^1
\]

\[- A_n (\lambda^{-\sigma} y) (\lambda^{\delta p + \kappa} l_{y_n}^1 + \lambda^{1+\delta q} \gamma) + L(\lambda)(y, D_y).\]

Remark that \(\kappa < 1\) and

\[
l_{y_j}^1 (\tilde{y}) = 0, \quad 1 \leq j \leq n.
\]

Write

\[
A^n (\lambda^{-\sigma} y) = \begin{bmatrix} \lambda^{-\alpha} (a(y) + O(\lambda^{-\theta})) & \lambda^{-\beta} (b(y) + O(\lambda^{-\theta})) \\ \lambda^{-\beta} (\bar{b}(y) + O(\lambda^{-\theta})) & \lambda^{-\alpha} (\bar{a}(y) + O(\lambda^{-\theta})) \end{bmatrix}
\]

where \(\alpha \geq \beta \geq 0\). If \(\delta p + \kappa\) is \(>, =, <\) of \(1 + O\) we have that the principal part is, respectively:

\[
l_{y_0}^1 I_2,
\]

\[
l_{y_0}^1 I_2 + \gamma \begin{bmatrix} \lambda^\beta - \alpha (\ast) & b(y) + O(\lambda^{-\theta}) \\ \bar{b}(y) + O(\lambda^{-\theta}) & \lambda^\beta - \alpha (\ast) \end{bmatrix},
\]

\[
\gamma \begin{bmatrix} \lambda^\beta - \alpha (\ast) & b(y) + O(\lambda^{-\theta}) \\ \bar{b}(y) + O(\lambda^{-\theta}) & \lambda^\beta - \alpha (\ast) \end{bmatrix},
\]

where \(\ast\) stands for some non vanishing term. Then it is clear that we can take \(v_0\) so that \(u_0(\tilde{y}) \neq 0\).

Let \(N\) be fixed. Then we can find a neighborhood \(W_N\) of \(\tilde{y}\) so that \(v_j (0 \leq j \leq N)\) is defined in \(W := W_N\). Set

\[
U_\lambda := \lambda^k \sum_{j=0}^{N} u_j \lambda^{-j \theta}.
\]

Take \(\chi \in C_0^\infty (W)\) which is equal to 1 near \(\tilde{y}\). Then it is easy to see that

\[
|(L + B)(\lambda) \chi E(y, \lambda) U_\lambda|_{C^r (W_{y_0})} = O(\lambda^{-\theta N + 2 \delta p + \kappa + r}),
\]

because

\[
3 \sum_{\nu=1}^{\nu} \tilde{v}(y) \lambda^{\theta (\nu + 1 - j)} \geq c_1 \left[ (\tilde{y}_0 - y_0) + |y' - \tilde{y}'|^2 \right] \lambda^{\delta \nu}
\]
in supp $\chi \cap \{ y \mid y_0 \leq \bar{y}_0 \}$ with some $c_1 > 0$. On the other hand we have

$$|\chi E(y, \lambda)U_\lambda|_{C^0(W_{\bar{y}_0})} \geq c_2 \lambda^k,$$

with some $c_2 > 0$. Then the estimate in Proposition 4.2 fails if we take $N$ large enough. This completes the proof.

5. The invariance of the conditions

The aim of this section is to prove the following

**Proposition 5.3.** Assume $\text{Ord}[h_\lambda] = 2 \delta p$. Then the condition (L) is invariant under real analytic change of bases in $\mathbb{C}^2$.

**Proof.** Assume that $\text{Ord}[h_\lambda] = 2 \delta p$ and there is a meromorphic matrix $C(x, \xi)$ with $\text{Ord} C_\lambda \leq \delta p$ such that

$$(\mathcal{L} + L_1 C)_\lambda = O(\lambda^{2\delta p}).$$

Let $S(x)$ be a smooth non-singular $2 \times 2$ matrix with analytic entries and define

$$\tilde{L}(x, D) = S(x)^{-1}L(x, D)S(x) = \tilde{L}_1(x, D) + \tilde{L}_0(x),$$

$$\tilde{L}_1(x, \xi) = S(x)^{-1}L_1(x, \xi)S(x).$$

It is enough to show that there is a $\tilde{C}$ with $\text{Ord}[\tilde{C}_\lambda] \leq \delta p$ such that

$$(\tilde{\mathcal{L}} + \tilde{L}_1 \tilde{C})_\lambda = O(\lambda^{2\delta p}).$$

After conjugation by $S(x)$, we have (see [17])

$$\tilde{\mathcal{L}} = S(x)^{-1}\mathcal{L}S(x) + S(x)^{-1}L_1 S(x)T$$

where $T$ is a smooth matrix. Thus, if we take $\tilde{C}(x, \xi) = S^{-1}(x)C(x, \xi)S(x) - T$ then $\text{Ord}[\tilde{C}_\lambda] \leq \delta p$, moreover we have

$$\text{Ord}[\tilde{\mathcal{L}} + \tilde{L}_1 \tilde{C}]_\lambda = \text{Ord}[\mathcal{L} + L_1 C]_\lambda \leq 2\delta p$$

i.e. the condition (L) is satisfied for $\tilde{L}(x, D)$, and we get the assertion. □

**Corollary 5.1.** Assume $\text{Ord}[h_\lambda] = 2 \delta p$. Then the condition (C) is invariant under real analytic changes of bases in $\mathbb{C}^2$.

**Proof.** It is clear that $\text{Ord}[\tilde{L}_1 \lambda] \leq 2 \delta p$ (the condition (C_1) is satisfied for $\tilde{L}(x, D)$). Note that

$$\tilde{L}^i = \tilde{\mathcal{L}} - \tilde{L}_0(x)^{co} \tilde{L}_1, \quad \mathcal{L}^i = \mathcal{L} - L_0(x)^{co} L_1.$$
Since $\text{Ord}[\tilde{L}_0(x)^{\alpha_0} L_1]\lambda \leq 2\delta p$, $\text{Ord}[L_0(x)^{\alpha_0} L_1]\lambda \leq 2\delta p$ the assertion follows from Proposition 5.3. □

References

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