2-graded decompositions of exceptional Lie algebras $\mathfrak{g}$
and group realizations of $\mathfrak{g}_{ev}, \mathfrak{g}_{0}$
Part III, $G = E_8$

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The 2-graded decompositions of simple Lie algebras $\mathfrak{g}$,

$$\mathfrak{g} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_{0} \oplus \mathfrak{g}_{1} \oplus \mathfrak{g}_{2}, \quad [\mathfrak{g}_{k}, \mathfrak{g}_{l}] \subset \mathfrak{g}_{k+l}$$

are classified and the types of subalgebras $\mathfrak{g}_{ev}, \mathfrak{g}_{0}$ are determined. The following table is the results of $\mathfrak{g}_{ev}, \mathfrak{g}_{0}$ for the exceptional Lie algebras of type $E_8$ (Kaneyuki [2]).

<table>
<thead>
<tr>
<th>$\mathfrak{g}$</th>
<th>$\mathfrak{g}_{ev}$</th>
<th>$\mathfrak{g}_{0}$</th>
<th>dim $\mathfrak{g}_{1}$</th>
<th>dim $\mathfrak{g}_{2}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mathfrak{e}_8^C$</td>
<td>$\mathfrak{sl}(2, C) \oplus \mathfrak{e}_7^C$</td>
<td>$C \oplus \mathfrak{e}_7^C$</td>
<td>56</td>
<td>1</td>
</tr>
<tr>
<td>$\mathfrak{e}_8(8)$</td>
<td>$\mathfrak{sl}(2, \mathbb{R}) \oplus \mathfrak{e}_7(7)$</td>
<td>$\mathbb{R} \oplus \mathfrak{e}_7(7)$</td>
<td>56</td>
<td>1</td>
</tr>
<tr>
<td>$\mathfrak{e}_8(-24)$</td>
<td>$\mathfrak{sl}(2, \mathbb{R}) \oplus \mathfrak{e}_7(-25)$</td>
<td>$\mathbb{R} \oplus \mathfrak{e}_7(-25)$</td>
<td>56</td>
<td>1</td>
</tr>
<tr>
<td>$\mathfrak{e}_8^C$</td>
<td>$\mathfrak{so}(16, C)$</td>
<td>$C \oplus \mathfrak{so}(14, C)$</td>
<td>64</td>
<td>14</td>
</tr>
<tr>
<td>$\mathfrak{e}_8(8)$</td>
<td>$\mathfrak{so}(8, 8)$</td>
<td>$\mathbb{R} \oplus \mathfrak{so}(7, 7)$</td>
<td>64</td>
<td>14</td>
</tr>
<tr>
<td>$\mathfrak{e}_8(-24)$</td>
<td>$\mathfrak{so}(4, 12)$</td>
<td>$\mathbb{R} \oplus \mathfrak{so}(3, 11)$</td>
<td>64</td>
<td>14</td>
</tr>
</tbody>
</table>

In the previous papers [7], [13], we gave group realizations of $\mathfrak{g}_{ev}, \mathfrak{g}_{0}$ for the exceptional universal linear Lie groups $G$ of type $G_2, F_4, E_6$ and $E_7$. Now, in this paper, for the exceptional universal linear Lie groups $G$ of type $E_8$, we realize the subgroups $G_{ev}, G_0$ of $G$ corresponding to $\mathfrak{g}_{ev}, \mathfrak{g}_{0}$ of $\mathfrak{g} = \text{Lie}G$. Our results are as follows. But as for the results of $(E_8^C)_{ev}, (E_8(8))_{ev}$ and $(E_8(-24))_{ev}$, we refer to [10] (cf. [12]).
This paper is a continuation of [7], [13] and we use the same notations as in [7], [13]. So the numbering of Sections and Theorems start from 5.1.

Group $E_8$

5.1. Lie groups of type $E_8$ and their Lie algebras

In a $C$-vector space $\mathfrak{e}_8^C$ and $R$-vector spaces $\mathfrak{e}_8(8)$, $\mathfrak{e}_8(-24)$,

\[
\begin{align*}
\mathfrak{e}_8^C &= \mathfrak{g}_7^C \oplus \mathfrak{p}_C \oplus \mathfrak{p}_C \oplus C \oplus C \oplus C, \\
\mathfrak{e}_8(8) &= \mathfrak{g}_{7(7)} \oplus \mathfrak{p}_7 \oplus \mathfrak{p}_7 \oplus R \oplus R \oplus R, \\
\mathfrak{e}_8(-24) &= \mathfrak{g}_{7(-25)} \oplus \mathfrak{p} \oplus \mathfrak{p} \oplus R \oplus R \oplus R,
\end{align*}
\]

we define the Lie bracket $[R_1, R_2]$ by

\[
[(\Phi_1, P_1, Q_1, r_1, u_1, v_1), (\Phi_2, P_2, Q_2, r_2, u_2, v_2)] = (\Phi, P, Q, r, u, v),
\]

\[
\begin{align*}
\Phi &= [\Phi_1, \Phi_2] + P_1 \times Q_2 - P_2 \times Q_1, \\
P &= \Phi_1 P_2 - \Phi_2 P_1 + r_1 P_2 - r_2 P_1 + u_1 Q_2 - u_2 Q_1, \\
Q &= \Phi_1 Q_2 - \Phi_2 Q_1 - r_1 Q_2 + r_2 Q_1 + v_1 P_2 - v_2 P_1, \\
r &= -\frac{1}{8} \{P_1, Q_2\} + \frac{1}{8} \{P_2, Q_1\} + u_1 v_2 - u_2 v_1, \\
u &= \frac{1}{4} \{P_1, P_2\} + 2r_1 u_2 - 2r_2 u_1, \\
v &= -\frac{1}{4} \{Q_1, Q_2\} - 2r_1 v_2 + 2r_2 v_1,
\end{align*}
\]

then these become the simple Lie algebras of type $E_8^C$, $E_8(8)$ and $E_8(-24)$, respectively. The Killing form $B_8(R_1, R_2)$ of the Lie algebra $\mathfrak{e}_8^C$ is given by

\[
B_8(R_1, R_2) = \frac{5}{3} B_7(\Phi_1, \Phi_2) + 15 \{Q_1, P_2\} - 15 \{P_1, Q_2\} + 120 r_1 r_2 + 60 v_1 u_2 + 60 u_1 v_2
\]
\[(R_k = (\Phi_k, P_k, Q_k, r_k, u_k, v_k) \in \mathfrak{e}_8^C), \text{ where } B_7(\Phi_1, \Phi_2) \text{ is the Killing form of the Lie algebra } \mathfrak{e}_7^C, \text{ which is given by}
\]
\[B_7(\Phi_1, \Phi_2) = \frac{3}{2} B_6(\phi_1, \phi_2) + 36(A_1, B_2) + 36(A_2, B_1) + 24\nu_1\nu_2\]

\[(\Phi_k = \Phi(\phi_k, A_k, B_k, \nu_k) \in \mathfrak{e}_7^C), \text{ where } B_6(\phi_1, \phi_2) \text{ is the Killing form of the Lie algebra } \mathfrak{e}_6^C, \text{ which is given by}
\]
\[B_6(\phi_1, \phi_2) = \frac{4}{3} B_4(\delta_1, \delta_2) + 12(T_1, T_2)\]

\[(\phi_k = \delta_k + \tilde{T}_k \in \mathfrak{e}_6^C), \text{ where } B_4(\delta_1, \delta_2) \text{ is the Killing form of the Lie algebra } \mathfrak{f}_4^C, \text{ which is given by}
\]
\[B_4(\delta_1, \delta_2) = 3\text{tr}(\delta_1\delta_2)\]

\[(\delta_k \in \mathfrak{f}_4^C), \text{ where } \text{tr}(\delta_1\delta_2) \text{ is the trace of } \delta_1\delta_2 \text{ in } \mathfrak{f}_4^C.\]

We define \(C\)-linear transformations \(\gamma, \lambda\) and \(\lambda'\) of \(\mathfrak{e}_8^C\) by

\[\gamma(\Phi, P, Q, r, u, v) = (\gamma \Phi \gamma, \gamma P, \gamma Q, r, u, v),\]
\[\lambda(\Phi, P, Q, r, u, v) = (\lambda \Phi \lambda^{-1}, \lambda P, \lambda Q, r, u, v),\]

where \(\gamma, \lambda\) of the right sides are the same as \(\gamma \in G_2^C \subset F_4^C \subset E_6^C \subset E_7^C, \\lambda \in E_7^C\) and

\[\lambda'(\Phi, P, Q, r, u, v) = (\Phi, Q, -P, -r, -v, -u),\]

respectively. Moreover we define \(\tilde{\lambda}\) by

\[\tilde{\lambda} = \lambda \lambda' = \lambda' \lambda.\]

The complex conjugation in \(\mathfrak{e}_8^C\) is denoted by \(\tau\):

\[\tau(\Phi, P, Q, r, u, v) = (\tau \Phi \tau, \tau P, \tau Q, \tau r, \tau u, \tau v).\]

The connected universal linear Lie groups of type \(E_8\) are given by

\[E_8^C = \{ \alpha \in \text{Iso}_C(\mathfrak{e}_8^C) \mid [\alpha[R, R'] = [\alpha R, \alpha R']\},\]
\[E_8 = \{ \alpha \in \text{Iso}_C(\mathfrak{e}_8^C) \mid [\alpha[R, R'] = [\alpha R, \alpha R'], (\alpha R, \alpha R') = (R, R')\},\]
\[E_8(8) = \{ \alpha \in \text{Iso}_R(\mathfrak{e}_8(8)) \mid [\alpha[R, R'] = [\alpha R, \alpha R']\},\]
\[E_8(-24) = \{ \alpha \in \text{Iso}_R(\mathfrak{e}_8(-24)) \mid [\alpha[R, R'] = [\alpha R, \alpha R']\},\]
where \( \langle R, R' \rangle = -B_8(\tau R, R') \). \( E_8 \) and \( E_8^C \) are simply connected. From the definitions of the groups above, we have

**Proposition 5.1.** \( E_8 = (E_8^C)_{\gamma - \lambda} \), \( E_{8(8)} = (E_8^C)_{\gamma + \lambda} \), \( E_{8(-24)} = (E_8^C)^{\gamma} \).

For \( \alpha \in E_7^C \), if the mapping \( \tilde{\alpha} : e_8^C \rightarrow e_8^C \) is defined by

\[
\tilde{\alpha}(\Phi, P, Q, r, u, v) = (\alpha \Phi \alpha^{-1}, \alpha P, \alpha Q, r, u, v),
\]

then \( \tilde{\alpha} \in E_8^C \), so \( \alpha \) and \( \tilde{\alpha} \) will be identified. The group \( E_8^C \) contains \( E_7^C \) as a subgroup by

\[
E_7^C = \{ \tilde{\alpha} \in E_8^C \mid \alpha \in E_7^C \}
= (E_8^C)_{(0,0,0,1,0,0),(0,0,0,0,1,0),(0,0,0,0,0,1)}.
\]

Similarly, \( E_7 \subset E_8 \). In particular, \( \gamma \in E_8 \subset E_7 \subset E_8 \subset E_8^C \), \( \lambda \in E_7 \subset E_8 \subset E_8^C \). Furthermore, \( \lambda', \lambda \in E_8 \subset E_8^C \) and \( \lambda'^2 = \lambda^2 = 1 \).

Hereafter, in \( \mathfrak{P}^C \) and \( e_8^C \), we use the following notations.

\[
\begin{align*}
\hat{X} &= (X, 0, 0, 0, 0), \quad Y = (0, Y, 0, 0), \quad \hat{\xi} = (0, 0, \xi, 0) \quad \eta = (0, 0, 0, \eta), \\
\Phi &= (\Phi, 0, 0, 0, 0, 0), \quad P^- = (0, P, 0, 0, 0, 0), \quad Q^- = (0, 0, Q, 0, 0, 0), \\
\tilde{r} &= (0, 0, 0, r, 0, 0), \quad u^- = (0, 0, 0, 0, u, 0), \quad v^- = (0, 0, 0, 0, 0, v).
\end{align*}
\]

**5.2. Subgroups of type \( A_1^C \oplus E_7^C \) and \( C \oplus E_7^C \) of \( E_8^C \)**

We define \( C \)-linear transformations \( v \) and \( v_3 \) of \( e_8^C \) by

\[
\begin{align*}
v(\Phi, P, Q, r, u, v) &= (\Phi, -P, -Q, r, u, v), \\
v_3(\Phi, P, Q, r, u, v) &= (\Phi, \omega^2 P, \omega Q, r, \omega u, \omega^2 v),
\end{align*}
\]

\( (\omega = -\frac{1}{2} + \frac{\sqrt{3}i}{2}, \) respectively. Now, in the Lie algebra \( e_8^C \), let \( Z = (0, 0, 0, -1, 0, 0) = -\tilde{1} \), then we can verify that

\[
v = \exp \frac{2\pi i}{2} \text{ad } Z, \quad v_3 = \exp \frac{2\pi i}{3} \text{ad } Z.
\]

**Theorem 5.2.** The 2-graded decomposition of \( e_8^C \) (\( e_{8(8)} = (e_8^C)^{\gamma + \lambda} \) or \( e_{8(-24)} = (e_8^C)^{\gamma} \)),

\[
e_8^C = g_{-2} \oplus g_{-1} \oplus g_0 \oplus g_1 \oplus g_2
\]
with respect to ad \( Z, Z = (0, 0, 0, -1, 0, 0) \), is given by

\[
\mathfrak{g}_0 = \{ \tilde{1}, \Phi \mid \Phi \in \mathfrak{e}_7^C (\mathfrak{e}_7(7) \text{ or } \mathfrak{e}_7(-25)) \} 134,
\]

\[
\mathfrak{g}_{-1} = \{ P^- \mid P \in \mathfrak{p}_C (\mathfrak{p}^* \text{ or } \mathfrak{p}) \} 56, \quad \mathfrak{g}_{-2} = \{ 1^- \} 1,
\]

\[
\mathfrak{g}_1 = \tilde{\lambda}(\mathfrak{g}_{-1}), \quad \mathfrak{g}_2 = \tilde{\lambda}(\mathfrak{g}_{-2}).
\]

**Proof.** We can easily obtain this theorem from the Lie bracket form

\[ [Z, (\Phi, P, Q, r, u, v)] = (0, -P, Q, 0, -2u, 2v) \]

in the Lie algebra \( \mathfrak{e}_8 \) (\( \mathfrak{e}_8(8) \) or \( \mathfrak{e}_8(-24) \)).

We shall determine the group structures of

\[
(E_8^C)_{ev} = (E_8^C)^v, \quad (E_8^C)_0 = (E_8^C)^{v3}.
\]

We define a mapping \( \psi : SL(2, C) \rightarrow E_8^C, A \rightarrow \psi(A) \), where \( \psi(A) \) is the \( C \)-linear transformation of \( \mathfrak{e}_8^C = \mathfrak{e}_7^C \oplus \mathfrak{p}^C \oplus \mathfrak{p}^C \oplus C \oplus C \oplus C \) defined by

\[
\psi \left( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\
0 & a & b & 1 & 0 & 0 \\
0 & c & d & 1 & 0 & 0 \\
0 & 0 & 0 & ad + bc & -ac & bd \\
0 & 0 & 0 & -2ab & a^2 & -b^2 \\
0 & 0 & 0 & 2cd & -c^2 & d^2 \end{pmatrix}.
\]

**Theorem 5.3.** (1) \( (E_8^C)_{ev} \cong (SL(2, C) \times E_7^C) / \mathbb{Z}_2, \mathbb{Z}_2 = \{(E, 1), (-E, -1)\} \).

(2) \( (E_8^C)_0 \cong (C^* \times E_7^C) / \mathbb{Z}_2, \mathbb{Z}_2 = \{(1, 1), (-1, -1)\} \).

**Proof.** (1) We define \( \varphi : SL(2, C) \times E_7^C \rightarrow (E_8^C)^v \) by

\[
\varphi(A, \beta) = \psi(A)\beta.
\]

\( \varphi \) is well-defined, because \( \psi(A) \in (E_8^C)^v \). Since \( \psi(A) \) and \( \beta \in E_7^C \) commute, \( \varphi \) is a homomorphism. \( \text{Ker} \varphi = \mathbb{Z}_2 \). Since \( (E_8^C)^v \) is connected and \( \text{dim}_C(sl(2, C) \oplus e_7^C) = 3 + 133 = 136 = 134 \times 2 \times 1 = \text{dim}_C((\mathfrak{e}_8^C)^{ev}) \) (Theorem 5.2), \( \varphi \) is onto. Therefore \( (E_8^C)_{ev} \cong (SL(2, C) \times E_7^C) / \mathbb{Z}_2 \).

(2) Let \( C^* \) be the subgroup \( \left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \mid a \in C^* \right\} \) of \( SL(2, C) \) and \( \varphi : C^* \times E_7^C \rightarrow (E_8^C)^{v3} \) be the restriction mapping of \( \varphi \) above. \( \varphi \) is well-defined and \( \text{Ker} \varphi = \mathbb{Z}_2 \). Since \( (E_8^C)^{v3} \) is connected and \( \text{dim}_C(C \oplus e_7^C) = 1 + 133 = 134 = \text{dim}_C(\mathfrak{e}_8^C)_0 \) (Theorem 5.2), \( \varphi \) is onto. Therefore \( (E_8^C)_0 \cong (C^* \times E_7^C) / \mathbb{Z}_2 \).
5.2.1. Subgroups of type $A_1 \oplus E_7(7)$ and $R \oplus E_7(7)$ of $E_8(8)$

We shall determine the group structures of

$$(E_8(8))_{ev} = (E_8^C)^v \cap (E_8^C)^\gamma, \quad (E_8(8))_0 = (E_8^C)^{v_3} \cap (E_8^C)^{\gamma_1}.$$

We define $\nu_1 \in E_8^C$ by $\nu_1 = \psi(iI)i$ ($i$ is found in 4.1). The explicit form of $\nu_1$ is

$$\nu_1(P, P, Q, r, u, v) = (\nu_i, -itP, -itQ, r, -u, -v).$$

**Lemma 5.4.** In the group $E_8^C$, we have

$$\tau \psi(A) \tau = \psi(\tau A), \quad \gamma \psi(A) \gamma = \psi(A).$$

**Theorem 5.5.** (1) $(E_8(8))_{ev} \cong (SL(2, R) \times E_7(7))/Z_2 \times \{1, \nu_1\}$, $Z_2 = \{(E, 1), (-E, -1)\}$.

(2) $(E_8(8))_0 \cong (R^+ \times E_7(7)) \times \{1, \nu_1\}$.

**Proof.** (1) For $\alpha \in (E_8(8))_{ev} \subset (E_8^C)^v$, there exist $A \in SL(2, C)$ and $\beta \in E_7^C$ such that $\alpha = \varphi(A, \beta) = \psi(A)\beta$ (Theorem 5.3.(1)). From $\gamma \tau \alpha \gamma = \alpha$, we have

$$\begin{cases} \gamma \tau \psi(A) \tau \gamma = \psi(A) & \quad \text{or} \quad \gamma \tau \psi(A) \tau \gamma = \psi(-A) \\ \gamma \tau \beta \tau \gamma = \beta \end{cases} \quad \begin{cases} \gamma \tau \psi(A) \tau \gamma = \psi(\alpha) \\ \gamma \tau \beta \tau \gamma = -\beta. \end{cases}$$

In the former case, from $\psi(\tau A) = \psi(A)$ (Lemma 5.4), we have $\tau A = A$, hence $A \in SL(2, R)$. The group $(E_8^C)^{\gamma_1}$ is $E_7(7)$ by definition. Hence the group of the former case is $(SL(2, R) \times E_7(7))/Z_2$. In the latter case, $A = iI$ and $\beta = \iota$ satisfy the conditions and $\varphi(iI, \iota) = \nu_1$. Therefore $(E_8(8))_{ev} \cong (SL(2, R) \times E_7(7))/Z_2 \times \{1, \nu_1\}$.

(2) For $\alpha \in (E_8(8))_0 \subset (E_8^C)^{v_3}$, there exist $A \in C^* \subset SL(2, C)$ and $\beta \in E_7^C$ such that $\alpha = \varphi(A, \beta) = \psi(A)\beta$ (Theorem 5.3.(2)). From $\gamma \tau \alpha \gamma = \alpha$, we have

$$\begin{cases} \gamma \tau \psi(A) \tau \gamma = \psi(A) & \quad \text{or} \quad \gamma \tau \psi(A) \tau \gamma = \psi(-A) \\ \gamma \tau \beta \tau \gamma = \beta \end{cases} \quad \begin{cases} \gamma \tau \psi(A) \tau \gamma = \psi(-A) \\ \gamma \tau \beta \tau \gamma = -\beta. \end{cases}$$

In the former case, from $\psi(\tau A) = \psi(A)$ (Lemma 5.4), we have $\tau A = A$, hence $A \in R^*$. The group $(E_7^C)^{\gamma_1}$ is $E_7(7)$. Hence the group of the former case is $(R^* \times E_7(7))/Z_2$ $(Z_2 = \{(1, 1), (-1, -1)\}) \cong R^+ \times E_7(7)$. In the latter case, $A = iI$ and $\beta = \iota$ satisfy the conditions and $\varphi(iI, \iota) = \nu_1$. Therefore $(E_8(8))_0 \cong (R^+ \times E_7(7)) \times \{1, \nu_1\}$. 

5.2.2. Subgroups of type $A_1 \oplus E_7(-25)$ and $R \oplus E_7(-25)$ of $E_8(-24)$

We shall determine the group structures of

$$
(E_{8(-24)})_{ev} = (E_8^C)^\nu \cap (E_8^C)^r, \quad (E_{8(-24)})_0 = (E_8^C)^\nu_3 \cap (E_8^C)^r.
$$

**Theorem 5.6.** (1) $$(E_{8(-24)})_{ev} \cong (SL(2, R) \times E_7(-24))/\mathbb{Z}_2 \times \{1, \nu_1\}, \quad \mathbb{Z}_2 = \{(E, 1), (-E, -1)\}.$$

(2) $$(E_{8(-24)})_0 \cong (R^+ \times E_7(-25)) \times \{1, \nu_1\}. $$

**Proof.** We can prove this theorem in a way similar to Theorem 5.5, using Theorem 5.2.

5.3. Subgroup of type $C \oplus D_7^C$ of $E_8^C$

We define $C$-linear transformations $\sigma$ and $\kappa_3$ of $\mathfrak{e}_8^C$ by

$$
\sigma(\Phi, P, Q, r, u, v) = (\sigma \Phi \sigma, \sigma P, \sigma Q, r, u, v),
$$

$$
\kappa_3(\Phi, P, Q, r, u, v) = (\kappa_3 \Phi \kappa_3^{-1}, \omega^2 \kappa_3 P, \omega \kappa_3 Q, r, \omega u, \omega^2 v),
$$

where $\sigma$ of the right side is the same as $\sigma \in F_4 \subset E_6 \subset E_7 \subset E_7^C$ and $\kappa_3$ is the mapping defined in 4.3. Now, in the Lie algebra $\mathfrak{e}_8^C$, let $Z = (\kappa, 0, 0, -1, 0, 0) = (\Phi(-2E_1 \vee E_1, 0, 0, -1), 0, 0, -1, 0, 0)$ and we denote $ad Z$ by $\tilde{\kappa}$. Then we can verify that

$$
\sigma = \exp \frac{2\pi i}{2} \tilde{\kappa}, \quad \kappa_3 = \exp \frac{2\pi i}{3} \tilde{\kappa}.
$$

**Theorem 5.7.** The 2-graded decomposition of $\mathfrak{e}_8^C$,

$$
\mathfrak{e}_8^C = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2
$$

with respect to $ad Z$, $Z = (\kappa, 0, 0, -1, 0, 0)$, is given by

$$
\mathfrak{g}_0 = \left\{ \begin{pmatrix} d_1 & 0 & 0 \\ 0 & 0 & \tau_1 \\ 0 & -\frac{d_1}{\tau_1} & 0 \end{pmatrix}, \begin{pmatrix} \tau_2 & t_1 \\ 0 & \tau_3 \end{pmatrix}, \begin{pmatrix} \alpha_2 & a_1 \\ a_1 & \alpha_3 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ \rho_2 & p_1 \\ \frac{1}{p_1} & \rho_3 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} \zeta & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right\},
$$

$$
\begin{pmatrix} 0 & \beta_2 & b_1 \\ \beta_1 & b_3 \end{pmatrix}, \begin{pmatrix} \rho_1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0, \rho \end{pmatrix}, \begin{pmatrix} 0, 0, 0 \end{pmatrix}, \begin{pmatrix} 0, 0, 0 \end{pmatrix},
$$

$$
\begin{pmatrix} 0, 0, 0 \end{pmatrix}, \begin{pmatrix} 0, 0, 0 \end{pmatrix}, \begin{pmatrix} 0, 0, 0 \end{pmatrix}.
$$
The element above of $g_0$ is simply denoted by

$$R_0(\Phi(D, d_1\sim, (\tau_1, \tau_2, \tau_3, t_1)\sim, (\alpha_2, \alpha_3, a_1), (\beta_2, \beta_3, b_1), \nu),$$

$$((\rho_2, \rho_3, p_1), \rho_1, \rho), (\zeta_1, (\zeta_2, \zeta_3, z_1), \zeta), r).$$

$$g_1 = \left\{ \begin{array}{l}
\Phi\left( \frac{0}{a_3} - \frac{a_2}{a_3}, 0 \right) + \frac{0}{a_2}, a_2 0 0 \right\}
\begin{pmatrix}
0 & x_2 & 0 0 \\
\overline{x_2} & 0 0 \\
x_2 & 0 0 \\
\end{pmatrix},
\begin{pmatrix}
0 & y_3 & 0 0 \\
\overline{y_3} & 0 0 \\
y_3 & 0 0 \\
\end{pmatrix},
\begin{pmatrix}
0 & z_3 & \overline{z_2} \\
\overline{z_3} & 0 0 \\
z_3 & 0 0 \\
\end{pmatrix},
\begin{pmatrix}
0, 0, 0, 0, 0, 0, 0, 0, 0, 0 \\
\end{pmatrix}
\right\} \quad | a_k, z_k, x_k, y_k \in \mathbb{C}^C \quad 64,
$$

$$g_2 = \left\{ \begin{array}{l}
\Phi\left( \frac{0}{a_3} - \frac{a_2}{a_3}, 0 \right) + \frac{0}{a_2}, a_2 0 0 \right\}
\begin{pmatrix}
0 & x_2 & 0 0 \\
\overline{x_2} & 0 0 \\
x_2 & 0 0 \\
\end{pmatrix},
\begin{pmatrix}
0 & y_3 & 0 0 \\
\overline{y_3} & 0 0 \\
y_3 & 0 0 \\
\end{pmatrix},
\begin{pmatrix}
0 & z_3 & \overline{z_2} \\
\overline{z_3} & 0 0 \\
z_3 & 0 0 \\
\end{pmatrix},
\begin{pmatrix}
0, 0, 0, 0, 0, 0, 0, 0, 0, 0 \\
\end{pmatrix}
\right\} \quad | a_k, z_k, x_k, y_k \in \mathbb{C}^C \quad 64,
$$
The element above of $g_2$ is simply denoted by

\[ R_2(\zeta_1, ((\xi_2, \xi_3, x_1), \eta_1, \eta), v). \]

**Proof.** We can easily obtain this theorem from the Lie bracket form

\[ [Z, (\Phi, P, Q, r, u, v)] = ([\kappa, \Phi], \kappa P - P, \kappa Q + Q, 0, -2u, 2v) \]

in the Lie algebra $e_8^C$, using Theorem 4.21.

To prove the following result $(E_8^C)_0 \cong (C^* \times \text{Spin}(14, C))/\mathbb{Z}_4$ (Theorem 5.10), we need to find a subgroup of $E_8^C$ which is isomorphic to the group $\text{Spin}(14, C)$. For this purpose, we define two $C$-linear mappings $\mu_1 : e_8^C \rightarrow e_8^C$ and $\delta : g_2 \rightarrow g_2$ by

\[ \tilde{\mu}_1 = \exp \frac{\pi i}{2} \text{ad} \bar{\mu}, \quad \bar{\mu} = (\mu, 0, 0, 0, 1, 1) \in e_8^C, \quad \mu = \Phi(0, E_1, E_1, 0) \in e_7^C \]

and

\[ \delta(R_2(\zeta_1, ((\xi_2, \xi_3, x_1), \eta_1, \eta), v)) = R_2(-v, ((\xi_2, \xi_3, x_1), \eta_1, \eta), -\zeta_1), \]

respectively. The action of $\tilde{\mu}_1$ on $e_8^C$ is given by

\[ \tilde{\mu}_1(\Phi, P, Q, r, u, v) = (\mu_1 \Phi \mu_1^{-1}, i\mu_1 Q, i\mu_1 P, -r, v, u), \]

where

\[ \mu_1(X, Y, \xi, \eta) = \begin{pmatrix} i\eta & x_3 & \bar{x}_2 \\ \bar{x}_3 & i\eta_3 & -iy_1 \\ x_2 & -iy_1 & i\eta_2 \end{pmatrix}, \quad \begin{pmatrix} i\xi & y_3 & \bar{y}_2 \\ \bar{y}_3 & i\xi_3 & -ix_1 \\ y_2 & -ix_1 & i\xi_2 \end{pmatrix}, \quad i\eta_1, i\xi_1 \].

In particular, the explicit form of the mapping $\tilde{\mu}_1 : g_{-2} \rightarrow g_2$ is given by

\[ \tilde{\mu}_1(R_{-2}(\zeta_1, (\xi_1, (\eta_2, \eta_3, y_1), \xi), u)) = R_2(\zeta_1, ((-\eta_3, -\eta_2, y_1), \xi, \xi_1), u). \]

The composition mapping $\delta \tilde{\mu}_1 : g_{-2} \rightarrow g_2$ of $\tilde{\mu}_1$ and $\delta$ is denoted by $\tilde{\mu}_5$:

\[ \tilde{\mu}_5(R_{-2}(\zeta_1, (\xi_1, (\eta_2, \eta_3, y_1), \xi), u)) = R_2(-u, ((-\eta_3, -\eta_2, y_1), \xi, \xi_1), -\xi_1). \]
Now, we define the inner product \((R, R')_\mu\) in \(\mathfrak{g}_{-2}\) by
\[
(R, R')_\mu = \frac{1}{30} B_8(\tilde{\mu}_\delta R, R').
\]
The explicit form of \((R, R)_\mu\) is
\[
(R, R)_\mu = -4\zeta_1 u - \eta_2 \eta_3 + y_1 \bar{y}_1 + \xi_1 \xi,
\]
where \(R = R_{-2}(\zeta_1, (\zeta_1, (\eta_2, \eta_3, y_1), \xi), u) \in \mathfrak{g}_{-2}\).

We define \(C\)-vector spaces \((V^C)^{14}, (V^C)^{13}\) and \((V^C)^{12}\) by
\[
(V^C)^{14} = \mathfrak{g}_{-2} = \{R_{-2}(\zeta_1, (\zeta_1, (\eta_2, \eta_3, y_1), \xi), u) \in \mathfrak{g}_{-2}\},
\]
\[
(V^C)^{13} = \{R \in (V^C)^{14} \mid (R, (\Phi_1, 0, 0, 0, 1, 0))_\mu = 0\}
\]
\[
= \{R = R_{-2}(\zeta_1, (\zeta_1, (\eta_2, \eta_3, y_1), \xi), -\zeta_1) \in \mathfrak{g}_{-2}\}
\]
where \(\Phi_1 = \Phi(0, E_1, 0, 0))\),
\[
(V^C)^{12} = \{P \in \mathfrak{P}^C \mid \kappa P = -P\}
\]
\[
= \{P = (\xi_1 E_1, \eta_2 E_2 + \eta_3 E_3 + F_1(y_1), \xi, 0) \in \mathfrak{P}^C\},
\]
respectively. \((V^C)^{12}\) will be identified with \(\{P = (0, P, 0, 0, 0, 0) \in (V^C)^{14} \mid P \in (V^C)^{12}\} \subset (V^C)^{13}\). So an element \(P = (\xi_1 E_1, \eta_2 E_2 + \eta_3 E_3 + F_1(y_1), \xi, 0)\) of \((V^C)^{12}\) is denoted by \(P = (\zeta_1, (\eta_2, \eta_3, y_1), \xi, 0)\). We define the inner product \((P, P)_\mu\) in \(\mathfrak{P}^C\) by
\[
(P, P)_\mu = \frac{1}{2} \{i\mu_1 P, P\} = -\eta_2 \eta_3 + y_1 \bar{y}_1 + \xi_1 \xi,
\]
where \(P = (\xi_1, (\eta_2, \eta_1, y_1), \xi, 0)\) which is the restriction of the inner product \((P, P)_\mu\) of \((V^C)^{14}\).

Now, we define subgroups \(G_{14}, G_{13}\) and \(G_{12}\) of \(E_8^C\) by
\[
G_{14} = \{\alpha \in E_8^C \mid \tilde{\kappa} \alpha = \alpha \tilde{\kappa}, \tilde{\mu}_5 \alpha R = \alpha \tilde{\mu}_5 R, \ R \in (V^C)^{14}\},
\]
\[
G_{13} = \{\alpha \in G_{14} \mid \alpha(\Phi_1, 0, 0, 0, 1, 0) = (\Phi_1, 0, 0, 0, 1, 0)\},
\]
\[
G_{12} = \{\alpha \in G_{13} \mid \alpha(\Phi_1, 0, 0, 0, -1, 0) = (\Phi_1, 0, 0, 0, -1, 0)\},
\]
respectively. Note that the group \(G_{14}\) leaves the inner product \((R, R)_\mu\) of \((V^C)^{14}\) invariant: \((\alpha R, \alpha R)_\mu = (R, R)_\mu, \alpha \in G_{14}, \ R \in (V^C)^{14}\).

**Lemma 5.8.1.** \(G_{12} \subset E_7^C\).

**Proof.** Let \(\tilde{1} = (0, 0, 0, 0, 1, 0)\), \(1^- = (0, 0, 0, 0, 1, 0)\) and \(1_- = (0, 0, 0, 0, 0, 1)\) in \(e_8^C\). We need to show that \(\alpha \in G_{12}\) satisfies
\[
\alpha \tilde{1} = \tilde{1}, \quad \alpha 1^- = 1^-, \quad \alpha 1_- = 1_-.
\]
\(a_1^- = 1^-\) is trivial, so we shall show the other two. Now, since \(1^- \in \mathfrak{g}_2\), let \(a_1^- = (\Phi, 0, Q, 0, 0, v) \in \mathfrak{g}_2\). \([a_1^-, 1^-] = [\alpha_1^-, \alpha_1^-] = \alpha[1^-, 1^-] = \alpha(\overline{1})\). On the other hand, \([\alpha_1^-, 1^-] = [(\Phi, 0, Q, 0, 0, v), (0, 0, 0, 0, 0, 1)] = (0, -Q, 0, -v, 0, 0)\).

Hence we have

\[
a \overline{1} = (0, Q, 0, v, 0, 0).
\]

\([a_1^-, \Phi_1] = [\alpha_1^-, \alpha \Phi_1] = \alpha[1^-, \Phi_1] = \alpha 0 = 0\). On the other hand, \([\alpha_1^-, \Phi_1] = [(\Phi, 0, Q, 0, 0, v), \Phi_1] = [(\Phi, \Phi_1), 0, -\Phi_1 Q, 0, 0, 0]\). If \(\Phi = \Phi(0, 0, \zeta_1 E_1, 0)\) and \(Q = (\xi_2 E_2 + \xi_3 E_3 + F_1(x_1), \eta_1 E_1, 0, \eta)\), then

\[
[\Phi, \Phi_1] = \Phi(-2 \zeta_1 E_1 \vee E_1, 0, 0, -\zeta_1), \quad \Phi_1 Q = (\eta E_1, \xi_3 E_2 + \xi_2 E_3 - F_1(x_1), \eta_1, 0).
\]

Therefore \(\Phi = 0, Q = 0\). Hence we have

\[
a_1^- = (0, 0, 0, 0, 0, v), \quad a \overline{1} = (0, 0, 0, v, 0, 0).
\]

Finally \([a \overline{1}, 1^-] = [a \overline{1}, a_1^-] = \alpha(0, 0, 0, 0, 2, 0) = (0, 0, 0, 0, 2, 0)\). On the other hand, \([a \overline{1}, 1^-] = [(0, 0, 0, v, 0, 0), (0, 0, 0, 0, 1)] = (0, 0, 0, 0, 2v, 0)\). Hence \(v = 1\). Thus we have \(a_1^- = 1^-, a \overline{1} = \overline{1}\).

**Proposition 5.8.2.** \(G_{12} = \text{Spin}(12, C)\).

**Proof.** Let \(\text{Spin}(12, C) = \{\alpha \in E_7^C | \kappa \alpha = \alpha \kappa, \mu \alpha = \alpha \mu\}\). Firstly we shall show that \(G_{12} \subset \text{Spin}(12, C)\). Since \(G_{12} \subset E_7^C\) (Lemma 5.8.1), it suffices to consider the actions on \(\mathfrak{p}^C\). Since \(\alpha \in G_{12}\) satisfies \(\tilde{\kappa} \alpha = \alpha \tilde{\kappa}\), from

\[
\tilde{\kappa} \alpha P = \kappa \alpha P - \alpha P \quad \text{and} \quad \alpha \tilde{\kappa} P = \alpha \kappa P - \alpha P, \quad P \in \mathfrak{p}^C,
\]

we have \(\kappa \alpha = \alpha \kappa\). Since \(\exp(\pi i \kappa) = \sigma, \alpha\) satisfies \(\sigma \alpha = \alpha \sigma\) too. Hence the \(C\)-vector space \(\mathfrak{p}^C\) is decomposable in \(\alpha\)-invariant \(C\)-vector subspaces:

\[
\mathfrak{p}^C = (\mathfrak{p}^C)_\sigma \oplus (\mathfrak{p}^C)_{-\sigma},
\]

\(\mathfrak{p}^C)_\sigma = \{P \in \mathfrak{p}^C | \sigma P = P\}, \quad (\mathfrak{p}^C)_{-\sigma} = \{P \in \mathfrak{p}^C | \sigma P = -P\},
\]

and the mappings \(\tilde{\mu}_1\) and \(\mu\) are related with

\[
\tilde{\mu}_1 = -\mu \quad \text{on} \quad (\mathfrak{p}^C)_\sigma, \quad \tilde{\mu}_1 = i1, \quad \mu = 0 \quad \text{on} \quad (\mathfrak{p}^C)_{-\sigma}.
\]

Now under the assumption \(\tilde{\mu}_1 \alpha = \alpha \tilde{\mu}_1\), for \(S \in (\mathfrak{p}^C)_\sigma, T \in (\mathfrak{p}^C)_{-\sigma}\), we have \(\mu \alpha (S + T) = \mu (\alpha S) + \mu (\alpha T) = -\tilde{\mu}_1 (\alpha S) = -\tilde{\mu}_1 \alpha S\). On the other hand \(\alpha \mu (S + T) = \alpha (\mu S) = \alpha (-\tilde{\mu}_1 S) = -\tilde{\mu}_1 \alpha S\). Hence \(\mu \alpha = \alpha \mu\), so \(\alpha \in \text{Spin}(12, C)\). Conversely \(\text{Spin}(12, C) \subset G_{12}\) is trivial. Thus the proof of \(G_{12} = \text{Spin}(12, C)\) is completed.
LEMMA 5.8.3. The Lie algebras $\mathfrak{g}_{14}$ and $\mathfrak{g}_{13}$ of the groups $G_{14}$ and $G_{13}$ are given by

\[
\mathfrak{g}_{14} = \{ R_0 \in \mathfrak{g}_0 \mid (\bar{\mu}_3(\text{ad } R_0)) R = ((\text{ad } R_0)\bar{\mu}_3) R, R \in (V^C)^{14} \}
\]
\[
= \left\{ R_0(\Phi(D, d_1\sim, (\tau_1, \tau_2, \tau_3, t_1)\sim, (\alpha_2, \alpha_3, \alpha_1), (\beta_2, \beta_3, b_1), \nu), ((\rho_2, \rho_3, p_1), \rho_1, \rho),
(\zeta_1, (\zeta_2, \zeta_3, z_1), \zeta), r) \mid \tau_1 + \frac{2}{3}\nu + 2r = 0 \right\},
\]
\[
\mathfrak{g}_{13} = \{ R_0 \in \mathfrak{g}_{14} \mid \text{ad } (R_0)(\Phi_1, 0, 0, 0, 1, 0) = 0 \}
\]
\[
= \left\{ R_0(\Phi(D, d_1\sim, (\tau_1, \tau_2, \tau_3, t_1)\sim, (\alpha_2, \alpha_3, \alpha_1), (\beta_2, \beta_3, b_1), \nu), ((\rho_2, \rho_3, p_1), \rho_1, \rho),
(\tau_1 + \frac{2}{3}\nu = 0) \right\},
\]

respectively.

REMARK. The condition $\tau_1 + \frac{2}{3}\nu + 2r = 0$ in $\mathfrak{g}_{14}$ is characterized by the orthogonality with $Z = (\kappa, 0, 0, -1, 0, 0)$ with respect to the Killing form $B_8$ of $\mathfrak{e}_8^C$, that is,

\[
\mathfrak{g}_{14} = \{ R \in \mathfrak{g}_0 \mid B_8(Z, R) = 0 \}.
\]

LEMMA 5.8.4. (1) For $a \in \mathbb{C}$, we define a $\mathbb{C}$-linear transformation $\epsilon_{13}(a)$ of $\mathfrak{e}_8^C$ by

\[
\epsilon_{13}(a) = \exp(\text{ad } (0, (F_1(a), 0, 0, 0), (0, F_1(a), 0, 0), 0, 0, 0)).
\]

Then $\epsilon_{13}(a) \in G_{13}$ (Lemma 5.8.3). The action of $\epsilon_{13}(a)$ on $(V^C)^{13}$ is given by

\[
\epsilon_{13}(a)(R_{-2}(\zeta_1, (\xi_1, (\eta_2, \eta_3, y_1), \xi), -\zeta_1)) = R_{-2}(\zeta_1', (\xi_1', (\eta_2', \eta_3', y_1'), \xi'), -\zeta_1'),
\]

\[
\begin{cases}
\zeta_1' = \zeta_1 \cos |a| - \frac{(a, y_1)}{2|a|} \sin |a| \\
\xi_1' = \xi_1 \\
\eta_2' = \eta_2 \\
\eta_3' = \eta_3 \\
y_1' = y_1 + \frac{2\zeta_1 a}{|a|} \sin |a| - \frac{2(a, y_1)a}{|a|^2} \sin^2 |a| \\
\xi' = \xi.
\end{cases}
\]

(2) For $t \in \mathbb{R}$, we define a $\mathbb{C}$-linear transformation $\theta_{13}(t)$ of $\mathfrak{e}_8^C$ by

\[
\theta_{13}(t) = \exp(\text{ad } (0, (0, -t E_1, 0, -t), (t E_1, 0, t, 0), 0, 0, 0)).
\]
Then $\theta_{13}(t) \in G_{13}$ (Lemma 5.8.3). The action of $\theta_{13}(t)$ on $(V^C)^{13}$ is given by

$$\theta_{13}(t)(R_{-2}(\zeta_1, (\eta_2, \eta_3, y_1), \xi, -\zeta_1)) = R_{-2}(\zeta'_1, (\eta_2', \eta_3', y_1'), \xi'), -\zeta_1'),$$

where

$$\begin{cases}
\zeta'_1 = \zeta_1 \cos t - \frac{1}{4}(\xi_1 + \xi) \sin t \\
\xi'_1 = \frac{1}{2}(\xi_1 - \xi) + \frac{1}{2}(\xi_1 + \xi) \cos t + 2\zeta_1 \sin t \\
\eta_2' = \eta_2 \\
\eta_3' = \eta_3 \\
y_1' = y_1 \\
\xi' = -\frac{1}{2}(\xi_1 - \xi) + \frac{1}{2}(\xi_1 + \xi) \cos t + 2\zeta_1 \sin t.
\end{cases}$$

**Proposition 5.8.5.** $G_{13}/G_{12} \simeq (S^C)^{12}$. In particular, $G_{13} \cong \text{Spin}(13, C)$.

**Proof.** Let $(S^C)^{12} = \{R \in (V^C)^{13} \mid (R, R) = 1\}$. The group $G_{13}$ acts on $(S^C)^{12}$. We shall show that this action is transitive. To prove this, it suffices to show that any $R \in (S^C)^{12}$ can be transformed to $\frac{1}{2}(\Phi_1, 0, 0, 0, -1, 0) \in (S^C)^{12}$.

For a given $R = R_{-2}(\zeta_1, (\eta_2, \eta_3, y_1), \xi, -\zeta_1) \in (S^C)^{12}$,

choose $a \in \mathcal{E}$ such that $|a| = \pi/2$, $(a, y_1) = 0$ and operate $e_{13}(a) \in G_{13}$ (Lemma 5.8.4.(1)) on $R$. Then we have

$$e_{13}(a)R = R_{-2}(0, (\eta_2, \eta_3, y_1'), \xi', 0) = R' \in (S^C)^{11},$$

where $(S^C)^{11} = \{R \in (V^C)^{12} \mid (R, R) = 1\}$. Since the group $G_{12}$ acts transitively on $(S^C)^{11}$ (Proposition 5.8.2), there exists $\beta \in G_{12}$ such that

$$\beta R' = R_{-2}(0, (1, 0, 0, 0, 1), 0) = R'' \in (S^C)^{11}.$$  

Finally operate $\theta_{13}(-\pi/2) \in G_{13}$ (Lemma 5.8.4.(2)) on $R''$, then we have

$$\theta_{13}(-\pi/2)R'' = \frac{1}{2}R_{-2}(1, (0, 0, 0, 0, -1, 0), -1) = \frac{1}{2} (\Phi_1, 0, 0, 0, -1, 0).$$

Thus the transitivity is proved. The isotropy subgroup at $\frac{1}{2}(\Phi_1, 0, 0, 0, -1, 0)$ of $G_{13}$ is obviously $G_{12}$. Therefore $G_{13}/G_{12} \simeq (S^C)^{12}$. Since the group $G_{13}$ is connected, we can define a homomorphism $\pi : G_{13} \to SO(13, C) = SO((V^C)^{13})$ by $\pi(\alpha) = \alpha(V^C)^{13}$. $\ker \pi = \{1, \sigma\} = Z_2$. Since $\dim_C(G_{13}) = \dim_C(Spin(12, C)) + \dim_C((S^C)^{12}) = 66 + 12 = 78 = \dim_C(SO(13, C))$, $\pi$ is onto. Hence $G_{13}/Z_2 \cong$
SO(13, C). Therefore $G_{13}$ is $Spin(13, C)$ as a covering group of $SO(13, C) = SO((V^C)_{13})$.

**Lemma 5.8.6.** (1) For $a \in \mathfrak{e}$, we define a $C$-linear transformation $\epsilon_{14}(a)$ of $\mathfrak{e}_8^C$ by

$$\epsilon_{14}(a) = \exp(\text{ad}(0, (-iF_1(a), 0, 0, 0), (0, iF_1(a), 0, 0), 0, 0, 0)).$$

Then $\epsilon_{14}(a) \in G_{14}$ (Lemma 5.8.3). The action of $\epsilon_{14}(a)$ on $(V^C)_{14}$ is given by

$$\epsilon_{14}(a)(R_{-2}(\xi_1, (\xi_1, (\eta_2, \eta_3, y_1), \xi), u)) = R_{-2}(\xi_1', (\xi_1', (\eta_2', \eta_3', y_1'), \xi'), u'),$$

\[
\begin{aligned}
\xi_1' &= \frac{1}{2}(\xi_1 - u) + \frac{1}{2}(\xi_1 + u)\cos|a| - \frac{1}{2}\frac{(a, y_1)}{|a|}\sin|a| \\
\eta_2' &= \eta_2 \\
\eta_3' &= \eta_3 \\
y_1' &= y_1 - i\frac{1}{2}\frac{(\xi_1 + u)a}{|a|}\sin|a| - \frac{2}{3}|a|^2\sin^2|a| \\
\xi' &= \xi \\
u' &= -\frac{1}{2}(\xi_1 - u) + \frac{1}{2}(\xi_1 + u)\cos|a| - \frac{1}{2}\frac{(a, y_1)}{|a|}\sin|a|.
\end{aligned}
\]

(2) For $t \in \mathbb{R}$, we define a $C$-linear transformation $\theta_{14}(t)$ of $\mathfrak{e}_8^C$ by

$$\theta_{14}(t) = \exp(\text{ad}(0, (0, itE_1, 0, it), (itE_1, 0, it, 0), 0, 0, 0)).$$

Then $\theta_{14}(t) \in G_{14}$ (Lemma 5.8.3). The action of $\theta_{14}(t)$ on $(V^C)_{14}$ is given by

$$\theta_{14}(t)(R_{-2}(\xi_1, (\xi_1, (\eta_2, \eta_3, y_1), \xi), u)) = R_{-2}(\xi_1', (\xi_1', (\eta_2', \eta_3', y_1'), \xi'), u'),$$

\[
\begin{aligned}
\xi_1' &= \frac{1}{2}(\xi_1 - u) + \frac{1}{2}(\xi_1 + u)\cos t - \frac{i}{4}(\xi_1 + \xi)\sin t \\
\xi_1' &= \frac{1}{2}(\xi_1 - \xi) + \frac{1}{2}(\xi_1 + \xi)\cos t - i(\xi_1 + u)\sin t \\
\eta_2' &= \eta_2 \\
\eta_3' &= \eta_3 \\
y_1' &= y_1 \\
\xi' &= -\frac{1}{2}(\xi_1 - \xi) + \frac{1}{2}(\xi_1 + \xi)\cos t - i(\xi_1 + u)\sin t \\
u' &= -\frac{1}{2}(\xi_1 - u) + \frac{1}{2}(\xi_1 + u)\cos t - \frac{i}{4}(\xi_1 + \xi)\sin t.
\end{aligned}
\]
PROPOSITION 5.8.7. \( G_{14}/G_{13} \cong (S^C)^{13} \). In particular, \( G_{14} \cong \text{Spin}(14, C) \).

PROOF. Let \( (S^C)^{13} = \{ R \in (V^C)^{14} \mid (R, R) = 1 \} \). The group \( G_{14} \) acts on \( (S^C)^{13} \). We shall show that this action is transitive. To prove this, it suffices to show that any \( R \in (S^C)^{13} \) can be transformed to \( \frac{1}{2}(\Phi_1, 0, 0, 0, 1, 0) \in (S^C)^{13} \). Now for a given

\[
R = R_{-2}(\zeta_1, (\xi_1, (\eta_2, \eta_1), \xi), u) \in (S^C)^{13},
\]

choose \( a \in \mathbb{C} \) such that \( \|a\| = \pi/2, (a, y_1) = 0 \) and operate \( \epsilon_{14}(a) \in G_{14} \) (Lemma 5.8.6.(1)) on \( R \). Then we have

\[
\epsilon_{14}(a)R = R_{-2}(\zeta', (\xi_1, (\eta_2, \eta_1'), \xi), ) = R' \in (S^C)^{12}.
\]

Since the group \( G_{13} \) acts transitively on \( (S^C)^{12} \) (Proposition 5.8.5), there exists \( \beta \in G_{13} \) such that

\[
\beta R' = R_{-2}(0, (1, (0, 0, 0), 1, 0) = R'' \in (S^C)^{12}.
\]

Finally operate \( \theta_{14}(-\pi/2) \in G_{14} \) (Lemma 5.8.6.(2)) on \( R'' \), then we have

\[
\theta_{14}(-\pi/2)R'' = \frac{i}{2}R_{-2}(1, (0, (0, 0, 0), 0, 1) = \frac{i}{2}(\Phi_1, 0, 0, 1, 0).
\]

Thus the transitivity is proved. The isotropy subgroup at \( \frac{1}{2}(\Phi_1, 0, 0, 1, 0) \) of \( G_{14} \) is obviously \( G_{13} \). Therefore \( G_{14}/G_{13} \cong (S^C)^{13} \). \( G_{14} \cong \text{Spin}(14, C) \) is proved in a way similar to Proposition 5.8.5.

We shall determine the group structure of

\[
(E_8^C)_{0} = (E_8^C)^{\delta_3}.
\]

We define a mapping \( \phi : C^* \to (E_8^C)^{\delta_3}, \) \( E_8^C \) by \( \phi(a) = \exp(\sqrt{\delta} \sqrt{a}) \), where \( a = e^{i\theta} \).

Lemma 5.9. The action of \( \phi(a), a \in C^*, \) on \( E_8^C \) is given by

\[
\phi(a)(P, Q, r, u, v) = (\psi(a)P, a^{-1}Q, a^{-1}P, a^{-1}u, a^{-2}v),
\]

where \( \psi(a) \in E_7^C \) is defined by

\[
\psi(a)(X, Y, \xi, \eta) = \begin{pmatrix}
\alpha \xi_1 & x_3 & \overline{x}_2 \\
\overline{x}_3 & \alpha^{-1} \xi_2 & \alpha^{-1} x_1 \\
x_2 & \alpha^{-1} \overline{x}_1 & \alpha^{-1} \xi_3
\end{pmatrix}, \begin{pmatrix}
\alpha^{-1} \eta_1 & y_3 & \overline{y}_2 \\
\overline{y}_3 & \alpha \eta_2 & \alpha y_1 \\
y_2 & \alpha \overline{y}_1 & \alpha^{-1} \eta_3
\end{pmatrix}, \alpha \xi, \alpha^{-1} \eta.
\]
In particular, the action of \( \phi(a) \) on \((V^C)^{14}\) is given by

\[
\phi(a)R = a^2R, \quad R \in (V^C)^{14}.
\]

**Proof.** The mapping \( \psi(a) : \mathfrak{P}^C \to \mathfrak{P}^C \) coincides with \( \psi\left( \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \right) \), where \( \psi : SL(2, C) \to E_7^C \) is defined in 4.2. The fact that \( \psi(a)\Phi_1\psi(a)^{-1} = a^2\Phi_1 \) in \( \phi(a)R = a^2R \) is proved as follows.

\[
\psi(a)\Phi_1\psi(a)^{-1} = \psi(a)(2(E_1,0,1,0) \times (E_1,0,1,0))\psi(a)^{-1} \\
= 2\psi(a)(E_1,0,1,0) \times \psi(a)(E_1,0,1,0) \\
= 2(aE_1,0,a,0) \times (aE_1,0,a,0) = a^2\Phi_1.
\]

**Theorem 5.10.** \( (E_8^C)^0 \cong (C^* \times \text{Spin}(14, C))/\mathbb{Z}_4, \mathbb{Z}_4 = \{(1,1), (-1, \phi(-1)), (i, \phi(-i)), (-i, \phi(i))\}. \)

**Proof.** Let \( \text{Spin}(14, C) = \{\alpha \in E_8^C \mid \tilde{\alpha}\alpha = \alpha\tilde{\alpha}, (\tilde{\mu}_\delta)\alpha R = \alpha(\tilde{\mu}_\delta)R, R \in (V^C)^{14}\} \) (Proposition 5.8.7). Define \( \varphi : C^* \times \text{Spin}(14, C) \to E_8^C \) by

\[
\varphi(a, \beta) = \phi(a)\beta.
\]

\( \varphi \) is well-defined. Since \( Z \) is a center of \( \mathfrak{g}_0, \phi(a) \) and \( \beta \in \text{Spin}(14, C) \) commute, hence \( \varphi \) is a homomorphism. \( \text{Ker} \varphi = \mathbb{Z}_4 \). In fact, let \( (a, \beta) \in C^* \times \text{Spin}(14, C) \) satisfy \( \phi(a)\beta = 1 \). Then, for \( R = (\tilde{\mu}_1,0,0,0,1,0) \), we have \( \beta R = \phi(a)^{-1}R = a^{-2}R \) (Lemma 5.9). Hence \( -4a^{-4} = (a^{-2}R, a^{-2}R)_\mu = (\beta R, \beta R)_\mu = (R, R)_\mu = -4 \). So \( a^4 = 1 \), that is, \( a = 1, -1, i, -i \) and \( \beta = 1, \phi(-1) = \alpha, \phi(-i), \phi(i) \), respectively. Therefore \( \text{Ker} \varphi = \mathbb{Z}_4 \). Since \( (E_8^C)^{\tilde{\delta}_3} \) is connected and \( \text{dim}_C(C \oplus \text{spin}(14, C)) = 1 + 91 = 92 = \text{dim}_C((E_8^C)^0) \) (Theorem 5.7), \( \varphi \) is onto. Therefore \( (E_8^C)^0 \cong (C^* \times \text{Spin}(14, C))/\mathbb{Z}_4 \).

**Remark.** We can prove that \( (E_8^C)^\sigma \cong S_8(16, C) \), but in this paper do not give its proof.

**5.3.1. Subgroup of type \( R \oplus D_{7(7)} \) of \( E_8(8) \)**

We shall determine the group structure of

\[
(E_8(8))^0 = (E_8^C)^{\tilde{\delta}_3} \cap (E_8^C)^{\varphi_7}.
\]

**Theorem 5.11.** The 2-graded decomposition of \( \varepsilon_8(8) = (E_8^C)^{\varphi_7}, \)

\[
\varepsilon_8(8) = \varepsilon_{-2} \oplus \varepsilon_{-1} \oplus \varepsilon_0 \oplus \varepsilon_1 \oplus \varepsilon_2
\]
2-graded decompositions of exceptional Lie algebras

with respect to \( \text{ad } Z, Z = (\kappa, 0, 0, -1, 0, 0) \), is given by

\[
\mathfrak{g}_0 = \begin{cases}
i \mathbb{B}_{ki}, 0 \leq k < 4 \leq l \leq 7, G_{kl}, \text{ otherwise}, \\
\mathbb{A}_1(e_k), \mathbb{F}_1(e_k), \mathbb{F}_1(e_k), \mathbb{F}_1(e_k), 0 \leq k \leq 3, \\
i \mathbb{A}_1(e_k), i \mathbb{F}_1(e_k), i \mathbb{F}_1(e_k), i \mathbb{F}_1(e_k), 4 \leq k \leq 7, \\
E_2 \lor E_1, (E_2 - E_3)^-, \mathbb{E}_2, \mathbb{E}_3, \mathbb{E}_2, \mathbb{E}_3, 1, \\
E_2^-, \mathbb{E}_3^-, \mathbb{F}_1(e_k)^-, 0 \leq k \leq 3, i \mathbb{F}_1(e_k)^-, 4 \leq k \leq 7, E_1^-, \mathbb{F}_1^-, \mathbb{E}_2^-, \mathbb{E}_3^-, \mathbb{F}_1(e_k)^-, 0 \leq k \leq 3, i \mathbb{F}_1(e_k)^-, 4 \leq k \leq 7, \mathbb{E}_1^-, \mathbb{F}_1^-, \mathbb{E}_2^-, \mathbb{E}_3^-.
\end{cases}
\]

\[
\mathfrak{g}_{-1} = \begin{cases}
\mathbb{A}_2(e_k) - \mathbb{F}_2(e_k), \mathbb{A}_3(e_k) + \mathbb{F}_3(e_k), \mathbb{F}_2(e_k), \mathbb{F}_3(e_k), 0 \leq k \leq 3, \\
i \mathbb{A}_2(e_k) - i \mathbb{F}_2(e_k), i \mathbb{A}_3(e_k) + i \mathbb{F}_3(e_k), i \mathbb{F}_2(e_k), i \mathbb{F}_3(e_k), 4 \leq k \leq 7, \\
i \mathbb{F}_2(e_k)^-, \mathbb{F}_3(e_k)^-, \mathbb{F}_2(e_k)^-, \mathbb{F}_3(e_k)^-, 0 \leq k \leq 3, \\
i \mathbb{F}_2(e_k)^-, i \mathbb{F}_3(e_k)^-, i \mathbb{F}_2(e_k)^-, i \mathbb{F}_3(e_k)^-, 4 \leq k \leq 7.
\end{cases}
\]

\[
\mathfrak{g}_{-2} = \begin{cases}
\mathbb{E}_1, \mathbb{E}_1^-, E_2^-, E_3^-, \mathbb{E}_1^-, \mathbb{E}_1^-, \mathbb{E}_2^-, \mathbb{E}_3^-, \mathbb{F}_1(e_k)^-, 0 \leq k \leq 3, i \mathbb{F}_1(e_k)^-, 4 \leq k \leq 7.
\end{cases}
\]

\[ \mathfrak{g}_1 = \lambda(\mathfrak{g}_{-1}), \quad \mathfrak{g}_2 = \lambda(\mathfrak{g}_{-2}). \]

**Proof.** We can prove this theorem in a way similar to Theorem 4.21.

**Lemma 5.12.** In the group \( E_8^C \), we have

\[
\tau \phi(a) \tau = \phi(\tau a), \quad \gamma \phi(a) \gamma = \phi(a).
\]

We define \( \rho_4 \in E_6 \subset E_7 \subset E_8 \subset E_8^C \) by

\[
\rho_4 \begin{pmatrix} \xi_1 & x_3 & \bar{x}_2 \\ \bar{x}_3 & \xi_2 & x_1 \\ x_2 & \bar{x}_1 & \xi_3 \end{pmatrix} = \begin{pmatrix} -\xi_1 & e_4 x_3 e_4 & -i \bar{x}_2 e_4 \\ e_4 x_3 e_4 & -\xi_2 & i e_4 x_1 \\ -i \bar{x}_2 e_4 & i e_4 x_1 & \xi_3 \end{pmatrix}, \quad (\text{cf. 3.4.1}).
\]

**Lemma 5.13.** The action of \( \rho_4 \) on \( (V^C)^{14} \) is given by

\[
\rho_4(R_2(\zeta_1, (\xi_1, (\eta_2, \eta_3, y_1)), (\xi, u)) = R_2(-\zeta_1, (-\xi_1, (-\eta_2, \eta_3, i e_4 y_1)), (\xi, u)).
\]

**Theorem 5.14.** \( (E_8(8))_0 \cong (R^+ \times \text{spin}(7, 7)) \times \{1, \rho_4\} \).
Proof. For $\alpha \in (E_8(0))_0 \subset (E_8^C)^{k_3}$, there exist $a \in C^*$ and $\beta \in Spin(14,C)$ such that $\alpha = \varphi(a, \beta) = \phi(a)\beta$ (Theorem 5.9). From $\gamma_\tau \sigma \gamma = \alpha$, we have

\[
\begin{align*}
(i) \quad \gamma \tau \phi(a) \tau \gamma &= \phi(a) \\
(ii) \quad \phi(-a) &= \sigma \beta \\
(iii) \quad \phi(ia) &= \phi(-i) \beta \\
(iv) \quad \phi(-ia) &= \phi(i) \beta
\end{align*}
\]

(i) From $\phi(\gamma a) = \phi(a)$ (Lemma 5.12), we have $\gamma a = a$, hence $a \in R^\ast$. The group $(Spin(14,C))^\tau$ acts on the $R$-vector space

\[
V_7 \cdot 7 = (V^C)^\tau = \{ R \in (V^C)^{14} | \tau \gamma R = R \}
\]

\[
= \{ R = R_{-2}(\xi_1, \xi_1, (\eta_2, \eta_3, y_1), \xi, u) | \xi_1, \xi_1, \eta_2, \eta_3, \xi, u \in R, y_1 \in (\mathcal{C}^C)^\tau = \mathcal{C} \}
\]

with the norm $(R,R)_u = -4\xi_1 u - \eta_2 \eta_3 + y_1 \overline{y_1} + \xi_1 \xi$. Hence the group $(Spin(14,C))^\tau$ is $spin(7,7)$, in a way similar to Theorem 4.10.(1), as a covering group of $O(7,7) = O(V_7 \cdot 7)_0$. Hence the group of the former case is $(R^\ast \times spin(7,7))/\mathbb{Z}_2 (\mathbb{Z}_2 = \{(1,1), (-1,0)\}) \cong R^+ \times spin(7,7)$. (ii) $a = i, \beta = \phi(-\overline{i})$ satisfy the conditions and $\varphi(i, \phi(-\overline{i})) = 1$. (iii) $a = \frac{1+i}{\sqrt{2}}, \beta = \rho_4 \phi(\frac{1+i}{\sqrt{2}})$ satisfy the conditions (Lemmas 5.9, 5.13) and $\varphi(\frac{1+i}{\sqrt{2}}, \rho_4 \phi(\frac{1+i}{\sqrt{2}})) = \rho_4$. (iv) $a = \frac{1+i}{\sqrt{2}}, \beta = \rho_4 \phi(\frac{1+i}{\sqrt{2}})$ satisfy the conditions and $\varphi(\frac{1+i}{\sqrt{2}}, \rho_4 \phi(\frac{1+i}{\sqrt{2}})) = \rho_4$. Therefore $(E_8(-24))_0 \cong (R^+ \times spin(7,7)) \times \{1, \rho_4\}$.

5.3.2. Subgroup of type $R \oplus D_7(-25)$ of $E_8(-24)$

We shall determine the group structure of

\[
(E_8(-24))_0 = (E_8^C)^{k_3} \cap (E_8^C)^\tau.
\]

Theorem 5.15. The 2-graded decomposition of $\varepsilon_{8(-24)} = (\varepsilon_8^C)^\tau$,

\[
\varepsilon_{8(-24)} = g_{-2} \oplus g_{-1} \oplus g_0 \oplus g_1 \oplus g_2
\]

with respect to $ad Z, Z = (\kappa, 0, 0, -1, 0, 0)$, is similar to the one in Theorem 5.7 given by replacing $C \rightarrow R$ and $\mathcal{C}^C \rightarrow \mathcal{C}$.

Theorem 5.16. $(E_8(-24))_0 \cong R^+ \times spin(3,11)$.

Proof. For $\alpha \in (E_8(-24))_0 \subset (E_8^C)^{k_3}$, there exist $a \in C^*$ and $\beta \in Spin(14,C)$ such that $\alpha = \varphi(a, \beta) = \psi(a)\beta$ (Theorem 5.9). From $\gamma_\tau \sigma \gamma = \alpha$, we have

\[
\begin{align*}
(i) \quad \tau \phi(a) \tau &= \phi(a) \\
(ii) \quad \phi(-a) &= \sigma \beta \\
(iii) \quad \phi(ia) &= \phi(-i) \beta \\
(iv) \quad \phi(-ia) &= \phi(i) \beta
\end{align*}
\]
From $\phi(\tau a) = \phi(a)$ (Lemma 5.12), we have $\tau a = a$, hence $a \in R^*$. The group $(\text{Spin}(14, C))^\tau$ acts on the $R$-vector space

$$V^{3,11} = (V^C)_\tau = \{ R \in (V^C)^{14} \mid \tau R = R \} = \{ R = R_{-2}(\zeta_1, (\xi_1, (\eta_2, \eta_3, y_1)), u) \mid \zeta_1, \xi_1, \eta_2, \eta_3, u \in R, y_1 \in C \}$$

with the norm $(R, R)_\mu = -4\zeta_1 u - \eta_2 \eta_3 + y_1 \overline{y_1} + \xi_1 \xi$. Hence the group $(\text{Spin}(14, C))^\tau$ is $\text{spin}(3,11)$, in a way similar to Theorem 4.10.(1), as a covering group of $O(3,11)^0 = O(V^{3,11})^0$. Hence the group of the former case is $(R^* \times \text{spin}(3,11))/\mathbb{Z}_2$ ($\mathbb{Z}_2 = \{(1,1), (-1, 0)\}$). (ii) $a = i$, $\beta = \phi(-i)$ satisfy the conditions and $\phi(i, \phi(-i)) = 1$. (iii) is impossible. In fact, for $R \in V^{3,11}$, we have $\tau \beta R = \tau \beta \tau R = \phi(-i) \beta R = \beta(\phi(-i) R = \beta(-R)$ (Lemma 5.9) $= -\beta R$. Hence there exists $R' \in V^{3,11}$ such that $\beta R = i R'$. So $(R, R)_\mu = (\beta R, \beta R)_\mu = (i R', i R')_\mu = -(R', R')_\mu$. But this is false, because the signatures of both sides are different. (iv) is also impossible. Therefore $(E_8(-24))^0 \cong R^* \times \text{spin}(3,11)$.

References
