Polynomials associated with an algebraic solution of the sixth Painlevé equation

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Introduction

For a non-negative integer n , let T_n be the rational function determined by the recurrence relation

$$(1) \quad T_{n-1}T_{n+1} = \left\{ \frac{1}{4}(-2b_1^2 - 2b_2^2 + (b_1^2 - b_2^2)v) + \left(n - \frac{1}{2}\right)^2 \right\} T_n^2 \\ + \frac{1}{4}(v^2 - 4)^2 \left\{ T_n \frac{d^2 T_n}{dv^2} - \left(\frac{dT_n}{dv} \right)^2 \right\} \\ + \frac{1}{4}(v^2 - 4)v T_n \frac{dT_n}{dv}$$

with initial conditions $T_0 = T_1 = 1$, where v is a variable and b_1 and b_2 are parameters. If you enjoy calculating a first few T_n 's by Mathematica, you will realize that they are polynomials. The polynomials T_n are, however quite complicated ([1]). A transformation given in [2] yields a surprising simplification. Namely, we put $z = (2 - v)/4$, $w = (2 + v)/4$, $c = -4b_1^2$ and $d = -4b_2^2$. For a positive integer k , we denote $\overline{c_k} = (c + (2k - 1)^2)$, $\overline{d_k} = (d + (2k - 1)^2)$ and define $c_k = \overline{c_1} \overline{c_2} \overline{c_3} \cdots \overline{c_k}$, $d_k = \overline{d_1} \overline{d_2} \overline{d_3} \cdots \overline{d_k}$. See the formulae (9) and (10) below.

We denote, for a positive integer n , by $[n]$ the set $[n] = \{1, 2, 3, \dots, n\}$. For a subset $I = \{i_1, i_2, \dots, i_p\} \subset [n - 1]$, we define the weight $||I||$ by

$$(2) \quad ||I|| = i_1 + i_2 + \cdots + i_p.$$

We put $c_\emptyset = 1$ and $d_\emptyset = 1$. For a subset $I = \{i_1, i_2, \dots, i_p\} \subset [n - 1]$, $I \neq \emptyset$, we set $c_I = c_{i_1} c_{i_2} \cdots c_{i_p}$ and $d_I = d_{i_1} d_{i_2} \cdots d_{i_p}$. For a partition $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_q)$, we define $p(\lambda) = \#\{i \in \mathbf{Z} : 1 \leq i \leq q, \lambda_i \geq i\}$, $\lambda'_i = \#\{j \in \mathbf{Z} : 1 \leq j \leq q, \lambda_j \geq i\}$, $\alpha_i = \lambda_i - i$ and $\beta_i = \lambda'_i - i$ for $1 \leq i \leq p(\lambda)$. See the example below. Since α_i, β_i ,

$1 \leq i \leq p(\lambda)$ determine the partition we write

$$(3) \quad \lambda = (\alpha_1, \alpha_2, \dots, \alpha_{p(\lambda)} \mid \beta_1, \beta_2, \dots, \beta_{p(\lambda)}),$$

which is called the Frobenius notation. For a partition $\lambda = (\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_q)$, we define the Young diagram $\bar{\lambda}$ by

$$(4) \quad \bar{\lambda} = \{(i, j) \in Z \times Z : 1 \leq j \leq \lambda_i, 1 \leq i \leq q\}.$$

The Young diagram is often illustrated with squares called cels. For a cell $x = (i, j) \in \bar{\lambda}$, we define its content $c(x)$ and hook-length $h(x)$ by

$$(5) \quad c(x) = j - i,$$

$$(6) \quad h(x) = \lambda_i + \lambda'_j - i - j + 1.$$

As this example shows for a partition $\lambda = (5 \geq 4 \geq 2 \geq 1)$ for which $\lambda_1 = 5$, $\lambda_2 = 4$, $\lambda_3 = 2$, $\lambda_4 = 1$, we have $\lambda'_1 = 4$, $\lambda'_2 = 3$, $\lambda'_3 = 2$, $\lambda'_4 = 2$, $\lambda'_5 = 1$, $p(\lambda) = 2$, $\alpha_1 = 4$, $\alpha_2 = 2$, $\beta_1 = 3$, $\beta_2 = 1$ and hence the Frobenius notation of λ is $\lambda = (4, 2 \mid 3, 1)$. The Young diagram of $\bar{\lambda}$ is

Figure

so that λ_i is the number of cells in the i -th row and λ'_i is the number of cells in the i -th column.

We associate to a strictly decreasing partition $I = \{i_1 > i_2 > \dots > i_p\} \neq \emptyset$ a partition

$$(7) \quad \lambda(I) = (i_1 - 1, i_2 - 1, \dots, i_p - 1 \mid i_1, i_2, \dots, i_p).$$

It is convenient to put $\lambda(\emptyset) = \emptyset$, $\prod_{x \in \emptyset} f(x) = 1$ and $\sum_{x \in \emptyset} f(x) = 0$. With this notation, we can state Conjecture (7.3) in [2].

THEOREM 1. *For an integer $n \geq 2$, we have*

$$(8) \quad T_n = 2^{-n(n-1)} \sum_{I \subset [n-1]} \left\{ \prod_{x \in \overline{\lambda(I)}} \left(\frac{n+c(x)}{h(x)} \right) c_I d_{[n-1]-I} z^{|I|} w^{|[n-1]-I|} \right\}.$$

The purpose of this paper is to prove Theorem 1.

For example, Theorem 1 tells us

$$(9) \quad T_2 = 2^{-1}(c_1 z + d_1 w),$$

$$(10) \quad T_3 = 2^{-6}(c_1 c_2 z^3 + 3d_1 c_2 z^2 w + 3c_1 d_2 z w^2 + d_1 d_2 w^3).$$

§1. Some Propositions

For an integer $n \geq 2$ and a subset $I \subset [n-1]$, we set

$$(11) \quad \mu_n(I) = \prod_{x \in \overline{\lambda(I)}} \frac{n+c(x)}{h(x)}.$$

We notice here that $\mu_n(I)$ is the dimension of the irreducible representation of GL_n associated to partition $\lambda(I)$. So Theorem 1 reveals a new aspect of Painlevé equations, an unexpected relation between combinatorics of the symmetric groups and the Painlevé equations. We first prove the following two Propositions.

PROPOSITION 1. *For an integer $n \geq 2$ and a subset $I \in [n-1]$, we have*

$$(12) \quad \mu_n(I) = \prod_{\nu \in I} \left(\frac{n-\nu}{n+\nu} \right) \mu_{n+1}(I).$$

PROOF. We put $I = \{i_1 > i_2 > \cdots > i_p\} \subset [n-1]$ and denote $\overline{\lambda_k(I)} = \{(i, j) \in \overline{\lambda(I)} : i = k, j \geq k \text{ or } j = k, i \geq k\}$. Then we have

$$(13) \quad \begin{aligned} \prod_{x \in \overline{\lambda(I)}} (n+c(x)) &= \prod_{k=1}^p \left\{ \prod_{x \in \overline{\lambda_k(I)}} (n+c(x)) \right\} \\ &= \prod_{k=1}^p \left\{ \prod_{j=0}^{2i_k-1} (n-i_k+j) \right\} \end{aligned}$$

$$\begin{aligned}
&= \prod_{k=1}^p \left\{ \left(\frac{n-i_k}{n+i_k} \right) \prod_{x \in \overline{\lambda_k(I)}} (n+1+c(x)) \right\} \\
&= \prod_{\nu \in I} \left(\frac{n-\nu}{n+\nu} \right) \prod_{x \in \overline{\lambda(I)}} (n+1+c(x)).
\end{aligned}$$

So the denominator of $\mu_n(I)$ is equal to that of $\mu_{n+1}(I)$, and hence Proposition 1 is proved.

PROPOSITION 2. *For an integer $n \geq 2$ and a subset $I \subset [n-1]$, we have*

$$(14) \quad \mu_n(I) = \mu_n([n-1] - I).$$

PROOF. We prove Proposition 2 by induction on n . As we have $\mu_2(\emptyset) = 1$ and $\mu_2(\{1\}) = 1$, Proposition 2 holds for $n = 2$. Suppose that Proposition 2 is verified for $n = N-1$, $N \geq 3$. We put $I = \{i_1 > i_2 > \cdots > i_p\} \subset [N-1]$ so that $\lambda_1 \neq N-1$. It follows from Proposition 1 that

$$\begin{aligned}
(15) \quad &\mu_N([N-1] - I) \\
&= \prod_{\nu \in [N-1] - I} \left(\frac{N-1+\nu}{N-1-\nu} \right) \mu_{N-1}([N-1] - I) \\
&= \prod_{\nu \in [N-1] - I} \left(\frac{N-1+\nu}{N-1-\nu} \right) \mu_{N-1}(I - \{N-1\}) \\
&= \prod_{\nu \in [N-1] - I} \left(\frac{N-1+\nu}{N-1-\nu} \right) \prod_{\nu \in I - \{N-1\}} \left(\frac{N-1-\nu}{N-1+\nu} \right) \mu_N(I - \{N-1\}).
\end{aligned}$$

Now we can see that the content and the hook-length of a cell $(i, j) \in \overline{\lambda(I)}$, $i \geq 2$, $j \geq 2$ is equal to the content and the hook-length of a cell $(i-1, j-1) \in \overline{\lambda(I - \{N-1\})}$. Moreover we have

$$(16) \quad \prod_{x \in \overline{\lambda_1(I)}} (N+c(x)) = (2N-2)!,$$

$$(17) \quad h((1, 1)) = 2N-2, \quad h((p+1, 1)) = N-1,$$

$$(18) \quad h((k, 1)) = h((1, k)) = N-1+i_k, \quad (2 \leq k \leq p).$$

We now put $[N-1] - I = \{j_1 > j_2 > \cdots > j_{N-1-p}\}$. Then we have

$$(19) \quad h((N+1-k, 1)) = h((1, N-k)) = N-1-j_k, \quad (1 \leq k \leq N-1-p).$$

Therefore, we have

$$\begin{aligned}
 (20) \quad & \mu_N(I - \{N - 1\}) \\
 &= \frac{1}{(2N - 2)!} (2N - 2)(N - 1) \left\{ \prod_{\nu \in [N-1]-I} (N - 1 - \nu) \right\}^2 \\
 &\quad \times \left\{ \prod_{\nu \in I - \{N-1\}} (N - 1 + \nu) \right\}^2 \mu_N(I).
 \end{aligned}$$

Combining the identity (20) with the identity (15), we have

$$(21) \quad \mu_N([N - 1] - I) = \mu_N(I),$$

which is equality (12) for $n = N$. Hence we have completed the mathematical induction and verified Proposition 2.

Now we need the following

DEFINITION 1. For an integer $n \geq 2$, subsets $I, J \in [n - 1]$, a positive integer $\lambda \in I \cup J$, we define $b_\lambda^{I,J}$ by the following partial fractional decomposition

$$\begin{aligned}
 (22) \quad & \prod_{\lambda \in I} \left(\frac{x + \lambda}{x - \lambda} \right) \prod_{\lambda \in J} \left(\frac{x - 1 - \lambda}{x - 1 + \lambda} \right) + \prod_{\lambda \in I} \left(\frac{x - 1 - \lambda}{x - 1 + \lambda} \right) \prod_{\lambda \in J} \left(\frac{x + \lambda}{x - \lambda} \right) \\
 &= 2 + \sum_{\lambda \in I \cup J} \frac{b_\lambda^{I,J}}{(x - \lambda)(x - 1 + \lambda)}.
 \end{aligned}$$

In particular, for an element $\lambda \in I \cap J$, we have

$$\begin{aligned}
 (23) \quad & b_\lambda^{I,J} = 2\lambda(2\lambda - 1) \left\{ \prod_{\nu \in I - \{\lambda\}} \left(\frac{\lambda + \nu}{\lambda - \nu} \right) \prod_{\nu \in J} \left(\frac{\lambda - 1 - \nu}{\lambda - 1 + \nu} \right) \right. \\
 &\quad \left. + \prod_{\nu \in I} \left(\frac{\lambda - 1 - \nu}{\lambda - 1 + \nu} \right) \prod_{\nu \in J - \{\lambda\}} \left(\frac{\lambda + \nu}{\lambda - \nu} \right) \right\}.
 \end{aligned}$$

For an element $\lambda \in I - I \cup J$, we have

$$(24) \quad b_\lambda^{I,J} = 2\lambda(2\lambda - 1) \prod_{\nu \in I - \{\lambda\}} \left(\frac{\lambda + \nu}{\lambda - \nu} \right) \prod_{\nu \in J} \left(\frac{\lambda - 1 - \nu}{\lambda - 1 + \nu} \right).$$

The following two Lemmas are easy consequences of Definition 1. The union $A \cup B$ of two subsets A, B of a set S will be sometimes denoted by $A + B$.

LEMMA 1. *We have*

$$(25) \quad \sum_{\lambda \in I \cup J} b_{\lambda}^{I,J} = 4(\|I\| - \|J\|)^2 - 2(\|I\| + \|J\|).$$

LEMMA 2. *For an element $\lambda \in I \cap J$, we have $b_{\lambda}^{I,J} = 0$ if and only if $\lambda - 1 \in I \cap J$. For an element $\lambda \in I - I \cap J$, we have $b_{\lambda}^{I,J} = 0$ if and only if $\lambda - 1 \in J$.*

To prove Theorem 1, we need also the following

DEFINITION 2. For an integer $n \geq 2$, $I, J \subset [n-1]$, an integer $\lambda \in I \cup J$, we define $FP(I, J)$ by

$$(26) \quad FP(I, J) = \mu_n(I)\mu_n(J) \left(1 + \sum_{\lambda \in I \cup J} \frac{b_{\lambda}^{I,J}}{2(n-\lambda)(n-1+\lambda)} \right)$$

and $SP(n, \lambda, I, J)$ by

$$(27) \quad SP(n, \lambda, I, J) = b_{\lambda}^{I,J} \mu_n(I) \mu_n(J).$$

We set

$$(28) \quad \begin{aligned} TP(n, 1, I + \{1\}; J + \{1\}) \\ = -2 \prod_{\nu \in I} \left(\frac{1+\nu}{1-\nu} \right) (-1)^{\|J\|} \mu_n(I + \{1\}) \mu_n(J + \{1\}) \end{aligned}$$

for an integer $n \geq 2$ and subsets $I, J \subset [n-1] - \{1\}$, and

$$(29) \quad \begin{aligned} TP(n, \lambda, I + \{\lambda\}; J + \{\lambda\}) \\ = 2\lambda(2\lambda-1) \prod_{\nu \in I} \left(\frac{\lambda+\nu}{\lambda-\nu} \right) \prod_{\nu \in J + \{\lambda\}} \left(\frac{\lambda-1-\nu}{\lambda-1+\nu} \right) \\ \times \mu_n(I + \{\lambda\}) \mu_n(J + \{\lambda\}) \end{aligned}$$

for an integer $n \geq 3$, an integer $2 \leq \lambda \leq n-1$ and subsets $I, J \subset [n-1] - \{\lambda\} - \{\lambda-1\}$.

We can see $FP(I, J) = FP(J, I)$ and $SP(n, \lambda, I, J) = SP(n, \lambda, J, I)$. From Proposition 1 and Definitions 1, 2, we immediately obtain the following

PROPOSITION 3. For an integer $n \geq 2$, we have

$$(30) \quad 2FP(I, J) = \begin{cases} 0, & \text{if } n-1 \in I \text{ and } n-1 \in J, \\ \mu_{n-1}(I)\mu_{n+1}(J) + \mu_{n-1}(J)\mu_{n+1}(I), & \text{if } n-1 \notin I \text{ and } n-1 \notin J, \\ \mu_{n-1}(I)\mu_{n+1}(J), & \text{if } n-1 \notin I \text{ and } n-1 \in J, \\ \mu_{n-1}(J)\mu_{n+1}(I), & \text{if } n-1 \in I \text{ and } n-1 \notin J. \end{cases}$$

We now prove the following three lemmas.

LEMMA 3. For an integer $n \geq 2$ and subsets $I, I', J, J' \subset [n-1] - \{1\}$ such that $I + I' = [n-1] - \{1\}$, $J + J' = [n-1] - \{1\}$, we have

$$(31) \quad TP(n, 1, I + \{1\}; J + \{1\}) + SP(n, 1, I' + \{1\}, J') = 0.$$

PROOF. We prove Lemma 3 by mathematical induction on n . As we have $TP(2, 1, \{1\}; \{1\}) = -2$, $SP(2, 1, \{1\}, \emptyset) = 2$. Lemma 3 holds for $n = 2$. Suppose that Lemma 3 is proved for $n = N-1$, $N \geq 3$. We shall prove Lemma 3 for $n = N$. Let $HP(N, 1, I, J)$ be the function defined by

$$(32) \quad HP(N, 1, I, J) = \begin{cases} \prod_{\nu \in I + \{1\}} \left(\frac{N-1+\nu}{N-1-\nu} \right) \prod_{\nu \in I' + \{1\}} \left(\frac{N-1+\nu}{N-1-\nu} \right), & \text{if } N-1 \notin I, N-1 \notin J, \\ - \prod_{\nu \in I' + \{1\}} \left(\frac{N-1+\nu}{N-1-\nu} \right) \prod_{\nu \in J + \{1\}} \left(\frac{N-1+\nu}{N-1-\nu} \right), & \text{if } N-1 \in I, N-1 \notin J, \\ - \prod_{\nu \in I + \{1\}} \left(\frac{N-1+\nu}{N-1-\nu} \right) \prod_{\nu \in J'} \left(\frac{N-1+\nu}{N-1-\nu} \right), & \text{if } N-1 \notin I, N-1 \in J, \\ \prod_{\nu \in I' + \{1\}} \left(\frac{N-1+\nu}{N-1-\nu} \right) \prod_{\nu \in J'} \left(\frac{N-1+\nu}{N-1-\nu} \right), & \text{if } N-1 \in I, N-1 \in J. \end{cases}$$

By Propositions 1, 2 and Definitions 1, 2, we have

$$(33) \quad \begin{aligned} TP(N, 1, I + \{1\}; J + \{1\}) \\ = TP(N-1, 1, I + \{1\}; J + \{1\})HP(N, 1, I, J), \end{aligned}$$

$$(34) \quad \begin{aligned} SP(N, 1, I' + \{1\}, J') \\ = SP(N-1, 1, I' + \{1\} - \{N-1\}, J' - \{N-1\})HP(N, 1, I, J). \end{aligned}$$

By the induction hypothesis, we have

$$(35) \quad \begin{aligned} TP(N, 1, I + \{1\}; J + \{1\}) + SP(N, 1, I' + \{1\}, J') \\ = HP(N, 1, I, J) \{ TP(N-1, 1, I + \{1\}; J + \{1\}) \\ + SP(N-1, 1, I' + \{1\} - \{N-1\}, J' - \{N-1\}) \} \\ = 0. \end{aligned}$$

We have verified Lemma 3 for $n = N$. Hence Lemma 3 is proved by mathematical induction on n .

LEMMA 4. *For an integer $n \geq 3$, an integer $2 \leq \lambda \leq n-1$ and subsets $I, I', J, J' \subset [n-1] - \{\lambda\} - \{\lambda-1\}$ such that $I + I' = [n-1] - \{\lambda\} - \{\lambda-1\}$, $J + J' = [n-1] - \{\lambda\} - \{\lambda-1\}$, we have*

$$(36) \quad TP(n, \lambda, I + \{\lambda\}; J + \{\lambda\}) + SP(n, 1, I' + \{\lambda-1\} + \{\lambda\}, J') = 0.$$

PROOF. We prove Lemma 4 by mathematical induction on n . As we have $TP(3, 2, \{2\}; \{2\}) = -36$ and $SP(3, 2, \{1, 2\}, \emptyset) = 36$, Lemma 4 holds for $n = 3$. Suppose that Lemma 4 is proved for $n = N-1$, $N \geq 4$. First, we shall prove Lemma 4 for $n = N$, $\lambda \neq N-1$. Let $HP(N, \lambda, I, J)$ be the function defined by

$$(37) \quad HP(N, \lambda, I, J) = \begin{cases} \prod_{\nu \in I + \{\lambda\}} \left(\frac{N-1+\nu}{N-1-\nu} \right) \prod_{\nu \in J + \{\lambda\}} \left(\frac{N-1+\nu}{N-1-\nu} \right), & \text{if } N-1 \notin I, N-1 \notin J, \\ - \prod_{\nu \in I' + \{\lambda-1\} + \{\lambda\}} \left(\frac{N-1+\nu}{N-1-\nu} \right) \prod_{\nu \in J + \{\lambda\}} \left(\frac{N-1+\nu}{N-1-\nu} \right), & \text{if } N-1 \in I, N-1 \notin J, \\ - \prod_{\nu \in I + \{\lambda\}} \left(\frac{N-1+\nu}{N-1-\nu} \right) \prod_{\nu \in J'} \left(\frac{N-1+\nu}{N-1-\nu} \right), & \text{if } N-1 \notin I, N-1 \in J, \\ \prod_{\nu \in I' + \{\lambda-1\} + \{\lambda\}} \left(\frac{N-1+\nu}{N-1-\nu} \right) \prod_{\nu \in J'} \left(\frac{N-1+\nu}{N-1-\nu} \right), & \text{if } N-1 \in I, N-1 \in J. \end{cases}$$

By Propositions 1, 2 and Definitions 1, 2, we have

$$(38) \quad \begin{aligned} TP(N, \lambda, I + \{\lambda\}; J + \{\lambda\}) \\ = TP(N - 1, \lambda, I + \{\lambda\}; J + \{\lambda\}) HP(N, \lambda, I, J), \end{aligned}$$

$$(39) \quad \begin{aligned} SP(N, 1, I' + \{\lambda - 1\} + \{\lambda\}, J') \\ = SP(N - 1, 1, I' + \{\lambda - 1\} + \{\lambda\} - \{N - 1\}, J' - \{N - 1\}) \\ \times HP(N, \lambda, I, J). \end{aligned}$$

By the induction hypothesis, we have

$$(40) \quad \begin{aligned} TP(N, \lambda, I + \{\lambda\}; J + \{\lambda\}) + SP(N, \lambda, I' + \{\lambda - 1\} + \{\lambda\}, J') \\ = HP(N, \lambda, I, J) \\ \times \{TP(N - 1, \lambda, I + \{\lambda\}; J + \{\lambda\}) \\ + SP(N - 1, \lambda, I' + \{\lambda - 1\} + \{\lambda\} - \{N - 1\}, J' - \{N - 1\})\} \\ = 0. \end{aligned}$$

Hence, we have verified Lemma 4 for $n = N$, $\lambda \neq N - 1$.

It remains to prove Lemma 4 for $n = N$, $\lambda = N - 1$. By Propositions 1, 2 and Definitions 1, 2, we have

$$(41) \quad \begin{aligned} TP(N, N - 1, I + \{N - 1\}; J + \{N - 1\}) \\ = -2(N - 1) \prod_{\nu \in I} \left(\frac{N - 1 + \nu}{N - 1 - \nu} \right) \prod_{\nu \in J} \left(\frac{N - 2 - \nu}{N - 2 + \nu} \right) \\ \times \mu_N(I + \{N - 1\}) \mu_N(J + \{N - 1\}). \end{aligned}$$

Now, we shall prove

$$(42) \quad \begin{aligned} \mu_N(I' + \{N - 2\} + \{N - 1\}) \\ = \prod_{\nu \in I} \left(\frac{N - 1 + \nu}{N - 1 - \nu} \right) \prod_{\nu \in I' + \{N - 2\}} \left(\frac{N - 1 - \nu}{N - 1 + \nu} \right) \mu_N(I + \{N - 1\}). \end{aligned}$$

It follows from Proposition 2 that

$$(43) \quad \mu_N(I' + \{N - 2\} + \{N - 1\}) = \mu_N(I)$$

by Proposition 1 for $n = N - 2$,

$$= \prod_{\nu \in I} \left(\frac{N - 1 + \nu}{N - 1 - \nu} \right) \mu_{N-1}(I)$$

by Proposition 2,

$$= \prod_{\nu \in I} \left(\frac{N-1+\nu}{N-1-\nu} \right) \mu_{N-1}(I' + \{N-2\})$$

by Proposition 1 for $n = N-1$,

$$= \prod_{\nu \in I} \left(\frac{N-1+\nu}{N-1-\nu} \right) \prod_{\nu \in I' + \{N-2\}} \left(\frac{N-1-\nu}{N-1+\nu} \right) \mu_N(I' + \{N-2\})$$

by Proposition 2,

$$= \prod_{\nu \in I} \left(\frac{N-1+\nu}{N-1-\nu} \right) \prod_{\nu \in I' + \{N-2\}} \left(\frac{N-1-\nu}{N-1+\nu} \right) \mu_N(I + \{N-1\}).$$

We hence have the identity (42).

Now, we shall prove

$$(44) \quad \mu_N(J') = \prod_{\nu \in J'} \left(\frac{N-1+\nu}{N-1-\nu} \right) \prod_{\nu \in J'} \left(\frac{N-2+\nu}{N-2-\nu} \right) \prod_{\nu \in J} \left(\frac{N-2-\nu}{N-2+\nu} \right) \\ \times \prod_{\nu \in J' + \{N-2\}} \left(\frac{N-1-\nu}{N-1+\nu} \right) \mu_N(J + \{N-1\}).$$

We apply Proposition 1 for $n = N-2$ and $n = N-1$ to get

$$(45) \quad \mu_N(J') = \prod_{\nu \in J'} \left(\frac{N-1+\nu}{N-1-\nu} \right) \prod_{\nu \in J'} \left(\frac{N-2+\nu}{N-2-\nu} \right) \mu_{N-2}(J')$$

by Proposition 2,

$$= \prod_{\nu \in J'} \left(\frac{N-1+\nu}{N-1-\nu} \right) \prod_{\nu \in J'} \left(\frac{N-2+\nu}{N-2-\nu} \right) \mu_{N-2}(J)$$

by Proposition 1 for $n = N-2$,

$$= \prod_{\nu \in J'} \left(\frac{N-1+\nu}{N-1-\nu} \right) \prod_{\nu \in J'} \left(\frac{N-2+\nu}{N-2-\nu} \right) \prod_{\nu \in J} \left(\frac{N-2-\nu}{N-2+\nu} \right) \mu_{N-1}(J)$$

by Proposition 2,

$$= \prod_{\nu \in J'} \left(\frac{N-1+\nu}{N-1-\nu} \right) \prod_{\nu \in J'} \left(\frac{N-2+\nu}{N-2-\nu} \right) \prod_{\nu \in J} \left(\frac{N-2-\nu}{N-2+\nu} \right) \mu_{N-1}(J' + \{N-2\})$$

by Proposition 1 for $n = N - 1$,

$$= \prod_{\nu \in J'} \left(\frac{N-1+\nu}{N-1-\nu} \right) \prod_{\nu \in J'} \left(\frac{N-2+\nu}{N-2-\nu} \right) \prod_{\nu \in J} \left(\frac{N-2-\nu}{N-2+\nu} \right) \\ \times \prod_{\nu \in J' + \{N-2\}} \left(\frac{N-1-\nu}{N-1+\nu} \right) \mu_N(J' + \{N-2\})$$

by Proposition 2,

$$= \prod_{\nu \in J'} \left(\frac{N-1+\nu}{N-1-\nu} \right) \prod_{\nu \in J'} \left(\frac{N-2+\nu}{N-2-\nu} \right) \prod_{\nu \in J} \left(\frac{N-2-\nu}{N-2+\nu} \right) \\ \times \prod_{\nu \in J' + \{N-2\}} \left(\frac{N-1-\nu}{N-1+\nu} \right) \mu_N(J + \{N-1\}).$$

We hence have the identity (44).

By the identities (42), (44) and Definitions 1, 2, we get

$$(46) \quad SP(N, N-1, I' + \{N-2\} + \{N-1\}, J') \\ = 2(N-1) \prod_{\nu \in I} \left(\frac{N-1+\nu}{N-1-\nu} \right) \prod_{\nu \in J} \left(\frac{N-2-\nu}{N-2+\nu} \right) \\ \times \mu_N(I + \{N-1\}) \mu_N(J + \{N-1\}).$$

Combining the identity (46) with the identity (41), we have

$$(47) \quad TP(N, N-1, I + \{N-1\}; J + \{N-1\}) \\ + SP(N, N-1, I' + \{N-2\} + \{N-1\}, J') = 0.$$

So, Lemma 4 holds for $n = N$, $\lambda = N - 1$. We have verified Lemma 4 for $n = N$ and Lemma 4 is proved by mathematical induction.

LEMMA 5. *For an integer $n \geq 3$, an integer $2 \leq \lambda \leq n - 1$ and subsets I, I', J, J' such that $I + I' = [n - 1] - \{\lambda\} - \{\lambda - 1\}$, $J + J' = [n - 1] - \{\lambda\} - \{\lambda - 1\}$, we have*

$$(48) \quad SP(n, \lambda, I + \{\lambda - 1\} + \{\lambda\}, J + \{\lambda\}) + SP(n, \lambda, I' + \{\lambda\}, J') = 0.$$

PROOF. We prove Lemma 5 as well as Lemma 4 by mathematical induction on n . As we have $SP(3, 2, \{1, 2\}, \{2\}) = -36$, $SP(3, 2, \{2\}, \emptyset) = 36$. Lemma 5 holds for $n = 3$. Suppose that Lemma 5 is proved for $n = N - 1$, $N \geq 3$. First, we shall

prove Lemma 5 for $n = N$, $\lambda \neq N - 1$. Let $IP(N, \lambda, I, J)$ be the function defined by

$$(49) \quad IP(N, \lambda, I, J) = \begin{cases} \prod_{\nu \in I + \{\lambda-1\} + \{\lambda\}} \left(\frac{N-1+\nu}{N-1-\nu} \right) \prod_{\nu \in J + \{\lambda\}} \left(\frac{N-1+\nu}{N-1-\nu} \right), & \text{if } N-1 \notin I, N-1 \notin J, \\ - \prod_{\nu \in I' + \{\lambda\}} \left(\frac{N-1+\nu}{N-1-\nu} \right) \prod_{\nu \in J + \{\lambda\}} \left(\frac{N-1+\nu}{N-1-\nu} \right), & \text{if } N-1 \in I, N-1 \notin J, \\ - \prod_{\nu \in I + \{\lambda-1\} + \{\lambda\}} \left(\frac{N-1+\nu}{N-1-\nu} \right) \prod_{\nu \in J'} \left(\frac{N-1+\nu}{N-1-\nu} \right), & \text{if } N-1 \notin I, N-1 \in J, \\ \prod_{\nu \in I' + \{\lambda\}} \left(\frac{N-1+\nu}{N-1-\nu} \right) \prod_{\nu \in J'} \left(\frac{N-1+\nu}{N-1-\nu} \right), & \text{if } N-1 \in I, N-1 \in J. \end{cases}$$

By Propositions 1, 2 and Definitions 1, 2, we have

$$(50) \quad \begin{aligned} SP(N, \lambda, I + \{\lambda-1\} + \{\lambda\}, J + \{\lambda\}) \\ = SP(N-1, \lambda, I + \{\lambda-1\} + \{\lambda\}, J + \{\lambda\})IP(N, \lambda, I, J), \end{aligned}$$

$$(51) \quad \begin{aligned} SP(N, 1, I' + \{\lambda\}, J') \\ = SP(N-1, 1, I' + \{\lambda\} - \{N-1\}, J' - \{N-1\})IP(N, \lambda, I, J). \end{aligned}$$

By the induction hypothesis, we have

$$(52) \quad \begin{aligned} SP(N, \lambda, I + \{\lambda-1\} + \{\lambda\}, J + \{\lambda\}) + SP(N, \lambda, I' + \{\lambda\}, J') \\ = IP(N, \lambda, I, J) \\ \times \{ SP(N-1, \lambda, I + \{\lambda\}, J + \{\lambda\}) \\ + SP(N-1, \lambda, I' + \{\lambda-1\} + \{\lambda\} - \{N-1\}, J' - \{N-1\}) \} \\ = 0. \end{aligned}$$

Hence, we have verified Lemma 5 for $n = N$, $\lambda \neq N - 1$. Now, we shall prove Lemma 5 for $n = N$, $\lambda = N - 1$. By Propositions 1, 2 and Definitions 1, 2, we have

$$(53) \quad \begin{aligned} SP(N, N-1, I + \{N-2\} + \{N-1\}, J + \{N-1\}) \\ = -2(N-1)(2N-3) \prod_{\nu \in I} \left(\frac{N-1+\nu}{N-1-\nu} \right) \prod_{\nu \in J} \left(\frac{N-2-\nu}{N-2+\nu} \right) \end{aligned}$$

$$\times \mu_N(I + \{N - 2\} + \{N - 1\})\mu_N(J + \{N - 1\}).$$

We shall prove

$$(54) \quad \mu_N(I' + \{N - 1\}) = \prod_{\nu \in I + \{N - 2\}} \left(\frac{N - 1 + \nu}{N - 1 - \nu} \right) \prod_{\nu \in I'} \left(\frac{N - 1 - \nu}{N - 1 + \nu} \right) \\ \times \mu_N(I + \{N - 2\} + \{N - 1\}).$$

It follows from Proposition 2 that

$$(55) \quad \mu(I' + \{N - 1\}) = \mu(I + \{N - 2\})$$

by Proposition 1 for $n = N - 1$,

$$= \prod_{\nu \in I + \{N - 2\}} \left(\frac{N - 1 + \nu}{N - 1 - \nu} \right) \mu_{N-1}(I + \{N - 2\})$$

by Proposition 2,

$$= \prod_{\nu \in I + \{N - 2\}} \left(\frac{N - 1 + \nu}{N - 1 - \nu} \right) \mu_{N-1}(I')$$

by Proposition 1 for $n = N - 1$,

$$= \prod_{\nu \in I + \{N - 2\}} \left(\frac{N - 1 + \nu}{N - 1 - \nu} \right) \prod_{\nu \in I'} \left(\frac{N - 1 - \nu}{N - 1 + \nu} \right) \mu_N(I')$$

by Proposition 2,

$$= \prod_{\nu \in I + \{N - 2\}} \left(\frac{N - 1 + \nu}{N - 1 - \nu} \right) \prod_{\nu \in I'} \left(\frac{N - 1 - \nu}{N - 1 + \nu} \right) \mu_N(I + \{N - 2\} + \{N - 1\}).$$

We hence have the identity (54). So it follows from the identities (54), (44) and Definitions 1, 2,

$$(56) \quad SP(N, N - 1, I' + \{N - 1\}, J') \\ = 2(N - 1)(2N - 3) \prod_{\nu \in I} \left(\frac{N - 1 + \nu}{N - 1 - \nu} \right) \prod_{\nu \in J} \left(\frac{N - 2 - \nu}{N - 2 + \nu} \right) \\ \times \mu_N(I + \{N - 2\} + \{N - 1\})\mu_N(J + \{N - 1\}).$$

Combining the identity (56) with the identity (53), we have

$$(57) \quad \begin{aligned} SP(N, N-1, I + \{N-2\} + \{N-1\}, J + \{N-1\}) \\ + SP(N, N-1, I' + \{N-1\}, J') = 0. \end{aligned}$$

Lemma 5 holds for $n = N$, $\lambda = N-1$. With these result, we have verified Lemma 5 for $n = N$ and hence obtain Lemma 5 by mathematical induction.

To prove Theorem 1, we use the following

PROPOSITION 4. *For an integer $n \geq 3$, an integer $2 \leq \lambda \leq n-1$ and subsets $I, J \subset [n-1] - \{\lambda\} - \{\lambda-1\}$, we have*

$$(58) \quad SP(n, \lambda, I + \{\lambda-1\} + \{\lambda\}, J + \{\lambda-1\} + \{\lambda\}) = 0,$$

$$(59) \quad SP(n, \lambda, I + \{\lambda\}, J + \{\lambda-1\}) = 0,$$

$$(60) \quad SP(n, \lambda, I + \{\lambda-1\} + \{\lambda\}, J + \{\lambda-1\}) = 0,$$

$$(61) \quad SP(n, \lambda, I + \{\lambda-1\}, J + \{\lambda\}) = 0,$$

$$(62) \quad SP(n, \lambda, I + \{\lambda-1\}, J + \{\lambda-1\} + \{\lambda\}) = 0.$$

For an integer $n \geq 2$, and subsets I, I', J, J' such that $I + I' = [n-1] - \{1\}$, $J + J' = [n-1] - \{1\}$, we have

$$(63) \quad \begin{aligned} SP(n, 1, I + \{1\}, J + \{1\}) + SP(n, 1, I', J' + \{1\}) \\ + SP(n, 1, I' + \{1\}, J') = 0. \end{aligned}$$

For an integer $n \geq 3$, an integer $2 \leq \lambda \leq n-1$, and subsets I, I', J, J' such that $I + I' = [n-1] - \{\lambda\} - \{\lambda-1\}$, $J + J' = [n-1] - \{\lambda\} - \{\lambda-1\}$, we have

$$(64) \quad \begin{aligned} SP(n, \lambda, I + \{\lambda\}, J + \{\lambda\}) + SP(n, \lambda, I', J' + \{\lambda-1\} + \{\lambda\}) \\ + SP(n, \lambda, I' + \{\lambda-1\} + \{\lambda\}, J') = 0, \end{aligned}$$

$$(65) \quad SP(n, \lambda, I + \{\lambda-1\} + \{\lambda\}, J + \{\lambda\}) + SP(n, \lambda, I' + \{\lambda\}, J') = 0,$$

$$(66) \quad SP(n, \lambda, I + \{\lambda\}, J + \{\lambda-1\} + \{\lambda\}) + SP(n, \lambda, I', J' + \{\lambda\}) = 0.$$

PROOF. From Lemma 2, we have the identities (58), (59), (60), (61) and (62). We have

$$(67) \quad \begin{aligned} SP(n, 1, I + \{1\}, J + \{1\}) \\ = TP(n, 1, I + \{1\}; J + \{1\}) + TP(n, 1, J + \{1\}; I + \{1\}) \end{aligned}$$

for an integer $n \geq 2$ and $I, J \subset [n-1] - \{1\}$. By Lemma 3, we hence have the identity (63). We have

$$(68) \quad \begin{aligned} SP(n, \lambda, I + \{\lambda\}, J + \{\lambda\}) \\ = TP(n, \lambda, I + \{\lambda\}; J + \{\lambda\}) + TP(n, \lambda, J + \{\lambda\}; I + \{\lambda\}) \end{aligned}$$

for an integer $n \geq 2$ and $I, J \subset [n-1] - \{1\}$. By Lemma 4, we hence have the identity (64). We proved already the identity (65) in Lemma 5. We note that the identity (66) is equivalent to the identity (65).

§2. Proof of Theorem 1

We put $v = e^x + e^{-x}$ and define the operators $l_x(f)$ and $l_x(f, g)$ by

$$(69) \quad l_x(f) = f \frac{d^2 f}{dx^2} - \left(\frac{df}{dx} \right)^2,$$

$$(70) \quad l_x(f, g) = f \frac{d^2 g}{dx^2} - 2 \frac{df}{dx} \frac{dg}{dx} + g \frac{d^2 f}{dx^2}.$$

Let c be a constant, f, g be functions of variable x and n, m be non-negative integers. Then we have

$$\begin{aligned} l_x(cf) &= c^2 l_x(f), \quad l_x(f, g) = l_x(g, f), \quad l_x(cf, g) = cl_x(f, g), \\ l_x(fg) &= f^2 l_x(g) + g^2 l_x(f), \quad l_x(f+g) = l_x(f) + l_x(g) + l_x(f, g). \end{aligned}$$

By $z = (2 - v)/4$, $w = (2 + v)/4$, $v = e^x + e^{-x}$, we have

$$(71) \quad zw l_x(z^n w^m) = ((n/2)w + (m/2)z) z^{2n} w^{2m},$$

and for $n_1 + m_1 = n_2 + m_2$

$$(72) \quad \begin{aligned} zw l_x(z^{n_1} w^{m_1}, z^{n_2} w^{m_2}) \\ = \left[\left\{ (1/2)(n_1 + n_2) - (n_1 - n_2)^2 \right\} w \right. \\ \left. + \left\{ (1/2)(m_1 + m_2) - (m_1 - m_2)^2 \right\} z \right] z^{n_1 + n_2} w^{m_1 + m_2}. \end{aligned}$$

We now define U_n for $n \geq 2$ by

$$(73) \quad U_n = \sum_{I \subset [n-1]} \mu_n(I) c_I d_{[n-1]-I} z^{|I|} w^{|[n-1]-I|},$$

and denote $U_0 = U_1 = 1$. We prove that, for a positive integer n , U_n satisfies the following recurrence relation

$$(74) \quad U_{n-1}U_{n+1} = (\overline{c_n}z + \overline{d_n}w)U_n^2 - 16zwl_x(U_n),$$

which is equivalent to the fact that T_n presented by Theorem 1 satisfies the recurrence relation (1). As we have $U_0 = U_1 = 1$, $U_2 = c_1z + d_1w$. The recurrence relation (74) holds for $n = 1$. We have to prove the recurrence relation (74) for $n \geq 2$. We then have for $n \geq 2$

$$(75) \quad (\overline{c_n}z + \overline{d_n}w)U_n^2 - 16zwl_x(U_n) \\ = \sum_{I, J \subset [n-1]} P(I, J) c_I d_{[n-1]-I} c_J d_{[n-1]-J} z^{|I|+|J|} w^{|2[n-1]||-||I|-||J||+1} \\ + \sum_{I, J \subset [n-1]} P^*(I, J) c_{[n-1]-I} d_I c_{[n-1]-J} d_J z^{|2[n-1]||-||I|-||J||+1} w^{|I|+|J|}$$

where $P(I, J)$, $P^*(I, J)$ for an integer $n \geq 2$ and subsets $I, J \subset [n-1]$ are defined by

$$(76) \quad P(I, J) = \mu_n(I)\mu_n(J) \left(\overline{d_n} - 4(|I| + |J|) + 8(|I| - |J|)^2 \right),$$

$$(77) \quad P^*(I, J) = \mu_n(I)\mu_n(J) \left(\overline{c_n} - 4(|I| + |J|) + 8(|I| - |J|)^2 \right).$$

Using $b_\lambda^{I,J}$ and Lemma 1, we deform the polynomial $P(I, J)$ by

$$(78) \quad P(I, J) = \mu_n(I)\mu_n(J) (\overline{d_n} - 4(|I| + |J|) + 8(|I| - |J|)^2) \\ = \mu_n(I)\mu_n(J) \left(\overline{d_n} + 2 \sum_{\lambda \in I \cup J} b_\lambda^{I,J} \right) \\ = \mu_n(I)\mu_n(J) \overline{d_n} \left(1 + \sum_{\lambda \in I \cup J} \frac{b_\lambda^{I,J}}{2(n-\lambda)(n-1+\lambda)} \right) \\ - \sum_{\lambda \in I \cup J} \frac{b_\lambda^{I,J} \mu_n(I)\mu_n(J) (\overline{d_n} - 4(n-\lambda)(n-1+\lambda))}{2(n-\lambda)(n-1+\lambda)}.$$

Combining the definitions of $FP(I, J)$ and $SP(n, \lambda, I, J)$ with the identity (78), we have by $(\overline{d_n} - 4(n-\lambda)(n-1+\lambda)) = \overline{d_\lambda}$

$$(79) \quad P(I, J) = FP(I, J) \overline{d_n} - \sum_{\lambda \in I \cup J} \frac{SP(n, \lambda, I, J) \overline{d_\lambda}}{2(n-\lambda)(n-1+\lambda)}.$$

Similarly, we have

$$(80) \quad P^*(I, J) = FP(I, J)\overline{c_n} - \sum_{\lambda \in I \cup J} \frac{SP(n, \lambda, I, J)\overline{c_\lambda}}{2(n-\lambda)(n-1+\lambda)}.$$

For an integer $n \geq 2$, subsets $I, J \subset [n-1]$ and an integer $\lambda \in I \cup J$, we define

$$(81) \quad E(n, \lambda, I, J) = \overline{d_\lambda} c_I d_{[n-1]-I} c_J d_{[n-1]-J} z^{|I|+|J|} w^{|2[n-1]|-|I|-|J|+1}$$

$$(82) \quad E^*(n, \lambda, I, J) = \overline{c_\lambda} c_{[n-1]-I} d_I c_{[n-1]-J} d_J z^{|2[n-1]|-|I|-|J|+1} w^{|I|+|J|}.$$

We then have the following

LEMMA 6. For an integer $n \geq 2$, and subsets I, I', J, J' such that $I + I' = [n-1] - \{1\}$, $J + J' = [n-1] - \{1\}$, we have

$$(83) \quad E(n, 1, I + \{1\}, J + \{1\}) = E^*(n, 1, I', J' + \{1\}) = E^*(n, 1, I' + \{1\}, J').$$

For an integer $n \geq 3$, an integer $2 \leq \lambda \leq n-1$, and subsets I, I', J, J' such that $I + I' = [n-1] - \{\lambda\} - \{\lambda-1\}$, $J + J' = [n-1] - \{\lambda\} - \{\lambda-1\}$, we have

$$(84) \quad \begin{aligned} E(n, \lambda, I + \{\lambda\}, J + \{\lambda\}) &= E^*(n, \lambda, I', J' + \{\lambda-1\} + \{\lambda\}) \\ &= E^*(n, \lambda, I' + \{\lambda-1\} + \{\lambda\}, J') \end{aligned}$$

$$(85) \quad E(n, \lambda, I + \{\lambda-1\} + \{\lambda\}, J + \{\lambda\}) = E^*(n, \lambda, I' + \{\lambda\}, J')$$

$$(86) \quad E(n, \lambda, I + \{\lambda\}, J + \{\lambda-1\} + \{\lambda\}) = E^*(n, \lambda, I', J' + \{\lambda\}).$$

We note the following

LEMMA 7. For an integer $n \geq 2$ and subsets $I, J \subset [n-1]$ such that $n-1 \notin I$, we have

$$(87) \quad E(n, n, I, J) = c_I d_{[n-2]-I} z^{|I|} w^{|[n-2]-I|} c_J d_{[n]-J} z^{|J|} w^{|[n]-J|},$$

$$(88) \quad E^*(n, n, I, J) = c_{[n-2]-I} d_I z^{|[n-2]-I|} w^{|I|} c_{[n]-J} d_J z^{|[n]-J|} w^{|J|}.$$

From Proposition 4 and Lemma 6, we have

$$(89) \quad \begin{aligned} &\sum_{I, J \subset [n-1]} \sum_{\lambda \in I \cup J} \frac{SP(n, \lambda, I, J)E(n, \lambda, I, J)}{2(n-\lambda)(n-1+\lambda)} \\ &+ \sum_{I, J \subset [n-1]} \sum_{\lambda \in I \cup J} \frac{SP(n, \lambda, I, J)E^*(n, \lambda, I, J)}{2(n-\lambda)(n-1+\lambda)} = 0. \end{aligned}$$

Combining the identities (75), (79), (80) with the identity (89) and combining Proposition 3 with Lemma 7, we have for an integer $n \geq 2$

$$\begin{aligned}
 (90) \quad & (\overline{c_n}z + \overline{d_n}w)U_n^2 - 16zwl_x(U_n) \\
 &= \sum_{I, J \subset [n-1]} FP(I, J)E(n, n, I, J) + \sum_{I, J \subset [n-1]} FP(I, J)E^*(n, n, I, J) \\
 &= U_{n-1}U_{n+1}.
 \end{aligned}$$

Consequently we have verified that U_n satisfy the recurrence relation (74). So, T_n given by Theorem 1 satisfies the recurrence relation (1). Since T_n is uniquely determined by the recurrence relation (1), Theorem 1 holds.

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