Conformally flat 3-manifolds with constant scalar curvature II

By Qing-Ming CHENG*, Susumu ISHIKAWA** and Katsuhiko SHIOHAMA**

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1. Introduction

A well known result in Riemannian geometry states that every compact n -dimensional Riemannian manifold can be deformed to a Riemannian manifold with constant scalar curvature by a conformal transformation (see [3], [5], [6], etc.). This result was obtained through the application of the so-called Yamabe equation. The Yamabe equation has a deep connection with locally conformally flat Riemannian manifolds (see [4]). Let us begin by recalling that M^n is by definition locally conformally flat if and only if every point on M^n admits a coordinate neighborhood with coordinates $u^1, \dots, u^n$ in which the Riemannian metric can be expressed as

$$ds^2 = \sum_{i=1}^n \lambda^2(u) (du^i)^2,$$

where $\lambda(u)$ is a positive function defined in the coordinate neighborhood. There are many examples of locally conformally flat Riemannian manifolds (see [4], etc.).

The study of global properties on locally conformally flat Riemannian manifolds was initiated by Kuiper, who studied their fundamental groups. Recently, Schoen and Yau also investigated the global properties on the locally conformally flat Riemannian manifolds by using the Yamabe equation. In the previous paper [1], we have classified in dimension 3 the class of such Riemannian manifolds with a certain additional properties. Let r and S be respectively the scalar curvature and the squared norm of Ricci curvature tensor of a complete locally conformally

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flat n -dimensional Riemannian manifold M^n . Let $M^n(c)$ be n -dimensional space form of constant sectional curvature c . The following results have been established in [1]:

Let M^3 be a complete locally conformally flat 3-dimensional Riemannian manifold with both r and S being constant. If $r \geq 0$, then

(1) M^3 is isometric to a space form of constant sectional curvature

or

(2) M^3 is isometric to a Riemannian product $M^2(c) \times N^1$ where $N^1 = \mathbf{R}^1$ or S^1 for some $c \geq 0$.

If $r < 0$, then

(3) M is isometric to a space form of constant negative curvature

or

(4) $\frac{r^2}{3} < S \leq \frac{r^2}{2}$.

We have also proved in [1] that there exists no compact locally conformally flat Riemannian 3-manifold M^3 with the following two conditions:

(a) r is a negative constant

(b) S is constant and all eigenvalues of the Ricci curvature tensor at each point of M^3 are distinct.

Typical examples are stated as follows.

EXAMPLE 1. $H^3(r/6)$ ($r < 0$) is a complete locally conformally flat 3-dimensional Riemannian manifold with $S = \frac{r^2}{3}$.

EXAMPLE 2. $H^2(c) \times \mathbf{R}$ ($c < 0$) is a complete locally conformally flat 3-dimensional Riemannian manifold with $S = \frac{r^2}{2}$.

However, it has not yet been known if there exists a complete locally conformally flat 3-dimensional Riemannian manifold on which r is a negative constant and S is constant such that $\frac{r^2}{3} < S < \frac{r^2}{2}$. The following conjecture has been proposed in [1]

CONJECTURE. Let M^3 be a complete locally conformally flat 3-dimensional Riemannian manifold with the constant negative scalar curvature and the Ricci curvature tensor of constant squared norm. Then, M^3 is isometric to either a space form of constant negative curvature or the Riemannian product $H^2(c) \times N^1$ ($c < 0$).

The purpose of present article is to study complete locally conformally flat 3-dimensional Riemannian manifolds with constant negative scalar curvature r possessing Ricci curvature tensor of constant squared norm S . Our Main Theorem strongly supports the validity of the above conjecture.

MAIN THEOREM. *There exist no complete locally conformally flat 3-dimensional Riemannian manifolds with negative constant scalar curvature r and the Ricci curvature tensor with constant squared norm S satisfying $\frac{r^2}{3} + \left(\frac{7}{113}\right)^2 \frac{2r^2}{3} < S < \frac{r^2}{3} + \left(\frac{35}{137}\right)^2 \frac{r^2}{6}$.*

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2. Preliminaries

Let M^n be an n -dimensional Riemannian manifold. We choose a local frame of orthonormal vector fields $\{e_1, \dots, e_n\}$ adapted to the Riemannian metric of M^n and the dual coframe $\{\omega_1, \dots, \omega_n\}$. Then the connection forms $\{\omega_{ij}\}$ of M^n are characterized by the structure equations:

$$d\omega_i = - \sum_j \omega_{ij} \wedge \omega_j, \quad \omega_{ij} + \omega_{ji} = 0,$$

$$d\omega_{ij} = - \sum_k \omega_{ik} \wedge \omega_{kj} + \frac{1}{2} \sum_{k,l} R_{ijkl} \omega_k \wedge \omega_l,$$

where R_{ijkl} are the components of curvature tensor of M^n .

Let C_{ijkl} denote the components of Weyl tensor of M^n , that is,

$$C_{ijkl} = R_{ijkl} - \frac{1}{n-2} (R_{ik} \delta_{jl} - R_{il} \delta_{jk} + R_{jl} \delta_{ik} - R_{jk} \delta_{il}) \\ + \frac{r}{(n-1)(n-2)} (\delta_{ik} \delta_{jl} - \delta_{il} \delta_{jk}),$$

where R_{ij} and r are the components of Ricci curvature tensor and the scalar curvature of M^n , respectively. Let ∇ denote the covariant differentiation on M^n . By writing $R_{ij,k} = \nabla_k R_{ij}$ and $R_{ij,kl} = \nabla_l \nabla_k R_{ij}$, etc., we have the Ricci formulas

$$R_{ij,kl} - R_{ij,lk} = \sum_t R_{tj} R_{tikl} + \sum_t R_{it} R_{tjkl},$$

$$R_{ij,klm} - R_{ij,kml} = \sum_t R_{tj,k} R_{tilm} + \sum_t R_{it,k} R_{tjlm} + \sum_t R_{ij,t} R_{tklm}.$$

The Bek tensor with the components C_{ijk} is defined by

$$C_{ijk} = \frac{1}{n-2} (R_{ij,k} - R_{ik,j}) - \frac{1}{2(n-1)(n-2)} (\delta_{ij} r_k - \delta_{ik} r_j),$$

where $r_k = \nabla_k r$.

For $n \geq 4$, we know that M^n is a locally conformally flat Riemannian manifold if and only if $C_{ijkl} = 0$, and that $C_{ijk} = 0$ on M^n in this case.

When $n = 3$, we always have $C_{ijkl} = 0$ on M^3 and that M^3 is a locally conformally flat Riemannian manifold if and only if $C_{ijk} = 0$. Hence, if M^n is a locally conformally flat n -dimensional Riemannian manifold, then

$$(2.1) \quad R_{ijkl} = \frac{1}{n-2}(R_{ik}\delta_{jl} - R_{il}\delta_{jk} + R_{jl}\delta_{ik} - R_{jk}\delta_{il}) \\ - \frac{r}{(n-1)(n-2)}(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}).$$

Moreover if the scalar curvature r is constant, then we obtain

$$(2.2) \quad R_{ij,k} = R_{ik,j}.$$

From now on, we assume $n = 3$. Let λ_i for $i = 1, 2, 3$ be the eigenvalues of Ricci curvature tensor (R_{ij}) of M^3 and define $\mu_i := \lambda_i - \frac{r}{3}$. We put $B = S - \frac{r^2}{3}$. The following has been proved in [1]:

PROPOSITION. *Let M^3 be a locally conformally flat Riemannian manifold of dimension 3. If the scalar curvature and the squared norm of Ricci curvature tensor of M^3 are constant, then we have*

$$(2.3) \quad \sum_{i,j,k} R_{ij,k}^2 = -3 \sum_i \mu_i^3 - \frac{r}{2} B,$$

$$(2.4) \quad \sum_{i,j,k,l} R_{ij,kl}^2 = -24 \sum_{i,j,k} \mu_i R_{ij,k}^2 - \frac{3}{4} r \sum_{i,j,k} R_{ij,k}^2 - \frac{9}{4} B \left(B - \frac{r^2}{6} \right).$$

In order to prove the Main Theorem, the Theorem 3 in [1] will play an important role. We restate it in the following:

THEOREM 3 (see [1]). *Let M^3 be a complete locally conformally flat 3-dimensional Riemannian manifold with constant scalar curvature r and constant squared norm S of the Ricci curvature tensor. If $r < 0$, then we have*

$$\sup \sum_{i=1}^3 \mu_i^3 = \frac{B^{\frac{3}{2}}}{\sqrt{6}}.$$

3. Proof of Main Theorem

In order to prove Main Theorem, first, we discuss the following special case:

THEOREM 3.0. *There exist no locally conformally flat 3-dimensional Riemannian manifolds with the negative constant scalar curvature r and Ricci curvature tensor of constant squared norm S satisfying $\frac{r^2}{3} + \left(\frac{7}{113}\right)^2 \frac{2r^2}{3} < S < \frac{r^2}{3} + \left(\frac{35}{137}\right)^2 \frac{r^2}{6}$ such that $\sum_i (\lambda_i - \frac{r}{3})^3 = \frac{(S - \frac{r^2}{3})^{\frac{3}{2}}}{\sqrt{6}}$ holds at some point p .*

From the Lemma 1 in [1], we know that

$$-\frac{B^{\frac{3}{2}}}{\sqrt{6}} \leq \sum_i \mu_i^3 \leq \frac{B^{\frac{3}{2}}}{\sqrt{6}},$$

where the equality holds if and only if at least two of μ_i are equal. From the assumption of Theorem 3.0, we have, at a point p , $\sum_i \mu_i^3 = \frac{B^{\frac{3}{2}}}{\sqrt{6}}$. Hence, for the purpose of proving this Theorem 3.0, we can assume that at the point p

$$(3.1) \quad \lambda_1 = \lambda_2 = \lambda \quad \text{and} \quad \lambda_3 = r - 2\lambda,$$

that is,

$$(3.2) \quad \mu_1 = \mu_2 = \mu \quad \text{and} \quad \mu_3 = -2\mu.$$

If $\mu = 0$ at the point p , then $\mu_1 = \mu_2 = \mu_3 = 0$ at p . Hence, $S - \frac{r^2}{3} = B = \sum_i \mu_i^2 = 0$ at p . Since $S > \frac{r^2}{3} + \left(\frac{7}{113}\right)^2 \frac{2r^2}{3}$, that is, $B > \left(\frac{7}{113}\right)^2 \frac{2r^2}{3}$, we can assume $\mu \neq 0$ at the point p . Then we obtain from (3.2) at p

$$(3.3) \quad B = 6\mu^2.$$

Let $\{e_1, e_2, e_3\}$ be an orthonormal basis of tangent space $T_p M$ at p . Thus we have at p

$$(R_{ij}) = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & r - 2\lambda \end{pmatrix}$$

and

$$(\tau_{ij}) = \begin{pmatrix} \mu & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & -2\mu \end{pmatrix},$$

where $\tau_{ij} = R_{ij} - \frac{r}{3}\delta_{ij}$. Because $\{e_1, e_2\}$ is a basis of the eigenspace corresponding to the λ -eigenvalue of (R_{ij}) at p , we can assume $R_{12,3} = 0$ at p , rotating it if necessary.

LEMMA 3.1. *At the point p , we have*

$$(3.4) \quad R_{22,k} = -R_{11,k}, \quad R_{33,k} = 0 \quad \text{for } k = 1, 2, 3,$$

$$(3.5) \quad \sum_{i,j,k} R_{ij,k}^2 = 6R_{11,3}^2 + 4(R_{11,1}^2 + R_{22,2}^2) = 3B \left(\mu - \frac{r}{6} \right).$$

PROOF. By differentiating $\sum_i R_{ii} = r$ and $\sum_{i,j} R_{ij}^2 = S$ at p , we have respectively

$$\sum_i R_{ii,k} = 0 \quad \text{and} \quad \sum_i R_{ii} R_{ii,k} = 0.$$

Hence, at p , we have $R_{22,k} = -R_{11,k}$ and $R_{33,k} = 0$ for $k = 1, 2, 3$. Therefore, by using (2.3), we obtain also at p

$$\sum_{i,j,k} R_{ij,k}^2 = 6R_{11,3}^2 + 4(R_{11,1}^2 + R_{22,2}^2) = 3B \left(\mu - \frac{r}{6} \right).$$

This completes the proof of Lemma 3.1. $\square$

Let us denote A_1 and A_2 by

$$(3.6) \quad A_1 = R_{11,3}^2 \quad \text{and} \quad A_2 = R_{11,1}^2 + R_{22,2}^2.$$

Then (3.5) in Lemma 3.1 means

$$(3.7) \quad 6A_1 + 4A_2 = 3B \left(\mu - \frac{r}{6} \right).$$

LEMMA 3.2. *If $\{i, j, k, l\}$ is not a permutation of $\{1, 1, 3, 3\}$ or $\{2, 2, 3, 3\}$, then, at p ,*

$$R_{ij,kl} = R_{ij,lk}.$$

PROOF. From the Ricci formulas and (2.1), we obtain at p

$$R_{ij,kl} - R_{ij,lk} = (\mu_i - \mu_j) \left(\mu_i + \mu_j + \frac{r}{6} \right) (\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}).$$

If $\{i, j, k, l\}$ is not a permutation of $\{1, 1, 3, 3\}$ or $\{2, 2, 3, 3\}$, we have at p

$$R_{ij,kl} = R_{ij,lk}.$$

Thus, we have finished the proof of Lemma 3.2. $\square$

LEMMA 3.3. *At the point p ,*

$$\begin{aligned} R_{11,11} &= R_{22,22}, \quad R_{33,33} = \frac{2A_1}{3\mu}, \quad R_{33,12} = 0, \quad R_{33,13} = \frac{2R_{11,1}R_{11,3}}{3\mu}, \\ R_{33,23} &= -\frac{2R_{22,2}R_{11,3}}{3\mu}, \quad R_{33,11} = R_{33,22} = \frac{2(A_1 + A_2)}{3\mu}, \\ R_{11,33} &= R_{22,33} = -\frac{A_1}{3\mu}. \end{aligned}$$

PROOF. Since $\sum_i R_{ii} = r$ and $\sum_{i,j} R_{ij}^2 = S$ are constants, we have, by differentiating them at p ,

$$(3.8) \quad \sum_i R_{ii,kl} = 0,$$

and

$$\sum_i \mu_i R_{ii,kl} + \sum_{i,j} R_{ij,k} R_{ij,l} = 0.$$

Hence, by direct calculation it follows that

$$R_{33,kl} = \frac{1}{3\mu} \sum_{i,j} R_{ij,k} R_{ij,l}.$$

That is, by using (3.7), it follows that

$$\begin{aligned} R_{33,12} &= 0, \quad R_{33,33} = \frac{1}{3\mu} \sum_{i,j} R_{ij,3}^2 = \frac{2A_1}{3\mu}, \\ R_{33,13} &= \frac{1}{3\mu} \sum_{i,j} R_{ij,1} R_{ij,3} = \frac{2R_{11,1} R_{11,3}}{3\mu}, \\ R_{33,23} &= \frac{1}{3\mu} \sum_{i,j} R_{ij,2} R_{ij,3} = -\frac{2R_{22,2} R_{11,3}}{3\mu}, \\ R_{33,11} &= R_{33,22} = \frac{1}{3\mu} \sum_{i,j} R_{ij,1}^2 = \frac{2(A_1 + A_2)}{3\mu}, \\ R_{11,33} &= R_{22,33} = R_{33,11} + (\mu_1 - \mu_3) \left(\mu_1 + \mu_3 + \frac{r}{6} \right) = -\frac{A_1}{3\mu} \end{aligned}$$

at the point p . According to (3.8), we have also at p

$$R_{11,11} = R_{22,22}.$$

This completes the proof of Lemma 3.3. $\square$

LEMMA 3.4. *At the point p , we have*

$$\begin{aligned} & \mu^2 \{ 2(R_{11,11}^2 + R_{11,12}^2) + 3(R_{11,23}^2 + R_{22,13}^2) + R_{11,13}^2 + R_{22,23}^2 \} \\ & + 2(A_1 + A_2)\mu R_{11,11} \\ & = -\frac{58A_1^2 + 112A_1A_2 + 48A_2^2}{36} - 4\mu B A_2 + \frac{3\mu^2}{16} (r^2 - 3r\mu - 3B)B. \end{aligned}$$

PROOF. By calculating $\sum_{i,j,k} R_{ij,k1}^2$, $\sum_{i,j,k} R_{ij,k2}^2$ and $\sum_{i,j,k} R_{ij,k3}^2$ separately, we have at p

$$\begin{aligned}
 (3.9) \quad \sum_{i,j,k} R_{ij,k1}^2 &= 6R_{12,31}^2 + \sum_i R_{ii,i1}^2 + 3 \sum_{i \neq k} R_{ii,k1}^2 \\
 &= 3(R_{11,21}^2 + R_{11,31}^2 + R_{22,11}^2 + R_{22,31}^2 + R_{33,11}^2 + R_{33,21}^2) \\
 &\quad + 6R_{11,23}^2 + R_{11,11}^2 + R_{22,21}^2 + R_{33,31}^2 \\
 &= 3(2R_{11,23}^2 + R_{11,13}^2 + R_{22,13}^2) + R_{33,31}^2 \\
 &\quad + 6(R_{11,11}R_{33,11} + R_{33,11}^2) + 4(R_{11,12}^2 + R_{11,11}^2),
 \end{aligned}$$

$$\begin{aligned}
 (3.10) \quad \sum_{i,j,k} R_{ij,k2}^2 &= 6R_{12,32}^2 + \sum_i R_{ii,i2}^2 + 3 \sum_{i \neq k} R_{ii,k2}^2 \\
 &= 3(R_{11,22}^2 + R_{11,32}^2 + R_{22,12}^2 + R_{22,32}^2 + R_{33,12}^2 + R_{33,22}^2) \\
 &\quad + 6R_{22,13}^2 + R_{11,12}^2 + R_{22,22}^2 + R_{33,32}^2 \\
 &= 3(2R_{11,11}R_{33,11} + R_{33,11}^2 + R_{33,22}^2) + 4(R_{11,12}^2 + R_{11,11}^2) \\
 &\quad + 3(2R_{22,13}^2 + R_{11,23}^2 + R_{22,23}^2) + R_{33,23}^2,
 \end{aligned}$$

$$\begin{aligned}
 (3.11) \quad \sum_{i,j,k} R_{ij,k3}^2 &= 6R_{12,33}^2 + \sum_i R_{ii,i3}^2 + 3 \sum_{i \neq k} R_{ii,k3}^2 \\
 &= 3(R_{11,23}^2 + R_{11,33}^2 + R_{22,13}^2 + R_{22,33}^2 + R_{33,13}^2 + R_{33,23}^2) \\
 &\quad + R_{11,13}^2 + R_{22,23}^2 + R_{33,33}^2 \\
 &= 3(R_{11,23}^2 + R_{22,13}^2) + (R_{11,13}^2 + R_{22,23}^2) + \left(\frac{2A_1}{3\mu}\right)^2 \\
 &\quad + 3\left\{2\left(\frac{-A_1}{3\mu}\right)^2 + \left(\frac{2R_{11,1}R_{11,3}}{3\mu}\right)^2 + \left(\frac{-2R_{22,2}R_{11,3}}{3\mu}\right)^2\right\}.
 \end{aligned}$$

Here, we used the relations $R_{11,11} + R_{22,11} + R_{33,11} = 0$, $R_{33,21} = 0$, $R_{22,21} = -R_{11,21}$ and so on in (3.8) and Lemma 3.3. Therefore, by summing up these equations (3.9), (3.10) and (3.11), we obtain at p

$$\begin{aligned}
 (3.12) \quad \sum_{i,j,k,l} R_{ij,kl}^2 &= 12(R_{11,23}^2 + R_{22,13}^2) + 4(R_{11,13}^2 + R_{22,23}^2) \\
 &\quad + 8(R_{11,12}^2 + R_{11,11}^2) + \frac{10A_1^2 + 12A_1A_2}{9\mu^2} \\
 &\quad + \frac{16(A_1 + A_2)^2}{3\mu^2} + \frac{4A_1A_2}{9\mu^2} + \frac{8(A_1 + A_2)}{\mu}R_{11,11}.
 \end{aligned}$$

The other hand, by using (2.4), we have at p

$$\begin{aligned}
 (3.13) \quad \sum_{i,j,k,l} R_{ij,kl}^2 &= -24(4\mu A_2) - \frac{3r}{4}3B\left(\mu - \frac{r}{6}\right) - \frac{9}{4}B\left(B - \frac{r^2}{6}\right) \\
 &= -96\mu A_2 + \frac{3B}{4}(r^2 - 3r\mu - 3B).
 \end{aligned}$$

Hence, from (3.12) and (3.13), we obtain at p

$$\begin{aligned}
 &\mu^2 \{2(R_{11,11}^2 + R_{11,12}^2) + 3(R_{11,23}^2 + R_{22,13}^2) + R_{11,13}^2 + R_{22,23}^2\} \\
 &\quad + 2(A_1 + A_2)\mu R_{11,11} \\
 &= -\frac{58A_1^2 + 112A_1A_2 + 48A_2^2}{36} - 4\mu BA_2 + \frac{3\mu^2}{16}(r^2 - 3r\mu - 3B)B.
 \end{aligned}$$

This proves Lemma 3.4. $\square$

LEMMA 3.5. *At the point p , it follows that*

$$\begin{aligned}
 \sum_{i,j,k} R_{ij,k} R_{ij,k33} &= -3(R_{11,23}^2 + R_{22,13}^2) - (R_{11,13}^2 + R_{22,23}^2) \\
 &\quad - \frac{9}{2} \left(2 \sum_{i,j} \mu_i R_{ij,3}^2 + \sum_i \mu_i^2 R_{ii,33} \right) - \frac{10A_1^2 + 12A_1A_2}{9\mu^2}.
 \end{aligned}$$

PROOF. By differentiating at p the both sides of

$$\sum_{i,j,k} R_{ij,k}^2 = -3 \sum_i \mu_i^3 - \frac{r}{2}B,$$

obtained in Proposition, we have

$$(3.14) \quad 2 \sum_{i,j,k} R_{ij,k} R_{ij,k33} + 2 \sum_{i,j,k} R_{ij,k3}^2 = -3\nabla_3 \nabla_3 \sum_i \mu_i^3$$

and we obtain at p , by directly calculating,

$$(3.15) \quad \nabla_3 \nabla_3 \sum_i \mu_i^3 = 6 \sum_{i,j} \mu_i R_{ij,3}^2 + 3 \sum_i \mu_i^2 R_{ii,33}.$$

Thus, from (3.11), (3.14) and (3.15), we conclude

$$\begin{aligned}
 \sum_{i,j,k} R_{ij,k} R_{ij,k33} &= -3(R_{11,23}^2 + R_{22,13}^2) - (R_{11,13}^2 + R_{22,23}^2) \\
 &\quad - \frac{9}{2} \left(2 \sum_{i,j} \mu_i R_{ij,3}^2 + \sum_{i,j} \mu_i^2 R_{ii,33} \right) - \frac{10A_1^2 + 12A_1A_2}{9\mu^2},
 \end{aligned}$$

that is, the proposed relation in Lemma 3.5 holds. $\square$

LEMMA 3.6. *At the point p , we have*

$$\sum_{i,j,k} R_{ij,k} R_{33,ijk} = \frac{4}{3\mu^2} (2A_1^2 - A_2^2) + \frac{4A_1}{\mu} R_{11,11}.$$

PROOF. By differentiating $\sum_i R_{ii} = r$ and $\sum_{i,j} R_{ij}^2 = S$ at p , we have,

$$(3.16) \quad \sum_i R_{ii,klm} = 0$$

and

$$\sum_{i,j} R_{ij} R_{ij,klm} + \sum_{i,j} R_{ij,m} R_{ij,kl} + \sum_{i,j} R_{ij,k} R_{ij,lm} + \sum_{i,j} R_{ij,l} R_{ij,km} = 0$$

respectively. Therefore, from (3.16), we obtain at p

$$(3.17) \quad \sum_{i,j} R_{ij} R_{ij,klm} = -3\mu R_{33,klm}.$$

Hence, it follows that at p

$$(3.18) \quad \begin{aligned} & \mu \sum_{k,l,m} R_{kl,m} R_{33,klm} \\ &= \sum_{i,j,k,l,m} R_{kl,m} R_{ij,m} R_{ij,kl} \\ &= \sum_{k,l,m} R_{kl,m} R_{11,k} (R_{11,ml} - R_{22,ml}) + 2 \sum_{k,l,m} R_{kl,m} R_{12,k} R_{12,ml} \\ & \quad + 2 \sum_{k,l,m} R_{kl,m} R_{13,k} R_{13,ml} + 2 \sum_{k,l,m} R_{kl,m} R_{23,k} R_{23,ml}. \end{aligned}$$

Now, by calculating the terms in (3.18) separately, we have at p respectively

$$(3.19) \quad \begin{aligned} & \sum_{k,l,m} R_{kl,m} R_{11,k} (R_{11,ml} - R_{22,ml}) \\ &= 2(A_1 + A_2)(R_{11,11} - R_{22,11}) + 2R_{11,1} R_{11,3} (R_{11,13} - R_{22,13}) \\ & \quad + 2R_{22,2} R_{11,3} (R_{11,23} - R_{22,23}), \end{aligned}$$

$$(3.20) \quad \begin{aligned} & 2 \sum_{k,l,m} R_{kl,m} R_{12,k} R_{12,ml} \\ &= 4A_2 R_{11,22} - 4R_{22,2} R_{11,3} R_{11,23} + 4R_{11,1} R_{11,3} R_{22,13}, \end{aligned}$$

$$\begin{aligned}
 (3.21) \quad & 2 \sum_{k,l,m} R_{kl,m} R_{13,k} R_{13,ml} \\
 &= 2A_1(R_{11,33} + R_{33,11}) + 2R_{11,1}R_{11,3}R_{11,13} \\
 &\quad - 2R_{11,1}R_{11,3}R_{22,13} - 4R_{22,2}R_{11,3}R_{11,23},
 \end{aligned}$$

$$\begin{aligned}
 (3.22) \quad & 2 \sum_{k,l,m} R_{kl,m} R_{23,k} R_{23,ml} \\
 &= 2A_1(R_{22,33} + R_{33,22}) + 2R_{22,2}R_{11,3}(R_{11,23} - R_{22,23}) \\
 &\quad + 4R_{11,1}R_{11,3}R_{22,13}.
 \end{aligned}$$

From (3.18), (3.19), (3.20), (3.21) and (3.22), we obtain at p , by making use of the relation $R_{11,11} + R_{22,11} + R_{33,11} = 0$,

$$(3.23) \quad \mu \sum_{kl,m} R_{kl,m} R_{33,klm} = \frac{4}{3\mu}(2A_1^2 - A_2^2) + 4A_1R_{11,11}.$$

That is, Lemma 3.6 holds. $\square$

LEMMA 3.7. *At the point p , we obtain*

$$\begin{aligned}
 (3.24) \quad & \mu^2 \{3(R_{11,23}^2 + R_{22,13}^2) + (R_{11,13}^2 + R_{22,23}^2)\} \\
 &= -\frac{10A_1^2 + 12A_1A_2}{9} - 27A_1\mu^3 - \frac{4}{3}(2A_1^2 - A_2^2) \\
 &\quad - 3r\mu^2A_1 - \left(4\mu + \frac{4}{3}r\right)\mu^2A_2 - 4A_1\mu R_{11,11}.
 \end{aligned}$$

PROOF. We write

$$R_{ijkl,m} = \nabla_m R_{ijkl}.$$

Then, since

$$\begin{aligned}
 & R_{33,ijk} - R_{ij,k33} \\
 &= \nabla_k \left(R_{ij,33} + \sum_m R_{mi} R_{m33j} + \sum_m R_{3m} R_{mi3j} \right) - R_{ij,k33} \\
 &= R_{ij,33k} + \sum_m R_{mi,k} R_{m33j} + \sum_m R_{mi} R_{m33j,k} \\
 &\quad + \sum_m R_{3m,k} R_{mi3j} + \sum_m R_{3m} R_{mi3j,k} - R_{ij,k33} \\
 &= R_{ij,3k3} + \sum_m R_{mj,3} R_{mi3k} + \sum_m R_{im,3} R_{mj3k}
 \end{aligned}$$

$$\begin{aligned}
& + \sum_m R_{ij,m} R_{m33k} + \sum_m R_{mi,k} R_{m33j} + \sum_m R_{mi} R_{m33j,k} \\
& + \sum_m R_{3m,k} R_{mi3j} + \sum_m R_{3m} R_{mi3j,k} - R_{ij,k33} \\
& = \nabla_3 \left(R_{ij,k3} + \sum_m R_{mj} R_{mi3k} + \sum_m R_{im} R_{mj3k} \right) + \sum_m R_{mj,3} R_{mi3k} \\
& + \sum_m R_{im,3} R_{mj3k} + \sum_m R_{ij,m} R_{m33k} + \sum_m R_{mi,k} R_{m33j} \\
& + \sum_m R_{mi} R_{m33j,k} + \sum_m R_{3m,k} R_{mi3j} + \sum_m R_{3m} R_{mi3j,k} - R_{ij,k33} \\
& = \sum_m R_{mj,3} R_{mi3k} + \sum_m R_{im,3} R_{mj3k} + \sum_m R_{mj} R_{mi3k,3} \\
& + \sum_m R_{im} R_{mj3k,3} + \sum_m R_{mj,3} R_{mi3k} + \sum_m R_{im,3} R_{mj3k} \\
& + \sum_m R_{ij,m} R_{m33k} + \sum_m R_{mi,k} R_{m33j} + \sum_m R_{mi} R_{m33j,k} \\
& + \sum_m R_{3m,k} R_{mi3j} + \sum_m R_{3m} R_{mi3j,k},
\end{aligned}$$

we obtain at p

$$\begin{aligned}
(3.25) \quad & \sum_{i,j,k} R_{ij,k} (R_{33,ijk} - R_{ij,k33}) \\
& = \sum_{i,j,k,m} R_{ij,k} \{ 5R_{mj,3} R_{mi3k} + 2R_{ij,m} R_{m33k} \\
& \quad + (\lambda_i - \lambda_j) R_{ij3k,3} + (\lambda_i - \lambda_3) R_{i33j,k} \}.
\end{aligned}$$

Since M^3 is a 3-dimensional Riemannian manifold, we have

$$(3.26) \quad R_{ijkl} = (R_{ik}\delta_{jl} - R_{il}\delta_{jk} + \delta_{ik}R_{jl} - \delta_{il}R_{jk}) - \frac{r}{2}(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}).$$

So, we obtain

$$(3.27) \quad R_{ijkl,m} = R_{ik,m}\delta_{jl} - R_{il,m}\delta_{jk} + \delta_{ik}R_{jl,m} - \delta_{il}R_{jk,m}.$$

Because $R_{ij} = \lambda_i\delta_{ij}$ at the point p , by the direct calculations, we also have the following relations

$$(3.28) \quad R_{ijkl} = \left(\lambda_i + \lambda_j - \frac{r}{2} \right) (\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}),$$

$$\begin{aligned}
 (3.29) \quad & \sum_{i,j,k,m} R_{ij,k} R_{mj,3} R_{mi3k} \\
 &= \sum_{i,j,k,m} R_{ij,k} R_{mj,3} \left(\lambda_m + \lambda_i - \frac{r}{2} \right) (\delta_{m3} \delta_{ik} - \delta_{mk} \delta_{i3}) \\
 &= - \sum_{i,j} R_{ij,3}^2 \left(\lambda_i + \lambda_3 - \frac{r}{2} \right),
 \end{aligned}$$

$$\begin{aligned}
 (3.30) \quad & \sum_{i,j,k,m} R_{ij,k} R_{ij,m} R_{m33k} \\
 &= \sum_{i,j,k,m} R_{ij,k} R_{ij,m} \left(\lambda_m + \lambda_3 - \frac{r}{2} \right) (\delta_{m3} \delta_{3k} - \delta_{mk} \delta_{33}) \\
 &= \sum_{i,j} R_{ij,3}^2 \left(2\lambda_3 - \frac{r}{2} \right) - \sum_{i,j,k} \lambda_i R_{ij,k}^2 + \frac{r}{2} \sum_{i,j,k} R_{ij,k}^2 - \lambda_3 \sum_{i,j,k} R_{ij,k}^2,
 \end{aligned}$$

$$\begin{aligned}
 (3.31) \quad & \sum_{i,j,k} R_{ij,k} (\lambda_i - \lambda_j) R_{ij3k,3} \\
 &= \sum_{i,j,k} R_{ij,k} (\lambda_j - \lambda_i) (R_{ik,3} \delta_{j3} - R_{i3,3} \delta_{jk} + \delta_{ik} R_{j3,3} - \delta_{i3} R_{jk,3}) \\
 &= 2\lambda_3 \sum_{i,j} R_{ij,3}^2 - 2 \sum_{i,j} \lambda_i R_{ij,3}^2,
 \end{aligned}$$

$$\begin{aligned}
 (3.32) \quad & \sum_{i,j,k} R_{ij,k} (\lambda_i - \lambda_3) R_{i33j,k} \\
 &= \sum_{i,j,k} R_{ij,k} (\lambda_i - \lambda_3) (R_{i3,k} \delta_{3j} - R_{ij,k} \delta_{33} + \delta_{i3} R_{3j,k} - \delta_{ij} R_{33,k}) \\
 &= \sum_{i,j} \lambda_i R_{ij,3}^2 - \lambda_3 \sum_{i,j} R_{ij,3}^2 - \sum_{i,j,k} \lambda_i R_{ij,k}^2 + \lambda_3 \sum_{i,j,k} R_{ij,k}^2.
 \end{aligned}$$

Therefore, by summing up (3.25), (3.29), (3.30), (3.31) and (3.32), we have at p ,

$$\begin{aligned}
 (3.33) \quad & \sum_{i,j,k} R_{ij,k} (R_{33,ijk} - R_{ij,k33}) \\
 &= -5 \sum_{i,j} R_{ij,3}^2 \left(\lambda_i + \lambda_3 - \frac{r}{2} \right) + 2 \sum_{i,j} R_{ij,3}^2 \left(2\lambda_3 - \frac{r}{2} \right) - 2 \sum_{i,j,k} \lambda_i R_{ij,k}^2 \\
 &\quad + r \sum_{i,j,k} R_{ij,k}^2 - 2\lambda_3 \sum_{i,j,k} R_{ij,k}^2 + 2\lambda_3 \sum_{i,j} R_{ij,3}^2 - 2 \sum_{i,j} \lambda_i R_{ij,3}^2 \\
 &\quad + \sum_{i,j} \lambda_i R_{ij,3}^2 - \lambda_3 \sum_{i,j} R_{ij,3}^2 - \sum_{i,j,k} \lambda_i R_{ij,k}^2 + \lambda_3 \sum_{i,j,k} R_{ij,k}^2
 \end{aligned}$$

$$\begin{aligned}
&= -12 \left(\mu + \frac{r}{3} \right) A_1 + 3rA_1 - 3 \left\{ 4\mu A_2 + \frac{r}{3} (6A_1 + 4A_2) \right\} \\
&\quad + \left(2\mu - \frac{r}{3} \right) (6A_1 + 4A_2) + r(6A_1 + 4A_2) \\
&= -3rA_1 - \left(4\mu + \frac{4}{3}r \right) A_2.
\end{aligned}$$

Here, we used the following relations at p

$$(3.34) \quad \sum_{i,j} R_{ij,3}^2 = 2R_{11,3}^2 = 2A_1,$$

$$(3.35) \quad \sum_{i,j,k} R_{ij,k}^2 = 6A_1 + 4A_2,$$

$$\begin{aligned}
(3.36) \quad \sum_{i,j,k} \lambda_i R_{ij,k}^2 &= \sum_{i,j,k} \mu_i R_{ij,k}^2 + \frac{r}{3} \sum_{i,j,k} R_{ij,k}^2 \\
&= 4\mu A_2 + \frac{r}{3} (6A_1 + 4A_2)
\end{aligned}$$

$$(3.37) \quad \sum_{i,j} \lambda_i R_{ij,3}^2 = 2 \left(\mu + \frac{r}{3} \right) A_1.$$

Therefore, from Lemma 3.5, Lemma 3.6 and (3.33), we conclude

$$\begin{aligned}
(3.38) \quad &3(R_{11,23}^2 + R_{22,13}^2) + (R_{11,13}^2 + R_{22,23}^2) + \frac{10A_1^2 + 12A_1A_2}{9\mu^2} \\
&+ 9 \sum_{i,j} \mu_i R_{ij,3}^2 + \frac{9}{2} \sum_{i,j} \mu_i^2 R_{ii,33} + \frac{4}{3\mu^2} (2A_1^2 - A_2^2) + \frac{4A_1}{\mu} R_{11,11} \\
&= -3rA_1 - \left(4\mu + \frac{4}{3}r \right) A_2.
\end{aligned}$$

Since we have at p by Lemma 3.3

$$\begin{aligned}
(3.39) \quad \sum_i \mu_i^2 R_{ii,33} &= \mu^2 (R_{11,33} + R_{22,33}) + 4\mu^2 R_{33,33} \\
&= 2\mu A_1,
\end{aligned}$$

it follows from (3.38) that

$$\begin{aligned}
(3.40) \quad &\mu^2 \{ 3(R_{11,23}^2 + R_{22,13}^2) + (R_{11,13}^2 + R_{22,23}^2) \} \\
&= -\frac{10A_1^2 + 12A_1A_2}{9} - 27A_1\mu^3 - \frac{4}{3} (2A_1^2 - A_2^2) \\
&\quad - 3r\mu^2 A_1 - \left(4\mu + \frac{4}{3}r \right) \mu^2 A_2 - 4\mu A_1 R_{11,11},
\end{aligned}$$

that is, Lemma 3.7 holds. $\square$

LEMMA 3.8. *At the point p , we have*

$$\begin{aligned} & 2\mu^2(R_{11,11}^2 + R_{11,12}^2) + 2(A_2 - A_1)\mu R_{11,11} \\ &= \frac{39A_1^2 - 32A_1A_2 - 48A_2^2}{18} + \left(\frac{9}{2}B\mu + 3r\mu^2\right)A_1 \\ &+ \left(\frac{2}{9}r - \frac{10\mu}{3}\right)BA_2 - \left(\frac{3}{8}r + \frac{9}{8}\mu\right)\mu^2(6A_1 + 4A_2). \end{aligned}$$

PROOF. According to (3.7), we have at p

$$\begin{aligned} (3.41) \quad \frac{3B\mu^2}{16}(r^2 - 3r\mu - 3B) &= -\left(\frac{9}{8}\mu + \frac{3}{8}r\right)\mu^2\left\{3B\left(\mu - \frac{r}{6}\right)\right\} \\ &= -\left(\frac{3}{8}r + \frac{9}{8}\mu\right)\mu^2(6A_1 + 4A_2). \end{aligned}$$

From Lemma 3.4 and Lemma 3.7, we also obtain at p

$$\begin{aligned} & 2\mu^2(R_{11,11}^2 + R_{11,12}^2) + 2(A_2 + A_1)\mu R_{11,11} - 4A_1\mu R_{11,11} \\ &= \frac{10A_1^2 + 12A_1A_2}{9} + 27A_1\mu^3 + \frac{4}{3}(2A_1^2 - A_2^2) + 3r\mu^2A_1 \\ &+ \left(4\mu + \frac{4}{3}r\right)\mu^2A_2 - \frac{58A_1^2 + 112A_1A_2 + 48A_2^2}{36} \\ &- 4B\mu A_2 + \frac{3B\mu^2}{16}(r^2 - 3r\mu - 3B). \end{aligned}$$

Together with (3.41), we get at p

$$\begin{aligned} & 2\mu^2(R_{11,11}^2 + R_{11,12}^2) + 2(A_2 - A_1)\mu R_{11,11} \\ &= \frac{39A_1^2 - 32A_1A_2 - 48A_2^2}{18} + \left(\frac{9}{2}B\mu + 3r\mu^2\right)A_1 \\ &+ \left(\frac{2}{9}r - \frac{10\mu}{3}\right)BA_2 - \left(\frac{3}{8}r + \frac{9}{8}\mu\right)\mu^2(6A_1 + 4A_2). \end{aligned}$$

This completes the proof of Lemma 3.8. $\square$

PROOF OF THEOREM 3.0. We suppose that there exists a complete locally conformally flat 3-dimensional Riemannian manifold which satisfies the assumptions in Theorem 3.0. According to the Lemma 1 in [1], we know that, at each point in M^3 ,

$$-\frac{B^{\frac{3}{2}}}{\sqrt{6}} \leq \sum_i \mu_i^3 \leq \frac{B^{\frac{3}{2}}}{\sqrt{6}},$$

where the equality holds if and only if at least two of μ_i are equal. From the assumptions of Theorem 3.0, we have at a point p ,

$$\sum_i \mu_i^3 = \frac{B^{\frac{3}{2}}}{\sqrt{6}}.$$

Hence, we can assume $\mu_1 = \mu_2 = \mu$ and $\mu_3 = -2\mu$. Since $\sum_i \mu_i^3 = -6\mu^3$, we have

$$\mu = -\frac{\sqrt{B}}{\sqrt{6}} < 0.$$

For any real numbers $a > 0$, x and y , the following inequality holds:

$$(3.42) \quad 2xy \leq ax^2 + \frac{1}{a}y^2.$$

By making use of the inequality (3.42) and Lemma 3.8, we have at p

$$\begin{aligned} -\frac{1}{2}(A_1 - A_2)^2 &\leq \frac{39A_1^2 - 32A_1A_2 - 48A_2^2}{18} + \left(\frac{9}{2}B\mu + 3r\mu^2\right)A_1 \\ &\quad + \left(\frac{2}{9}r - \frac{10\mu}{3}\right)BA_2 - \left(\frac{3}{8}r + \frac{9}{8}\mu\right)\mu^2(6A_1 + 4A_2). \end{aligned}$$

This implies

$$\begin{aligned} (3.43) \quad 0 &\leq \frac{48A_1^2 - 50A_1A_2 - 39A_2^2}{18} + \left(\frac{9}{2}B\mu + 3r\mu^2\right)A_1 \\ &\quad + \left(\frac{2}{9}r - \frac{10\mu}{3}\right)BA_2 - \left(\frac{3}{8}r + \frac{9}{8}\mu\right)\mu^2(6A_1 + 4A_2). \end{aligned}$$

From (3.7), we have at p

$$\begin{aligned} (3.44) \quad &\frac{-48A_1^2 + 50A_1A_2 + 39A_2^2}{18} \\ &= \frac{-8(6A_1 + 4A_2)A_1 + \frac{39}{4}(6A_1 + 4A_2)A_2 + \frac{47}{2}A_1A_2}{18} \\ &= -\frac{4}{3}B\left(\mu - \frac{r}{6}\right)A_1 + \frac{13}{8}B\left(\mu - \frac{r}{6}\right)A_2 + \frac{47}{36}A_1A_2. \end{aligned}$$

Therefore, (3.43) and (3.44) yield the following at p

$$(3.45) \quad 0 \leq -\frac{47}{36}A_1A_2 + \left(\frac{113}{24}\mu - \frac{7}{72}r\right)BA_1 + \left(-\frac{137}{24}\mu + \frac{35}{144}r\right)BA_2.$$

Since $\mu < 0$, $r < 0$ and $\frac{r^2}{3} + (\frac{7}{113})^2 \frac{2r^2}{3} < S < \frac{r^2}{3} + (\frac{35}{137})^2 \frac{r^2}{6}$, then $\frac{113}{24}\mu - \frac{7}{72}r < 0$ and $-\frac{137}{24}\mu + \frac{35}{144}r < 0$ hold. From $A_1 \geq 0$, $A_2 \geq 0$, $B > 0$, we infer $A_1 = A_2 = 0$. But (3.7) and $A_1 = A_2 = 0$ say that $B = 0$ or $\mu = \frac{r}{6}$. This is impossible. Hence, we can conclude that Theorem 3.0 holds. Thus we have completed the proof of Theorem 3.0.

PROOF OF MAIN THEOREM. If we suppose that there exists a complete locally conformally flat 3-dimensional Riemannian manifold which satisfies the conditions in the Main Theorem, from the Theorem 3 in the Section 2, we have

$$\sup \sum_{i=1}^3 \mu_i^3 = \frac{B^{\frac{3}{2}}}{\sqrt{6}}.$$

From the proof of Theorem 3 in [1], we know that there exists a sequence $\{p_m\}$ of points in M^3 such that

$$\lim_{m \rightarrow \infty} \sum_{i=1}^3 \mu_i^3(p_m) = \sup \sum_{i=1}^3 \mu_i^3 = \frac{B^{\frac{3}{2}}}{\sqrt{6}}.$$

Since B is constant, we have that μ_i for any i is bounded. From formulas (2.3) and (2.4) in Proposition, we know that $R_{ij,k}$ and $R_{ij,kl}$ are bounded for any i, j, k and l . Thus, by taking a subsequence if necessary, we can assume $\lim_{m \rightarrow \infty} \mu_i(p_m) = \mu_i^*$, $\lim_{m \rightarrow \infty} R_{ij,k}(p_m) = R_{ij,k}^*$ and $\lim_{m \rightarrow \infty} R_{ij,kl}(p_m) = R_{ij,kl}^*$. Since $\sum_i (\mu_i^*)^3 = \sup \sum_i \mu_i^3 = \frac{B^{\frac{3}{2}}}{\sqrt{6}}$ holds if and only if at least two of μ_i^* are equal. Hence, we can put $\mu_1^* = \mu_2^* = \mu$ and $\mu_3^* = -2\mu$. By replacing the point p which was used in the proofs of Lemmas and Theorem 3.0 with the above limit and making use of the same proofs, we can conclude that the Main Theorem holds. In fact, although we do not know whether $\lim_{m \rightarrow \infty} R_{ij,klt}(p_m) = R_{ij,klt}^*$ holds, from Ricci formulas in the Section 2, we know that $\lim_{m \rightarrow \infty} (R_{ij,klt} - R_{ij,ktl})(p_m)$ exists. From the proof of Lemma 3.5, we can infer $\lim_{m \rightarrow \infty} \sum_{i,j,k} R_{ij,k} R_{ij,k33}(p_m)$ exists. Therefore, we have $\lim_{m \rightarrow \infty} \sum_{i,j,k} R_{ij,k} R_{33,ijk}(p_m)$ also exists because $\lim_{m \rightarrow \infty} \sum_{i,j,k} R_{ij,k} (R_{33,ijk} - R_{ij,k33})(p_m)$ exists. Thus, formula (3.18) in the proof of Lemma 3.6 can also be obtained when we replace the point p with the above limit. Making use of these facts above, these Lemmas in the section can be proved by the same methods. Hence, we can conclude that the Main Theorem is true.

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QING-MING CHENG
DEPARTMENT OF MATHEMATICS
FACULTY OF SCIENCE
JOSAI UNIVERSITY
SAITAMA, SAKADO 350-0295, JAPAN

SUSUMU ISHIKAWA
DEPARTMENT OF MATHEMATICS
SAGA UNIVERSITY
SAGA 840-0027, JAPAN

KATSUHIRO SHIOHAMA
DEPARTMENT OF MATHEMATICS
SAGA UNIVERSITY
SAGA 840-0027, JAPAN