Zero sets of holomorphic functions in the Nevanlinna type class on convex domains in $\mathbb{C}^2$

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Abstract. We study a problem concerning zero sets of holomorphic functions in the Nevanlinna type class on convex domains (not necessarily finite type) in $\mathbb{C}^2$. We give a Blaschke type condition for the zero set defined by a function in the Nevanlinna type class on convex domains in $\mathbb{C}^2$. Moreover, we present an example of a convex domain of exponentially-flat infinite type on which it is possible to give sufficient conditions for zero sets of holomorphic functions in the classical Nevanlinna class.

1. Introduction and statement of the results

In this paper, we give a Blaschke type condition for the zero set of holomorphic functions in the Nevanlinna type class on convex domains in $\mathbb{C}^2$ without the finite type condition of domains.

Let $D = \{\rho < 0\}$ be a bounded domain in $\mathbb{C}^n$, $\rho \in C^2(\mathbb{C}^n)$ and $d\rho \neq 0$ on ∂D . For $\alpha > 0$ the Nevanlinna class of order α denoted by $\mathcal{N}_\alpha(D)$ consists of the holomorphic functions f in D satisfying

$$\int_D |\rho(z)|^{\alpha-1} |\log |f(z)|| dV(z) < \infty.$$

In particular, let $\mathcal{N}(D) = \mathcal{N}_0(D)$ be the Nevanlinna class consisting of the holomorphic functions f in D satisfying

$$\sup_{\epsilon > 0} \int_{\partial D_\epsilon} |\log |f(z)|| d\sigma_\epsilon(z) < \infty.$$

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Here $\partial D_\epsilon = \{\rho = -\epsilon\}$ and $d\sigma_\epsilon$ denotes the Lebesgue measure on ∂D_ϵ . Let X be a zero set of a holomorphic function on D . Let X_k be the irreducible components of X and n_k the corresponding multiplicities of X_k . Classically the data $X = \sum_k n_k X_k$ is called a divisor. For strictly pseudoconvex domains with some topological condition, in the mid 1970's, Henkin [Hen], Skoda [Sko] and Dautov-Henkin [DaHe] proved that a divisor X satisfies the Blaschke condition of order α ,

$$\sum_k n_k \int_{X_k} |\rho(z)|^{\alpha+1} d\sigma_{X_k}(z) < \infty$$

if and only if X is the zero set of a function of class $\mathcal{N}_\alpha(D)$, $\alpha \geq 0$, where $d\sigma_{X_k}$ denotes the Lebesgue measure on the regular part of X_k . Bonami-Charpentier [BoCh] proved the same results on complex ellipsoids in 1982. In 1988, by using estimates for the $\bar{\partial}$ -Neumann problem on pseudoconvex domains in $\mathbb{C}^2$ of finite type, Chang-Nagel-Stein [CNS] proved that the Blaschke condition is sufficient for the zero set of a function in the Nevanlinna class. Bruna-Charpentier-Dupain [BCD] proved the same result as Chang-Nagel-Stein's for the case of smoothly bounded convex domains of finite type in $\mathbb{C}^n$ which are strict type. Without the condition of finite type for convex domains, if we strengthen the Blaschke type condition we can prove the following theorem.

THEOREM 1.1. *Let $D = \{\rho < 0\}$ be a bounded convex domain in $\mathbb{C}^2$, $\rho \in C^2(\mathbb{C}^2)$, $\partial D = \{\rho = 0\}$ and $d\rho \neq 0$ on ∂D . Let $\alpha > 0$. Assume that a divisor X in D satisfies the Blaschke condition of order $\alpha - 1/2$, i.e.,*

$$\sum_k n_k \int_{X_k} |\rho(z)|^{\alpha+1/2} d\sigma_{X_k} < \infty.$$

Then there is a function $h \in \mathcal{N}_\alpha(D)$ satisfying $X = \{z \in D; h(z) = 0\}$.

Recently Cumenge [Cum] and Diederich-Mazzilli [DiMa] obtained independently the characterization of the zero sets of functions in the Nevanlinna class of order $\alpha \geq 0$, in smoothly bounded convex domains of finite type in $\mathbb{C}^n$ without the strict type condition.

For $s > 0$, define

$$D_s = \left\{ (z_1, z_2) \in \mathbb{C}^2; |z_1|^2 + c \cdot \exp\left(-\frac{1}{|z_2|^s}\right) < 1 \right\},$$

where $c = \exp(1 + 2/s)$. Then D_s is a bounded convex domain of C^∞ class and any point of the form $(e^{i\theta}, 0)$ in the boundary of D_s is a typical example of a non-Levi flat infinite type boundary point. Except these boundary points all boundary points are strongly pseudoconvex.

The proof of Theorem 1.1 depends essentially on weighted $L^1(D)$ -estimates for $\bar{\partial}$. For the case of $\alpha = 0$ in Theorem 1.1 we need the $L^1(\partial D)$ -estimate for $\bar{\partial}$. However, we cannot prove the $L^1(\partial D)$ -estimate for $\bar{\partial}$ on convex domains without finite type condition. Quite surprisingly, we discovered that even though D_s is exponentially-flat infinite type, the $L^1(\partial D)$ -estimate for $\bar{\partial}$ depends on the grade of flatness of D_s . If $0 < s < 1$, we prove the $L^1(\partial D)$ -estimate for $\bar{\partial}$ on D_s in Theorem 4.1. Thus we obtain the following result.

THEOREM 1.2. *Let $0 < s < 1$. Let X be a divisor in D_s that satisfies the Blaschke condition of order $-1/2$. Then X is defined by a function in $\mathcal{N}(D_s)$.*

REMARK 1.3. In Theorems 1.1 and 1.2 our Blaschke type conditions are only sufficient ones. It is well-known that for every smoothly bounded domain D the divisor X of any function $\mathcal{N}_\alpha(D)$, $\alpha \geq 0$, satisfies the Blaschke condition of order α ([DaHe], [Sko]).

2. Weighted L^1 -estimate for the $\bar{\partial}$ -equation

Let $D = \{\rho < 0\}$ be a bounded convex domain in $\mathbb{C}^2$, $\rho \in C^2(\mathbb{C}^2)$, $\partial D = \{\rho = 0\}$ and $d\rho \neq 0$ on ∂D . In this section we prove the following weighted L^1 -estimates for $\bar{\partial}$ which are essential for proving Theorem 1.1. In the case $p > 1$, L^p estimates for $\bar{\partial}$ was proved by Polking [Pol]. The $L^1(D)$ -estimate for $\bar{\partial}$ on convex domains in $\mathbb{C}^2$ was one of the problems presented by Range (see Problem 4 of §6 in [Ra3]).

THEOREM 2.1. *Let $\alpha > 0$. Then there is a solution u to $\bar{\partial}u = f$ in D such that*

$$\int_D |\rho(z)|^{\alpha-1} |u(z)| dV(z) \leq C_\alpha \int_D |\rho(z)|^\alpha \left(|f(z)| + \frac{|f(z) \wedge \bar{\partial}\rho(z)|}{|\rho(z)|} \right) dV(z)$$

provided that the right-hand side is finite for $(0,1)$ -form f in D with $\bar{\partial}f = 0$. Here C_α is a constant depending only on α .

One of the crucial points for proving Theorem 2.1 is to construct a certain weighted solution formula. We define

$$\tilde{\phi}(\zeta, z) = \sum_{j=1}^2 \frac{\partial \rho}{\partial \zeta_j}(\zeta)(\zeta_j - z_j) - \rho(\zeta).$$

The following is a well-known consequence of the convexity of D :

$$(2.1) \quad 2 \operatorname{Re} \tilde{\phi}(\zeta, z) \geq -\rho(\zeta) - \rho(z) \quad \text{for all } \zeta, z \in \bar{D}.$$

For any $r > 1$ we can define a kernel

$$\begin{aligned} K^r(\zeta, z) = & c_{0,r} \frac{\rho(\zeta)^r}{|\zeta - z|^4 \tilde{\phi}(\zeta, z)^r} \partial_\zeta |\zeta - z|^2 \wedge \bar{\partial}_\zeta \partial_\zeta |\zeta - z|^2 \\ & + c_{1,r} \frac{\rho(\zeta)^{r+1}}{|\zeta - z|^2 \tilde{\phi}(\zeta, z)^{r+1}} \partial_\zeta |\zeta - z|^2 \wedge \partial \bar{\partial} \log \frac{1}{-\rho(\zeta)} \end{aligned}$$

which induces a solution operator

$$Sf(z) = \int_{\zeta \in D} f(\zeta) \wedge K^r(\zeta, z), \quad z \in D$$

such that

$$f = \bar{\partial}(Sf),$$

for a continuous $(0, 1)$ -form f in $\bar{D}$ with $\bar{\partial}f = 0$ (see [BeAn]). For a smooth form f , this formula holds for any $r > 0$. Note that

$$\partial \bar{\partial} \log \frac{1}{-\rho} = \frac{\partial \rho \wedge \bar{\partial} \rho}{\rho^2} - \frac{\partial \bar{\partial} \rho}{\rho}.$$

Thus

$$K^r(\zeta, z) = |\rho(\zeta)|^r K_1^r(\zeta, z) + \bar{\partial} \rho(\zeta) \wedge |\rho(\zeta)|^{r-1} K_2^r(\zeta, z) + |\rho(\zeta)|^r K_3^r(\zeta, z),$$

where

$$|K_1^r(\zeta, z)| \lesssim \frac{1}{|\zeta - z|^3 |\tilde{\phi}(\zeta, z)|^r}$$

and

$$|K_j^r(\zeta, z)| \lesssim \frac{1}{|\zeta - z| |\tilde{\phi}(\zeta, z)|^{r+1}}$$

for $j = 2, 3$.

LEMMA 2.2 ([Pol]). *Let $(\zeta_0, z_0) \in \partial D \times \partial D$ such that $\tilde{\phi}(\zeta_0, z_0) = 0$. Then there exist neighborhoods V of z_0 and W of ζ_0 such that for each $\zeta \in W$, there exists a C^1 local coordinate system $z \mapsto t^{(\zeta)}(z) = (t_1, t_2, t_3, t_4)$ on V with the following properties:*

$$(2.2) \quad t_1(z) = \rho(\zeta) - \rho(z), \quad t_2(z) = \operatorname{Im} \tilde{\phi}(\zeta, z), \quad t_3(\zeta) = t_4(\zeta) = 0;$$

$$(2.3) \quad |t^{(\zeta)}(z) - t^{(\zeta)}(z')| \sim |z - z'|$$

for all $z, z' \in V$ with the constants in (2.3) independent of $\zeta \in W$.

LEMMA 2.3. *Let r be sufficiently large. If $\alpha > 0$ and $\alpha - \epsilon > -1$, then there exists a constant $C_{\alpha, \epsilon}$ such that for $j = 1, 2$*

$$\int_{z \in D} |\rho(z)|^{\alpha - \epsilon} |K_j^r(\zeta, z)| dV(z) \leq C_{\alpha, \epsilon} |\rho(\zeta)|^{\alpha - \epsilon - r + 1} \quad \text{for } \zeta \in D.$$

PROOF. We fix $(\zeta_0, z_0) \in \partial D \times \partial D$ such that $\tilde{\phi}(\zeta_0, z_0) = 0$ and use the coordinate system $t = (t_1, t_2, t_3, t_4)$ of (2.2). It is enough to prove the inequality

$$\int_{z \in D \cap V} |\rho(z)|^{\alpha - \epsilon} |K_j^r(\zeta, z)| dV(z) \lesssim |\rho(\zeta)|^{\alpha - \epsilon - r + 1} \quad \text{for } \zeta \in D \cap W.$$

Put $w = t_3 + it_4$. For the case $j = 1$ it follows from (2.1) that

$$\begin{aligned} J_1(\zeta) &= \int_{z \in D \cap V} |\rho(\zeta)|^{\alpha - \epsilon} |K_1^r(\zeta, z)| dV(z) \\ &\lesssim \int_{\substack{w \in \mathbb{C} \\ |w| < R}} \int_{-T_2}^{T_2} \int_0^{T_1} \frac{t_1^{\alpha - \epsilon} dw dt_1 dt_2}{(|w| + t_1 + |t_2|)^3 (t_1 + |t_2| + |\rho(\zeta)|)^r} \\ &\lesssim \int_{-T_2}^{T_2} \int_0^{T_1} \frac{t_1^{\alpha - \epsilon} dt_1 dt_2}{(t_1 + |t_2|)(t_1 + |t_2| + |\rho(\zeta)|)^r}. \end{aligned}$$

We make the change of variables $t_1 = |\rho|t'_1$ and $t_2 = |\rho|t'_2$, and omit the primes, this becomes

$$J_1(\zeta) \lesssim |\rho(\zeta)|^{\alpha - \epsilon - r + 1} \int_{(t_1, t_2) \in \mathbb{R}^2} \frac{|t_1|^{\alpha - \epsilon} dt_1 dt_2}{(|t_1| + |t_2|)(|t_1| + |t_2| + 1)^r}.$$

If $\alpha - \epsilon \geq 0$, choose r such that $r - \alpha + \epsilon > 1$. We have

$$\begin{aligned} J'_1 &= \int_{(t_1, t_2) \in \mathbb{R}^2} \frac{|t_1|^{\alpha - \epsilon} dt_1 dt_2}{(|t_1| + |t_2|)(|t_1| + |t_2| + 1)^r} \\ &\lesssim \int_0^\infty \frac{d\tau}{(\tau + 1)^{r - \alpha + \epsilon}} \lesssim 1. \end{aligned}$$

Now if $-1 < \alpha - \epsilon < 0$, then

$$\begin{aligned} J'_1 &= \int_{(t_1, t_2) \in \mathbb{R}^2} \frac{dt_1 dt_2}{|t_1|^{\epsilon - \alpha} (|t_1| + |t_2|)(|t_1| + |t_2| + 1)^r} \\ &\lesssim \int_0^\infty \int_0^\infty \frac{dt_1 dt_2}{|t_1|^{\epsilon - \alpha + \eta} |t_2|^{1 - \eta} (|t_1| + |t_2| + 1)^r} \\ &\lesssim \int_0^\infty \frac{dt_1}{|t_1|^{\epsilon - \alpha + \eta} (|t_1| + 1)^{r/2}} \int_0^\infty \frac{dt_2}{|t_2|^{1 - \eta} (|t_2| + 1)^{r/2}}. \end{aligned}$$

If we choose sufficiently small η and large r so that $0 < \epsilon - \alpha + \eta < 1$ and $r > 2$, then we have $J'_1 \lesssim 1$.

For the case $j = 2$ we have

$$\begin{aligned}
 J_2(z) &= \int_{z \in D \cap V} |\rho(z)|^{\alpha-\epsilon} |K_2^r(\zeta, z)| \\
 &\lesssim \int_{z \in D \cap V} |\rho(z)|^{\alpha-\epsilon} \frac{1}{|\zeta - z| |\tilde{\phi}(\zeta, z)|^{r+1}} dV(z) \\
 &\lesssim \int_{\substack{w \in \mathbb{C} \\ |w| < R}} \int_{-T_2}^{T_2} \int_0^{T_1} t_1^{\alpha-\epsilon} \frac{dt_1 dt_2 dw}{|w|(t_1 + |t_2| + |\rho(\zeta)|)^{r+1}} \\
 &\lesssim \int_{-T_2}^{T_2} \int_0^{T_1} t_1^{\alpha-\epsilon} \frac{dt_1 dt_2}{(t_1 + |t_2| + |\rho(\zeta)|)^{r+1}} \\
 &\lesssim |\rho(\zeta)|^{\alpha-r+1-\epsilon} \int_{-\infty}^{\infty} \int_0^{\infty} t_1^{\alpha-\epsilon} \frac{dt_1 dt_2}{(t_1 + |t_2| + 1)^{r+1}}.
 \end{aligned}$$

Now the integral

$$\int_{-\infty}^{\infty} \int_0^{\infty} t_1^{\alpha-\epsilon} \frac{dt_1 dt_2}{(t_1 + |t_2| + 1)^{r+1}}$$

converges, if $\alpha - \epsilon > -1$ and $r - \alpha + \epsilon > 1$. Thus we have the result. $\square$

Now we can prove Theorem 2.1. For $j = 1, 3$ we define

$$I_j^r f(z) = \int_{\zeta \in D} f(\zeta) \wedge |\rho(\zeta)|^r K_j^r(\zeta, z)$$

and

$$I_2^r f(z) = \int_{\zeta \in D} f(\zeta) \wedge \bar{\partial} \rho(\zeta) \wedge |\rho(\zeta)|^{r-1} K_2^r(\zeta, z).$$

Then Theorem 2.1 is a direct consequence of the next proposition.

PROPOSITION 2.4. *Let $\alpha > 0$. Then there is a constant C_α such that for $j = 1, 2, 3$*

$$\int_D |\rho(z)|^{\alpha-1} |I_j^r f(z)| dV(z) \leq C_\alpha \int_D |\rho(z)|^\alpha \left(|f(z)| + \frac{|f(z) \wedge \bar{\partial} \rho(z)|}{|\rho(z)|} \right) dV(z).$$

PROOF. For the case $j = 1$ it follows from Lemma 2.3 that

$$\begin{aligned}
 &\int_{z \in D} |\rho(z)|^{\alpha-1} |I_1^r f(z)| dV(z) \\
 &\lesssim \int_{\zeta \in D} |f(\zeta)| |\rho(\zeta)|^r \int_{z \in D} |\rho(z)|^{\alpha-1} |K_1^r(\zeta, z)| dV(z)
 \end{aligned}$$

$$\lesssim \int_{\zeta \in D} |f(\zeta)| |\rho(\zeta)|^\alpha dV(\zeta).$$

In the case $j = 2$, we have

$$\begin{aligned} & \int_{z \in D} |\rho(z)|^{\alpha-1} |I_2^r f(z)| dV(z) \\ & \lesssim \int_{\zeta \in D} |f(\zeta) \wedge \bar{\rho}(\zeta)| |\rho(\zeta)|^{r-1} \int_{z \in D} |\rho(z)|^{\alpha-1} |K_2^r(\zeta, z)| dV(z) \\ & \lesssim \int_{\zeta \in D} |f(\zeta) \wedge \bar{\rho}(\zeta)| |\rho(\zeta)|^{\alpha-1} dV(\zeta). \end{aligned}$$

The case $j = 3$ easily can be reduced to the case $j = 2$. $\square$

3. Poincaré-Lelong equation and zero sets for the Nevanlinna class of order $\alpha > 0$

In this section we prove Theorem 1.1. Let X be a divisor of D . Lelong showed that X is represented by a d -closed, positive $(1,1)$ -current Θ_X ([Le1], [Le2]). We assume that X satisfies the Blaschke condition of order $\alpha - 1/2$, $\alpha > 0$. Let t_{Θ_X} be the trace measure defined by $t_{\Theta_X} = \Theta_X \wedge \beta$, where $\beta = i\partial\bar{\partial}|z|^2$. Then the Blaschke condition of order $\alpha - 1/2$ can be written as

$$\int_D |\rho|^{\alpha+1/2} t_{\Theta_X} =: B\left(\Theta_X, \alpha - \frac{1}{2}\right) < +\infty.$$

The proof of Theorem 1.1 is reduced to finding real valued solutions of the so-called Poincaré-Lelong equation $i\partial\bar{\partial}v = \Theta_X$ with control of the growth of v , since any solution is $v = \log|g|$ for a holomorphic function g that defines X . In fact, to prove Theorem 1.1 it suffices to show that for a d -closed, positive $(1,1)$ -current Θ in D satisfying the Blaschke condition of order $\alpha - 1/2$, there is a real valued solution v to $i\partial\bar{\partial}v = \Theta$ such that

$$\int_D |\rho|^{\alpha-1} |v| dV \lesssim B\left(\Theta, \alpha - \frac{1}{2}\right).$$

We will prove this result in Theorem 3.3. By a standard regularization and approximation argument we may assume that Θ has smooth coefficients in $\bar{D}$. First we show that the L^1 -norms of $|\rho|^{\alpha-1/2} |\Theta \wedge \partial\rho \wedge \bar{\partial}\rho|$ and $|\rho|^\alpha \{|\Theta \wedge \partial\rho| + |\Theta \wedge \bar{\partial}\rho|\}$ are controlled by $B(\Theta, \alpha - 1/2)$.

THEOREM 3.1. *Let Θ be a d -closed, positive smooth $(1,1)$ -form satisfying the Blaschke condition of order $\alpha - 1/2$. Let $\alpha \geq 0$. Then the following assertions are true.*

(i)

$$\int_D |\rho(z)|^{\alpha-1/2} |\Theta(z) \wedge \partial\rho(z) \wedge \bar{\partial}\rho(z)| dV \lesssim B\left(\Theta, \alpha - \frac{1}{2}\right).$$

(ii)

$$\int_D |\rho(z)|^\alpha (|\Theta(z) \wedge \partial\rho(z)| + |\Theta(z) \wedge \bar{\partial}\rho(z)|) dV \lesssim B\left(\Theta, \alpha - \frac{1}{2}\right).$$

PROOF. The results for the case $\alpha > 1/2$ where stated in [DaHe].

(i) Since $|\rho|^{\alpha-1/2}\Theta \wedge \partial\rho \wedge \bar{\partial}\rho/i$ is positive, it is sufficient to verify that

$$\left| \int_D |\rho|^{\alpha-1/2} \Theta \wedge \partial\rho \wedge \bar{\partial}\rho \right| \lesssim B\left(\Theta, \alpha - \frac{1}{2}\right).$$

By Stokes' theorem, it follows that

$$\int_D d(|\rho|^{\alpha+1/2}\Theta \wedge d^c\rho) = \int_{\partial D} |\rho|^{\alpha+1/2}\Theta \wedge d^c\rho = 0.$$

Since Θ is d -closed, we have

$$d(|\rho|^{\alpha+1/2}\Theta \wedge d^c\rho) = -\left(\alpha + \frac{1}{2}\right)|\rho|^{\alpha-1/2}\Theta \wedge d\rho \wedge d^c\rho + |\rho|^{\alpha+1/2}\Theta \wedge dd^c\rho.$$

Hence we have

$$\left(\alpha + \frac{1}{2}\right) \int_D |\rho|^{\alpha-1/2}\Theta \wedge d\rho \wedge d^c\rho = \int_D |\rho|^{\alpha+1/2}\Theta \wedge dd^c\rho$$

and

$$\begin{aligned} (3.1) \quad \int_D |\rho|^{\alpha-1/2}\Theta \wedge \partial\rho \wedge \bar{\partial}\rho &\sim \int_D |\rho|^{\alpha-1/2}\Theta \wedge d\rho \wedge d^c\rho \\ &\sim \int_D |\rho|^{\alpha+1/2}\Theta \wedge dd^c\rho. \end{aligned}$$

The result follows from (3.1) and the Blaschke condition of order $\alpha - 1/2$.

(ii) Introduce a new coordinate system (z_1, z_2) in a neighborhood V_ζ of each $\zeta \in \partial D$ such that the $(1, 0)$ -forms dz_1, dz_2 constitute an orthogonal basis and $dz_1 = \lambda\partial\rho$, where λ is a normalizing factor. Let $\Theta = \sum \Theta^{jk}$, where $\Theta^{jk} = \Theta_{jk}dz_j \wedge d\bar{z}_k$. Note that the decomposition of Θ does not depend on the choice of the basis dz_1, dz_2 . Hence the decomposition is valid globally in a neighborhood of ∂D .

The inequalities in (i) and (ii) of Theorem 3.1 can be written in the equivalent forms respectively

$$(3.2) \quad \int_D |\rho|^{\alpha-1/2} |\Theta_{22}| dV \lesssim B\left(\Theta, \alpha - \frac{1}{2}\right)$$

$$(3.3) \quad \int_D |\rho|^\alpha (|\Theta_{12}| + |\Theta_{21}|) dV \lesssim B\left(\Theta, \alpha - \frac{1}{2}\right).$$

Since $\Theta = \bar{\Theta}$, we have $\Theta_{kj} = -\bar{\Theta}_{jk}$. Put $\omega = |\rho|^{\alpha/2-1/4} dz_1 + \theta |\rho|^{\alpha/2+1/4} dz_2$, where θ is a complex valued function with $|\theta| = 1$. Then

$$\omega \wedge \bar{\omega} = |\rho|^{\alpha-1/2} dz_1 \wedge d\bar{z}_1 + |\rho|^{\alpha+1/2} dz_2 \wedge d\bar{z}_2 + |\rho|^\alpha \theta dz_2 \wedge d\bar{z}_1 + |\rho|^\alpha \bar{\theta} dz_1 \wedge d\bar{z}_2$$

and

$$(3.4) \quad \begin{aligned} \Theta \wedge \frac{\omega \wedge \bar{\omega}}{i} &= |\rho|^{\alpha-1/2} \Theta^{22} \wedge \frac{dz_1 \wedge d\bar{z}_1}{i} + |\rho|^{\alpha+1/2} \Theta^{11} \wedge \frac{dz_2 \wedge d\bar{z}_2}{i} \\ &\quad + |\rho|^\alpha \theta \Theta^{12} \wedge \frac{dz_2 \wedge d\bar{z}_1}{i} + |\rho|^\alpha \bar{\theta} \Theta^{21} \wedge \frac{dz_1 \wedge d\bar{z}_2}{i}. \end{aligned}$$

Take $\theta = \frac{1}{i} \exp(\frac{1}{i} \arg \Theta_{12})$. Then

$$\theta \Theta_{12} = |\Theta_{12}| \frac{1}{i} \quad \text{and} \quad \bar{\theta} \Theta_{21} = |\Theta_{21}| \frac{1}{i}.$$

Substituting this in (3.4), we get

$$\begin{aligned} \Theta \wedge \frac{\omega \wedge \bar{\omega}}{i} &= |\rho|^{\alpha-1/2} \Theta^{22} \frac{dz_1 \wedge d\bar{z}_1}{i} + |\rho|^{\alpha+1/2} \Theta^{11} \wedge \frac{dz_2 \wedge d\bar{z}_2}{i} \\ &\quad - 4|\rho|^\alpha |\Theta_{12}| dV - 4|\rho|^\alpha |\Theta_{21}| dV. \end{aligned}$$

Since $\Theta \wedge \omega \wedge \bar{\omega}/i$ is a positive measure, we get

$$\int_D |\rho|^\alpha (|\Theta_{12}| + |\Theta_{21}|) dV \lesssim \int_D |\rho|^{\alpha+1/2} |\Theta_{11}| dV + \int_D |\rho|^{\alpha-1/2} |\Theta_{22}| dV.$$

By (i) and the Blaschke condition of order $\alpha - 1/2$, we get the result. $\square$

LEMMA 3.2. *Let g be a continuous function in $\bar{D}$ satisfying $g \equiv 0$ in some neighborhood of the origin. We define*

$$(Kg)(z) = \int_0^1 g(tz) dt, \quad z \in \bar{D}.$$

(i) *If $\alpha > 0$, there is a constant $C_\alpha > 0$ such that*

$$\int_D |\rho(z)|^{\alpha-1} |Kg(z)| dV \leq C_\alpha \int_D |\rho(z)|^\alpha |g(z)| dV.$$

(ii) *There is a constant $C > 0$ such that*

$$\int_{\partial D} |Kg(z)| d\sigma \leq C \int_D |g(z)| dV.$$

PROOF. (i) See Lemma 3.1 in the chapter II of [Sko].

(ii) Let $t \cdot \partial D = \{w; w = tz, z \in \partial D\}$. Then for some $c > 0$,

$$\begin{aligned} \int_{\partial D} |Kg(z)| d\sigma &= \int_{\partial D} \left| \int_0^1 g(tz) dt \right| d\sigma \\ &\leq \int_c^1 dt \int_{\partial D} |g(tz)| d\sigma(z), \quad \text{by Fubini's Theorem} \\ &\lesssim \int_c^1 \frac{dt}{t^3} \int_{w \in t \cdot \partial D} |g(w)| d\sigma(w) \\ &\lesssim \int_D |g(z)| dV(z), \quad \text{by integrating by parts.} \end{aligned}$$

□

THEOREM 3.3. Let $\alpha > 0$. Let Θ be a d -closed, positive smooth $(1, 1)$ -form satisfying the Blaschke condition of order $\alpha - 1/2$. Then there exists a real valued solution v to $i\partial\bar{\partial}v = \Theta$ satisfying

$$(3.5) \quad \int_D |\rho(z)|^{\alpha-1} |v(z)| dV \lesssim B\left(\Theta, \alpha - \frac{1}{2}\right).$$

PROOF. Assuming without loss of generality that D contains the origin, we apply the Poincaré-Cartan homotopy operator P to Θ . Then $dP\Theta = \Theta$. Note that if $\Theta = \sum \Theta_{jk} dz_j \wedge d\bar{z}_k$, then $P\Theta$ can be written by

$$\begin{aligned} P\Theta(z) &= \sum_{j=1}^2 \left(\sum_{k=1}^2 \int_0^1 tz_k \Theta_{kj}(tz) dt \right) d\bar{z}_j - \sum_{k=1}^2 \left(\sum_{j=1}^2 \int_0^1 t\bar{z}_j \Theta_{kj}(tz) dt \right) dz_k \\ &= f + \bar{f} = P'\Theta + P''\Theta. \end{aligned}$$

From $\Theta = \bar{\Theta}$, we see $\Theta_{kj} = -\bar{\Theta}_{jk}$. Thus $\bar{\partial}f = 0$. It suffices to show that

$$(3.6) \quad \int_D |\rho|^\alpha \left(|f| + \frac{|\bar{\partial}\rho \wedge f|}{|\rho|} \right) dV \lesssim B\left(\Theta, \alpha - \frac{1}{2}\right).$$

Indeed if f satisfies (3.6), then we can apply Theorem 2.1. Thus there exists a solution u to $\bar{\partial}u = f$ such that

$$\int_D |\rho(z)|^{\alpha-1} |u(z)| dV \lesssim B\left(\Theta, \alpha - \frac{1}{2}\right).$$

Then $v = 2 \operatorname{Re} u$ is a real valued solution of the equation $i\partial\bar{\partial}v = \Theta$ satisfying the inequality (3.5). Now we will show (3.6). For sufficiently small $\eta > 0$ choose a smooth cutoff function ψ_η , $0 \leq \psi_\eta \leq 1$ satisfying

$$\psi_\eta(z) = \begin{cases} 1, & z \in \{z \in \mathbb{C}^2; |\rho(z)| \leq \eta/2\} \\ 0, & z \in \{z \in \mathbb{C}^2; |\rho(z)| \geq \eta\}. \end{cases}$$

Decompose $\Theta = (1 - \psi_\eta)\Theta + \psi_\eta\Theta$. Let $P'\Theta(z) = f(z)$. Since $\psi_\eta\Theta_1$ has compact support in D , (3.6) is clear. Hence we may assume that $\Theta \equiv 0$ in $D_\eta = \{z \in D : \rho(z) < -\eta\}$. By (i) of Lemma 3.2, we obtain

$$\begin{aligned} (3.7) \quad \int_D |\rho(z)|^\alpha |f(z)| dV &\lesssim \int_D |\rho(z)|^\alpha \sum_{j,k} \left(\int_0^1 |\Theta_{jk}(tz)| dt \right) dV \\ &\lesssim \int_D |\rho(z)|^{\alpha+1} |\Theta(z)|. \end{aligned}$$

Since $|\bar{\partial}\rho \wedge f| = |\bar{\partial}\rho \wedge P'\Theta| \lesssim |P'(\Theta^{12} + \Theta^{22})|$, (3.6) will follow from (3.7) and

$$\int_D |\rho(z)|^{\alpha-1} |P'(\Theta^{12} + \Theta^{22})(z)| dV \lesssim B\left(\Theta, \alpha - \frac{1}{2}\right) dV.$$

Let $|\Theta^2(z)| = |\Theta^{12}(z)| + |\Theta^{22}(z)|$. Then by (i) of Lemma 3.2, (3.2) and (3.3), we obtain

$$\begin{aligned} \int_D |\rho(z)|^{\alpha-1} |P'(\Theta^{12} + \Theta^{22})(z)| dV &\lesssim \int_D |\rho(z)|^{\alpha-1} \left(\int_0^1 |\Theta^2(tz)| dt \right) dV \\ &\lesssim \int_D |\rho(z)|^\alpha |\Theta^2(z)| dV \\ &\lesssim B\left(\Theta, \alpha - \frac{1}{2}\right) dV. \end{aligned}$$

□

4. The proof of Theorem 1.2

We recall that

$$D_s = \left\{ (z_1, z_2) \in \mathbb{C}^2; \rho(z_1, z_2) = |z_1|^2 + c \cdot \exp\left(-\frac{1}{|z_2|^s}\right) - 1 < 0 \right\},$$

where $0 < s < 1$ and $c = \exp(1 + 2/s)$.

Let Θ be a d -closed, positive smooth (1,1)-form satisfying the Blaschke condition of order $-1/2$. By the same argument as in the proof of Theorem 1.1, it is

enough to show that there exists a real valued function v such that $i\partial\bar{\partial}v = \Theta$ in D_s and

$$(4.1) \quad \int_{\partial D_s} |v(z)| d\sigma(z) \lesssim B\left(\Theta, -\frac{1}{2}\right).$$

As in §3, the essential part of the proof of (4.1) is the following $L^1(\partial D_s)$ -estimates for $\bar{\partial}$.

THEOREM 4.1. *Let $f \in C_{(0,1)}^\infty(\bar{D}_s)$ and $\bar{\partial}f = 0$ in $\bar{D}_s$. There is a solution $u \in C(\bar{D}_s)$ to $\bar{\partial}u = f$ in D_s satisfying*

$$(4.2) \quad \int_{\partial D_s} |u(z)| d\sigma \leq C \int_{\partial D_s} |f(z) \wedge \bar{\partial}\rho(z)| d\sigma.$$

PROOF. Let U be a small neighborhood of $\bar{D}_s$. Define

$$\phi(\zeta, z) = \sum_{j=1}^2 \frac{\partial \rho}{\partial \zeta_j}(\zeta)(\zeta_j - z_j), \quad (\zeta, z) \in U \times \mathbb{C}^2.$$

By Lemma 2 of [Ver], we see that there are constants $\eta, c > 0$ depending only on s such that for $M = 1/32$, we have

$$(4.3) \quad 2 \operatorname{Re} \phi(\zeta, z) \geq \rho(\zeta) - \rho(z) + ce^{-1/M|\zeta-z|^{2s}},$$

for $\zeta \in U$, $|\zeta - z| < \eta$.

Now we will construct a solution operator for the $\bar{\partial}$ -equation. Because we need the $L^1(\partial D_s)$ -estimate for $\bar{\partial}$, we use the well-known formula obtained from the Cauchy-Fantappiè kernel.

Let $W(\zeta, z) = \partial \rho(\zeta)/\phi(\zeta, z)$ and $\hat{W}(\zeta, z, \lambda) = \lambda W(\zeta, z) + (1-\lambda)B(\zeta, z)$, where $0 \leq \lambda \leq 1$ and $B(\zeta, z) = \partial_\zeta |\zeta - z|^2/|\zeta - z|^2$. The Cauchy-Fantappiè kernel $\Omega_0(\hat{W})$ of order 0 generated by $\hat{W}$ is defined by

$$\Omega_0(\hat{W}) = \frac{1}{(2\pi i)^2} \hat{W} \wedge \bar{\partial}_{\zeta, \lambda} \hat{W}.$$

$\Omega_0(W)$ is defined in the same way, with W instead of $\hat{W}$. For more details see Range [Ra1]. For $f \in C_{(0,1)}^\infty(\bar{D}_s)$, define

$$\mathbb{B}f(z) = \int_{\zeta \in D_s} f(\zeta) \wedge \Omega_0(B), \quad z \in D_s$$

and

$$\mathbb{T}^+ f(z) = \int_{\partial D_s \times [0,1]} f(\zeta) \wedge \Omega_0(\hat{W}), \quad z \in D_s.$$

Then we obtain a solution $u(z) = (\mathbb{T}^+ - \mathbb{B})f(z)$ for the $\bar{\partial}$ -equation.

Since $f \in C_{(0,1)}^\infty(\bar{D}_s)$, $\mathbb{T}^+ f$ and $\mathbb{B}f$ can be continuously extended up to the boundary of D_s [Ra2]. In order to estimate the boundary values of $u = (\mathbb{T}^+ - \mathbb{B})f$ in terms of the boundary values of f , we have to express the integral $\mathbb{B}f$ in a different form using the dual form of $W(\zeta, z)$. Define $\phi^*(\zeta, z) = -\phi(z, \zeta)$ and $W^*(\zeta, z) = \partial_\zeta(z)/\phi^*(\zeta, z)$. Note that ϕ^* is well-defined for $(\zeta, z) \in \bar{D}_s \times (U \setminus D_s)$. We define

$$\mathbb{T}^- f(z) = \int_{\partial D_s \times [0,1]} f(\zeta) \wedge \Omega_0(\hat{W}^*), \quad z \in U \setminus D_s.$$

We see that if $z \in \partial D_s$, then the solution $u(z) = \mathbb{T}^+ f(z) - \mathbb{T}^- f(z)$ [Sha]. Integrating over λ , we have for some constants a, a^* ,

$$\begin{aligned} \mathbb{T}^+ f(z) &= \frac{a}{(2\pi i)^2} \int_{\zeta \in \partial D_s} f(z) \wedge E(\zeta, z), \\ \mathbb{T}^- f(z) &= \frac{a^*}{(2\pi i)^2} \int_{\zeta \in \partial D_s} f(z) \wedge E^*(\zeta, z), \end{aligned}$$

where

$$E(\zeta, z) = \frac{\partial_\zeta \rho(\zeta) \wedge \partial_\zeta |\zeta - z|^2}{\phi(\zeta, z)|\zeta - z|^2}, \quad E^*(\zeta, z) = \frac{\partial_\zeta \rho(z) \wedge \partial_\zeta |\zeta - z|^2}{\phi^*(\zeta, z)|\zeta - z|^2}.$$

Let $f = f_1 + f_2$, where f_1 is the normal component and f_2 the tangential component of f . Then $f_1 = g\bar{\partial}\rho$ for some function $g \in C^\infty(\partial D_s)$. Thus we have

$$\begin{aligned} \mathbb{T}^+ f_1(z) &= a \int_{\partial D_s} f_1 \wedge E \\ &= a \int_{\partial D_s} g\bar{\partial}\rho \wedge E \\ &= a \int_{\partial D_s} g d\rho \wedge E \\ &= 0, \quad \text{by Stokes' Theorem.} \end{aligned}$$

Thus we have $\mathbb{T}^+ f(z) = \mathbb{T}^+ f_2(z)$ on ∂D_s . Similarly $\mathbb{T}^- f(z) = \mathbb{T}^- f_2(z)$ on ∂D_s . Therefore we obtain

$$u(z) = \mathbb{T}^+ f_2(z) - \mathbb{T}^- f_2(z) \quad \text{for } z \in \partial D_s.$$

The inequality (4.2) can be written in the equivalent form

$$(4.4) \quad \int_{\partial D_s} |u(z)| d\sigma(z) \lesssim \int_{\partial D_s} |f_2(z)| d\sigma(z).$$

To prove the inequality (4.4), by the Fubini theorem, it suffices to show that the following integrals,

$$(4.5) \quad \int_{\partial D_s} |E(\zeta, z)| dV(z), \quad \int_{\partial D_s} |E^*(\zeta, z)| dV(z),$$

are uniformly bounded for each $\zeta \in \partial D_s$. For the first integral of (4.5), we use the coordinate (2.2) and the inequality (4.3). By the compactness of ∂D_s and the smoothness of the kernel E on $\bar{D}_s \times \bar{D}_s \setminus \{(\zeta, \zeta) : \zeta \in \partial D_s\}$, it suffices to show that for each $\zeta \in \partial D_s$ and sufficiently small ball $B_\eta(\zeta)$,

$$\begin{aligned} \int_{\partial D_s \cap B_\eta(\zeta)} |E(\zeta, z)| d\sigma(z) &\lesssim \int_{\partial D_s \cap B_\eta(\zeta)} \frac{d\sigma(z)}{|\zeta - z| |\phi(\zeta, z)|} \\ &\lesssim \int_{\substack{|t'| \leq 1 \\ t=(t_2, t'')}} \frac{dt_2 dt''}{|t''|(|t_2| + e^{-M/|t''|^s})} \\ &\lesssim \int_{|t''| < 1, t'' \in \mathbb{R}^2} \frac{dt''}{|t''| |t''|^s} \\ &\lesssim \int_0^1 \frac{dr}{r^s} \lesssim 1. \end{aligned}$$

On the other hand, since $\phi^*(\zeta, z) = -\phi(z, \zeta)$, it follows that

$$2 \operatorname{Re} \phi^*(\zeta, z) \leq -\rho(\zeta) + \rho(z) - ce^{-M/|\zeta-z|^s}$$

for $\zeta \in \bar{D}_s$, $|\zeta - z| < \eta$. Hence the second integral of (4.5) can be estimated as the first one. $\square$

Now we return to the proof of (4.1). Let $dP\Theta = \Theta$, $P\Theta = f + \bar{f}$ and $P'\Theta = f$ as in the proof of Theorem 3.3. Then it is enough to show that

$$\int_{\partial D_s} |f_2(z)| d\sigma(z) \lesssim B\left(\Theta, -\frac{1}{2}\right).$$

Note that $|f_2| \lesssim |P'\Theta^2|$. By (ii) of Lemma 3.2, (3.2) and (3.3), we have

$$\begin{aligned} \int_{\partial D_s} |P'(\Theta^2)(z)| d\sigma(z) &\lesssim \int_{\partial D_s} \int_0^1 |\Theta^2(tz)| dt d\sigma(z) \\ &\lesssim \int_{D_s} |\Theta^2| dV \\ &\lesssim B\left(\Theta, -\frac{1}{2}\right). \end{aligned}$$

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