Proximate order and lower type of entire gap power series

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Abstract. The paper deals with the characterization of lower (p, q) -type with respect to a proximate order of an entire function in terms of coefficients and exponents of its power series expansion. In particular, the problem proposed by Bajpai *et al.* [Trans. Amer. Math. Soc. (1975)], and subsequently, left open by Juneja *et al.* also [J. Reine Angew. Math. (1977)] has been solved.

Introduction

Let $f(z) = \sum_{n=0}^{\infty} a_n z^{\lambda_n}$ be a nonconstant entire function, where $\lambda_0 = 0$ and $\{\lambda_n\}_{n=1}^{\infty}$ is a strictly increasing sequence of positive integers such that $a_n \neq 0$ for $n = 1, 2, \dots$. We set $M(r) \equiv M(r, f) = \max_{|z|=r} |f(z)|$; $\mu(r) \equiv \mu(r, f) = \max_{n \geq 1} \{|a_n| r^{\lambda_n}\}$ and $\nu(r) \equiv \nu(r, f) = \max\{\lambda_n : \mu(r) = |a_n| r^{\lambda_n}\}$. Then $M(r)$, $\mu(r)$ and $\nu(r)$ are called the maximum modulus, maximum term and rank of the maximum term, respectively of $f(z)$ for $|z| = r$. The concepts of *index-pair* (p, q) , $p \geq q \geq 1$, (p, q) -orders and (p, q) -types were introduced by Juneja *et al.* ([2], [3]). The growth of an entire function $f(z)$ can be studied in terms of its (p, q) -orders and (p, q) -types. However, these concepts are inadequate for comparing the growth of those entire functions which are of the same (p, q) -orders but of infinite (p, q) -types. For a refinement of the above scale we utilize the concept of *proximate order* defined as follows:

DEFINITION 1. A positive function $\rho(r)$ defined on $[r_0, \infty)$, $r_0 > \exp^{[q-1]} 1$ is said to be a *proximate order* of an entire function with index-pair (p, q) if the following conditions are satisfied, where $\exp^{[0]} x = x$ and $\exp^{[q]} x = \exp(\exp^{[q-1]} x)$.

- (i) $\rho(r) \rightarrow \rho(p, q) \equiv \rho$ as $r \rightarrow \infty$, $b < \rho < \infty$;
- (ii) $\Lambda_{[q]}(r) \rho'(r) \rightarrow 0$ as $r \rightarrow \infty$, where $\rho'(r)$ denotes the derivative of $\rho(r)$, $b = 1$ if $p = q$, $b = 0$ if $p > q$ and, for convenience, $\Lambda_{[q]}(r) = \prod_{i=0}^q \log^{[i]} r$, and $\log^{[i]} x$ is defined by $\log^{[0]} x = x$ and $\log^{[q]} x = \log(\log^{[q-1]} x)$.

DEFINITION 2. With index-pair (p, q) , $p \geq q \geq 1$, the *generalized (p, q) -type* T^* and the *generalized lower (p, q) -type t^** of $f(z)$ in reference to a proximate order $\rho(r)$ (satisfying (i) and (ii)) are defined as

$$\lim_{r \rightarrow \infty} \frac{\sup \log^{[p-1]} M(r)}{\inf (\log^{[q-1]} r)^{\rho(r)}} = \frac{T^*(p, q) \equiv T^*}{t^*(p, q) \equiv t^*} ; \quad 0 \leq t^* \leq T^* \leq \infty.$$

Clearly, the proximate order and generalized lower (p, q) -type of an entire function are not uniquely determined. The existence theorem for such comparison functions on (p, q) -scale has been established in [4] and [5].

DEFINITION 3. An entire function $f(z)$ is said to be of perfectly regular (p, q) -growth in reference to its proximate order if $0 < t^* = T^* < \infty$.

Juneja *et al.* [3] proved the following:

THEOREM A. Let $f(z)$ be an entire function having (p, q) -order ρ ($b < \rho < \infty$) and lower (p, q) -type t such that

- (i) $\psi(n) = |a_n/a_{n+1}|^{1/(\lambda_{n+1}-\lambda_n)}$ forms a nondecreasing function of n for $n > n_0$,
- (ii) $\log^{[p-2]} \lambda_{n-1} \simeq \log^{[p-2]} \lambda_n$ as $n \rightarrow \infty$.

Then

$$\frac{t}{M} = \liminf_{n \rightarrow \infty} \frac{\log^{[p-2]} \lambda_{n-1}}{(\log^{[q-1]} |a_n|^{-1/\lambda_n})^{\rho-A}},$$

where $A = 1$ if $q = 2$, $A = 0$ if $q \neq 2$, and

$$M \equiv M(p, q) = \begin{cases} (\rho - 1)^{\rho-1}/\rho^\rho & \text{if } (p, q) = (2, 2), \\ 1/e\rho & \text{if } (p, q) = (2, 1), \\ 1 & \text{otherwise.} \end{cases}$$

This theorem was proved by Bajpai *et al.* [1] basically for the entire functions of Sato growth [9], *i.e.*, $(p, q) = (p, 1)$ and they raised a question whether the asymptotic condition (ii) on exponents in Theorem A can be weakened. In a subsequent paper, Juneja *et al.* [3] extended the theorem for index-pair (p, q) and again left this question open. In this paper we characterize lower (p, q) -type with respect to a proximate order in terms of coefficients and exponents of power series (with gaps) which proves Theorem A under weaker conditions.

1. Auxiliary results

This section consists of some auxiliary results which will be utilized in the sequel.

Since $(\log^{[q-1]} r)^{\rho(r)-A}$ is a monotonically increasing function [6] of r for $r > r_o$, we can define $\phi(x)$ to be the unique solution of the equation

$$(1.1) \quad x = (\log^{[q-1]} r)^{\rho(r)-A} \iff \phi(x) = \log^{[q-1]} r,$$

where $A = 1$, if $(p, q) = (2, 2)$ and $A = 0$ otherwise.

LEMMA 1. *For the function $\phi(x)$ in (1.1), we have*

$$(1.2) \quad \lim_{x \rightarrow \infty} \frac{\phi(\eta x)}{\phi(x)} = \eta^{1/(\rho-A)}, \quad \text{uniformly for } 0 < \eta < \infty.$$

PROOF. The proof of this lemma is available in [8].

LEMMA 2. *Let $f(z)$ be an entire function of (p, q) -order ρ ($b < \rho < \infty$) and generalized lower (p, q) -type t^* , in reference to a proximate order $\rho(r)$. Then*

$$\liminf_{r \rightarrow \infty} \frac{\log^{[p-1]} \mu(r)}{(\log^{[q-1]} r)^{\rho(r)}} = t^*.$$

PROOF. This lemma is an outcome of Definition 2 as well as the following inequality which is due to Lockhart and Straus [7],

$$\mu(r) < M(r) < \left(\frac{4r + \varepsilon}{\varepsilon} \right) \left(1 + \log \frac{\mu(r + \varepsilon)}{\mu(r)} \right) \mu(r).$$

LEMMA 3. *Let $f(z)$ be an entire function of (p, q) -order ρ ($b < \rho < \infty$) and generalized lower (p, q) -type t^* , in reference to a proximate order $\rho(r)$. Then, for $p \geq 3$,*

$$\liminf_{r \rightarrow \infty} \frac{\log^{[p-2]} \nu(r)}{(\log^{[q-1]} r)^{\rho(r)}} = t^*.$$

PROOF. It is well known that $\log \mu(r)$ is an increasing convex function of $\log r$ for $r > r_o$ and has the representation

$$(1.3) \quad \log \mu(r) = \log \mu(r_o) + \int_{r_o}^r \frac{\nu(x)}{x} dx.$$

As $\nu(r)$ is an increasing function of r , we observe that

$$\log \mu(r) \leq O(1) + \nu(r) \log r,$$

and by the property, $\log^{[p]}(x+y) \leq \log^{[p]} x + \log^{[p]} y$, for $x, y > \exp^{[p]} 1$,

$$\log^{[p-1]} \mu(r) \leq \log^{[p-2]} \nu(r) + \log^{[p-1]} r, \quad \text{for } p \geq 3.$$

Taking limits and considering Lemma 2, we have

$$(1.4) \quad t^* \leq \liminf_{r \rightarrow \infty} \frac{\log^{[p-2]} \nu(r)}{(\log^{[q-1]} r)^{\rho(r)}}.$$

Also, (1.3) implies $\log \mu(er) \geq \nu(r)$, which gives the reverse inequality to (1.4) and hence the lemma.

2. Main results

In this section we state and prove our results. To avoid repetitions, we shall assume throughout that $f(z) = \sum_{n=0}^{\infty} a_n z^{\lambda_n}$ is an entire function with index-pair (p, q) , (p, q) -order ρ ($b < \rho < \infty$) and generalized lower (p, q) -type t^* with respect to a proximate order $\rho(r)$ satisfying (i) and (ii). We prove

PROPOSITION 1. *For an entire function $f(z)$ and any increasing sequence $\{n_k\}$ of positive integers such that $n_k \rightarrow \infty$ as $k \rightarrow \infty$, we have*

$$(2.1) \quad \frac{t^*}{M} \geq G \liminf_{k \rightarrow \infty} \left[\frac{\phi(\log^{[p-2]} \lambda_{n_{k-1}})}{\log^{[q-1]} |a_{n_k}|^{-1/\lambda_{n_k}}} \right]^{\rho-A},$$

where t^* is the generalized lower (p, q) -type with respect to a proximate order $\rho(r)$, A and M have the same meaning as in Theorem A and

$$G \equiv G(\{n_k\}) = \begin{cases} \liminf_{k \rightarrow \infty} (\lambda_{n_{k-1}} / \lambda_{n_k})^{1/(\rho-1)} & \text{if } (p, q) = (2, 2), \\ 1 & \text{if } (p, q) \neq (2, 2). \end{cases}$$

PROOF. Let us put

$$\liminf_{k \rightarrow \infty} \left[\frac{\phi(\log^{[p-2]} \lambda_{n_{k-1}})}{\log^{[q-1]} |a_{n_k}|^{-1/\lambda_{n_k}}} \right]^{\rho-A} = B.$$

Obviously, $B \geq 0$. If $B = 0$ the proof of (2.1) is trivial. Hence, let $B > 0$. Then, for any arbitrary ε , $0 < \varepsilon < B$ and $\forall k > k_o(\varepsilon)$, we have

$$\frac{\phi(\log^{[p-2]} \lambda_{n_{k-1}})}{\log^{[q-1]} |a_{n_k}|^{-1/\lambda_{n_k}}} > (B - \varepsilon)^{1/(\rho-A)}$$

or

$$\log |a_{n_k}| > -\lambda_{n_k} \exp^{[q-2]} \left(\frac{\phi(\log^{[p-2]} \lambda_{n_{k-1}})}{(B - \varepsilon)^{1/(\rho-A)}} \right).$$

Keeping in view $k > k_o(\varepsilon)$ and $r > 0$, we use Cauchy's inequality in above and derive

$$(2.2) \quad \log M(r) > \lambda_{n_k} \log r - \lambda_{n_k} \exp^{[q-2]} \left(\frac{\phi(\log^{[p-2]} \lambda_{n_{k-1}})}{(B - \varepsilon)^{1/(\rho-A)}} \right).$$

Now, for $(p, q) = (2, 1)$, we choose a sequence $\{r_k\}$ satisfying

$$(2.3) \quad r_k = \frac{\phi(e\lambda_{n_{k-1}})}{(B - \varepsilon)^{1/\rho}}, \quad k > k_o.$$

When $r_k \leq r < r_{k+1}$, (2.2) leads to

$$\log M(r) > \lambda_{n_k} \log r_k - \lambda_{n_k} \log \left(\frac{\phi(\lambda_{n_{k-1}})}{(B - \varepsilon)^{1/\rho}} \right) \simeq \frac{\lambda_{n_k}}{\rho},$$

because $\phi(e\lambda_{n_{k-1}}) \simeq \phi(\lambda_{n_{k-1}})e^{1/\rho}$ as $k \rightarrow \infty$ by (1.2). Using the value of λ_{n_k} from (2.3) and definition of $\phi(x)$, we have

$$\begin{aligned} \log M(r) &> \left(\frac{B - \varepsilon}{e} \right)^{\rho(r)/\rho} \frac{r_{k+1}^{\rho(r)}}{\rho} \\ &> (B - \varepsilon)^{\rho(r)/\rho} \frac{r^{\rho(r)}}{e\rho} \quad \text{for } r_k \leq r < r_{k+1}. \end{aligned}$$

Dividing by $r^{\rho(r)}$ and proceeding to limits as $r \rightarrow \infty$, we get

$$(2.4) \quad \liminf_{r \rightarrow \infty} \frac{\log M(r)}{r^{\rho(r)}} = t^* \geq \frac{B}{e\rho}.$$

For the case $(p, q) = (2, 2)$, we choose the sequence $\{r_k\}$ satisfying the relation

$$(2.5) \quad \log r_k = \frac{\rho}{\rho - 1} \cdot \frac{\phi(\lambda_{n_{k-1}})}{(B - \varepsilon)^{1/(\rho-1)}}.$$

When $r_k \leq r < r_{k+1}$, we have from (2.2)

$$(2.6) \quad \begin{aligned} \log M(r) &> \lambda_{n_k} \log r_k - \lambda_{n_k} \frac{\phi(\lambda_{n_{k-1}})}{(B - \varepsilon)^{1/(\rho-1)}} \\ &= \frac{\lambda_{n_k} \phi(\lambda_{n_{k-1}})}{(\rho - 1)(B - \varepsilon)^{1/(\rho-1)}}. \end{aligned}$$

From (2.5), we have

$$\phi\left(\frac{\lambda_{n_k}}{B - \varepsilon}\right) = \log r_{k+1}^{(\rho-1)/\rho}.$$

Taking (1.1) into consideration this yields

$$(2.7) \quad \frac{\lambda_{n_k}}{B - \varepsilon} = \left(\log r_{k+1}^{(\rho-1)/\rho}\right)^{\rho(r_{k+1}^{(\rho-1)/\rho})-1}.$$

The asymptotic property of $\rho(r)$ when $(p, q) = (2, 2)$,

$$(\log r^k)^{\rho(r^k)} \simeq k^\rho (\log r)^{\rho(r)} \quad \text{as } r \rightarrow \infty,$$

coupled with (2.7), yields

$$\lambda_{n_k} \simeq (B - \varepsilon)^{\rho/(\rho-1)} \left(\frac{\rho - 1}{\rho}\right)^\rho \frac{(\log r_{k+1})^{\rho(r_{k+1})}}{\phi(\lambda_{n_k})}.$$

Using the above estimate in (2.6), when $r_k \leq r < r_{k+1}$, we find that

$$\log M(r) > (B - \varepsilon) \frac{(\rho - 1)^{\rho-1} (\log r)^{\rho(r)} \phi(\lambda_{n_{k-1}})}{\rho^\rho \phi(\lambda_{n_k})}$$

or

$$\frac{\log M(r)}{(\log r)^{\rho(r)}} > (B - \varepsilon) \frac{(\rho - 1)^{\rho-1} \phi(\lambda_{n_{k-1}})}{\rho^\rho \phi(\lambda_{n_k})}.$$

Proceeding to limits as $k \rightarrow \infty$ and hence $r \rightarrow \infty$ (in view of definition of G),

$$(2.8) \quad t^* \geq G \frac{B(\rho - 1)^{\rho-1}}{\rho^\rho}.$$

Let us now consider the case when $(p, q) \neq (2, 1)$ and $(p, q) \neq (2, 2)$. For this we define the sequence $\{r_k\}$ as

$$\log r_k = 1 + \exp^{[q-2]} \left(\frac{\phi(\log^{[p-2]} \lambda_{n_{k-1}})}{(B - \varepsilon)^{1/\rho}} \right).$$

From (2.2), we get

$$\log M(r) > \lambda_{n_k}$$

or

$$\log^{[p-1]} M(r) > (B - \varepsilon)^{\rho(r)/\rho} \left(\log^{[q-1]} \left(\frac{r_{k+1}}{e} \right) \right)^{\rho(r)}$$

or

$$\frac{\log^{[p-1]} M(r)}{(\log^{[q-1]}(r/e))^{\rho(r)}} > (B - \varepsilon)^{\rho(r)/\rho}.$$

Keeping in view, $\rho(r) \rightarrow \rho$ as $r \rightarrow \infty$ the above inequality gives $t^* \geq B$. Combining this with the inequalities (2.4) and (2.8), we get the required result.

PROPOSITION 2. For an entire function $f(z)$, if $\psi(n) = |a_n/a_{n+1}|^{1/(\lambda_{n+1}-\lambda_n)}$ forms a nondecreasing function of $n \forall n > n_o$, then

$$(2.9) \quad t^* \leq \liminf_{n \rightarrow \infty} \left[\frac{\phi(\log^{[p-2]} \lambda_{n-1})}{\log^{[q-1]} |a_n|^{-1/\lambda_n}} \right]^\rho, \quad p \geq 3,$$

and, for $p = 2$, we have

$$(2.10) \quad \frac{t^*}{M} \leq \liminf_{n \rightarrow \infty} \left[\frac{\phi(\lambda_n)}{\log^{[A]} |a_n|^{-1/\lambda_n}} \right]^{\rho-A},$$

where M is defined in Theorem A and t^* is the generalized lower (p, q) -type as in Proposition 1.

PROOF. From Lemma 2, for given $\varepsilon > 0$ and $r > r_o$, we have

$$(2.11) \quad \log^{[p-1]} \mu(r) > (t^* - \varepsilon)(\log^{[q-1]} r)^{\rho(r)}.$$

Since $\psi(n)$ is a nondecreasing function of n , the maximum term of $f(z)$ is

$$(2.12) \quad \mu(r) = |a_n| r^{\lambda_n} \quad \text{when } \psi(n-1) \leq r < \psi(n).$$

Hence (2.11) can be written as

$$(2.13) \quad \log |a_n| > \exp^{[p-2]} \left((t^* - \varepsilon)(\log^{[q-1]} r)^{\rho(r)} \right) - \lambda_n \log r.$$

For $(p, q) = (2, 1)$, (2.13) is reduced to

$$\log |a_n| > (t^* - \varepsilon) r^{\rho(r)} - \lambda_n \log r.$$

Now r is chosen such that

$$r^{\rho(r)} = \lambda_n / \rho(t^* - \varepsilon) \iff r = \phi\left(\frac{\lambda_n}{\rho(t^* - \varepsilon)}\right).$$

Hence

$$\log |a_n| > \frac{\lambda_n}{\rho} - \lambda_n \log \phi\left(\frac{\lambda_n}{\rho(t^* - \varepsilon)}\right)$$

or

$$\begin{aligned} \log |a_n|^{-1/\lambda_n} &< \log \left(\frac{\phi(\lambda_n / \rho(t^* - \varepsilon))}{e^{1/\rho}} \right) \\ &\simeq \log \phi(\lambda_n / e\rho(t^* - \varepsilon)) \end{aligned}$$

or for all large values of n ,

$$|a_n|^{-1/\lambda_n} < \phi\left(\frac{\lambda_n}{e\rho(t^* - \varepsilon)}\right) \simeq \frac{\phi(\lambda_n)}{(e\rho(t^* - \varepsilon))^{1/\rho}}.$$

On proceeding to limits, we have

$$(2.14) \quad e\rho t^* \leq \liminf_{n \rightarrow \infty} \left(\frac{\phi(\lambda_n)}{|a_n|^{-1/\lambda_n}} \right)^\rho.$$

In case of $(p, q) = (2, 2)$, r is chosen so that

$$(\log r)^{\rho(r)-1} = \frac{\lambda_n}{\rho(t^* - \varepsilon)} \iff \log r = \phi\left(\frac{\lambda_n}{\rho(t^* - \varepsilon)}\right).$$

Using this in (2.13) when $(p, q) = (2, 2)$, we get

$$\begin{aligned} \log |a_n| &> (t^* - \varepsilon) \log r^{\rho(r)} - \lambda_n \log r \\ &= \left(\frac{\lambda_n}{\rho} - \lambda_n \right) \log r = - \left(\frac{\rho - 1}{\rho} \right) \lambda_n \log r. \end{aligned}$$

Thus,

$$\begin{aligned} \log |a_n|^{-1/\lambda_n} &< \left(\frac{\rho - 1}{\rho} \right) \phi\left(\frac{\lambda_n}{\rho(t^* - \varepsilon)}\right) \\ &\simeq \left(\frac{\rho - 1}{\rho} \right) \frac{\phi(\lambda_n)}{(\rho(t^* - \varepsilon))^{1/(\rho-1)}}. \end{aligned}$$

Proceeding to limits as $n \rightarrow \infty$, we have

$$(2.15) \quad \rho t^* \leq \left(\frac{\rho - 1}{\rho} \right)^{\rho-1} \liminf_{n \rightarrow \infty} \left(\frac{\phi(\lambda_n)}{\log |a_n|^{-1/\lambda_n}} \right)^{\rho-1}.$$

The inequalities (2.14) and (2.15) combine to give (2.10).

For the case $p \geq 3$, let $|a_{n_1}|r^{\lambda_{n_1}}$ and $|a_{n_2}|r^{\lambda_{n_2}}$ be two consecutive maximum terms of $f(z)$. Then $n_1 \leq n_2 - 1$ and by (2.12), $|a_{n_1}|r^{\lambda_{n_1}}$ is maximum term of $f(z)$ for $\psi(n_1 - 1) \leq r < \psi(n_1)$. Further, in view of Lemma 3, for these values of r and for every $c > 0$, we have

$$\log^{[p-2]} \nu(r) = \log^{[p-2]} \lambda_{n_1} > (t^* - \varepsilon) \left(\log^{[q-1]}(\psi(n_1) - c) \right)^{\rho(r)}.$$

Thus, for $n_1 < n \leq n_2$, $\psi(n_1) = \psi(n_1 + 1) = \dots = \psi(n - 1)$ and

$$(2.16) \quad \log^{[p-2]} \lambda_{n-1} > (t^* - \varepsilon) \left(\log^{[q-1]}(\psi(n - 1) - c) \right)^{\rho(r)}.$$

As ψ is increasing it can be proved that

$$(2.17) \quad \log |a_n|^{-1/\lambda_n} < (1 + o(1)) \log \psi(n - 1).$$

Operating the function ϕ on (2.16), it follows that

$$\frac{\phi(\log^{[p-2]} \lambda_{n-1})}{\log^{[q-1]} \psi(n - 1)} > (t^* - \varepsilon)^{1/\rho}.$$

The above inequality in conjunction with (2.17) gives (2.9).

Now, Propositions 1 and 2 lead to the following result.

THEOREM 1. *Let $f(z)$ be an entire function with index-pair (p, q) , $p \geq q \geq 1$ and generalized lower (p, q) -type t^* with respect to a proximate order $\rho(r)$ such that $\psi(n) = |a_n/a_{n+1}|^{1/(\lambda_{n+1} - \lambda_n)}$ forms a nondecreasing function of n for $n > n_0$. Then*

$$(2.18) \quad t^* = \liminf_{n \rightarrow \infty} \left[\frac{\phi(\log^{[p-2]} \lambda_{n-1})}{\log^{[q-1]} |a_n|^{-1/\lambda_n}} \right]^\rho, \quad p \geq 3.$$

The relation (2.18) holds for $p = 2$ also, if $\lambda_{n-1} \simeq \lambda_n$ as $n \rightarrow \infty$, we have

$$(2.19) \quad \frac{t^*}{M} = \liminf_{n \rightarrow \infty} \left[\frac{\phi(\lambda_{n-1})}{\log^{[A]} |a_n|^{-1/\lambda_n}} \right]^{\rho-A},$$

where M has the same meaning as in Propositions 1 and 2.

REMARK. Let us define $\rho(r) = \rho \forall r \in [\exp^{[q-1]} 1, \infty)$ and $\phi(x) = x^{1/(\rho-A)}$. Then, the following corollary is an answer to the queries raised in Theorem A.

COROLLARY. Let $f(z)$ be an entire function with index-pair (p, q) , $p \geq q \geq 1$ and lower (p, q) -type t such that $\psi(n) = |a_n/a_{n+1}|^{1/(\lambda_{n+1}-\lambda_n)}$ forms a nondecreasing function of n for $n > n_0$. Then

$$\frac{t}{M} = \liminf_{n \rightarrow \infty} \frac{\log^{[p-2]} \lambda_{n-1}}{(\log^{[q-1]} |a_n|^{-1/\lambda_n})^\rho}, \quad p \geq 3,$$

and this holds for $p = 2$ also, if $\lambda_{n-1} \simeq \lambda_n$ as $n \rightarrow \infty$.

Finally, we prove the result for which the condition on $\psi(n)$ in Theorem 1 is not required.

THEOREM 2. For an entire function $f(z)$ with index-pair (p, q) , $p \geq q \geq 1$ and generalized (p, q) -type t^* in reference to a proximate order $\rho(r)$, we have

$$(2.20) \quad t^* = \max_{\{m_k\}} \left\{ \liminf_{k \rightarrow \infty} \left[\frac{\phi(\log^{[p-2]} \lambda_{m_{k-1}})}{\log^{[q-1]} |a_{m_k}|^{-1/\lambda_{m_k}}} \right]^\rho \right\}, \quad \text{for } p \geq 3.$$

Further, if $\{\lambda_{n_k}\}$ be the sequence of principal indices such that $\lambda_{n_{k-1}} \simeq \lambda_{n_k}$ as $k \rightarrow \infty$, then for $p = 2$,

$$(2.21) \quad \frac{t^*}{M} = \max_{\{m_k\}} \left\{ \liminf_{k \rightarrow \infty} \left[\frac{\phi(\lambda_{m_{k-1}})}{\log^{[A]} |a_{m_k}|^{-1/\lambda_{m_k}}} \right]^{\rho-A} \right\}.$$

The maximum is taken over all increasing sequences of positive integers $\{m_k\}$.

PROOF. Let $\{\lambda_{n_k}\}$ be the sequence of principal indices of an entire function $f(z)$. Consider the function

$$g(z) = \sum_{k=0}^{\infty} a_{n_k} z^{\lambda_{n_k}}.$$

Since the Taylor expansion of $g(z)$ contains coefficients and exponents as the subsequences of $\{a_n\}_{n=0}^{\infty}$ and $\{\lambda_n\}_{n=0}^{\infty}$, respectively, it follows that $g(z)$ is an entire function and moreover, as the exponents are the sequence of principal indices of $f(z)$ the function $g(z)$ has the same maximum term and rank as that of $f(z)$ for every value of r . Hence $g(z)$ is of index-pair (p, q) , (p, q) -order ρ and generalized lower (p, q) -type t^* (in view of Lemma 2). It can be easily seen that

$\psi(n_k) = |a_{n_k}/a_{n_{k+1}}|^{1/(\lambda_{n_{k+1}} - \lambda_{n_k})}$ forms a nondecreasing function of k . Hence from (2.18), we have, for $p \geq 3$,

$$t^* = \liminf_{k \rightarrow \infty} \left[\frac{\phi(\log^{[p-2]} \lambda_{n_{k-1}})}{\log^{[q-1]} |a_{n_k}|^{-1/\lambda_{n_k}}} \right]^\rho.$$

Since (2.1) holds for every increasing sequence $\{m_k\}$, it follows that

$$t^* \geq \max_{\{m_k\}} \left\{ \liminf_{k \rightarrow \infty} \left(\frac{\phi(\log^{[p-2]} \lambda_{m_{k-1}})}{\log^{[q-1]} |a_{m_k}|^{-1/\lambda_{m_k}}} \right)^\rho \right\}.$$

Comparing these two results we get (2.20).

Similarly, for $p = 2$, taking (2.19) and Proposition 1 into account, (2.21) follows.

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